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B. Eynard

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Lectures on compact Riemann surfaces.

B. Eynard^{1 2}

Paris–Saclay’s IPHT doctoral school Lecture given in winter 2018.



This is an introduction to the geometry of compact Riemann surfaces. We largely follow the books [8, 9, 10]. 1) Defining Riemann surfaces with atlases of charts, and as locus of solutions of algebraic equations. 2) Space of meromorphic functions and forms, we classify them with the Newton polygon. 3) Abel map, the Jacobian and Theta functions. 4) The Riemann–Roch theorem that computes the dimension of spaces of functions and forms with given orders of poles and zeros. 5) The moduli space of Riemann surfaces, with its combinatorial representation as Strebel graphs, and also with the uniformization theorem that maps Riemann surfaces to hyperbolic surfaces. 6) An application of Riemann surfaces to integrable systems, more precisely finding sections of an eigenvector bundle over a Riemann surface, which is known as the ”algebraic reconstruction” method in integrable systems, and we mention how it is related to Baker-Akhiezer functions and Tau functions.

¹Institut de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette Cedex, France.

²CRM, Centre de recherches mathématiques de Montréal, Université de Montréal, QC, Canada.

Notations

- $D(x, r)$ is the open disc of center x and radius r in \mathbb{C} , or the ball of center x and radius r in \mathbb{R}^n .
- $\mathcal{C}(x, r) = \partial D(x, r)$ is the circle (resp. the sphere) of center x and radius r in \mathbb{C} (resp. in \mathbb{R}^n).
- \mathcal{C}_x is a "small" circle around x in \mathbb{C} , or a small circle in a chart around x on a surface, small meaning that it is a circle of radius sufficiently small to avoid all other special points.
- $T_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is the 2-torus of modulus τ , obtained by identifying $z \equiv z + 1 \equiv z + \tau$.
- $\mathbb{C}P^1 = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- \mathbb{C}_+ is the upper complex half-plane $= \{z \mid \Im z > 0\}$, it is identified with the Hyperbolic plane, with the metric $\frac{|dz|^2}{(\Im z)^2}$, of constant curvature -1 , and whose geodesics are the circles or lines orthogonal to the real axis.
- $\mathfrak{M}^1(\Sigma)$ the \mathbb{C} vector space of meromorphic forms on Σ ,
- $\mathcal{O}^1(\Sigma)$ the \mathbb{C} vector space of holomorphic forms on Σ .
- $H_1(\Sigma, \mathbb{Z})$ (resp. $H_1(\Sigma, \mathbb{C})$) the \mathbb{Z} -module (resp. \mathbb{C} -vector space) generated by homology cycles (equivalence classes of closed Jordan arcs, $\gamma_1 \equiv \gamma_2$ if there exists a 2-cell A whose boundary is $\partial A = \gamma_1 - \gamma_2$, with addition of Jordan arcs by concatenation) on Σ .
- $\pi_1(\Sigma)$ is the fundamental group of a surface (the set of homotopy classes of closed curves on a Riemann surface with addition by concatenation).

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Chapter 1

Riemann surfaces

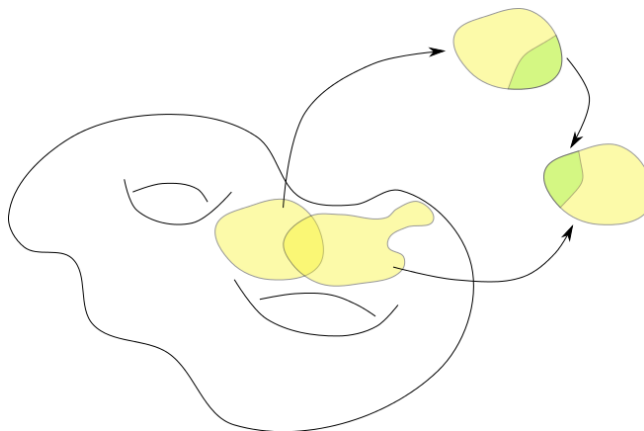
1 Manifolds, atlases, charts, surfaces

Definition 1.1 (Topological Manifold) A **manifold** M is a second countable (the topology can be generated by a countable basis of open sets) topological separated space (distinct points have disjoint neighborhoods, also called **Hausdorff space**), locally **Euclidian** (each point has a neighborhood homeomorphic to an open subset of \mathbb{R}^n for some integer n).

Definition 1.2 (Charts and atlas) A **chart** on M is a pair (V, ϕ_V) , of a neighborhood V , together with an homeomorphism $\phi_V : V \rightarrow U \subset \mathbb{R}^n$, called the **coordinate** or the **local coordinate**. For each intersecting pair $V \cap V' \neq \emptyset$, the **transition function** is the map: $\psi_{U \rightarrow U'} : \phi_V(V \cap V') \rightarrow \phi_{V'}(V \cap V')$, $x \mapsto \phi_{V'} \circ \phi_V^{-1}(x)$, it is a homeomorphism of Euclidian subspaces, with inverse

$$\psi_{U' \rightarrow U}^{-1} = \psi_{U \rightarrow U'} \tag{1-1}$$

A countable set of charts that cover the manifold M is called an **atlas** of M . Two atlases are said equivalent iff their union is an atlas.



Definition 1.3 (Various types of manifold) M is a topological (resp. smooth, resp. k -differentiable, resp. complex) **manifold** if it has an atlas for which all transition maps are continuous (resp. C^∞ , resp. C^k , resp. holomorphic).

An equivalence class of atlases with transition functions in the given class (smooth, resp. k -differentiable, resp. complex) is called a smooth, resp. k -differentiable, resp. complex **structure** on M .

The dimension n must be constant on each connected part of M . We shall most often restrict ourselves to connected manifolds, thus having fixed dimension.

- A surface is a manifold of dimension $n = 2$.
- A surface is a **Riemann surface** if, identifying $\mathbb{R}^2 = \mathbb{C}$, each transition map is analytic with analytic inverse. An equivalence class of analytic atlases on M is called a **complex structure** on M .
- A differentiable manifold is **orientable** if, all transition maps $\psi : (x_1, \dots, x_n) \mapsto (\psi_1(x_1, \dots, x_n), \dots, \psi_n(x_1, \dots, x_n))$, have positive Jacobian $\det(\partial\psi_i/\partial x_j) > 0$. Thanks to Cauchy-Riemann equations, a Riemann surface is always orientable.
- A manifold is **compact** if it has an atlas made of a finite number of bounded (by a ball in \mathbb{R}^n) charts. Every sequence of points $\{p_n\}_{n \in \mathbb{N}}$ on M , admits at least one adherence value, or also every Cauchy sequence on M is convergent.

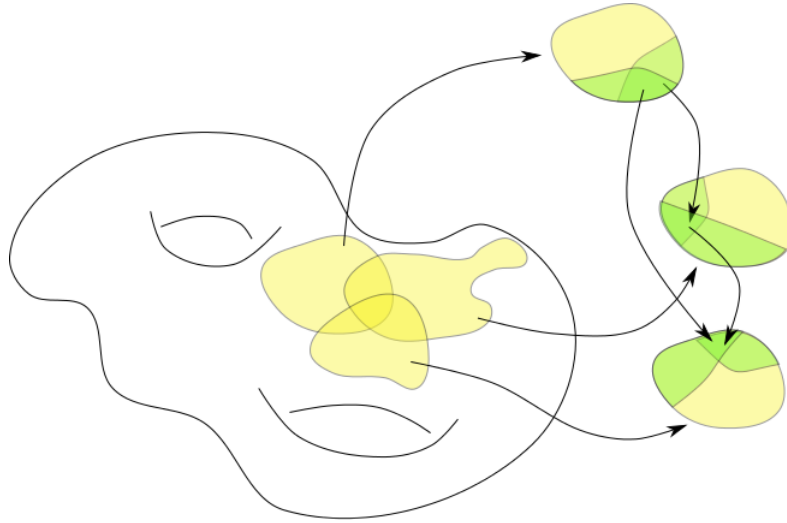
Defining a manifold from an atlas

Definition 1.4 An abstract atlas is the data of

- a countable set I ,
- a collection $\{U_i\}_{i \in I}$ of open subsets of \mathbb{R}^n ,
- a collection $\{U_{i,j}\}_{i,j \in I \times I}$ of (possibly empty) open subsets of \mathbb{R}^n such that $U_{i,j} \subset U_i$, and such that $U_{i,j}$ is homeomorphic to $U_{j,i}$, i.e. –if not empty– there exists an homeomorphism $\psi_{i,j} : U_{i,j} \rightarrow U_{j,i}$ and an homeomorphism $\psi_{j,i} : U_{j,i} \rightarrow U_{i,j}$ such that $\psi_{i,j} \circ \psi_{j,i} = Id$. Moreover we require that $U_{i,i} = U_i$ and $\psi_{i,i} = Id$. Moreover we require that $U_{j,i} \cap U_{j,k} = \psi_{i,j}(U_{i,j} \cap U_{i,k})$ and that $\psi_{j,k} = \psi_{i,k} \circ \psi_{j,i}$ on $U_{j,i} \cap U_{j,k}$ (if not empty):

$$\begin{cases} \psi_{i,j} \circ \psi_{j,i} = Id \\ \psi_{j,k} = \psi_{i,k} \circ \psi_{j,i} \end{cases} \quad (1-2)$$

Depending on the type of manifold (topological, smooth, k -differentiable, complex), we require all homeomorphisms to be in the corresponding class.



From an abstract atlas we can define a manifold as a subset of $\mathbb{R}^n \times I$ quotiented by an equivalence relation:

Proposition 1.1

$$M = \frac{\{(z, i) \in \mathbb{R}^n \times I \mid z \in U_i\}}{(z, i) \equiv (z', j) \text{ iff } z \in U_{i,j}, z' \in U_{j,i}, \psi_{i,j}(z) = z'} \quad (1-3)$$

with the topology induced by that of \mathbb{R}^n , is a manifold (resp. smooth, resp. complex, depending on the class of homeomorphisms $\psi_{i,j}$).

proof: It is easy to see that this satisfies the definition of a manifold. Notice that in order for M to be a well defined quotient, we need to prove that \equiv is a well defined equivalence relation, and this is realized thanks to relations (1-2). Then we need to show that the topology is well defined on M , this is easy and we leave it to the reader.

□

All manifolds can be obtained in this way.

1.1 Classification of surfaces

We shall admit the following classical theorem:

Theorem 1.1 (Classification of topological compact surfaces) *Topological compact connected surfaces are classified by:*

- *the orientability: orientable or non-orientable*
- *the Euler characteristics*

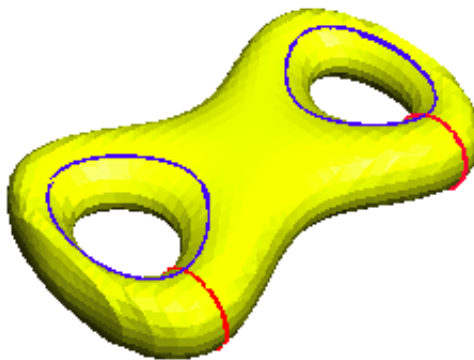
This means that 2 surfaces having the same orientability and Euler characteristic are isomorphic.

- An orientable surface Σ has an even Euler characteristic of the form

$$\chi = 2 - 2\mathbf{g} \quad (1-4)$$

where $\mathbf{g} \geq 0$ is called the genus, and is isomorphic to a surface with \mathbf{g} holes. Its fundamental group (non-contractible cycles) is generated by $2\mathbf{g}$ cycles:

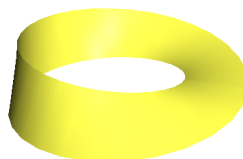
$$\pi_1(\Sigma) \sim \mathbb{Z}^{2\mathbf{g}}. \quad (1-5)$$



- A non-orientable surface Σ has an Euler characteristic

$$\chi = 2 - k \quad (1-6)$$

with $k \geq 1$, it is isomorphic to a sphere from which we have removed k disjoint discs, and glued k Möbius strips at the k boundaries (this is called k crosscaps).



If $\chi = 1$, it is isomorphic to the real projective plane $\mathbb{R}P^2$.

If $\chi = 0$, it is isomorphic to the Klein bottle.

2 Examples of Riemann surfaces

2.1 The Riemann sphere

- Consider the Euclidian unit sphere in \mathbb{R}^3 , the set $\{(X, Y, Z) \mid X^2 + Y^2 + Z^2 = 1\}$. Define the 2 charts:

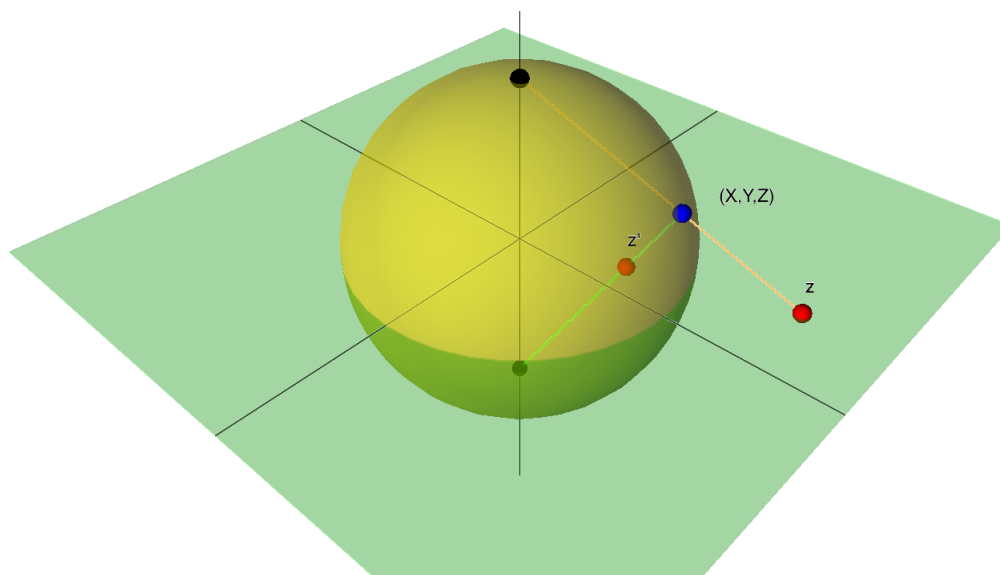
$$V_1 = \{(X, Y, Z) \mid X^2 + Y^2 + Z^2 = 1, Z > -\frac{3}{5}\}, \quad \phi_1 : (X, Y, Z) \mapsto \frac{X + iY}{1 + Z}$$

$$V_2 = \{(X, Y, Z) \mid X^2 + Y^2 + Z^2 = 1, Z < \frac{3}{5}\}, \quad \phi_2 : (X, Y, Z) \mapsto \frac{X - iY}{1 - Z}. \quad (2-1)$$

U_1 (resp. U_2), the image of ϕ_1 (resp. ϕ_2), is the open disc $D(0, 2) \subset \mathbb{C}$. The image by ϕ_1 (resp. ϕ_2) of $V_1 \cap V_2$ is the annulus $\frac{1}{2} < |z| < 2$ in U_1 (resp. U_2). On this annulus, the transition map

$$\phi_2 \circ \phi_1^{-1} = \psi : z \mapsto 1/z \quad (2-2)$$

is analytic, bijective, and its inverse is analytic. This defines the **Riemann sphere**, which is a compact (the 2 charts are bounded discs $D(0, 2)$), connected (obvious) and simply connected Riemann surface (easy). The map ϕ_1 (resp. $\overline{\phi_2}$) is called the *stereographic projection* from the south (resp. north) pole of the sphere to the Euclidian plane $Z = 0$ in \mathbb{R}^3 , identified with \mathbb{C} .



- Another definition of the Riemann sphere is the **complex projective plane** $\mathbb{C}P^1$:

$$\mathbb{C}P^1 = \frac{\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid (z_1, z_2) \neq (0, 0)\}}{(z_1, z_2) \equiv (\lambda z_1, \lambda z_2), \forall \lambda \in \mathbb{C}^*}. \quad (2-3)$$

It has also an atlas of 2 charts, $V_1 = \{[(z_1, z_2)] \mid z_2 \neq 0\}$, $\phi_1 : [(z_1, z_2)] \mapsto z_1/z_2$ and $V_2 = \{[(z_1, z_2)] \mid z_1 \neq 0\}$, $\phi_2 : [(z_1, z_2)] \mapsto z_2/z_1$, with transition map $z \mapsto 1/z$ (everything is well defined on the quotient by \equiv).

$\mathbb{C}P^1$ is analytically isomorphic to the Riemann sphere previously defined.

- Another definition of the Riemann sphere is from an abstract atlas of 2 charts $U_1 = D(0, R_1) \subset \mathbb{C}$ and $U_2 = D(0, R_2) \subset \mathbb{C}$ whose radius satisfy $R_1 R_2 > 1$. The 2 discs are glued by the analytic transition map $\psi : z \mapsto 1/z$ from the annulus $\frac{1}{R_2} < |z| < R_1$ in U_1 to the annulus $\frac{1}{R_1} < |z| < R_2$ in U_2 .

In other words, consider the following subset of $\mathbb{C} \times \{1, 2\}$

$$\frac{\{(z, i) \in \mathbb{C} \times \{1, 2\} \mid z \in U_i\}}{\equiv} \quad (2-4)$$

quotiented by the equivalence relation

$$(z, i) \equiv (\tilde{z}, j) \quad \text{iff} \quad i = j \text{ and } z = \tilde{z} \quad \text{or} \quad i + j = 3 \text{ and } z\tilde{z} = 1. \quad (2-5)$$

This Riemann surface is analytically isomorphic to the Riemann sphere previously defined.

- Notice that one can choose R_1 very large, and R_2 very small, and even consider a projective limit $R_1 \rightarrow \infty$ and $R_2 \rightarrow 0$, in other words glue the whole $U_1 = \mathbb{C}$ to the single point $U_2 = \{0\}$. Notice that the point $z' = 0$ in U_2 should correspond to the point $z = 1/z' = \infty$ in $U_1 = \mathbb{C}$. In this projective limit, by adding a single point to \mathbb{C} , we turn it into a compact Riemann surface $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The topology of $\overline{\mathbb{C}}$ is generated by the open sets of \mathbb{C} , as well as all the sets $V_R = \{\infty\} \cup \{z \in \mathbb{C} \mid |z| > R\}$ for all $R \geq 0$. These open sets form a basis of neighborhoods of ∞ . With this topology, $\overline{\mathbb{C}}$ is compact.

This justifies that the Riemann sphere is called a compactification of \mathbb{C} :

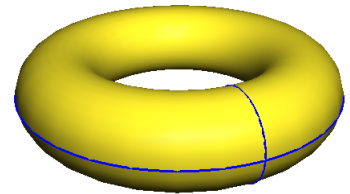
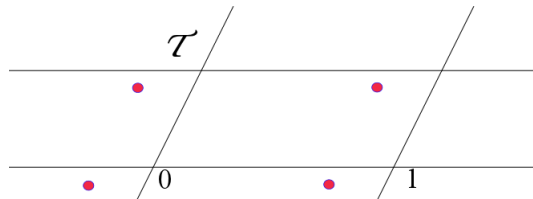
$$\mathbb{C}P^1 = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}. \quad (2-6)$$

2.2 The torus

Consider $\tau \in \mathbb{C}$ with $\Im\tau > 0$. Let

$$T_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \quad (2-7)$$

in other words, we identify $z \equiv z + 1 \equiv z + \tau$.



Each point has a neighborhood homeomorphic to a disc $\subset \mathbb{C}$. Transition maps are of the form $z \mapsto z + a + \tau b$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, they are translations, they are analytic, invertible with analytic inverse.

The torus is a Riemann surface, compact, connected, but not simply connected.

3 Compact Riemann surface from an algebraic equation

The idea is to show that the locus of zeros (in $\mathbb{C} \times \mathbb{C}$) of a polynomial equation $P(x, y) = 0$ is a Riemann surface. This is morally true for all generic polynomials, but there are some subtleties. Let us start by an example where it works directly, and then see why this assertion has to be slightly adapted.

3.1 Example

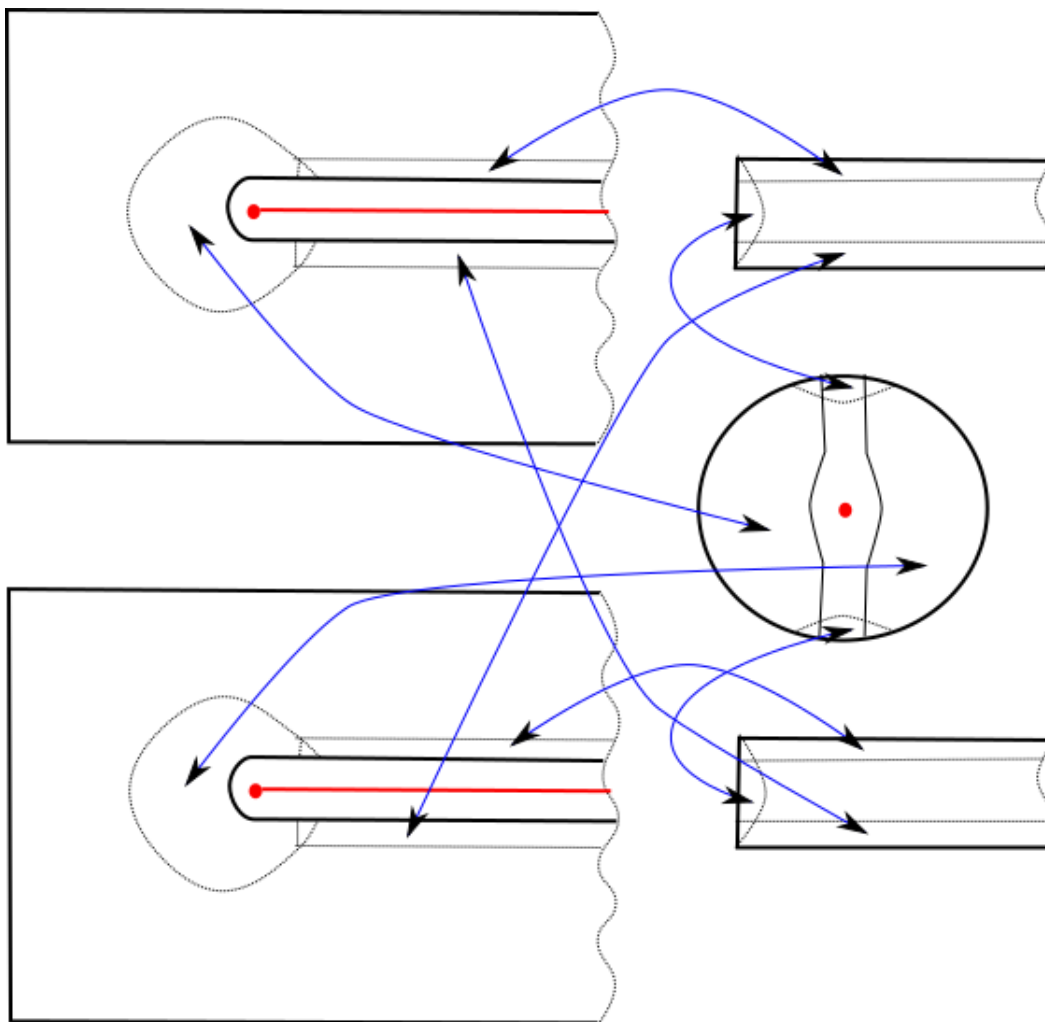
We start with the polynomial

$$P(x, y) = y^2 - x^2 + 4. \tag{3-1}$$

Consider

$$\tilde{\Sigma} = \{(x, y) \mid y^2 - x^2 + 4 = 0\} \subset \mathbb{C} \times \mathbb{C} \tag{3-2}$$

which is a smooth submanifold of $\mathbb{C} \times \mathbb{C}$.



We can cover it with an atlas of 6 charts as follows (we choose the square root such that $\sqrt{\mathbb{R}_+} = \mathbb{R}_+$, and with the cut on \mathbb{R}_-):

$$\begin{aligned}
V_+ &= \{(x, +\sqrt{x^2 - 4}) \mid x \in U_+ = \mathbb{C} \setminus [-2, 2]\} & , & \quad \phi_+ : (x, y) \mapsto x \\
V_- &= \{(x, -\sqrt{x^2 - 4}) \mid x \in U_- = \mathbb{C} \setminus [-2, 2]\} & , & \quad \phi_- : (x, y) \mapsto x \\
V_1 &= \{(2 + z^2, z\sqrt{4 + z^2}) \mid z \in U_1 = D(0, 1)\} & , & \quad \phi_1 : (x, y) \mapsto \sqrt{x - 2} \\
V_{-1} &= \{(-2 + z^2, iz\sqrt{4 - z^2}) \mid z \in U_{-1} = D(0, 1)\} & , & \quad \phi_{-1} : (x, y) \mapsto \sqrt{x + 2} \\
V_{+-} &= \{(x, i\sqrt{4 - x^2}) \mid x \in U_{+-} = \left[\frac{-3}{2}, \frac{3}{2}\right] \times \left[\frac{-1}{2}, \frac{1}{2}\right]\} & , & \quad \phi_- : (x, y) \mapsto x \\
V_{-+} &= \{(x, -i\sqrt{4 - x^2}) \mid x \in U_{-+} = \left[\frac{-3}{2}, \frac{3}{2}\right] \times \left[\frac{-1}{2}, \frac{1}{2}\right]\} & , & \quad \phi_- : (x, y) \mapsto x
\end{aligned} \tag{3-3}$$

We have $V_+ \cap V_- = \emptyset$ and $V_1 \cap V_{-1} = \emptyset$. The transition maps on $V_{+-} \cap V_{\pm}$ (resp. $V_{-+} \cap V_{\pm}$) are $x \mapsto x$. The transition maps on $V_{\pm 1} \cap V_{\pm}$ are:

$$z \mapsto \pm 2 + z^2 \tag{3-4}$$

with inverse

$$x \mapsto \pm\sqrt{x \mp 2}. \tag{3-5}$$

All points of $\tilde{\Sigma}$ are covered by a chart, this defines a Riemann surface, it is connected, but it is not simply connected (it has the topology of a cylinder). However, it is not compact, because two of the charts (V_+ and V_-) are not bounded in \mathbb{C} .

We shall define a compact Riemann surface Σ by adding two points, named $+(\infty, \infty)$ and $-(\infty, \infty)$ to $\tilde{\Sigma}$, with two charts as their neighborhoods:

$$V_{\pm\infty} = \{(x, \pm\sqrt{x^2 - 4}) \mid |x| > 4\} \cup \{\pm(\infty, \infty)\} \quad , \quad \phi_{\pm\infty} : \begin{array}{l} (x, y) \mapsto 1/x \\ \pm(\infty, \infty) \mapsto 0 \end{array} . \tag{3-6}$$

Their images $U_{\pm\infty} = D(0, \frac{1}{4})$ are discs in \mathbb{C} .

$V_{+\infty}$ (resp. $V_{-\infty}$) intersects V_+ (resp. V_-), and for both, the transition map is

$$x \mapsto 1/x. \tag{3-7}$$

The Riemann surface Σ is then compact, connected and simply connected. Therefore topologically it is a sphere. Indeed there is a holomorphic bijection (with holomorphic inverse) with the Riemann sphere:

$$\begin{aligned}
\mathbb{C}P^1 &\rightarrow \Sigma \\
z &\mapsto (z + 1/z, z - 1/z).
\end{aligned} \tag{3-8}$$

In fact, there is the theorem (that we admit here, proved below as theorem 3.5):

Theorem 3.5 (Genus zero = Riemann sphere) *Every simply connected (i.e. genus zero) compact Riemann surface is isomorphic to the Riemann sphere.*

3.2 General case

For every polynomial $P(x, y) \in \mathbb{C}[x, y]$, let $\tilde{\Sigma}$ be the locus of its zeros in $\mathbb{C} \times \mathbb{C}$:

$$\tilde{\Sigma} = \{(x, y) \mid P(x, y) = 0\} \subset \mathbb{C} \times \mathbb{C}. \quad (3-9)$$

The idea is that we need to map every neighborhood in $\tilde{\Sigma}$ to a neighborhood in \mathbb{C} , and for most of the points of $\tilde{\Sigma}$, we can use x as a coordinate, provided that x is locally invertible. This works almost everywhere on $\tilde{\Sigma}$ except at the point where x^{-1} is not locally analytic. Near those special points we can't use x as a coordinate, and we shall describe how to proceed.

- First let us consider the (finite) set of singular points

$$\tilde{\Sigma}_{\text{sing}} = \{(x, y) \mid P(x, y) = 0 \text{ and } P'_y(x, y) = 0\}, \quad (3-10)$$

and the set of their x coordinates, to which we add the point ∞ :

$$x_{\text{sing}} = x(\tilde{\Sigma}_{\text{sing}}) \cup \{\infty\} \subset \mathbb{C}P^1. \quad (3-11)$$

Remark that $x_{\text{sing}} - \{\infty\}$ is the set of solutions of a polynomial equation

$$x \in x_{\text{sing}} - \{\infty\} \quad \Leftrightarrow \quad 0 = \Delta(x) = \text{Discriminant}(P(x, \cdot)) = \text{Resultant}(P(x, \cdot), P'_y(x, \cdot)), \quad (3-12)$$

which implies that it is a finite set of isolated points.

- Then choose a connected simply connected set of non-intersecting Jordan arcs in $\mathbb{C}P^1$, linking the points of x_{sing} , i.e. a simply connected graph $\Gamma \subset \mathbb{C}P^1$ (a tree) whose vertices are the points of x_{sing} . Define $\Sigma_0 = \tilde{\Sigma} \setminus x^{-1}(\Gamma)$ by removing the preimage of Γ . Let $d = \deg_y P$, we define d charts as d identical copies of $\mathbb{C} \setminus \Gamma$ as

$$U_1 = U_2 = \cdots = U_d = \{x \mid (x, y) \in \Sigma_0\} = \mathbb{C} \setminus \Gamma. \quad (3-13)$$

Each U_i is open, connected and simply connected. The U_i s play the same role as U_+ and U_- in the previous example with $d = 2$. Let x_0 be a generic interior point in $\mathbb{C} \setminus \Gamma$. The equation $P(x_0, y) = 0$ has d distinct solutions, let us label them (arbitrarily) $Y_1(x_0), \dots, Y_d(x_0)$. For each $i = 1, \dots, d$, Y_i can be unambiguously analytically extended to the whole U_i (because it is simply connected), and thus there is an analytic map on U_i : $x \mapsto Y_i(x)$, $i = 1, \dots, d$. We then define the charts $V_i \subset \tilde{\Sigma}$ by

$$V_i = \{(x, Y_i(x)) \mid x \in U_i\} \quad , \quad \phi_i : \begin{array}{ccc} V_i & \rightarrow & U_i \\ (x, Y_i(x)) & \mapsto & x \end{array} . \quad (3-14)$$

This generalizes the two charts V_{\pm} in the previous example. Then we need to define charts that cover the neighborhood of singular points, and the neighborhood of edges of the graph Γ .

- Consider the most generic sort of singular point (a, b) , such that $P'_y(a, b) = 0$, but $P'_x(a, b) \neq 0$ and $P''_{yy}(a, b) \neq 0$. (a, b) is called a regular **ramification point** and a is called a **branch point**.

In that case, there are 2 charts, let us say V_i, V_j with $i \neq j$, that have (a, b) at their boundary. Due to our most-generic-assumption, the map $V_i \rightarrow \mathbb{C}$, $(x, y) \mapsto \sqrt{x - a}$ (resp. $V_j \rightarrow \mathbb{C}$, $(x, y) \mapsto -\sqrt{x - a}$) is analytic in a neighborhood of (a, b) in V_i (resp. V_j). We thus define a new chart for each singular point (a, b) , as a neighborhood of this point. It intersects V_i (resp. V_j) with transition map $z \mapsto (a + z^2, Y_i(a + z^2))$ (resp. $z \mapsto (a + z^2, Y_j(a + z^2))$). In other words we choose $\sqrt{x - a}$ as a local coordinate near (a, b) .

Also, this defines the "**deck transformation**" at the singular point: the permutation (here a transposition) $\sigma_a = (i, j)$.

- Consider an open edge e of Γ (open means excluding the vertices), its boundary consists of the x -images a, a' of 2 singular points, each with a permutation $\sigma_a, \sigma_{a'}$. In the generic case the 2 transpositions have to coincide, and we associate this transposition $\sigma_e = \sigma_a = \sigma_{a'}$ to the edge e .

Now consider a tubular neighborhood U_e of e in $\mathbb{C} \setminus x_{\text{sing}}$, and that contains no other edges. $U_e \cap U_i$ is disconnected and consists of 2 pieces $U_{e,i,\pm} \subset U_i$. By pulling back to Σ by x^{-1} , we get $V_{e,i,\pm} \subset V_i$, and we define

$$V_{e,i} = V_{e,i,+} \cup V_{e,\sigma_e(i),-} \cup \{(x, Y_i(x)) \mid x \in e\}, \quad \phi_{e,i} : (x, y) \mapsto x. \quad (3-15)$$

the chart $V_{e,i}$ is an open connected domain of $\tilde{\Sigma}$ and $\phi_{e,i}$ is analytic. The transition maps $x \mapsto x$ are analytic with analytic inverse.

This is the generalization of the charts V_{+-} and V_{-+} in the example above. It consists of gluing neighborhoods of the 2 sides of an edge, to neighborhoods obtained by the permutation σ_e .

- For generic polynomials P , all singular points are of that type, and we get a Riemann surface, non-compact (this was the case for the example $y^2 - x^2 + 4 = 0$).

- We can make it compact by adding new points at ∞ , as we did for the example above, but many subtleties can occur at ∞ .

This shows that algebraic curves are generically Riemann surfaces, that can be compactified.

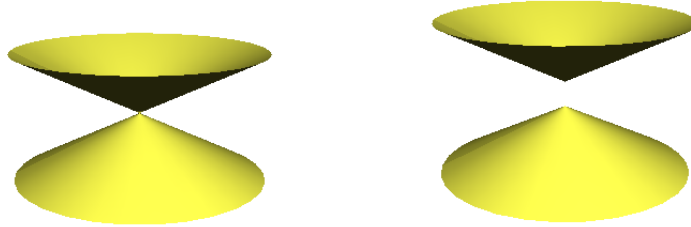
In fact we shall see below in section 3.4 that the converse is almost true: every compact Riemann surface can be algebraically immersed into $\mathbb{C}P^2$ (we have to replace \mathbb{C}^2 by $\mathbb{C}P^2$ to properly compactify at ∞). Generically this immersion is in fact an embedding.

3.3 Non-generic case: desingularization

Sometimes the singular points are not generic, this can also be the case near ∞ . Like we did in the example, where we added new points to Σ to make it compact in neighborhoods of ∞ , we can desingularize all singular points by adding new points, and defining a new surface $\Sigma = \tilde{\Sigma} \cup \{\text{new points}\}$, which is a smooth compact Riemann surface.

- **Nodal points.** A slightly less (than ramification points) generic type of singular points (a, b) , is where both $P'_y(a, b)$ and $P'_x(a, b)$ vanish, to the lowest possible order, i.e. we assume that the second derivative Hessian matrix is invertible $\det \begin{pmatrix} P''_{xx}(a, b) & P''_{xy}(a, b) \\ P''_{yx}(a, b) & P''_{yy}(a, b) \end{pmatrix} \neq 0$. The intersection of $\tilde{\Sigma}$ with a small ball $D((a, b), r) \subset \mathbb{C} \times \mathbb{C}$, is not homeomorphic to a Euclidian disc, instead it is homeomorphic to a union of 2 discs, which have a common point (a, b) . This implies that $\tilde{\Sigma}$ is in fact not a manifold, it has points whose neighborhoods are not homeomorphic to Euclidian discs. We say that the surface $\tilde{\Sigma}$ has a **self intersection**, this is called a **nodal point**.

Nodal points can be desingularized by first removing the point (a, b) from $\tilde{\Sigma}$, and adding a 2 new points to $\tilde{\Sigma}$, called $(a, b)_+$ and $(a, b)_-$, and we define the neighborhoods of $(a, b)_\pm$ by one of the 2 punctured discs of $\tilde{\Sigma} \cap D((a, b), r)^*$, so that the neighborhoods are now 2 Euclidian discs, as illustrated below:



- In a similar manner, by adding new points to $\tilde{\Sigma}$, all other types of singular points (including neighborhoods of ∞ , and higher order singular points, at which the Hessian can vanish) can be "desingularized", leading to a Riemann surface Σ , which is a smooth compact Riemann surface.

- There is a holomorphic map:

$$\begin{aligned} \Sigma &\rightarrow \tilde{\Sigma} \\ p &\mapsto (x(p), y(p)), \end{aligned} \tag{3-16}$$

However this map is not always invertible (it is not invertible at nodal points, since a nodal point is the image of 2 (or more) distinct points of Σ).

The map defines 2 holomorphic maps $x : \Sigma \rightarrow \mathbb{C}P^1$ and $y : \Sigma \rightarrow \mathbb{C}P^1$. Since they can reach $\infty \in \mathbb{C}P^1$, we say that they are meromorphic, i.e. they can have poles.

Eventually this implies that an algebraic equation $P(x, y) = 0$ defines a compact Riemann surface, and we have the following theorem

Theorem 3.1 *There exists a smooth compact Riemann surface Σ , and 2 meromorphic maps $x : \Sigma \rightarrow \mathbb{C}P^1$ and $y : \Sigma \rightarrow \mathbb{C}P^1$, such that*

$$\tilde{\Sigma} = \{(x, y) \mid P(x, y) = 0\} = \{(x(p), y(p)) \mid p \in \Sigma \setminus x^{-1}(\infty) \cup y^{-1}(\infty)\}. \quad (3-17)$$

The map

$$\begin{aligned} \Sigma &\rightarrow \tilde{\Sigma} \\ p &\mapsto (x(p), y(p)), \end{aligned} \quad (3-18)$$

is meromorphic.

3.4 Projective algebraic curves

Consider a homogeneous polynomial $P(x, y, z) \in \mathbb{C}[x, y, z]$ of some degree d , write its coefficients

$$P(x, y, z) = \sum_{(i,j,k), i+j+k=d} P_{i,j,k} x^i y^j z^k. \quad (3-19)$$

Now consider the subset of

$$\mathbb{C}P^2 = \frac{\{(x, y, z) \neq (0, 0, 0)\}}{(x, y, z) \equiv (\lambda x, \lambda y, \lambda z) \forall \lambda \in \mathbb{C}^*} \quad (3-20)$$

annihilated by P

$$\Sigma = \{[(x, y, z)] \in \mathbb{C}P^2 \mid P(x, y, z) = 0\} \quad (3-21)$$

(it is well defined on equivalence classes thanks to the homogeneity of P .) Locally $\mathbb{C}P^2 \sim \mathbb{C}^2$, indeed in a neighborhood of a point where at least 1 of the 3 coordinates is not 0 (assume $z \neq 0$), then $(x, y, z) \equiv (x/z, y/z, 1)$. In other words, away from neighborhoods of $z = 0$, we can choose $z = 1$ and write the equation as

$$P(x, y, 1) = 0. \quad (3-22)$$

The points of Σ where one of the 3 coordinates x , y or z vanishes are called **punctures**.

Following the same procedure as above, we can find an atlas of Σ , by first removing a graph containing all singular points and punctures, with local coordinate x , and transition maps are $x \mapsto x$. Except for charts around singular points, or charts around punctures, where we have to find another local parameter, typically $(x - a)^{1/d_a}$, and possibly desingularize by adding new points to Σ .

We shall admit the following:

Theorem 3.2 *Every compact Riemann surface can be algebraically immersed into $\mathbb{C}P^2$, with at most simple nodal points.*

Moreover, every compact Riemann surface can be algebraically embedded into $\mathbb{C}P^3$ (embedding=bijjective, no nodal points).

This is why algebraic is almost synonymous to compact for Riemann surfaces

Algebraic = compact

Chapter 2

Functions and forms on Riemann surfaces

1 Definitions

Definition 1.1 (Functions) *An analytic function, f on an atlas of a Riemann surface Σ , is the data of a holomorphic function $f_U : U \rightarrow \mathbb{C}P^1$ in each chart, satisfying for every transition:*

$$f_U = f_{U'} \circ \psi_{U \rightarrow U'}. \quad (1-1)$$

This allows to define unambiguously for every point of Σ :

$$f(\phi_U^{-1}(z)) = f_U(z). \quad (1-2)$$

*A holomorphic function that takes values in $\mathbb{C}P^1$ is called **meromorphic** if it reaches the value ∞ (it has poles), and **holomorphic** otherwise. We shall denote*

- $\mathfrak{M}^0(\Sigma)$ *the vector space of meromorphic functions on Σ ,*
- $\mathcal{O}(\Sigma)$ *(or sometimes $\mathcal{O}^0(\Sigma)$) the vector space of holomorphic functions on Σ .*

Definition 1.2 (Forms) *A meromorphic 1-form, ω on an atlas of a Riemann surface Σ , is the data of a holomorphic function $\omega_U : U \rightarrow \mathbb{C}P^1$ in each chart, that satisfies for every transition:*

$$\omega_U(z) = \omega_{U'}(\psi_{U \rightarrow U'}(z)) \frac{d}{dz} \psi_{U \rightarrow U'}(z). \quad (1-3)$$

This allows to define unambiguously

$$\omega(\phi_U^{-1}(z)) = \omega_U(z) dz \quad (1-4)$$

on every point of Σ . A 1-form such that ω_U takes values in $\mathbb{C}P^1$ is called meromorphic if ω_U reaches the value ∞ (it has poles), and holomorphic otherwise.

A meromorphic 1-form ω is called *exact* iff there exists a meromorphic function f such that $\omega = df$, i.e. in each chart $\omega_U(z) = df_U(z)/dz$.

We shall denote

- $\mathfrak{M}^1(\Sigma)$ the vector space of meromorphic forms on Σ .
- $\mathcal{O}^1(\Sigma)$ the vector space of holomorphic forms on Σ .

These are in fact special cases of

Definition 1.3 (Higher order forms) A k^{th} order holomorphic (resp. meromorphic) form on an atlas of a Riemann surface Σ , is the data of a holomorphic function $f_U : U \rightarrow \mathbb{C}P^1$ in each chart, that satisfies for every transition:

$$f_U(z) = f_{U'}(\psi_{U \rightarrow U'}(z)) \left(\frac{d}{dz} \psi_{U \rightarrow U'}(z) \right)^k. \quad (1-5)$$

This allows to define unambiguously

$$f(\phi_U^{-1}(z)) = f_U(z) dz^k \quad (1-6)$$

on every point of Σ .

- If $k = 0$ this is called a holomorphic (resp. meromorphic) function.
- If $k = 1$ this is called a holomorphic (resp. meromorphic) 1-form.
- If $k = 2$ this is called a holomorphic (resp. meromorphic) **quadratic differential**.
- It can also be defined for half-integer $k \in \frac{1}{2}\mathbb{Z}$, and then only $\pm f$ is well defined globally on Σ . This is called a **spinor form**.

We shall denote the vector space of holomorphic and meromorphic order k forms on Σ as

$$\mathcal{O}^k(\Sigma) \subset \mathfrak{M}^k(\Sigma). \quad (1-7)$$

1.1 Examples

- On the Riemann sphere, the function $f(z) = z$ is meromorphic, it has a pole at $z = \infty$. Also on the Riemann sphere, the 1-form $\omega(z) = dz$ is meromorphic, it has a double pole at ∞ . Indeed in the chart which is the neighborhood of ∞ , we use the coordinate $z' = 1/z$ and we have

$$\omega(z) = dz = d(1/z') = \frac{-1}{z'^2} dz' \quad (1-8)$$

it has a double pole at $z' = 0$, i.e. a double pole at $z = \infty$.

- On the torus $T_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, the 1-form

$$dz \tag{1-9}$$

is a holomorphic 1-form. Indeed it satisfies the transition condition with $\psi(z) = z' = z + a + \tau b$, we have $dz = dz'$. Moreover, it has no pole.

The following series

$$\wp(z) = \frac{1}{z^2} + \sum_{n,m \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(z + n + \tau m)^2} - \frac{1}{(n + \tau m)^2} \tag{1-10}$$

is absolutely convergent for all $z \notin \mathbb{Z} + \tau\mathbb{Z}$, it is clearly bi-periodic $\wp(z+1) = \wp(z + \tau) = \wp(z)$, so that it satisfies the transition conditions, it is thus a meromorphic function on the torus. It has a unique pole at $z = 0$, of degree 2. It is called the **Weierstrass function**.

Definition 1.4 (Order) *The order_p $f = k$ (resp. order_p $\omega = k$) of a function f (resp. a form) at a point $p \in \Sigma$ is:*

- the order of vanishing of f_U if p is not a pole, i.e. in any chart U , with coordinate z , $f_U(z) \sim C_U(z - \phi_U(p))^k$. In this case $k > 0$.
- or minus the degree of the pole of f_U if p is a pole, i.e. in any chart U , with coordinate z , $f_U(z) \sim C_U(z - \phi_U(p))^{-|k|}$. In this case $k < 0$.
- For generic points (neither poles nor zeros) we define order_p $f = 0$.

The order is independent of a choice of chart and coordinate. A holomorphic function (resp. form) has non-negative orders at all points.

Definition 1.5 (Residue of a form) *Let ω a meromorphic 1-form, and p one of its poles. We define its **residue** in any chart U that contains p , where $\omega(\phi_U^{-1}(z)) = f_U(z)dz$, as*

$$\text{Res}_p \omega = c_{-1} \quad \text{where} \quad f_U(z) = \sum_{0 > j \geq -\text{order}_p \omega} c_j z^j + \text{analytic} \tag{1-11}$$

It is independent of a choice of chart and coordinate.

Notice that only the coefficient c_{-1} is independent of a choice of chart and coordinates, all other c_j with $j \neq -1$ do depend on that choice. Similarly, for a k^{th} order meromorphic form, the residue is defined as the coefficient (independent of chart and coordinate) c_{-k} .

Definition 1.6 (Jordan arcs) A Jordan arc γ on Σ , is a continuous map $\gamma : [a, b] \rightarrow \Sigma$, such that there exist a finite partition of $[a, b] \subset \mathbb{R}$

$$[a, b] = [a_1, a_2] \cup [a_2, a_3] \cup \cdots \cup [a_{n-1}, b] \quad , \quad a = a_1 < a_2 < a_3 < \cdots < a_n = b, \quad (1-12)$$

such that each $\gamma([a_i, a_{i+1}])$ is included in a single chart V_i , and such that the map $\gamma_i : [a_i, a_{i+1}] \rightarrow U_i$ defined by $\gamma_i(t) = \phi_{V_i}(\gamma(t))$ is a Jordan arc in $U_i \subset \mathbb{C}$.

There is a notion of homotopic deformations of Jordan arcs on Σ inherited from that in \mathbb{C} , and of concatenation of Jordan arcs.

A closed Jordan arc is called a Jordan curve, or a contour, it is such that $\gamma(b) = \gamma(a)$.

Definition 1.7 (Integral of a form) Let ω a meromorphic 1-form. Let γ a Jordan arc on Σ , not containing any pole of ω , represented by a collection $\gamma_1, \dots, \gamma_n$ of Jordan arcs in charts $\gamma_i \subset U_i$. We define

$$\int_{\gamma} \omega = \sum_i \int_{\gamma_i} \omega_{U_i}(z) dz \quad (1-13)$$

It is independent of a choice of charts and local coordinates.

Moreover it is invariant under homotopic deformations of γ .

If γ is closed, we write \oint_{γ} rather than \int_{γ} , which is then invariant under change of initial point of the Jordan curve.

Definition 1.8 (Integral of a form on a chain or cycle) We define an integer (resp. complex) chain $\hat{\gamma}$ as a \mathbb{Z} (resp. \mathbb{C}) linear combination of homotopy classes of Jordan arcs modulo boundaries of open surfaces. A chain is a cycle iff its boundary vanishes.

Integration defines the **Poincaré pairing** between chains (resp. cycles) and 1-forms

$$\langle \gamma, \omega \rangle = \int_{\hat{\gamma}} \omega. \quad (1-14)$$

Theorem 1.1 (Cauchy) We have

$$\text{Res}_p \omega = \frac{1}{2\pi i} \oint_{C_p} \omega \quad (1-15)$$

with C_p a small anti-clockwise contour around p .

proof: It holds in every chart because it holds on \mathbb{C} . \square

Definition 1.9 (Divisors) A **divisor** is a formal linear combination of points of the surface. Let $D = \sum_i \alpha_i .p_i$ a divisor. We define its degree as:

$$\deg D = \sum_i \alpha_i. \quad (1-16)$$

We shall say that the divisor is an integer divisor if $\alpha_i \in \mathbb{Z}$ and complex if $\alpha_i \in \mathbb{C}$.

If f is a meromorphic function, not identically vanishing, we denote the **divisor of f** (it is an integer divisor):

$$(f) = \sum_{p \in \Sigma} \text{order}_p f .p. \quad (1-17)$$

Similarly, if ω is a meromorphic 1-form not indetically zero, we denote

$$(\omega) = \sum_{p \in \Sigma} \text{order}_p \omega .p. \quad (1-18)$$

If $f = 0$ (resp. $\omega = 0$) we define

$$(0) = 0. \quad (1-19)$$

We define similarly the divisors of any k^{th} order forms.

1.2 Classification of 1-forms

1-forms have been customarily divided into the following classes

Definition 1.10 (Classification of 1-forms) A meromorphic 1-form ω is called

- **1st kind** iff it is holomorphic (it has no poles),
- **3rd kind** iff it has poles of degree at most 1,
- **2nd kind** iff it has some poles of degree ≥ 2 .
- **exact** iff there exists a meromorphic function f such that $\omega = df$, i.e. in each chart $\omega_U(z) = df_U(z)/dz$. In fact all exact forms must be 2nd kind.

2 Some theorems about forms and functions

Theorem 2.1 (finite number of poles) On a compact Riemann surface, each meromorphic function (resp. form) has at most a finite number of poles. Also each non-vanishing meromorphic function (resp. form) has only finitely many points with non-vanishing order, so the divisor is a finite sum.

proof: Compactness implies that any infinite sequence of points of Σ must have accumulation points. If the number would be infinite there would be an accumulation point of poles, and the function (resp. form) would not be analytic at the accumulation point. If there is an accumulation of points of strictly positive orders, i.e. an accumulation of zeros, then the analytic function (resp. form) has to vanish identically in a neighborhood of this point, and thus vanishes identically on Σ . In all cases the divisor is finite. \square

Theorem 2.2 (exact forms) *A 1-form ω is exact if and only if*

$$\forall C = \text{cycle} \quad \oint_C \omega = 0. \quad (2-1)$$

proof:

Let o a generic point of Σ . The function

$$f(p) = \int_o^p \omega \quad (2-2)$$

seems to be ill defined on Σ as it seems to depend on a choice of Jordan arc from o to p . However, thanks to (2-1), its value is independent of the choice of arc and thus depends only on p , so it is a well defined function on Σ .

ω may have poles, its integral can have poles and logs, but the condition (2-1) implies that the residues of ω at all poles vanish. This implies that f , can't have logarithmic terms, it is thus a meromorphic function on Σ , and satisfies

$$df = \omega. \quad (2-3)$$

The converse is obvious. \square

Theorem 2.3 (vanishing total residue) *For a meromorphic form ω on a compact Riemann surface:*

$$\sum_{p=\text{poles}} \text{Res}_p \omega = 0. \quad (2-4)$$

proof: Let us admit here that Σ can be polygonized, i.e. that there is a graph Γ on Σ , whose faces are polygons entirely contained in a chart, and such that the poles of ω are not on Γ . By homotopic deformation, the sum of residues inside each polygon is the integral along edges. Each edge is spanned twice, in each direction, so the sum of integral along edges is zero.

\square

Corollary 2.1 (More than 1 simple pole) *There is no meromorphic 1-form with only one simple pole. It has either several simple poles, or poles of higher orders, or no pole at all.*

Theorem 2.4 *If $g \geq 1$, there is no meromorphic function with only 1 simple pole.*

proof: If f is a meromorphic function with 1 simple pole p . Then, consider a holomorphic 1-form ω (it is possible if $g > 0$, we anticipate on the next section). Assume that ω either doesn't vanish at p ($k = 0$) or has a zero of order k at p . Then $f^{k+1}\omega$ is a meromorphic form, and has only 1 simple pole at p , which is impossible. \square

Theorem 2.5 (Functions: #poles = #zeros) *Let f a meromorphic function, not identically vanishing, then the number of poles (with multiplicity) equals the number of zeros:*

$$\#\text{zeros} - \#\text{poles} = \deg(f) = 0. \quad (2-5)$$

proof: Use theorem 2.3 with $\omega = d \log f = \frac{1}{f}df$. \square

Theorem 2.6 (Holomorphic function = constant) *Any holomorphic function is constant. This implies that*

$$\mathcal{O}(\Sigma) = \mathbb{C} \quad , \quad \dim \mathcal{O}(\Sigma) = 1. \quad (2-6)$$

proof: Let f a holomorphic function. Let $p_0 \in \Sigma$ a given generic point, define the function $g(p) = f(p) - f(p_0)$. This function has no pole and has at least one zero. This would contradict theorem 2.5, unless g is identically vanishing, i.e. f is constant. \square

Theorem 2.7 (any 2 meromorphic functions are algebraically related) *Let f and g be two meromorphic functions on Σ . Then, there exists a bivariate polynomial Q such that*

$$Q(f, g) = 0. \quad (2-7)$$

proof: The proof is nothing but the Lagrange interpolation polynomial.

Let us call $d = \deg f$ the total degree of f , i.e. the sum of degrees of all its poles. Let $x \in \mathbb{C}P^1$, then $f^{-1}(x)$ has generically a cardinal equal to d . Let us define $Q_0 = 1$ and for $k = 1, \dots, d$:

$$Q_k(x) = \sum_{I \subset_k f^{-1}(x)} \prod_{p \in I} g(p), \quad (2-8)$$

where we sum over all subsets of $f^{-1}(x)$ of cardinal k , and we count preimages with multiplicities when x is not generic. Q_k is clearly a meromorphic function $\mathbb{C}P^1 \rightarrow \mathbb{C}P^1$, therefore it is a rational function $Q_k(x) \in \mathbb{C}(x)$. We then define:

$$Q(x, y) = \sum_{k=0}^d (-1)^k Q_k(x) y^k \in \mathbb{C}(x)[y]. \quad (2-9)$$

It is also equal to

$$Q(x, y) = \prod_{p \in f^{-1}(x)} (y - g(p)). \quad (2-10)$$

Therefore, for any $p \in \Sigma$ we have

$$Q(f(p), g(p)) = 0. \quad (2-11)$$

□

Theorem 2.8 (Riemann-Hurwitz) *Let ω a meromorphic 1-form not identically vanishing, then*

$$\deg(\omega) = 2\mathfrak{g} - 2 \quad (2-12)$$

where \mathfrak{g} is the genus of Σ . In other words

$$\#\text{zeros} - \#\text{poles} = 2\mathfrak{g} - 2. \quad (2-13)$$

proof:

First remark that the ratio of 2 meromorphic 1-forms is a meromorphic function, for which $\#\text{zeros} = \#\text{poles}$, therefore $\#\text{zeros} - \#\text{poles}$ is the same for all 1-forms.

In particular, let us assume that there exist some non-constant meromorphic function $f : \Sigma \rightarrow \mathbb{C}P^1$ (we anticipate on the next sections. In the case of an algebraic curve $P(x, y) = 0$, one can choose $f = x$), then $\omega = df$ is an exact meromorphic 1-form.

Let R be the set of zeros of $\omega = df$, and P the set of poles.

Choose an arbitrary cellular (all faces are homeomorphic to discs) graph Γ on $\mathbb{C}P^1$, whose vertices are the points of $f(R) \cup \{\infty\}$.

Let F the number of faces, E the number of edges and V the number of vertices. Let d the degree of f , i.e. the sum of degrees of all its poles, which is also $d = \#f^{-1}(x)$ for x in a neighborhood of ∞ , and thus is the number of preimages of generic x .

The Riemann sphere has Euler characteristic 2:

$$\chi(\mathbb{C}P^1) = 2 = F - E + V. \quad (2-14)$$

Now consider the graph $\Gamma' = f^{-1}(\Gamma)$ on Σ . Since the faces of Γ contain no zero of df , then the preimages of each face is homeomorphic to a disc too, so that Γ' is a cellular graph on Σ . Its Euler characteristic is

$$\chi(\Sigma) = 2 - 2\mathbf{g} = F' - E' + V'. \quad (2-15)$$

The number of faces $F' = dF$ and edges $E' = dE$ because they are made of generic points. The points of R and P are by definition not generic. For $x \in f(R)$ we have

$$\#f^{-1}(x) = d - \sum_{r \in f^{-1}(x)} \text{order}_r df. \quad (2-16)$$

Their sum is

$$\sum_{x \in f(R)} \#f^{-1}(x) = d(V - 1) - \deg(df)_+ \quad (2-17)$$

where $(df)_+$ is the divisor of zeros of df . Similarly for $x = \infty$ we have

$$\#f^{-1}(\infty) = \sum_{p \in P} (-1 - \text{order}_p df). \quad (2-18)$$

This implies

$$V' = d(V - 1) - d - \deg(df). \quad (2-19)$$

Putting all together we get

$$\deg(df) = 2\mathbf{g} - 2. \quad (2-20)$$

Since $\deg(\omega)$ is the same for all 1-forms, it must be the same as for the exact form df and the theorem is proved.

In fact all what remains to prove is the existence of at least one non-constant meromorphic function. For Riemann surfaces coming from an algebraic equation $P(x, y) = 0$, the function $(x, y) \mapsto x$ can play this role. More generally the existence of non-constant meromorphic functions will be established below (and won't use this theorem). \square

3 Existence of meromorphic forms

Before going further, we need to make sure that meromorphic functions and forms do actually exist. First let us define the Hodge star:

Definition 3.1 (Hodge star, harmonic forms) *Let ω a C^∞ 1-form (real or complex) on Σ viewed as a smooth manifold of dimension 2 rather than a complex manifold. In a local coordinate $z = x + iy$ it can be written*

$$\omega = pdx + qdy = \frac{p - iq}{2} dz + \frac{p + iq}{2} d\bar{z}. \quad (3-1)$$

p and q can be viewed as real valued or complex valued C^∞ functions on Σ , they are not assumed to be analytic.

We define the **Hodge star** of a differential 1-form

$$*\omega = -qdx + pdy = -i \frac{p - iq}{2} dz + i \frac{p + iq}{2} d\bar{z}. \quad (3-2)$$

We have

$$\omega \wedge *\bar{\omega} = (|p|^2 + |q|^2) dx \wedge dy. \quad (3-3)$$

The Hilbert space $L^2(\Sigma)$ of real (resp. $L^2(\Sigma, \mathbb{C})$ of complex) square integrable 1-forms, is equipped with the norm (it is positive definite)

$$\|\omega\|^2 = \int_{\Sigma} \omega \wedge *\bar{\omega}. \quad (3-4)$$

A 1-form ω is called **closed** (resp. **co-closed**) iff $d\omega = 0$ (resp. $d*\omega = 0$). A 1-form ω is called **harmonic** iff it is closed and co-closed. Let \mathcal{H} the set of real harmonic forms.

If f is a function, then its Laplacian is

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = d*df = -2i \bar{\partial}\partial f. \quad (3-5)$$

Let E (resp. E^*) the closure of the set of exact (resp. co-exact) 1-forms, i.e. the set of differentials df (resp. $*df$) and limits (with respect to the topology induced by the $L^2(\Sigma)$ norm) of sequences $\lim_{n \rightarrow \infty} df_n$ (resp. $\lim_{n \rightarrow \infty} *df_n$).

There is the following theorem (that we shall not use in these lectures, in fact it will be reproved case by case corresponding to our needs):

Theorem 3.1 (Hodge decomposition theorem) *Every real square integrable 1-form can be uniquely decomposed as a harmonic+exact+co-exact:*

$$L^2(\Sigma) = \mathcal{H} \oplus E \oplus E^*. \quad (3-6)$$

proof: Admitted, and in fact not needed in these lectures. \square

The following theorem is the key to existence of holomorphic and meromorphic 1-forms:

Theorem 3.2 (Harmonic forms) *Let Σ a Riemann surface of genus $g \geq 1$, and let $\mathcal{A}_1, \dots, \mathcal{A}_{2g}$ a basis of $H_1(\Sigma, \mathbb{Z})$ (non-contractible cycles).*

Given $(\epsilon_1, \dots, \epsilon_{2g}) \in \mathbb{R}^{2g}$, there exists a unique real harmonic form ν on Σ such that

$$\forall i = 1, \dots, 2g, \quad \oint_{\mathcal{A}_i} \nu = \epsilon_i. \quad (3-7)$$

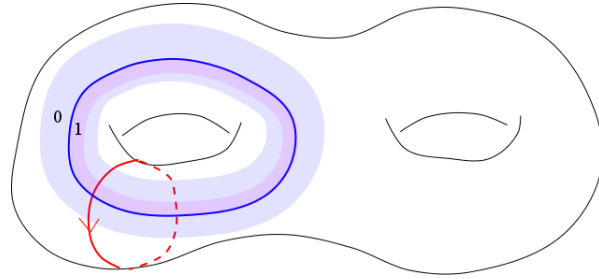
Moreover, \mathcal{H} is a vector space over \mathbb{R} of dimension

$$\dim \mathcal{H} = 2g. \quad (3-8)$$

proof: Let us prove it for $(\epsilon_1, \dots, \epsilon_{2g}) = (1, 0, \dots, 0)$, the general case will then hold by linearity and relabelling.

The constraints (3-7) define an affine subspace $V \subset L^2(\Sigma)$. Let us first show that this affine subspace is non-empty.

Let us choose some Jordan arcs representative of the \mathcal{A}_i s, and let \mathcal{B} a closed Jordan arc that intersects \mathcal{A}_1 once and no other \mathcal{A}_j : $\mathcal{A}_j \cap \mathcal{B} = \delta_{1,j}$. Choose a tubular neighborhood U of \mathcal{B} . Choose a real function θ , not continuous, but such that θ is C^∞ on $\Sigma \setminus \mathcal{B}$, is constant equal to 1 in a left neighborhood of \mathcal{B} , is constant equal to 0 in a right neighborhood of \mathcal{B} , and is identically zero outside U (left and right refer to the orientation of the arc \mathcal{A}_1). Such a function exists in the annulus of \mathbb{R}^2 , and is easy to define with the help of the $C^\infty(\mathbb{R})$ function $x \mapsto e^{-1/x}$ if $x > 0$ and $x \mapsto 0$ if $x \leq 0$.



Then $d\theta$ is a C^∞ 1-form on Σ , it belongs to $L^2(\Sigma)$, and it satisfies (3-7), so it belongs to V .

In order to satisfy (3-7), one can only add exact forms (and their limits), and thus

$$V = d\theta + E, \quad (3-9)$$

which is a closed set since E is closed. The Hilbert projection theorem ensures that there exists at least one element $\nu \in V$ whose norm $\|\nu\|$ is minimal.

This means that for any C^∞ function f , one has $\|\nu + df\|^2 \geq \|\nu\|^2$, and this implies that $(\nu, df) = 0$, i.e. $\int_\Sigma f d * \nu = 0$. Since this has to hold for all f , this implies that $d * \nu = 0$ and thus ν is a harmonic form $\nu \in V \cap \mathcal{H}$.

The space \mathcal{H} is clearly a real vector space, and we have just seen that the morphism

$$\begin{aligned} \mathcal{H} &\rightarrow \mathbb{R}^{2g} \\ \nu &\mapsto \left(\int_{\mathcal{A}_1} \nu, \int_{\mathcal{A}_2} \nu, \dots, \int_{\mathcal{A}_{2g}} \nu \right) \end{aligned} \quad (3-10)$$

is surjective. It is also injective, because if ν is in the kernel, then all its cycle integrals vanish, implying that it is an exact form, $\nu = df$, and since $d * \nu = 0$, f must be a harmonic function $\Delta f = 0$. Stokes theorem implies

$$\|\nu\|^2 = - \int_{\Sigma} f \Delta f = 0, \quad (3-11)$$

which implies $\nu = 0$. This proves that there is an isomorphism $\mathcal{H} \sim \mathbb{R}^{2g}$.

□

In other words, for $g \geq 1$, there exists harmonic forms. Notice that if ν is a real harmonic form, then $\omega = \nu + i * \nu$ is a holomorphic form on Σ .

Remark that we have obtained a harmonic form by an "extremization" procedure, this is often called "**Dirichlet Principle**".

This method can be generalized to get "meromorphic forms" (and then this works also for genus 0), with some adaptations, and we get

Theorem 3.3 (Harmonic forms with 2 simple poles) *Given q_+, q_- distinct in Σ , there exists a unique real harmonic form ν on $\Sigma \setminus \{q_+, q_-\}$, such that in a neighborhood U_+ (resp. U_-) of q_+ (resp. q_-):*

$$\nu(p) \sim_{p \rightarrow q_{\pm}} \pm d \text{Arg} |\phi_{U_{\pm}}(p) - \phi_{U_{\pm}}(q_{\pm})| + \text{harmonic at } q_{\pm} \quad (3-12)$$

and such that all its cycle integrals vanish

$$\forall i = 1, \dots, 2g, \quad \oint_{\mathcal{A}_i} \nu = 0 \quad (3-13)$$

In other words it satisfies

$$\oint_{\mathcal{C}} \nu = \begin{cases} \pm 2\pi & \text{if } \mathcal{C} \text{ surrounds } q_{\pm} \\ 0 & \text{otherwise} \end{cases} \quad (3-14)$$

proof: We shall denote z (resp. z') the coordinate of a point p in a neighborhood chart U_+ (resp. U_-) of q_+ (resp. q_-):

$$z = \phi_{U_+}(p) - \phi_{U_+}(q_+) \quad , \text{ resp. } \quad z' = \phi_{U_-}(p) - \phi_{U_-}(q_-) \quad , \quad (3-15)$$

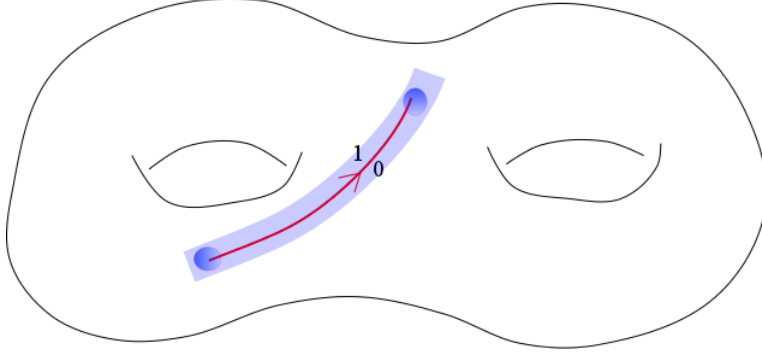
vanishing at $p = q_+$ (resp $p = q_-$).

Let us choose a Jordan arc γ with boundary $\partial\gamma = q_+ - q_-$, and let U a tubular neighborhood of γ . Let us choose a function $\theta \in \mathcal{C}^{\infty}(\Sigma \setminus \gamma)$, such that:

$$\begin{aligned} \theta &= \text{Arg} z && \text{in } U_+ \\ \theta &= 2\pi - \text{Arg}(-z') && \text{in } U_- \\ \theta &= 0 && \text{in } U_{\text{left}} - U_+ - U_- \end{aligned}$$

$$\theta = 2\pi \text{ in } U_{\text{right}} - U_+ - U_- \quad (3-16)$$

$d\theta$ is then a \mathcal{C}^∞ 1-form on $\Sigma - \{q_+\} - \{q_-\}$.



However $d\theta \notin L^2(\Sigma)$ because it has poles at q_+ and q_- , so its norm is infinite. Let us define the "regularized norm" on $d\theta + E$ as

$$\|d\theta + df\|^2 - \|d\theta\|^2 \stackrel{\text{def}}{=} \|df\|^2 - 2 \int_{\Sigma \setminus \gamma} \theta \Delta f + 2\pi(f(q_+) - f(q_-)), \quad (3-17)$$

which is a strictly convex functional on the closed space E . The Hilbert projection theorem says that there is an element ν of $d\theta + E$ with minimal norm, i.e. for any smooth f compactly supported on $\Sigma - \{q_+\} - \{q_-\}$, we have

$$\|\nu + df\|^2 - \|\nu\|^2 = -2 \int_{\Sigma} f d * \nu + \|df\|^2 \geq 0. \quad (3-18)$$

This having to be positive for all f implies that $d * \nu = 0$, and thus ν is harmonic on $\Sigma - \{q_+\} - \{q_-\}$. Moreover, it has the correct behaviors near q_+ and q_- , and the correct cycle integrals.

ν is unique because if there would exist another $\tilde{\nu}$, then the difference $\nu - \tilde{\nu}$ would be harmonic, with all cycle integrals vanishing, so it would be exact $\nu - \tilde{\nu} = df$ with f a harmonic function, i.e. it would have to vanish. \square

Corollary 3.1 (Green function) *Given q_+, q_- distinct in Σ , there exists a real function \mathcal{G}_{q_+, q_-} harmonic on $\Sigma \setminus \{q_+, q_-\}$, such that in a neighborhood U_+ (resp. U_-) of q_+ (resp. q_-):*

$$\mathcal{G}_{q_+, q_-}(p) \sim_{p \rightarrow q_{\pm}} \pm \log |\phi_{U_{\pm}}(p) - \phi_{U_{\pm}}(q_{\pm})| + \text{harmonic}. \quad (3-19)$$

It is unique up to adding a constant.

Moreover it satisfies the Poisson equation

$$\Delta \mathcal{G}_{q_+, q_-}(p) = 4\pi (\delta^{(2)}(p, q_+) - \delta^{(2)}(p, q_-)). \quad (3-20)$$

proof: We choose \mathcal{G}_{q_+,q_-} an integral of $*\nu$, to which we add the unique linear combination of harmonic forms that ensure that all its cycle integrals vanish. \square

Corollary 3.2 (Third kind forms) *Given any 2 distinct points q_+, q_- on Σ , there exists a unique meromorphic 1-form ω_{q_+,q_-} that has a simple pole at q_+ and a simple pole at q_- , and such that*

$$\operatorname{Res}_{q_+} \omega_{q_+,q_-} = 1 = - \operatorname{Res}_{q_-} \omega_{q_+,q_-}. \quad (3-21)$$

and such that, for any non-contractible cycle \mathcal{C} one has

$$\Re \oint_{\mathcal{C}} \omega_{q_+,q_-} = 0. \quad (3-22)$$

proof: Choose $\omega_{q_+,q_-} = i\nu - *\nu$, and add the unique linear combination of holomorphic forms, that cancels the real parts of all cycle integrals. \square

Corollary 3.3 (Higher order poles) *Let U a chart, and let ϕ a coordinate in U . Let $q \in U$ and let $k \geq 1$. There exists a unique meromorphic 1-form $\omega_{q,k}$ that has a pole of order $k+1$ at q and no other pole, and that behaves near q like*

$$\omega_{q,k}(z) \sim \frac{d\phi(z)}{(\phi(z) - \phi(q))^{k+1}} + \text{analytic at } q, \quad (3-23)$$

and such that, for any non-contractible cycle \mathcal{C} one has

$$\Re \oint_{\mathcal{C}} \omega_{q,k} = 0. \quad (3-24)$$

proof: Choose $q' \neq q$ generic in U , and choose $\omega_{q,k} = \frac{1}{k!} \frac{d^k \omega_{q,q'}}{dq^k}$. It is unique (and in particular independent of q'), because the difference of 2 such forms would have no pole and all real parts of its cycle integrals vanishing so it would vanish. \square

Fundamental second kind form

However, the form $\omega_{q,k}$, and in particular $\omega_{q,1}$ depends on the choice of chart neighborhood of q , and on the choice of coordinate in that chart. As we shall see now, $\omega_{q,1}$ in fact transforms as a 1-form of q under chart transitions, i.e. $B_S(z, q) = \omega_{q,1}(z)d\phi(q)$ is a well defined 1-form (chart independent) of both variables z and q . We call it a bilinear differential, the product is a tensor product, that we write

$$B_S(z, q) = \omega_{q,1}(z)d\phi(q) = \omega_{q,1}(z) \otimes d\phi(q) = \omega_{q,1}(z) \boxtimes d\phi(q). \quad (3-25)$$

The tensor product notation \otimes just means that this is a linear combination of 1-forms of z , whose coefficients are themselves 1-forms of q . The box tensor product notation

\boxtimes means that this is a differential form on the product $\Sigma \times \Sigma$, which is a 1-form on the first factor Σ of the product, tensored by a 1-form of the second factor Σ of the product.

We have the following theorem:

Proposition 3.1 (Schiffer kernel) *Define the following bi-differential of the Green function*

$$B_S(p, q) = \partial_p \partial_q \mathcal{G}_{q,q}(p) \quad dp \otimes dq \quad (3-26)$$

where, if z is a coordinate in a chart, $\partial_z = \frac{1}{2} \left(\frac{\partial}{\partial \Re z} - i \frac{\partial}{\partial \Im z} \right)$, and $dz = d\Re z + i d\Im z$. It is independent of a choice of chart and coordinate, and it has the following properties:

- it is a meromorphic bilinear differential (meromorphic 1-form of p , tensored by a meromorphic 1-form of q),
- it is independent of q' ,
- it has a double pole on the diagonal, such that in any chart and coordinate

$$B_S(p, q) \underset{p \rightarrow q}{\sim} \frac{d\phi_U(p) \boxtimes d\phi_U(q)}{(\phi_U(p) - \phi_U(q))^2} + \text{analytic at } p = q, \quad (3-27)$$

- for any pair of non-contractible cycles $\mathcal{C}, \mathcal{C}'$, one has

$$\Re \oint_{p \in \mathcal{C}} \oint_{q \in \mathcal{C}'} B_S(p, q) = 0. \quad (3-28)$$

Moreover we shall see later in section 5 of chap. 3 that B_S is in fact symmetric $B_S(p, q) = B_S(q, p)$.

The Schiffer kernel is a special example of the following notion:

Definition 3.2 (Fundamental form of second kind) *A fundamental form of second kind $B(z, z')$ is a symmetric $1 \boxtimes 1$ form on $\Sigma \times \Sigma$ that has a double pole at $z = z'$ and no other pole, and such that in any chart*

$$B(z, z') \sim \frac{d\phi_U(z) d\phi_U(z')}{(\phi_U(z) - \phi_U(z'))^2} + \text{analytic at } z = z'. \quad (3-29)$$

If the genus is ≥ 1 , a fundamental form of second kind is not unique since we can add to it any symmetric bilinear combination of holomorphic 1-forms:

$$B(z, z') \rightarrow B(z, z') + \sum_{i,j=1}^g \kappa_{i,j} \omega_i(z) \otimes \omega_j(z'). \quad (3-30)$$

Therefore, and due to the existence of at least one such form (the Schiffer kernel), we have

Theorem 3.4 *The space of fundamental second kind differentials is not empty, and is an affine space, with linear space $\mathcal{O}^1(\Sigma) \overset{sym}{\otimes} \mathcal{O}^1(\Sigma)$.*

3.1 Uniqueness of Riemann surfaces of genus 0

As a corollary of corollary 3.3, we have

Theorem 3.5 (Genus 0 = Riemann sphere) *Every compact simply connected (genus zero) Riemann surface is isomorphic to the Riemann sphere.*

proof: Let U a chart with its coordinate, and $p \neq o$ two distinct points in U . The 1-form $\omega_{p,1}$ as defined in corollary 3.3 has a double pole at p and no other pole, so it has vanishing residue. The function $f(z) = \int_o^z \omega_{p,1}$ is meromorphic on Σ , with a simple pole at p and no other pole, and a zero at o (and no other zeros since $\#\text{zeros}=\#\text{poles}$). The map $f : \Sigma \rightarrow \mathbb{C}P^1$ is injective, indeed if there would exist $q \neq q'$ with $f(q) = f(q')$, then the function $z \mapsto f(z) - f(q)$ would have 2 zeros q and q' , which is impossible. f is continuous, and since Σ is open and compact, its image must be $\mathbb{C}P^1$, it is thus surjective. f is an isomorphism. \square

4 Newton's polygon

So far we have been building 1-forms by Dirichlet principle, which is not explicit and not algebraic.

Newton's polygon's method allows to build and classify functions and forms on algebraic Riemann surfaces, by combinatorial and algebraic methods.

Let $P \in \mathbb{C}[x, y]$ a bivariate polynomial, and

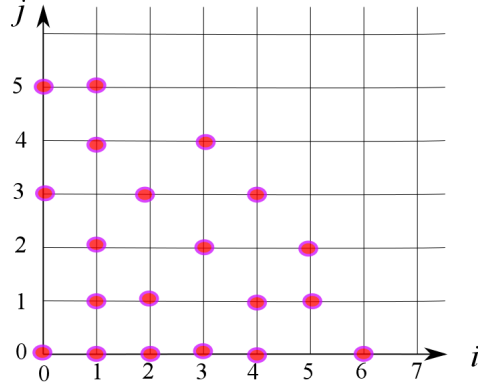
$$\tilde{\Sigma} = \{(x, y) \mid P(x, y) = 0\} \subset \mathbb{C} \times \mathbb{C}, \quad (4-1)$$

and the compact Riemann surface Σ its desingularization.

Definition 4.1 (Newton's polytope) *Let $\mathcal{N} \subset \mathbb{Z} \times \mathbb{Z}$ be the finite set of pairs (i, j) such that $P_{i,j} \neq 0$:*

$$P(x, y) = \sum_{(i,j) \in \mathcal{N}} P_{i,j} x^i y^j. \quad (4-2)$$

*The set of integer points $\mathcal{N} \subset \mathbb{Z} \times \mathbb{Z}$, is called the **Newton's polytope** of P .*



4.1 Meromorphic functions and forms

The maps $(x, y) \xrightarrow{x} x$ and $(x, y) \xrightarrow{y} y$ are meromorphic functions on Σ . In fact, they provide an algebraic basis for all meromorphic functions:

Proposition 4.1 *Any meromorphic function f on Σ is a rational function of x and y , there exists $R \in \mathbb{C}(x, y)$ such that*

$$f = R(x, y). \tag{4-3}$$

More precisely, if $d = \deg_y P$, there exists rational fractions $Q_0, Q_1, \dots, Q_{d-1} \in \mathbb{C}(x)$ such that

$$f = \sum_{k=0}^{d-1} Q_k(x) y^k. \tag{4-4}$$

proof:

This is Lagrange interpolating polynomial. Let us define

$$Q(x, y) = \sum_{Y \mid P(x, Y)=0} f(x, Y) \frac{P(x, y)}{P'_y(x, Y) (y - Y)} \tag{4-5}$$

it clearly satisfies $f(x, y) = Q(x, y)$ whenever $P(x, y) = 0$, i.e. on Σ . Moreover it is clearly a polynomial of y of degree at most $d - 1$, it can be written

$$Q(x, y) = \sum_{k=0}^{d-1} Q_k(x) y^k. \tag{4-6}$$

Its coefficients $Q_k(x)$ are analytic and meromorphic functions of x , defined on $\mathbb{C}P^1 \rightarrow \mathbb{C}P^1$. Therefore they are rational fractions: $Q_k(x) \in \mathbb{C}(x)$.

□

Since dx is a 1-form, it immediately follows that

Corollary 4.1 *Any meromorphic 1-form ω on Σ can be written as*

$$\omega(x, y) = R(x, y)dx \quad (4-7)$$

with $R(x, y) \in \mathbb{C}(x, y)$.

Up to changing $R \rightarrow R/P'_y$, it is more usual to write it in Poincaré form:

$$\omega(x, y) = R(x, y) \frac{dx}{P'_y(x, y)} \quad (4-8)$$

with $R(x, y) \in \mathbb{C}(x, y)$.

4.2 Poles and slopes of the convex envelope

The meromorphic functions x and y have poles and zeros. Let p a pole or zero of x and/or y , and let z a coordinate in a chart containing p , and such that p has coordinate $z = 0$.

If p is a pole or zero of x (resp. y) of order $-\alpha$ (resp. $-\beta$) we have

$$x(z) \sim c z^{-\alpha} \quad (\text{resp. } y(z) \sim \tilde{c} z^{-\beta}). \quad (4-9)$$

We assume that $(\alpha, \beta) \neq (0, 0)$.

Let $D_{\alpha, \beta, m}$ the line of equation

$$D_{\alpha, \beta, m} = \{(i, j) \mid \alpha i + \beta j = m\}. \quad (4-10)$$

There exists $m_{\alpha, \beta} \in \mathbb{Z}$ such that

$$m_{\alpha, \beta} = \max\{m \mid D_{\alpha, \beta, m} \cap \mathcal{N} \neq \emptyset\}. \quad (4-11)$$

In other words, the line $D_{\alpha, \beta, m_{\alpha, \beta}}$ is the leftmost line parallel to the vector $(\beta, -\alpha)$ (left with respect to the orientation of this vector), i.e. such that all the Newton's polytope lies to its right.

Theorem 4.1 (Poles and convex envelope) *The line $D_{\alpha, \beta, m_{\alpha, \beta}}$ is tangent to the convex envelope of the Newton's polytope, i.e. it contains an edge of the convex envelope of the polytope, or equivalently it contains at least 2 distinct vertices. There is a 1-1 correspondance between poles/zeros of x and/or y , and tangent segments to the convex envelope.*

proof: By definition, the line $D_{\alpha, \beta, m_{\alpha, \beta}}$ intersects the Newton's polytope, and is such that all points of the Newton's polytope lie on the right side of that line. It remains to prove that it intersects the Newton's polytope in at least 2 points. Near p we have:

$$0 = P(x, y) = \sum_{(i, j) \in \mathcal{N}} P_{i, j} x^i y^j$$

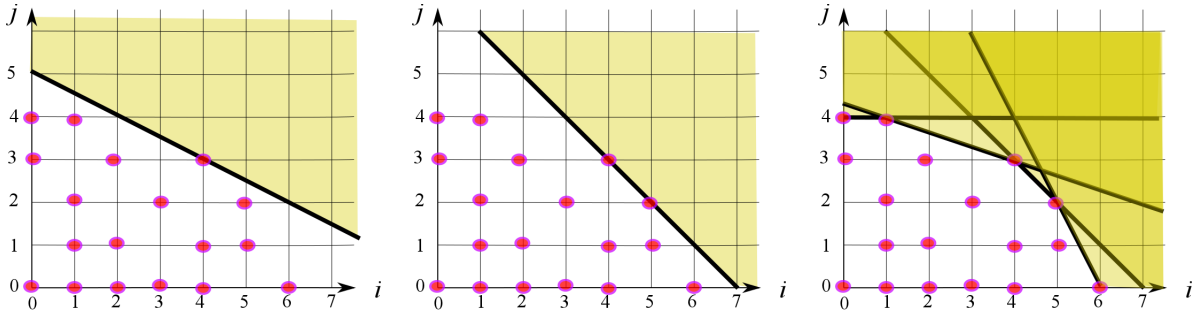
$$\begin{aligned}
&= \sum_m \sum_{(i,j) \in \mathcal{N} \cap D_{\alpha,\beta,m}} P_{i,j} x^i y^j \\
&\underset{z \rightarrow 0}{\sim} \sum_{m \leq m_{\alpha,\beta}} z^{-m} \sum_{(i,j) \in \mathcal{N} \cap D_{\alpha,\beta,m}} P_{i,j} c^i \tilde{c}^j (1 + O(z)) \\
&\underset{z \rightarrow 0}{\sim} z^{-m_{\alpha,\beta}} \sum_{(i,j) \in \mathcal{N} \cap D_{\alpha,\beta,m_{\alpha,\beta}}} P_{i,j} c^i \tilde{c}^j + O(z^{1-m_{\alpha,\beta}})
\end{aligned}
\tag{4-12}$$

If $D_{\alpha,\beta,m_{\alpha,\beta}} \cap \mathcal{N} = \{(i_p, j_p)\}$ would be a single point, we would have

$$0 \underset{z \rightarrow 0}{\sim} z^{-m_{\alpha,\beta}} P_{i_p, j_p} c^{i_p} \tilde{c}^{j_p} (1 + O(z)) \tag{4-13}$$

which can't be zero, leading to a contradiction.

□

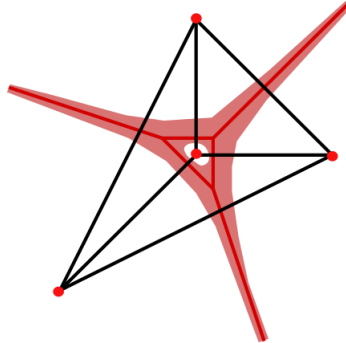


Definition 4.2 (Amoeba) The **Amoeba**, is the set $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}$ defined by:

$$\mathcal{A} = \{(\log |x|, \log |y|) \mid P(x, y) = 0\}. \tag{4-14}$$

The amoeba has asymptotic lines in directions where x and/or y have poles or zeros, these lines have slope $\frac{\beta}{\alpha}$, and are orthogonal to the tangents to the convex envelope. In fact, the amoeba looks like a thickening of a graph dual to a complete triangulation (with triangles whose vertices are on \mathbb{Z}^2 and of area $\frac{1}{2}$) of the Newton's polytope.

Example: for the equation $P(x, y) = 1 + cxy + x^2y + xy^2$, the amoeba looks like



4.3 Holomorphic forms

Since x is a meromorphic function, then dx is a meromorphic 1-form. It has poles and zeros. Near a generic branchpoint $x = a, y = b$, a local coordinate is $z = y - b$, and we have

$$x \sim a - \frac{1}{2} \frac{P''_{yy}(a, b)}{P'_x(a, b)} (y - b)^2 + O(y - b)^3, \quad (4-15)$$

so that

$$dx \sim - \frac{P''_{yy}(a, b)}{P'_x(a, b)} (y - b) dy + O(y - b)^2 dy. \quad (4-16)$$

And we have

$$P'_y(x, y) \sim (y - b) P''_{yy}(a, b) + O(y - b)^2, \quad (4-17)$$

therefore the 1-form

$$\frac{dx}{P'_y(x, y)} = - \frac{dy}{P'_x(x, y)} \quad (4-18)$$

is analytic at generic branchpoints. Can it be a holomorphic 1-form ?

Proposition 4.2 (Holomorphic forms and Newton's polytope) *We denote $\mathring{\mathcal{N}}$ the **interior** of the convex envelop of the Newton's polytope, i.e. the set of all integer points in $\mathbb{Z} \times \mathbb{Z}$ strictly inside the convex envelope of \mathcal{N} .*

Assume that the coefficients $P_{i,j}$ are generic so that $\tilde{\Sigma} = \{(x, y) \mid P(x, y) = 0\}$ has only generic branchpoints and no nodal points.

For every $(k, l) \in \mathbb{Z} \times \mathbb{Z}$, let

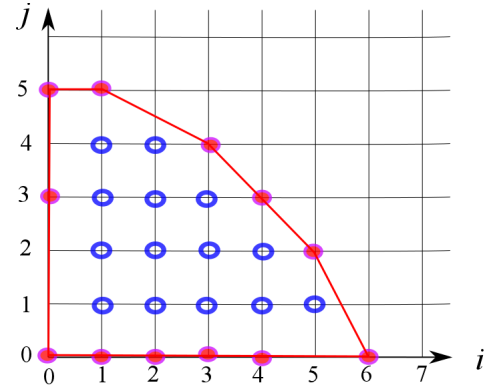
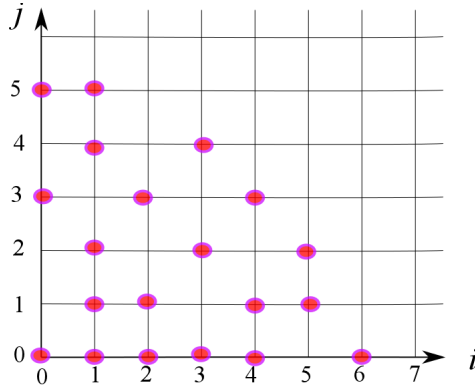
$$\omega_{(k,l)} = \frac{x^{k-1} y^{l-1}}{P'_y(x, y)} dx \quad (4-19)$$

Then

$$\omega_{(k,l)} \in \mathcal{O}^1(\Sigma) \iff (k, l) \in \mathring{\mathcal{N}} \quad (4-20)$$

It follows that

$$\mathfrak{g} = \dim \mathcal{O}^1(\Sigma) = \# \mathring{\mathcal{N}}. \quad (4-21)$$



Remark that if P is not generic, there can be nodal points, and thus some zeros of the denominator $P'_y(x, y)$ are not ramification points, and are not zeros of dx , which means that $\omega_{(k,l)}$ can have poles at nodal points. However, by taking linear combinations $\sum_{k,l} c_{k,l} \omega_{(k,l)}$, one can cancel the poles at nodal points, and thus $\mathcal{O}^1(\Sigma)$ is generated by a linear subspace of $\mathbb{C}^{\mathring{\mathcal{N}}}$, of codimension equal to the number of nodal points, i.e.:

$$\dim \mathcal{O}^1(\Sigma) = \# \mathring{\mathcal{N}} - \#\text{nodal points.} \quad (4-22)$$

On the other hand, if P is not generic, nodal points can be seen as degeneracies of cycles that have been pinched, therefore the genus has been decreased compared to the generic case, in such a way that

$$\mathfrak{g} = \# \mathring{\mathcal{N}} - \#\text{nodal points.} \quad (4-23)$$

Eventually we always have

$$\dim \mathcal{O}^1(\Sigma) = \mathfrak{g}. \quad (4-24)$$

proof: Let ω a holomorphic 1-form on Σ . ω/dx is a function, it may have poles at the zeros of dx . If we assume P generic, then dx has simple zeros, and thus ω/dx can have at most simple poles at the zeros of dx . Moreover, the function $P'_y(x, y)$ vanishes at the ramification points, therefore $P'_y(x, y)\omega/dx$ is a meromorphic function on Σ with no poles at ramification points. Its only poles can be at points where $P'_y(x, y) = \infty$, i.e. at poles of x and/or y . It can thus be written (the sum of non-vanishing terms is finite)

$$P'_y(x, y) \frac{\omega}{dx} = \sum_{(k,l) \in \mathbb{Z} \times \mathbb{Z}} c_{k,l} x^{k-1} y^{l-1}. \quad (4-25)$$

Let us see how it behaves at a pole p of x and/or y , corresponding to a tangent $D_{\alpha,\beta,m_{\alpha,\beta}}$ of the convex envelope. We have

$$P'_y(x, y) \sim z^{\beta-m_{\alpha,\beta}} \sum_{(i,j) \in D_{\alpha,\beta,m_{\alpha,\beta}} \cap \mathcal{N}} j P_{i,j} c^i \tilde{c}^{j-1}. \quad (4-26)$$

We also have $dx \sim -\alpha c z^{-\alpha-1} dz$ and therefore

$$\omega \sim \sum_{k,l} c_{k,l} z^{m_{\alpha,\beta} - k\alpha - l\beta - 1} dz \left(\frac{-1}{\alpha} \sum_{(i,j) \in D_{\alpha,\beta,m_{\alpha,\beta}} \cap \mathcal{N}} j P_{i,j} c^{i-1} \tilde{c}^{j-1} \right)^{-1}. \quad (4-27)$$

Since ω has no pole at p , i.e. at $z = 0$, we must have

$$k\alpha + l\beta \leq m_{\alpha,\beta} - 1 < m_{\alpha,\beta}. \quad (4-28)$$

In other words, the point (k, l) must be strictly below the line $D_{\alpha, \beta, m_{\alpha, \beta}}$. Since this must be true for all poles, i.e. all tangents to the convex envelope, we deduce that

$$(k, l) \in \overset{\circ}{\mathcal{N}}. \quad (4-29)$$

□

As an immediate corollary of the proof we have

Theorem 4.2 (classification) *Let $(k, l) \in \mathbb{Z}^2$ two integers. The 1-form*

$$\omega_{k, l} = \frac{x^{k-1} y^{l-1} dx}{P'_y(x, y)} \quad (4-30)$$

is

- 1st kind iff (k, l) is strictly interior,
- 3rd kind iff (k, l) belongs to the boundary, it is thus at the intersection of 2 tangent segments, corresponding to 2 poles p_1, p_2 . It is then a 3rd kind form with simple poles only at p_1 and p_2 .
- 2nd kind if (k, l) is strictly exterior. It has poles at all tangents that are to its right (it is to the left of). The degree of the pole of $\omega_{k, l}$ at a pole $p = (\alpha, \beta)$ of corresponding tangent line $D_{\alpha, \beta, m_{\alpha, \beta}}$ is

$$- \text{order}_p \omega_{k, l} = \alpha k + \beta l - m_{\alpha, \beta} + 1 \geq 2. \quad (4-31)$$

4.4 Fundamental second kind form

We defined fundamental second kind forms in def. 3.2. We recall that $B(z, z')$ is a symmetric $1 \otimes 1$ form on $\Sigma \times \Sigma$ that has a double pole at $z = z'$ and no other pole, and such that

$$B(z, z') \sim \frac{dz dz'}{(z - z')^2} + \text{analytic at } z = z'. \quad (4-32)$$

If the genus is ≥ 1 , it is not unique since we can add to it any symmetric bilinear combination of holomorphic 1-forms:

$$B(z, z') \rightarrow B(z, z') + \sum_{i, j=1}^g \kappa_{i, j} \omega_i(z) \otimes \omega_j(z'). \quad (4-33)$$

Remark that the $1 \otimes 1$ form

$$- \frac{\frac{P(x', y)P(x, y')}{(x-x')^2(y-y')^2}}{P'_y(x, y) P'_y(x', y')} dx dx' \quad (4-34)$$

is a symmetric $1 \otimes 1$ form, it has a double pole at $(x, y) = (x', y')$, with the behavior $\frac{dx dx'}{(x-x')^2}$, and it has no pole if $x = x'$ and $y \neq y'$ nor if $y = y'$ but $x \neq x'$, so it is a good candidate for B . However, it can have poles where P has poles, i.e. at x or x' or y or $y' = \infty$. Therefore the form

$$\tilde{B}((x, y), (x', y')) = B((x, y), (x', y')) + \frac{\frac{P(x', y)P(x, y')}{(x-x')^2(y-y')^2}}{P'_y(x, y) P'_y(x', y')} dx dx' \quad (4-35)$$

must be a symmetric 1-form \otimes 1-form, whose poles can be only at $x, x', y, y' = \infty$. There must exist a polynomial

$$P'_y(x, y) P'_y(x', y') \frac{\tilde{B}((x, y), (x', y'))}{dx dx'} = Q(x, y; x', y') \in \mathbb{C}[x, y; x', y']. \quad (4-36)$$

At fixed (x', y') , this polynomial must be such that

$$\left(\frac{P(x', y)P(x, y')}{(x-x')^2(y-y')^2} \right)_+ - Q(x, y; x', y') \quad (4-37)$$

has monomials $x^{u-1}y^{v-1}$ only inside the Newton's polygon, i.e. only if $(u, v) \in \mathring{\mathcal{N}}$.

General case

Proposition 4.3 (Fundamental 2nd kind form) *the following $1 \otimes 1$ form*

$$B_0((x, y); (x', y')) = - \frac{\frac{P(x, y')P(x', y)}{(x-x')^2(y-y')^2} - Q(x, y; x', y')}{P_y(x, y)P_y(x', y')} dx dx' \quad (4-38)$$

where $Q \in \mathbb{C}[x, y, x', y']$ is a polynomial

$$\begin{aligned} Q(x, y; x', y') &= \sum_{(i,j) \in \mathcal{N}} \sum_{(i',j') \in \mathcal{N}} P_{i,j} P_{i',j'} \sum_{(u,v) \in \mathbb{Z}^2 \cap \text{triangle } (i,j),(i',j') \text{ and } (i,j')} |u-i| |v-j'| \\ &\left(\delta_{(u,v) \notin \mathring{\mathcal{N}} \cup [(i,j),(i',j')]} x^{u-1} y^{v-1} x^{i+i'-u-1} y^{j+j'-v-1} \right. \\ &+ \delta_{(u,v) \notin \mathring{\mathcal{N}} \text{ and } (i+i'-u, j+j'-v) \in \mathring{\mathcal{N}}} x^{u-1} y^{v-1} x^{i+i'-u-1} y^{j+j'-v-1} \\ &\left. + \frac{1}{2} \delta_{(u,v) \in [(i,j),(i',j')]} x^{u-1} y^{v-1} x^{i+i'-u-1} y^{j+j'-v-1} \right). \quad (4-39) \end{aligned}$$

has a double pole at $(x, y) = (x', y')$ with behaviour (4-32), it has no pole if $x = x'$ and $y \neq y'$, it has no pole if $y = y'$ and $x \neq x'$, and it has no pole at the poles/zeros of x and y .

Its only possible poles could be at zeros of $P'_y(x, y)$ that are not zeros of dx , if these exist, i.e. these are common zeros of $P'_y(x, y)$ and $P'_x(x, y)$, and these are nodal points.

Generically, there is no nodal point, and the expression above is the fundamental second kind differential. If nodal points exist, one can add to Q a symmetric bilinear

combination of monomials belonging to the interior of Newton's polygon, that would exactly cancel these unwanted poles.

proof: When $x \rightarrow x'$ and $y \rightarrow y'$, we have $P(x, y') \sim (y' - y)P'_y(x, y)$ and $P(x', y) \sim (y - y')P'_y(x', y')$, so that expression (4-38) has a pole of the type (4-32).

When $y \rightarrow y'$ and $x \neq x'$, we have $P(x, y') \sim (y' - y)P'_y(x, y)$ and $P(x', y) \sim (y - y')P'_y(x', y)$, so that expression (4-38) behaves as

$$\frac{-P'_y(x, y)P'_y(x', y)}{(x - x')^2} \quad (4-40)$$

which has no pole at $x \neq x'$. Same thing for $x \rightarrow x'$ with $y \neq y'$.

It remains to study the behaviors at poles/zeros of x and/or y . Let us consider a point where both x and y have a pole (the other cases, can be obtained by changing $x \rightarrow 1/x$ and/or $y \rightarrow 1/y$, and remarking that expression (4-38) is unchanged under these changes). At a pole $x \rightarrow \infty$ and $y \rightarrow \infty$, we have

$$\begin{aligned} \frac{P(x, y')P(x', y)}{(x - x')^2(y - y')^2} &\sim \sum_{(i,j) \in \mathcal{N}} \sum_{(i',j') \in \mathcal{N}} \sum_{k \geq 1} \sum_{l \geq 1} P_{i,j} P_{i',j'} kl x^{i-k-1} y'^{j+l-1} x'^{i'+k-1} y^{j'-l-1} \\ &\sim \sum_{(u,v) \in \mathbb{Z}_+^2} x^{u-1} y^{v-1} \left(\sum_{(i,j) \in \mathcal{N}} \sum_{(i',j') \in \mathcal{N}} \sum_{k \geq 1} \sum_{l \geq 1} P_{i,j} P_{i',j'} kl \right. \\ &\quad \left. \delta_{u,i-k} \delta_{v,j'-l} x'^{i'+k-1} y'^{j+l-1} \right) \end{aligned} \quad (4-41)$$

where the last bracket contains in fact a finite sum. All the monomials such that $(u, v) \notin \overset{\circ}{\mathcal{N}}$ and (u, v) is at the NE of the Newton's polygon, would yield a pole, and must be compensated by a term in Q .

Let us consider such an (u, v) monomial that enters in Q . Notice that (u, v) at the NE of the Newton's polygon implies that $u = i - k \geq i'$ and $v = j' - l \geq j$, which implies in particular that this can occur only if $i > i'$ and $j' > j$. Moreover, since all the line $[(i, j), (i', j')]$ is contained in the Newton's polygon, we see that (u, v) must belong to the triangle $((i, j), (i', j'), (i', j))$.

Consider the point $(u', v') = (i' + k, j + l) = (i + i' - u, j + j' - v)$, which is the symmetric of (u, v) with respect to the middle of $[(i, j), (i', j')]$.

Let us thus assume that $u > i'$ and $j' > j$ and (u, v) belongs to the triangle $((i, j), (i', j'), (i', j))$, and let us consider different cases:

- $(u, v) \notin [(i, j), (i', j')]$. If $(u, v) \notin \overset{\circ}{\mathcal{N}}$, then the monomial $P_{i,j} P_{i',j'} kl x^{u-1} y^{v-1} x'^{u'-1} y'^{v'-1}$ should appear in Q and is indeed the first term in (4-38). Notice that in that case the point (u', v') can't be at the NE of Newton's polygon. There are then 2 sub-cases:

- $(u', v') \in \overset{\circ}{\mathcal{N}}$, then we can add to Q a monomial proportional to $x^{u'-1} y^{v'-1}$ without

changing the pole property of B . In particular we can add

$$P_{i,j}P_{i',j'} kl x^{u'-1}y^{v'-1}x^{u-1}y^{v-1} \quad (4-42)$$

which is the second term in (4-38). It allows to make Q symmetric under the exchange $(x, y) \leftrightarrow (x', y')$.

•• $(u', v') \notin \mathcal{N}$. Notice that since $(u, v) \notin [(i, j), (i', j')]$, we also have $(u', v') \notin [(i, j), (i', j')]$. Moreover, if $(u', v') \notin \mathring{\mathcal{N}}$, this implies that (u', v') is a SW of Newton's polygon. This means that the monomial $P_{i,j}P_{i',j'}klx^{u'-1}y^{v'-1}x^{u-1}y^{v-1}$ will appear in Q in the contribution with $(i, j) \leftrightarrow (i', j')$.

• $(u, v) \in [(i, j), (i', j')]$. This implies that $(u', v') \in [(i, j), (i', j')]$ as well. Remark- ing that if $(u, v) \in [(i, j), (i', j')]$, we have $kl = (i - u)(j' - v) = (u - i')(v - j)$, we have

$$P_{i,j}P_{i',j'}(i - u)(j' - v)x^{u-1}y^{v-1}x^{u'-1}y^{v'-1} = P_{i',j'}P_{i,j}(i' - u)(j - v)x^{u-1}y^{v-1}x^{u'-1}y^{v'-1} \quad (4-43)$$

i.e. this monomial appears twice in the sum (4-38) because it also appears in the term $(i, j) \leftrightarrow (i', j')$, and this is why it has to be multiplied by $\frac{1}{2}$.

Also, if $(u, v) \in [(i, j), (i', j')]$, we have $kl = (i - u)(j' - v) = (i - u')(j' - v')$, the monomial $P_{i,j}P_{i',j'}klx^{u'-1}y^{v'-1}x^{u-1}y^{v-1}$ also appears in (4-38).

Also, if $(u, v) \in \mathring{\mathcal{N}} \cap [(i, j), (i', j')]$, this implies that $(u', v') \in \mathring{\mathcal{N}} \cap [(i, j), (i', j')]$, and thus this monomial and its symmetric under $(x, y) \leftrightarrow (x', y')$ are both inside Newton's polygon, so don't contribute to poles of B .

Eventually we have shown that the polynomial of (4-38) is symmetric under $(x, y) \leftrightarrow (x', y')$, and up to monomials inside $\mathring{\mathcal{N}}$, it compensates all the terms of $\frac{P(x,y)P(x',y)}{(x-x')^2(y-y')^2}$ that could possibly diverge.

This concludes the proof. \square

Hyperelliptical case

Proposition 4.4 (Hyperelliptical curves) *Consider the case $P(x, y) = y^2 - Q(x)$, with $Q(x) \in \mathbb{C}[x]$ a polynomial of even degree, whose zeros are all distinct. Let $U(x) = (\sqrt{Q(x)})_+$ be the polynomial part near ∞ of its square-root, and let $V(x) = Q(x) - U(x)^2$. We then have*

$$B((x, y); (x', y')) = \frac{yy' + U(x)U(x') + \frac{1}{2}V(x) + \frac{1}{2}V(x')}{2yy'(x - x')^2} dx dx'. \quad (4-44)$$

It is a 1-form of $z = (x, y) \in \Sigma$, with a double pole at $(x, y) = (x', y')$, and no other pole, in particular no pole at $(x, y) = (x', -y')$.

Chapter 3

Abel map, Jacobian and Theta function

1 Holomorphic forms

We recall that we called $\mathcal{O}^1(\Sigma)$ the vector space of holomorphic 1-forms (no poles) on Σ . We also call $H_1(\Sigma, \mathbb{Z})$ the \mathbb{Z} -module (resp. $H_1(\Sigma, \mathbb{C})$ the \mathbb{C} -vector space) of cycles, and for a surface of genus g we have

$$\dim H_1(\Sigma, \mathbb{Z}) = \dim H_1(\Sigma, \mathbb{C}) = 2g. \quad (1-1)$$

1.1 Symplectic basis of cycles

We admit that, if $g \geq 1$, it is always possible to choose a

Definition 1.1 (symplectic basis of cycles) of $H_1(\Sigma, \mathbb{Z})$:

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j} \quad , \quad \mathcal{A}_i \cap \mathcal{A}_j = 0 \quad , \quad \mathcal{B}_i \cap \mathcal{B}_j = 0 . \quad (1-2)$$

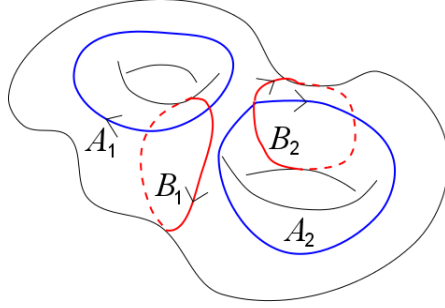
A choice of symplectic basis of cycles, is called a **marking** or **Torelli marking** of Σ .

In this definition, the **intersection numbers** are counted algebraically (taking the orientation into account):

$$\gamma \cap \gamma' = \sum_{p \in \gamma \cap \gamma'} \pm 1 \quad (1-3)$$

where at a crossing point p , ± 1 is $+1$ if the oriented γ' crosses γ from its right to its left, and -1 otherwise. The intersection number is invariant under homotopic deformations, is compatible with addition by concatenation, and thus descends to the homology classes by linearity.

We insist that a choice of symplectic basis is *not unique*.



1.2 Small genus

Theorem 1.1 (Riemann sphere) *There is no non-identically-vanishing holomorphic 1-form on the Riemann sphere:*

$$\mathcal{O}^1(\mathbb{C}P^1) = \{0\} \quad , \quad \dim \mathcal{O}^1(\mathbb{C}P^1) = 0. \quad (1-4)$$

proof: write $\omega(z) = f(z)dz = -z'^{-2}f(1/z')dz'$ with $z' = 1/z$. We want $f(z)$ to have no pole in the chart \mathbb{C} , so $f(z)$ could only be a polynomial, and we want $z'^{-2}f(1/z')$ to have no pole at $z' = 0$, which implies the polynomial should be of degree ≤ -2 which is not possible. \square

In fact this applies to every genus zero curve (but as we shall see later, every genus zero Riemann surface is isomorphic to the Riemann sphere):

Theorem 1.2 (Genus zero) *There is no non-identically-vanishing holomorphic 1-form on a curve Σ of genus 0:*

$$\mathcal{O}^1(\Sigma) = \{0\} \quad , \quad \dim \mathcal{O}^1(\Sigma) = 0. \quad (1-5)$$

proof: Let ω a holomorphic 1-form on Σ . Choose a base point $o \in \Sigma$, and define the function

$$f(p) = \int_o^p \omega. \quad (1-6)$$

The function f is well defined, in particular is independent of the integration path chosen to go from o to p , since Σ is simply connected. The function f is then a holomorphic function on Σ , and from theorem II-2.6, it must be constant, which implies $\omega = df = 0$. \square

Theorem 1.3 (Torus) *On the torus $T_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$*

$$\mathcal{O}^1(\Sigma) \sim \mathbb{C} \quad , \quad \dim \mathcal{O}^1(\Sigma) = 1. \quad (1-7)$$

proof: We know that dz is a holomorphic 1-form. If $\omega(z) = f(z)dz$ is another holomorphic 1-form, we must have $f(z) = f(z+1) = f(z+\tau)$, and f can have no pole, from theorem II-2.6 it must be a constant, and thus $\mathcal{O}^1(\Sigma) \sim \mathbb{C}$. \square

1.3 Higher genus ≥ 1

Let $\mathcal{A}_1, \dots, \mathcal{A}_{2g}$ be a basis of $H_1(\Sigma, \mathbb{Z})$.

Theorem 1.4 *The space of real harmonic 1-forms $\mathcal{H}(\Sigma)$ has real dimension:*

$$\dim_{\mathbb{R}} \mathcal{H}(\Sigma) = 2g. \quad (1-8)$$

The space of complex holomorphic 1-forms $\mathcal{O}^1(\Sigma)$ has complex dimension:

$$\dim_{\mathbb{C}} \mathcal{O}^1(\Sigma) = g. \quad (1-9)$$

proof: We have already proved that the dimension of the space of real harmonic forms is $2g$. If ν is a real harmonic form, then $\omega = \nu + i * \nu$ is a complex holomorphic 1-form, such that $\Re \omega = \nu$. Therefore the map

$$\begin{aligned} \mathcal{H}(\Sigma) &\rightarrow \mathcal{O}^1(\Sigma) \\ \nu &\mapsto \nu + i * \nu \\ \Re \omega &\leftarrow \omega \end{aligned} \quad (1-10)$$

is an invertible isomorphism of real vector spaces. This implies that $\dim_{\mathbb{R}} \mathcal{O}^1(\Sigma) = 2g$. Moreover, $\mathcal{O}^1(\Sigma)$ is clearly a complex vector space, and thus its dimension over \mathbb{C} is half of its dimension over \mathbb{R} , therefore it is g . \square

1.4 Riemann bilinear identity

The **Riemann bilinear identity** is the key to many theorems, let us state it and prove it here.

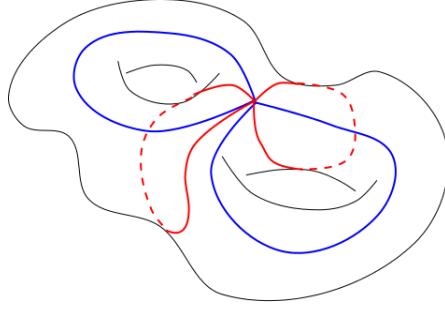
Let ω and $\tilde{\omega}$ be 2 meromorphic forms.

Let $\mathcal{A}_i, \mathcal{B}_i$ be Jordan arcs representative of a symplectic basis of cycles, chosen in a way that they all intersect (transversally) at the same unique point. We admit that it is always possible. Also up to homotopic deformations, we chose them to avoid all singularities of ω and $\tilde{\omega}$. Let

$$\Sigma_0 = \Sigma \setminus \cup_i \mathcal{A}_i \cup_i \mathcal{B}_i. \quad (1-11)$$

By definition of a basis of non-contractible cycles, Σ_0 is a simply connected domain of Σ , called a **fundamental domain**, it is bounded by the cycles $\mathcal{A}_i, \mathcal{B}_i$, its boundary is

$$\partial \Sigma_0 = \sum_i \mathcal{A}_i \text{ left} - \mathcal{A}_i \text{ right} + \sum_i \mathcal{B}_i \text{ left} - \mathcal{B}_i \text{ right}. \quad (1-12)$$



Lemma 1.1 *Let U a tubular neighborhood of $\partial\Sigma_0$ in Σ_0 , that contains no pole of ω , and let $o \in U$ a generic point. Then $f(p) = \int_o^p \omega$ is independent of a choice of integration path from o to p in U . $f(p)$ is a holomorphic function on U , such that*

$$df = \omega \quad \text{on } U. \quad (1-13)$$

Moreover f can be analytically continued to the boundary of U .

proof: A priori the integral $\int_o^p \omega$ depends on the path from o to p in U . Topologically Σ_0 is a disc, its boundary is a circle and its tubular neighborhood U is an annulus. There are 2 homotopically independent paths γ_+, γ_- from o to p in U . The difference between the integrals along the 2 independent paths, is

$$\begin{aligned} \int_{\gamma_+} \omega - \int_{\gamma_-} \omega &= \oint_{\gamma_+ - \gamma_-} \omega \\ &= 2\pi i \sum_{q=\text{poles of } \omega} \text{Res}_q \omega \\ &= 0, \end{aligned} \quad (1-14)$$

thanks to theorem II-2.3. Therefore $f(p)$ is independent of the path chosen, it defines a function on U . It clearly satisfies $df = \omega$ and is thus holomorphic on U . \square

Theorem 1.5 (Riemann bilinear identity) *Let ω and $\tilde{\omega}$ be 2 meromorphic forms on Σ , and let f, \tilde{f} be 2 functions, holomorphic on a tubular neighborhood of $\partial\Sigma_0$ in $\Sigma_0 - \text{poles}$, such that*

$$df = \omega, \quad d\tilde{f} = \tilde{\omega}. \quad (1-15)$$

Then we have

$$\oint_{\partial\Sigma_0} f\tilde{\omega} = - \oint_{\partial\Sigma_0} \tilde{f}\omega = \sum_{i=1}^g \oint_{\mathcal{A}_i} \omega \oint_{\mathcal{B}_i} \tilde{\omega} - \oint_{\mathcal{B}_i} \omega \oint_{\mathcal{A}_i} \tilde{\omega}. \quad (1-16)$$

Remark: Depending on our choice of ω and $\tilde{\omega}$, the contour integral on the left hand side can often be contracted to surround only singularities of f or of $\tilde{\omega}$, and eventually reduced to a sum of residues.

proof: Observe that $\tilde{\omega}$ is continuous across any cycles, it takes the same value on left and right, whereas f can have a discontinuity:

$$\begin{aligned} \text{on } \mathcal{A}_i \quad f_{\text{left}} - f_{\text{right}} &= - \oint_{\mathcal{B}_i} \omega \\ \text{on } \mathcal{B}_i \quad f_{\text{left}} - f_{\text{right}} &= \oint_{\mathcal{A}_i} \omega. \end{aligned} \quad (1-17)$$

Inserted into (1-12) this immediately yields the theorem. \square

Corollary 1.1 *If ω and $\tilde{\omega}$ are both holomorphic, then the left hand side can be contracted to 0, and*

$$\sum_i \oint_{\mathcal{A}_i} \omega \oint_{\mathcal{B}_i} \tilde{\omega} - \oint_{\mathcal{B}_i} \omega \oint_{\mathcal{A}_i} \tilde{\omega} = 0. \quad (1-18)$$

Theorem 1.6 (Riemann bilinear inequality) *If ω is a non-identically-vanishing holomorphic 1-form we have*

$$2i \left(\sum_i \oint_{\mathcal{A}_i} \omega \oint_{\mathcal{B}_i} \bar{\omega} - \oint_{\mathcal{B}_i} \omega \oint_{\mathcal{A}_i} \bar{\omega} \right) > 0. \quad (1-19)$$

proof: Observe that the $L^2(\Sigma)$ norm of ω is

$$\|\omega\|^2 = 2i \int_{\Sigma} \omega \wedge \bar{\omega} > 0. \quad (1-20)$$

Use Stokes theorem on the fundamental domain Σ_0 :

$$\int_{\Sigma} \omega \wedge \bar{\omega} = \int_{\Sigma_0} \omega \wedge \bar{\omega} = - \int_{\partial \Sigma_0} \bar{f} \omega \quad (1-21)$$

with $\omega = df$ on Σ_0 . Using (1-12) and (1-17) gives

$$\int_{\partial \Sigma_0} \bar{f} \omega = \sum_i \oint_{\mathcal{B}_i} \omega \oint_{\mathcal{A}_i} \bar{\omega} - \oint_{\mathcal{A}_i} \omega \oint_{\mathcal{B}_i} \bar{\omega}. \quad (1-22)$$

\square

2 Normalized basis

Theorem 2.1 (Normalized basis of holomorphic forms) *Given a symplectic basis of cycles, there exists a unique basis $\omega_1, \dots, \omega_{\mathfrak{g}}$ of $\mathcal{O}^1(\Sigma)$ such that*

$$\forall i = 1, \dots, \mathfrak{g}, \quad \oint_{\mathcal{A}_i} \omega_j = \delta_{i,j}. \quad (2-1)$$

proof: Define the map

$$\begin{aligned} \epsilon : \mathcal{O}^1(\Sigma) &\rightarrow \mathbb{C}^{\mathfrak{g}} \\ \omega &\mapsto \epsilon_i = \oint_{\mathcal{A}_i} \omega. \end{aligned} \quad (2-2)$$

We shall prove that it is an isomorphism. Since the dimension of the spaces on both sides are the same, it suffices to prove that the kernel vanishes. Let us also denote

$$\tilde{\epsilon}_i = \oint_{\mathcal{B}_i} \omega. \quad (2-3)$$

The Riemann bilinear inequality of theorem 1.6 implies that if $\omega \neq 0$ we have

$$\Im\left(\sum_{i=1}^{\mathfrak{g}} \epsilon_i \tilde{\epsilon}_i\right) < 0, \quad (2-4)$$

and therefore the vector $(\epsilon_1, \dots, \epsilon_{\mathfrak{g}})$ can't vanish. This implies that $\text{Ker } \epsilon = 0$, and thus ϵ is invertible.

The normalized basis is:

$$\omega_i = \epsilon^{-1}(\{\delta_{i,j}\}_{j=1,\dots,\mathfrak{g}}). \quad (2-5)$$

□

Then we define

Definition 2.1 (Riemann matrix of periods) *The $\mathfrak{g} \times \mathfrak{g}$ matrix*

$$\tau_{i,j} = \oint_{\mathcal{B}_i} \omega_j \quad (2-6)$$

*is called the **Riemann matrix of periods**.*

We shall now prove that the matrix τ is a **Siegel matrix**: τ is symmetric and $\Im\tau$ is positive definite. The proof relies on the Riemann bilinear identity.

Corollary 2.1 (Period \implies Siegel matrix) *The $\mathfrak{g} \times \mathfrak{g}$ matrix of periods $\tau_{i,j}$ is symmetric and its imaginary part is positive definite.*

Remark 2.1 The converse is not true, not all Siegel matrices are Riemann periods of Riemann surfaces. The subset of Siegel matrices that are periods of Riemann surfaces is characterized by the Krichever–Novikov conjecture, later proved by T. Shiota [11]. For genus $\mathfrak{g} = 1$, every 1×1 Siegel matrix τ (i.e. a complex number whose imaginary part is > 0) is a Riemann period, namely the Riemann period of the torus T_τ . This starts being wrong for $\mathfrak{g} > 2$.

proof: Indeed Choosing $\omega = \omega_i$ and $\tilde{\omega} = \omega_j$ in (1-18) gives

$$\tau_{i,j} - \tau_{j,i} = 0 \quad (2-7)$$

and thus τ is a symmetric matrix. Choosing $\omega = \omega_i$ in (1-19) gives

$$2i(\bar{\tau}_{i,i} - \tau_{i,i}) > 0. \quad (2-8)$$

More generally, let $c \in \mathbb{R}^g - \{0\}$, then choosing $\omega = \sum_i c_i \omega_i$ in (1-19) yields

$$\sum_{i,j} c_i \Im \tau_{i,j} c_j > 0 \quad (2-9)$$

i.e.

$$\Im \tau > 0. \quad (2-10)$$

□

3 Abel map and Theta functions

Let Σ^{univ} a universal cover of Σ , and Σ_0 a fundamental domain.

Definition 3.1 We define the map

$$\begin{aligned} \mathbf{u} : \Sigma^{\text{univ}} &\rightarrow \mathbb{C}^g \\ p &\mapsto \mathbf{u}(p) = (u_1(p), \dots, u_g(p)) \quad , \quad u_i(p) = \int_o^p \omega_i. \end{aligned} \quad (3-1)$$

We also denote, by the same name \mathbf{u} , the quotient modulo $\mathbb{Z}^g + \tau\mathbb{Z}^g$, which is then defined on Σ rather than Σ^{univ}

$$\begin{aligned} \mathbf{u} : \Sigma &\rightarrow \mathbb{J} = \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g) \\ p &\mapsto \mathbf{u}(p) = (u_1(p), \dots, u_g(p)) \pmod{\mathbb{Z}^g + \tau\mathbb{Z}^g}. \end{aligned} \quad (3-2)$$

It is called the **Abel map**. The $2g$ -dimensional torus $\mathbb{J} = \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g)$ is called the **Jacobian** of Σ .

Definition 3.2 (Abel map for divisors) If $D = \sum_i \alpha_i p_i$ is a divisor, we define by linearity

$$\mathbf{u}(D) = \sum_i \alpha_i \mathbf{u}(p_i). \quad (3-3)$$

Definition 3.3 (Riemann Theta function) If τ belongs to the g -dimensional **Siegel** space (symmetric complex matrices whose imaginary part is positive definite), then the map

$$\Theta : \mathbb{C}^g \rightarrow \mathbb{C}$$

$$\mathbf{u} \mapsto \Theta(\mathbf{u}, \tau) \stackrel{\text{def}}{=} \sum_{\mathbf{n} \in \mathbb{Z}^g} e^{2\pi i(\mathbf{n}, \mathbf{u})} e^{\pi i(\mathbf{n}, \tau \mathbf{n})} \quad (3-4)$$

is analytic on \mathbb{C}^g (the sum is absolutely convergent for all $u \in \mathbb{C}^g$).

Lemma 3.1 *It satisfies*

$$\begin{aligned} \Theta(-\mathbf{u}) &= \Theta(\mathbf{u}) \\ \mathbf{n} \in \mathbb{Z}^g, \quad \Theta(\mathbf{u} + \mathbf{n}) &= \Theta(\mathbf{u}) \\ \mathbf{n} \in \mathbb{Z}^g, \quad \Theta(\mathbf{u} + \tau \mathbf{n}) &= \Theta(\mathbf{u}) e^{-2\pi i(\mathbf{n}, \mathbf{u})} e^{-\pi i(\mathbf{n}, \tau \mathbf{n})} \\ \mathbf{a}, \mathbf{b} \in \mathbb{Z}^g \times \mathbb{Z}^g, \quad (\mathbf{a}, \mathbf{b}) \in 2\mathbb{Z} + 1 &\implies \Theta\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\tau\mathbf{b}\right) = 0 \end{aligned} \quad (3-5)$$

proof: Easy computations. \square

By composing the Abel map $\Sigma^{\text{univ}} \rightarrow \mathbb{C}^g$ together with the Theta function $\mathbb{C}^g \rightarrow \mathbb{C}$, we can build complex functions on Σ^{univ} , and if we take "good" combinations, they can sometimes be defined on Σ rather than Σ^{univ} .

Theta functions will serve as building blocks for any meromorphic functions.

As we shall see, Theta functions will be to meromorphic functions, what linear functions are to rational fractions (ratios of products of linear functions) for genus 0.

We define

Definition 3.4 *Let q a generic point of Σ , and $\zeta \in \mathbb{C}^g$. Define the map $\Sigma_0 \rightarrow \mathbb{C}$:*

$$p \mapsto g_{\zeta, q}(p) = \Theta(\mathbf{u}(p) - \mathbf{u}(q) + \zeta). \quad (3-6)$$

Lemma 3.2 *Let ζ a zero of the function Θ , i.e. $\Theta(\zeta) = 0$. If $(\Theta'_1(\zeta), \dots, \Theta'_g(\zeta)) = 0$, then ζ is called a singular zero.*

If ζ is a singular zero, then the function $g_{\zeta, q}$ vanishes identically on Σ_0 for any q . If ζ is not singular, then the map $\Sigma_0 \rightarrow \mathbb{C}$, $p \mapsto g_{\zeta, q}(p)$ has g zeros on Σ_0 . One of these zeros is q , and let P_1, \dots, P_{g-1} the other zeros. The Abel map of the divisor $P_1 + \dots + P_{g-1}$ is

$$\mathbf{u}(P_1) + \dots + \mathbf{u}(P_{g-1}) = K - \zeta \quad (3-7)$$

where K is the **Riemann's constant**:

$$K_k = \frac{-\tau_{k,k}}{2} - \sum_i \oint_{\mathcal{A}_i} u_i du_k. \quad (3-8)$$

proof: It again uses Riemann bilinear identity. Let Σ_0 a fundamental domain. If $g_{\zeta,a}$ is not identically vanishing, we have

$$\frac{1}{2\pi i} \int_{\partial\Sigma_0} d \log g_{\zeta,q}(p) = \#\text{zeros of } g_{\zeta,q}(p). \quad (3-9)$$

Then, as in the Riemann bilinear identity, we decompose the boundary as (1-12), and use the fact that $g_{\zeta,q}(p)$ is continuous across the cycle \mathcal{B}_i and $\log g_{\zeta,q}(p)$ has a discontinuity across the \mathcal{A}_i boundary given by (3-5), thus equal to

$$\log g_{\zeta,q}(p)_{\mathcal{A}_i \text{ left}} - \log g_{\zeta,q}(p)_{\mathcal{A}_i \text{ right}} = 2\pi i (u_i(p) - u_i(q) + \zeta_i + \frac{1}{2}\tau_{i,i}) \quad (3-10)$$

and thus taking the differential:

$$d \log g_{\zeta,q}(p)_{\mathcal{A}_i \text{ left}} - d \log g_{\zeta,q}(p)_{\mathcal{A}_i \text{ right}} = 2\pi i du_i(p). \quad (3-11)$$

It follows that

$$\#\text{zeros of } g_{\zeta,q} = \sum_{i=1}^g \oint_{\mathcal{A}_i} du_i = \mathfrak{g}. \quad (3-12)$$

Clearly, since $\Theta(\zeta) = 0$ we have $g_{\zeta,q}(q) = 0$. We call $P_{\mathfrak{g}} = q$ and $P_1, \dots, P_{\mathfrak{g}-1}$ the other zeros.

Moreover, we have

$$\frac{1}{2\pi i} \int_{\partial\Sigma_0} u_k(p) d \log g_{\zeta,q}(p) = \sum_{j=1}^g u_k(P_j) = u_k(q) + u_k\left(\sum_{j=1}^{\mathfrak{g}-1} P_j\right). \quad (3-13)$$

Integrating by parts

$$\frac{1}{2\pi i} \int_{\partial\Sigma_0} u_k(p) d \log g_{\zeta,q}(p) = -\frac{1}{2\pi i} \int_{\partial\Sigma_0} \log g_{\zeta,q}(p) du_k(p) \quad (3-14)$$

and using the discontinuity of $\log g_{\zeta,q}$ across \mathcal{A}_i , we have

$$\begin{aligned} u_k\left(\sum_{j=1}^g P_j\right) &= -\sum_i \oint_{p \in \mathcal{A}_i} du_k(p) (u_i(p) - u_i(q) + \zeta_i + \frac{1}{2}\tau_{i,i}) \\ &= u_k(q) - \zeta_k - \frac{\tau_{k,k}}{2} - \sum_i \oint_{p \in \mathcal{A}_i} u_i du_k \\ &= u_k(q) + K_k - \zeta_k \end{aligned} \quad (3-15)$$

where K is the Riemann's constant. \square

Lemma 3.3 (Theta divisor) *The set (Θ) of zeros of Θ , called the **Theta divisor** is a submanifold of \mathbb{C}^g of dimension $\mathfrak{g} - 1$. Let $W_{\mathfrak{g}-1}$ be the set of divisors sums of $\mathfrak{g} - 1$ points. The map*

$$W_{\mathfrak{g}-1} \rightarrow (\Theta)$$

$$D \mapsto K - \mathbf{u}(D) \quad (3-16)$$

is an isomorphism. In other words, every zero ζ of Θ can be uniquely written (modulo $\mathbb{Z}^g + \tau\mathbb{Z}^g$) as (minus) the Abel map of a sum of $g-1$ points shifted by K , and vice-versa, the Abel map of any sum of $g-1$ points, shifted by K is a zero of Θ .

$$\forall \zeta \mid \Theta(\zeta) = 0 \quad \exists! D = P_1 + \cdots + P_{g-1} \mid \zeta = K - \mathbf{u}(D). \quad (3-17)$$

$$\forall D = P_1 + \cdots + P_{g-1} \in W_{g-1} \quad , \quad \zeta = K - \mathbf{u}(D) \in (\Theta). \quad (3-18)$$

proof: Choose q a generic point of Σ . Let ζ a non-singular zero of Θ . Since one of the zeros of $g_{\zeta,q}$ is q , the others are P_1, \dots, P_{g-1} , and we have

$$\zeta = K - \sum_{i=1}^{g-1} \mathbf{u}(P_i). \quad (3-19)$$

A priori, the P_i s are functions of ζ and q .

At fixed q , we thus have a map

$$\begin{aligned} (\Theta) &\rightarrow W_{g-1} \\ \zeta &\mapsto P_1 + \cdots + P_{g-1} \end{aligned} \quad (3-20)$$

where W_{g-1} is the set of divisors sums of $g-1$ points. The inverse map is:

$$\begin{aligned} W_{g-1} &\rightarrow (\Theta) \\ D &\mapsto K - \mathbf{u}(D) \end{aligned} \quad (3-21)$$

which is clearly independent of q . So the map is invertible, and independent of q . We thus have

$$(\Theta) = K - \mathbf{u}(W_{g-1}). \quad (3-22)$$

□

Theorem 3.1 (Divisors of meromorphic functions) *If $f \neq 0$ is a meromorphic function, with divisor $(f) = \sum_i \alpha_i \cdot p_i$, then*

$$\deg(f) = 0 \quad , \quad \mathbf{u}((f)) = 0 \quad (3-23)$$

and, for any non-singular choice of $\zeta \in (\Theta)$, there exists $C \in \mathbb{C}^*$, such that

$$f(p) = C \prod_i g_{\zeta, p_i}(p)^{\alpha_i}. \quad (3-24)$$

Reciprocally, if $D = \sum_i \alpha_i \cdot p_i$, is a divisor such that

$$\deg D = 0 \quad , \quad \mathbf{u}(D) = 0 \quad (3-25)$$

then D is the divisor of a meromorphic function.

proof: If D is a divisor of degree 0 with $\mathbf{u}(D) = 0$, then (3-24) clearly defines a meromorphic function on Σ , proving the last part of the theorem.

Vice versa, let f a meromorphic function, let ζ a regular zero of Θ , and $D = (f) = \sum_i \alpha_i \cdot p_i$. We already know that $\deg D = \sum_i \alpha_i = 0$. Define on Σ_0 :

$$g(p) = f(p) \prod_i g_{\zeta, p_i}(p)^{-\alpha_i}. \quad (3-26)$$

It has no pole nor zeros in Σ_0 , therefore $\log g$ is holomorphic on Σ_0 and $d \log g$ is a holomorphic 1-form on Σ_0 .

g has no monodromy around \mathcal{A}_i , and around \mathcal{B}_i it gets multiplied by a phase independent of p :

$$g(p + \mathcal{A}_i) = g(p) \quad , \quad g(p + \mathcal{B}_i) = g(p) e^{2\pi i u_i(D)}, \quad (3-27)$$

which implies that $d \log g$ is analytic across the boundaries of Σ_0 , it thus defines a holomorphic 1-form on Σ . Therefore there exists $\lambda_1, \dots, \lambda_g$ such that

$$d \log g = \sum_{i=1}^g \lambda_i du_i \quad (3-28)$$

and thus there exists $C \in \mathbb{C}^*$ such that

$$g(p) = C e^{\sum_{i=1}^g \lambda_i u_i(p)}. \quad (3-29)$$

This last expression has monodromy $e^{\lambda_i} = 1$ around \mathcal{A}_i implying $\lambda_i \in 2\pi i \mathbb{Z}$, and it has monodromy $e^{\sum_k \tau_{i,k} \lambda_k} = e^{2\pi i u_i(D)}$ around \mathcal{B}_i , implying

$$\mathbf{u}(D) \in \mathbb{Z}^g + \tau \mathbb{Z}^g \equiv 0 \text{ in } \mathbb{J}. \quad (3-30)$$

This implies the theorem. \square

Corollary 3.1 *The Abel map $\Sigma \rightarrow \mathbb{J}$ is injective.*

proof: If there would be p_1, p_2 distinct and such that $\mathbf{u}(p_1) - \mathbf{u}(p_2) = 0$ in \mathbb{J} , this would imply that $D = p_1 - p_2$ would be a divisor of degree 0, and such that $\mathbf{u}(D) = 0$, and there would thus exist a meromorphic function f with only one simple pole at p_2 and no other pole. First assume that there exists a holomorphic 1-form ω that doesn't vanish at p_2 , then $f\omega$ would contradict corollary II-2.1. Therefore assume that every holomorphic 1-form in $\mathcal{O}^1(\Sigma)$ vanishes at p_2 . Let

$$k = \min\{\text{order}_{p_2} \omega \mid \omega \in H_1(\Sigma)\} \geq 1. \quad (3-31)$$

Then if $\omega \in \mathcal{O}^1(\Sigma)$ is a holomorphic 1-form that vanishes at order k at p_2 , then $f\omega$ would vanish at order $k - 1$, that would contradict the minimality of k . Therefore this is impossible, proving the corollary. \square

Theorem 3.2 (Jacobi inversion theorem) *There is a bijection*

$$\begin{aligned} W_{\mathfrak{g}} &\rightarrow \mathbb{J} \\ D &\mapsto \mathbf{u}(D) - K \end{aligned} \quad (3-32)$$

where $W_{\mathfrak{g}}$ is the set of divisors that are sums of \mathfrak{g} points. In other words any point $v \in \mathbb{J}$, can be uniquely written as

$$v = -K + \mathbf{u}(P_1) + \cdots + \mathbf{u}(P_{\mathfrak{g}}). \quad (3-33)$$

proof: Let $v \in \mathbb{C}^{\mathfrak{g}}$. By the same-as-usual Riemann bilinear identity argument, the function $p \mapsto g(p) = \Theta(\mathbf{u}(p) - v)$ on Σ_0 , has \mathfrak{g} zeros in Σ_0 . Let us call them $q_1, \dots, q_{\mathfrak{g}}$. By definition $\mathbf{u}(q_1) - v = \zeta$ is a zero of Θ , and therefore there exists a unique divisor $D' = P_1 + \cdots + P_{\mathfrak{g}-1}$ in $W_{\mathfrak{g}-1}$ such that

$$\mathbf{u}(q_1) - v = K - \mathbf{u}(P_1 + \cdots + P_{\mathfrak{g}-1}) \quad (3-34)$$

This implies, after defining $D = q_1 + D' \in W_{\mathfrak{g}}$ that

$$v = -K + \mathbf{u}(D). \quad (3-35)$$

The map $v \mapsto D$ seems to depend on the choice q_1 of zero of g , but we can see that the map is invertible, with inverse map $W_{\mathfrak{g}} \rightarrow \mathbb{J}$, $D \mapsto -K + \mathbf{u}(D)$ independent of this choice. This ends the proof. \square

Lemma 3.4 *Let $\zeta \in (\Theta)$. Since Θ is an even function then $-\zeta \in (\Theta)$, and*

$$\omega_{\zeta} = \sum_{i=1}^{\mathfrak{g}} \Theta'_i(\zeta) \omega_i = -\omega_{-\zeta} \quad (3-36)$$

is a holomorphic 1-form with divisor

$$(\omega_{\zeta}) = \sum_{i=1}^{\mathfrak{g}} P_i + \sum_{i=1}^{\mathfrak{g}} \tilde{P}_i \quad (3-37)$$

where $D = P_1 + \cdots + P_{\mathfrak{g}-1}$ is the unique divisor such that $\mathbf{u}(D) = K - \zeta$, and $\tilde{D} = \tilde{P}_1 + \cdots + \tilde{P}_{\mathfrak{g}-1}$ is the unique divisor such that $\mathbf{u}(\tilde{D}) = K + \zeta$.

proof: The function $g_{\zeta,q}(p) = \Theta(\mathbf{u}(p) - \mathbf{u}(q) + \zeta)$ has a zero at $p = q$, and at $P_1, \dots, P_{\mathfrak{g}-1}$. Near $q = p$ it behaves, in a chart coordinate, as

$$g_{\zeta,q}(p) \sim_{p \rightarrow q} (p - q) \frac{\omega_{\zeta}(q)}{dq}. \quad (3-38)$$

We can choose $q = P_i$, and then $g_{\zeta, P_i}(p)$ has a zero of order at least 2 at P_i , implying that $\omega_{\zeta}(P_i)$ must vanish (in fact it must vanish at the same order as the multiplicity of P_i in D). This also holds for $\omega_{-\zeta}(\tilde{P}_i) = -\omega_{\zeta}(\tilde{P}_i)$ by the same reason. Therefore ω_{ζ} vanishes at $D + \tilde{D}$. Since a holomorphic 1-form has $2\mathfrak{g} - 2$ zeros, these are the only zeros, and thus

$$(\omega_{\zeta}) = D + \tilde{D}. \quad (3-39)$$

□

Theorem 3.3 *Let ω a meromorphic 1-form, and $D = (\omega) = \sum_i \alpha_i \cdot p_i$ its divisor, then:*

$$\deg D = 2\mathfrak{g} - 2 \quad , \quad \mathbf{u}(D) = 2K. \quad (3-40)$$

proof: We have already proved (Riemann-Hurwitz theorem) that a meromorphic 1-form is such that $\deg D = 2\mathfrak{g} - 2$. Moreover, if ω_1 and ω_2 are meromorphic forms, then ω_1/ω_2 is a meromorphic function, its divisor is $(\omega_1) - (\omega_2)$ and it satisfies $\mathbf{u}((\omega_1) - (\omega_2)) = 0$, therefore $\mathbf{u}(D)$ is the same for all 1-forms.

The previous lemma shows that $\mathbf{u}((\omega_{\zeta})) = 2K$ for ζ a zero of Θ . □

Theorem 3.4 *Let ω a holomorphic 1-form, and $D = (\omega) = \sum_i \alpha_i \cdot p_i$ its divisor, then:*

$$\deg D = 2\mathfrak{g} - 2 \quad , \quad \mathbf{u}(D) = 2K. \quad (3-41)$$

Vice-versa, if D is a positive divisor satisfying (3-41), then there exists a unique (up to scalar multiplication) holomorphic 1-form whose divisor is D .

proof: The first part of the theorem is already proved. Now let D a positive divisor of degree $2\mathfrak{g} - 2$ that satisfies (3-41):

$$D = \sum_{i=1}^{2\mathfrak{g}-2} P_i. \quad (3-42)$$

Let $D' = \sum_{i=1}^{\mathfrak{g}-1} P_i$, then $\zeta = K - \mathbf{u}(D') \in (\Theta)$, which implies that ω_{ζ} is a holomorphic 1-form whose divisor is $D' + \tilde{D}'$, where \tilde{D}' is the unique positive divisor of degree $\mathfrak{g} - 1$ such that $K - \mathbf{u}(\tilde{D}') = -\zeta = -(K - \mathbf{u}(D')) = K - \mathbf{u}(D - D')$, i.e. $\tilde{D}' = D - D'$. This implies that $(\omega_{\zeta}) = D$.

To prove uniqueness (up to scalar multiplication), observe that if two holomorphic forms have the same divisor of zeros, their ratio is a meromorphic function without poles, therefore it is a constant. □

3.1 Divisors, classes, Picard group

Definition 3.5 A divisor D is called *principal*, iff there exists a meromorphic function f such that $(f) = D$. From theorem 3.1, principal divisors are those such that:

$$\deg D = 0 \quad , \quad \mathbf{u}(D) = 0. \quad (3-43)$$

The \mathbb{Z} -module of divisors, quotiented by principal divisors is called the *Picard group*:

$$\text{Pic}(\Sigma) = \text{Div}(\Sigma) / \text{Principal divisors}(\Sigma). \quad (3-44)$$

The degree and Abel maps are morphisms (they can be pushed to the quotient)

$$\deg : \text{Pic}(\Sigma) \rightarrow \mathbb{Z}, \quad (3-45)$$

$$\mathbf{u} : \text{Pic}(\Sigma) \rightarrow \mathbb{J}(\Sigma). \quad (3-46)$$

Definition 3.6 (Canonical divisor) The canonical divisor class $\mathfrak{K} \in \text{Pic}(\Sigma)$ is the divisor class (modulo principal divisors) of any meromorphic 1-form. (It is well defined since the ratio of 2 meromorphic forms is a meromorphic function). We have

$$\deg \mathfrak{K} = 2\mathfrak{g} - 2 \quad , \quad \mathbf{u}(\mathfrak{K}) = 2K. \quad (3-47)$$

4 Prime form

Let us consider special zeros of Θ as follows: Let $\mathbf{c} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\tau\mathbf{b}$ be an **half-integer characteristic**, i.e. $\mathbf{c} = -\mathbf{c}$. We say it is odd iff the scalar product (\mathbf{a}, \mathbf{b}) is odd

$$\mathbf{c} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\tau\mathbf{b} \text{ odd} \quad \Leftrightarrow \quad (\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{\mathfrak{g}} a_i b_i \in 2\mathbb{Z} + 1, \quad (4-1)$$

it is then a zero of Θ . Let us admit that there exists non-singular half-integer odd characteristics.

Lemma 4.1 Let \mathbf{c} a non-singular half-integer odd characteristics. The 1-form

$$h_{\mathbf{c}} = \sum_{i=1}^{\mathfrak{g}} \Theta'_i(\mathbf{c}) \omega_i \quad (4-2)$$

has $\mathfrak{g} - 1$ double zeros, located at the unique positive integer divisor of degree $\mathfrak{g} - 1$ such that $\mathbf{u}(D) = K - \mathbf{c}$.

The square root $\sqrt{h_{\mathbf{c}}}$ is well defined and analytic on a fundamental domain Σ_0 , it is a $\frac{1}{2}$ spinor form.

proof: We have seen that the zeros of h_c have divisor $D + \tilde{D}$, with D and \tilde{D} the unique divisors such that $u(D) = K - \mathbf{c}$ and $u(\tilde{D}) = K + \mathbf{c}$, but since $\mathbf{c} = -\mathbf{c}$ in \mathbb{J} , we have $D = \tilde{D}$, and thus all zeros are double zeros. \square

Definition 4.1 (Prime form)

$$E_{\mathbf{c}}(p, q) = \frac{\Theta(\mathbf{u}(p) - \mathbf{u}(q) + \mathbf{c})}{\sqrt{h_{\mathbf{c}}(p) h_{\mathbf{c}}(q)}} \quad (4-3)$$

It is a $\frac{-1}{2} \otimes \frac{-1}{2}$ bi-spinor defined on a fundamental domain $\Sigma_0 \times \Sigma_0$, it vanishes at $p = q$ and nowhere else, it behaves near $p = q$, in any local coordinate ϕ_U , as

$$E_{\mathbf{c}}(p, q) \sim \frac{\phi_U(p) - \phi_U(q)}{\sqrt{d\phi_U(p) d\phi_U(q)}} (1 + O(\phi_U(p) - \phi_U(q))). \quad (4-4)$$

It has monodromies (up to a sign):

$$\begin{aligned} E_{\mathbf{c}}(p + \mathcal{A}_i, q) &= \pm E_{\mathbf{c}}(p, q) \\ E_{\mathbf{c}}(p + \mathcal{B}_i, q) &= \pm E_{\mathbf{c}}(p, q) e^{-2\pi i (u_i(p-q) + c_i + \frac{1}{2}\tau_{i,i})} \end{aligned} \quad (4-5)$$

Under a change of \mathbf{c} , we have

$$E_{\mathbf{c}'}(p, q) = \pm E_{\mathbf{c}}(p, q) e^{\pi i (\mathbf{b} - \mathbf{b}', \mathbf{u}(p) - \mathbf{u}(q))}. \quad (4-6)$$

proof: Notice that $g_{\mathbf{c},q}(p)$ has \mathbf{g} zeros, one of them is q , and the others $P_1, \dots, P_{\mathbf{g}-1}$ are such that $u(\sum_i P_i) = K - D$, therefore they are the same as the zeros of $\sqrt{h_{\mathbf{c}}}$. This shows that as a function of p , $E_{\mathbf{c}}(p, q)$ vanishes only at $p = q$ and nowhere else. The other properties are rather obvious.

\square

Theorem 4.1 (Green function) *The Green function defined in cor II-3.1, is (up to an additive constant C independent of p):*

$$\mathcal{G}_{q_+, q_-}(p) = C + \log \left| \frac{\Theta(\mathbf{u}(p) - \mathbf{u}(q_+) + \mathbf{c})}{\Theta(\mathbf{u}(p) - \mathbf{u}(q_-) + \mathbf{c})} \right| - \pi \Im \mathbf{u}(p)^T (\Im \tau)^{-1} \Im(\mathbf{u}(q_+) - \mathbf{u}(q_-)). \quad (4-7)$$

This can also be written with the prime form

$$\mathcal{G}_{q_+, q_-}(p) = C + \log \left| \frac{E_{\mathbf{c}}(p, q_+)}{E_{\mathbf{c}}(p, q_-)} \frac{\nu(q_+)}{\nu(q_-)} \right| - \pi \sum_{i,j=1}^{\mathbf{g}} \Im \mathbf{u}_i(p) ((\Im \tau)^{-1})_{i,j} \Im(\mathbf{u}_j(q_+) - \mathbf{u}_j(q_-)). \quad (4-8)$$

where ν is any meromorphic 1-form. Observe that changing the 1-form ν , or the odd characteristic c , or the origin o for the definition of the Abel map, or the basis of symplectic cycles, amount to changing \mathcal{G} by an additive constant C independent of p .

5 Fundamental form

The following object is probably the most useful form that can be defined on a Riemann surface, it allows to reconstruct everything. Riemann already introduced it, it then received several names, and we shall call it the **Fundamental second kind differential**.

Although it differs from another object named Bergman kernel in operator theory, it is sometimes called also **Bergman kernel**, because it was extensively studied by Bergman and Schiffer [2], and this name was used in a series of Korotkin-Kokotov seminal articles [5, 6]. It is also very close to other objects, called Schiffer kernel, or Klein kernel, that were introduced, and that we shall see below.

Definition 5.1 (Fundamental second kind differential)

$$B(p, q) = d_p d_q (\log \Theta(\mathbf{u}(p) - \mathbf{u}(q) + \mathbf{c})). \quad (5-1)$$

It is independent of a choice of \mathbf{c} . B is a meromorphic symmetric bilinear $1 \boxtimes 1$ form on $\Sigma \times \Sigma$, it has a double pole at $p = q$ and no other pole, and near $p \rightarrow q$ it behaves (in any choice of local coordinate ϕ_U) as

$$B(p, q) = \frac{d\phi_U(p) d\phi_U(q)}{(\phi_U(p) - \phi_U(q))^2} + \text{analytic at } q. \quad (5-2)$$

It also satisfies

$$B(q, p) = B(p, q) \quad (5-3)$$

$$\oint_{q \in \mathcal{A}_i} B(p, q) = 0 \quad (5-4)$$

$$\oint_{q \in \mathcal{B}_i} B(p, q) = 2\pi i \omega_i(p). \quad (5-5)$$

*It is sometimes called **Bergman kernel**.*

This definition contains assertions that need to be proved.

proof: It is clearly symmetric because Θ is even.

It has a double pole at $p = q$ because it is the second derivative of the log of something that vanishes linearly. There is no simple pole contribution at $p = q$ because of parity, or because of the same reason, it is the second derivative of a log.

It is globally meromorphic on Σ , because when we go around a cycle, Θ gets multiplied by a phase, $\log \Theta$ receives an additive contribution which is linear in $\mathbf{u}(p) - \mathbf{u}(q) + c$, and thus is killed by taking the second derivative $d_p d_q$.

Θ also has zeros at P_1, \dots, P_{g-1} , but since those points are independent of p and q , taking derivatives with respect to p and q kills these poles.

The values of the \mathcal{A} -cycle and \mathcal{B} -cycle integrals are easy from the quasiperiodicity properties of Θ , indeed integrating a derivative just gives the difference of values of $\log \Theta$ between the end and start of the integration path, i.e. the phase shift of $\log \Theta$.

The fact that it is independent of \mathbf{c} follows from uniqueness. Indeed If B and B' are 2 fundamental forms of the second kind, for instance corresponding to \mathbf{c} and \mathbf{c}' , their difference has no pole at all, and has vanishing \mathcal{A} -cycle integrals, so it is a vanishing holomorphic 1-form, showing that B is unique. \square

As a corollary we get

Definition 5.2 (Third kind forms) *For any distinct points $q_1, q_2 \in \Sigma_0 \times \Sigma_0$, the following 1-form of $p \in \Sigma$*

$$\omega_{q_1, q_2}(p) = d_p \left(\log \frac{\Theta(\mathbf{u}(p) - \mathbf{u}(q_1) + \mathbf{c})}{\Theta(\mathbf{u}(p) - \mathbf{u}(q_2) + \mathbf{c})} \right) = \int_{q_2}^{q_1} B(p, \cdot) \quad (5-6)$$

is independent of a choice of \mathbf{c} , it is meromorphic on Σ , it has a simple pole at $p = q_1$ with residue +1 and a simple pole at $p = q_2$ with residue -1 and no other pole, and it is normalized on \mathcal{A} -cycles

$$\begin{aligned} \oint_{p \in \mathcal{A}_i} \omega_{q_1, q_2}(p) &= 0 \\ \operatorname{Res}_{q_1} \omega_{q_1, q_2} &= 1 \\ \operatorname{Res}_{q_2} \omega_{q_1, q_2} &= -1. \end{aligned} \quad (5-7)$$

Other kernels

There exist other classical bilinear differentials, slightly different from B . First notice that for every symmetric $\mathfrak{g} \times \mathfrak{g}$ matrix κ , then

$$B_\kappa(p, q) = B(p, q) + 2\pi i \sum_{i, j} \kappa_{i, j} \omega_i(p) \omega_j(q) \quad (5-8)$$

is also a meromorphic symmetric bilinear $1 \otimes 1$ form on $\Sigma \times \Sigma$, with a double pole at $p = q$ and no other pole with behavior (5-2), the only difference is that it satisfies

$$\begin{aligned} \oint_{q \in \mathcal{A}_i} B_\kappa(p, q) &= 2\pi i \sum_j \kappa_{i, j} \omega_j(p) \\ \oint_{q \in \mathcal{B}_i} B_\kappa(p, q) &= 2\pi i \left(\omega_i(p) + \sum_{j, l} \tau_{i, j} \kappa_{j, l} \omega_l(p) \right). \end{aligned} \quad (5-9)$$

In particular we have special choices of κ as follows:

Definition 5.3 (Klein kernel) Let $\zeta \in \mathbb{C}^g$, and define the **Klein kernel**

$$B_\zeta(p, q) = B_{\frac{1}{\pi i} \log \Theta''(\zeta)}(p, q) = B(p, q) + 2 \sum_{i,j} (\log \Theta)''(\zeta)_{i,j} \omega_i(p) \omega_j(q) \quad (5-10)$$

where

$$(\log \Theta)''(\zeta)_{i,j} = \left(\frac{\partial^2}{\partial u_i \partial u_j} \log \Theta(u) \right)_{u=\zeta}. \quad (5-11)$$

Definition 5.4 (Schiffer kernel) We have already defined the Schiffer kernel, it corresponds to the choice $\kappa = \frac{i}{2} \Im \tau^{-1}$:

$$B_S(p, q) = B_{\frac{i}{2} \Im \tau^{-1}}(p, q) \quad (5-12)$$

5.1 Modular transformations

A choice of symplectic basis of cycles (marking) is not unique, and some notions, like the normalized basis of holomorphic forms, the Riemann matrix of periods, and the fundamental second kind differential depend on the marking. The group of changes of symplectic basis, that keeps the intersection matrix constant, is the **symplectic group** $Sp_{2g}(\mathbb{Z})$, of $2g \times 2g$ matrices (written as 4 block matrices of size $g \times g$)

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (5-13)$$

satisfying

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & \text{Id}_g \\ -\text{Id}_g & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = \begin{pmatrix} 0 & \text{Id}_g \\ -\text{Id}_g & 0 \end{pmatrix}. \quad (5-14)$$

Now consider 2 markings, related by a $Sp_{2g}(\mathbb{Z})$ symplectic transformation:

$$\begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{A}} \\ \tilde{\mathcal{B}} \end{pmatrix}. \quad (5-15)$$

Theorem 5.1 (Modular transformations) Under a symplectic transformation, the Riemann matrix of periods changes as

$$\tilde{\tau} = (\delta - \tau\beta)^{-1}(\tau\alpha - \gamma), \quad (5-16)$$

and the normalized basis of holomorphic 1-forms as

$$\tilde{\omega}_i = \sum_j J_{i,j} \omega_j \quad , \quad J = (\delta - \tau\beta)^{-1} = \alpha^T + \tilde{\tau}\beta^T. \quad (5-17)$$

The Klein kernel and the Schiffer kernel are $Sp_{2g}(\mathbb{Z})$ modular invariant.

The kernel B_κ of def. 5.1 changes as

$$B_\kappa = \tilde{B}_{\tilde{\kappa}} \quad , \quad \tilde{\kappa} = (\alpha + \tilde{\tau}\beta)^{-1}\kappa(\delta - \tau\beta) - (\alpha + \tilde{\tau}\beta)^{-1}\beta. \quad (5-18)$$

In particular for $\tilde{\kappa} = 0$:

$$\tilde{B}_0 = B_{\beta(\delta - \tau\beta)^{-1}}, \quad (5-19)$$

i.e.

$$\tilde{B}_0(p, q) = B_0(p, q) + 2\pi i \sum_{i,j=1}^{\mathfrak{g}} (\beta(\delta - \tau\beta)^{-1})_{i,j} \omega_i(p) \omega_j(q). \quad (5-20)$$

Remark that if $\mathfrak{g} = 1$, the matrices $\alpha, \beta, \gamma, \delta$ are scalar elements of \mathbb{Z} , and we have

$$\tilde{\tau} = \frac{\alpha\tau - \gamma}{\delta - \beta\tau}. \quad (5-21)$$

5.2 Meromorphic forms and generalized cycles

Theorem 5.2 (All meromorphic forms are generated by integrating B) *A basis of $\mathfrak{M}^1(\Sigma)$ can be constructed by integrating B as follows:*

- 1st kind forms

$$\omega_i(p) = \frac{1}{2\pi i} \oint_{q \in \mathcal{B}_i} B(p, q) \quad (5-22)$$

It is the unique holomorphic 1-form normalized on \mathcal{A} -cycles as

$$\oint_{\mathcal{A}_j} \omega_i = \delta_{i,j}. \quad (5-23)$$

We call \mathcal{A}_i and \mathcal{B}_i 1st kind cycles.

- 3rd kind forms

$$\omega_{q_1, q_2}(p) = \int_{q=q_2}^{q_1} B(p, q) \quad (5-24)$$

where the integration chain from q_2 to q_1 is the unique one that doesn't intersect any \mathcal{A}_i -cycles nor \mathcal{B}_i -cycles. It is the unique 1-form normalized on \mathcal{A} -cycles that has a simple pole with residue 1 at q_1 and a simple pole with residue -1 at q_2 and no other poles, such that

$$\text{Res}_{q_1} \omega_{q_1, q_2} = 1$$

$$\text{Res}_{q_2} \omega_{q_1, q_2} = -1$$

$$\oint_{\mathcal{A}_i} \omega_{q_1, q_2} = 0. \quad (5-25)$$

We call a chain $q_2 \rightarrow q_1$ a 3rd kind cycle (we shall call it a generalized cycle below).

- *2nd kind forms.* For $q \in \Sigma$, choose ϕ_U a local coordinate in a neighborhood of q , and let f a function, meromorphic in the neighborhood U of q , and let $d = \text{order}_q f - f(q)$.

For $k \in \mathbb{Z}_+$, if $d > 0$, define

$$\omega_{f,q,k}(p) = \frac{1}{2\pi i} \frac{d}{k} \oint_{q' \in \mathcal{C}_q} (f(q') - f(q))^{-\frac{k}{d}} B(p, q'). \quad (5-26)$$

It is the unique 1-form normalized on \mathcal{A} -cycles that has a pole of degree $k + 1$ at q , that behaves as

$$\omega_{f,q,k}(p) \sim \frac{df(p)}{(f(p) - f(q))^{1+\frac{k}{d}}} + \text{analytic at } q \quad , \quad \oint_{\mathcal{A}_i} \omega_{f,q,k} = 0. \quad (5-27)$$

If $d < 0$, i.e. q is a pole of f , of degree $-d$ we define

$$\omega_{f,q,k}(p) = \frac{1}{2\pi i} \frac{d}{k} \oint_{q' \in \mathcal{C}_q} f(q')^{-\frac{k}{d}} B(p, q'). \quad (5-28)$$

It is the unique 1-form normalized on \mathcal{A} -cycles that has a pole of degree $k + 1$ at q , that behaves as

$$\omega_{f,q,k}(p) \sim \frac{df(p)}{(f(p))^{1+\frac{k}{d}}} + \text{analytic at } q \quad , \quad \oint_{\mathcal{A}_i} \omega_{f,q,k} = 0. \quad (5-29)$$

We call a pair (\mathcal{C}_q, f) , that we denote $\mathcal{C}_q.f$, made of a small cycle \mathcal{C}_q around q , together with a function f meromorphic in a neighborhood of \mathcal{C}_q with poles only at q , a **2nd kind cycle** (we shall call it a *generalized cycle* below).

In this theorem, we have implicitly defined a notion of "generalized cycles". B allows to define a form-cycle duality as follows:

through integration (Poincaré pairing), cycles can be viewed as acting linearly on the space of forms, and are thus elements of the dual of the space of 1-forms:

$$\langle \gamma, \omega \rangle = \oint_{\gamma} \omega \in \mathbb{C}. \quad (5-30)$$

In other words

$$H_1(\Sigma) \subset \mathfrak{M}^1(\Sigma)^*. \quad (5-31)$$

However, the dual $\mathfrak{M}^1(\Sigma)^*$ is much larger than $H_1(\Sigma)$, it may contain for instance integrals on a cycle, together with multiplication by a function (as in (5-26)) or a distribution, or integrals on open chains with boundaries as in (5-24), and also 2-dimensional integrals, and many other things. Since B is a $1 \boxtimes 1$ form, acting on its second variable by an element of the dual, yields a 1-form of the first variable, i.e. a 1-form. However, if we chose an arbitrary element of $\mathfrak{M}^1(\Sigma)^*$, this 1-form is not necessarily meromorphic. Let us define:

Definition 5.5 (Generalized cycles) We define the space of generalized cycles $\mathfrak{M}_1(\Sigma)$, as the subspace of $\mathfrak{M}^1(\Sigma)^*$, whose integral of B is a meromorphic 1-form. B defines a map \hat{B} from generalized cycles to meromorphic 1-forms:

$$\begin{aligned} \hat{B} : \mathfrak{M}_1(\Sigma) &\rightarrow \mathfrak{M}^1(\Sigma) \\ \gamma &\mapsto \hat{B}(\gamma) = \langle \gamma, B \rangle, \quad \hat{B}(\gamma)(p) = \oint_{q \in \gamma} B(p, q) \end{aligned} \quad (5-32)$$

From theorem 5.2, this map is surjective. It is not injective.

Definition 5.6 The intersection of generalized cycles is the symplectic form on $\mathfrak{M}_1(\Sigma) \times \mathfrak{M}_1(\Sigma)$ defined by

$$\gamma_1 \cap \gamma_2 = -\gamma_2 \cap \gamma_1 = \frac{1}{2\pi i} \left(\oint_{\gamma_1} \oint_{\gamma_2} B - \oint_{\gamma_2} \oint_{\gamma_1} B \right) \quad (5-33)$$

One can check that on $H_1(\Sigma) \times H_1(\Sigma)$ this is indeed the usual intersection.

Notice that $\text{Ker } \hat{B}$ is obviously a Lagrangian submanifold.

To go further

To learn more about generalized cycles, their symplectic structure, their Lagrangian submanifolds, and Hodge structures, see for instance [4].

Chapter 4

Riemann-Roch

The question of interest is: given a set of points p_i and orders $\alpha_i \in \mathbb{Z}$ (i.e. an integer divisor $D = \sum_i \alpha_i \cdot p_i$), is it possible to find a meromorphic function (resp. 1-form) with poles at p_i of degree at most $-\alpha_i$ if $\alpha_i < 0$, and zeros of order at least α_i if $\alpha_i > 0$? And how many linearly independent such functions (resp. 1-forms) is there?

1 Spaces and dimensions

Definition 1.1 (Positive divisors) A divisor $D = \sum_i \alpha_i \cdot p_i$ is said *positive*

$$\sum_i \alpha_i \cdot p_i \geq 0 \quad \text{iff } \forall i, \alpha_i \geq 0. \quad (1-1)$$

$$D > 0 \quad \text{iff } D \geq 0 \text{ and } D \neq 0. \quad (1-2)$$

This induces a (partial) order relation in the set of divisors:

$$D_1 \geq D_2 \quad \text{iff } D_1 - D_2 \geq 0. \quad (1-3)$$

$$D_1 > D_2 \quad \text{iff } D_1 - D_2 > 0. \quad (1-4)$$

Definition 1.2 Given a divisor D , we define the vector spaces (over \mathbb{C})

$$\mathcal{L}(D) = \{f \in \mathfrak{M}^0(\Sigma) \mid (f) \geq D\} \quad , \quad r(D) = \dim \mathcal{L}(D), \quad (1-5)$$

$$\Omega(D) = \{\omega \in \mathfrak{M}^1(\Sigma) \mid (\omega) \geq D\}. \quad , \quad i(D) = \dim \Omega(D). \quad (1-6)$$

Theorem 1.1 $r(D)$ and $i(D)$ depend only on the divisor class (modulo principal divisors):

$$r(D) = r([D]) \quad , \quad i(D) = i([D]). \quad (1-7)$$

proof: If $D_2 - D_1 = (f)$ is a principal divisor, then the maps

$$\begin{aligned}\Omega(D_1) &\rightarrow \Omega(D_2) \\ \omega_1 &\mapsto f\omega_1\end{aligned}\tag{1-8}$$

$$\begin{aligned}\mathcal{L}(D_1) &\rightarrow \mathcal{L}(D_2) \\ f_1 &\mapsto ff_1\end{aligned}\tag{1-9}$$

are isomorphisms (they are obviously invertible: the inverse is dividing by f), therefore $i(D_1) = i(D_2) = i([D_1])$ and $r(D_1) = r(D_2) = r([D_1])$. \square

The Riemann-Roch theorem, that we shall prove in this chapter, is:

Theorem 1.2 (Riemann-Roch) *For every divisor D we have*

$$r(-D) = \deg D + 1 - \mathfrak{g} + i(D).\tag{1-10}$$

This theorem is extremely useful and powerful. One reason is that most often one of the two indices $r(-D)$ or $i(D)$ is easy to compute and the other is more difficult, so the theorem gives it without effort.

Also, using positivity $i(D) \geq 0$ (resp. $r(-D) \geq 0$) we easily get a lower bound for $r(-D)$ (resp. $i(D)$), and thus the Riemann-Roch theorem is often used to prove the existence of functions or forms with given poles and degrees.

2 Special cases

Let us first prove it in special cases:

- If $D = 0$, then from theorem II-2.6 we have $\mathcal{L}(0) = \mathcal{O}(\Sigma) = \mathbb{C}$ and $r(0) = 1$. On the other hand, $\Omega(0) = \mathcal{O}^1(\Sigma)$ and $i(0) = \mathfrak{g}$. The Riemann-Roch theorem is satisfied.
- If $-D > 0$, then $\mathcal{L}(-D) = \{0\}$ and $r(-D) = 0$, and $\deg D < 0$. We have (using the basis of theorem III-5.2)

$$\Omega(D) = \left\{ \sum_i \sum_{k=2}^{-\alpha_i} t_{p_i,k} \omega_{p_i,k} + \sum_i t_{p_i,1} \omega_{p_i,o} + \sum_{i=1}^{\mathfrak{g}} \epsilon_i \omega_i \mid \sum_i t_{p_i,1} = 0 \right\}\tag{2-1}$$

Therefore

$$i(D) = \sum_i (-\alpha_i) + \mathfrak{g} - 1 = -\deg D + \mathfrak{g} - 1.\tag{2-2}$$

The Riemann-Roch theorem is satisfied.

- $\mathfrak{g} \geq 2$ and $D > 0$ and $\deg D = \mathfrak{g} - 1$. $\Omega(D)$ is then the set of holomorphic forms ω with zeros at the $\mathfrak{g} - 1$ point of D . $\zeta = K - u(D)$ is a zero of Θ , and by

parity, $-\zeta = u(D) - K$ is also a zero of Θ , therefore there is a unique positive divisor $\tilde{D} > 0$ of degree $\deg \tilde{D} = \mathfrak{g} - 1$ with Abel map $\mathbf{u}(\tilde{D}) = K + \zeta = 2K - \mathbf{u}(D)$. From theorem III-3.4, there exists a unique (up to scalar multiplication) holomorphic one form ω whose divisor of zeros is $D + \tilde{D}$, which implies that $i(D) = 1$. It also implies that $\Omega(D) = \Omega(D + \tilde{D}) = \Omega(\tilde{D})$.

For all $f \in \mathcal{L}(-D)$, the 1-form $f\omega$ has no pole, and its divisor of zeros is $\geq \tilde{D}$, so it belongs to $\Omega(\tilde{D})$. Reciprocally, if $\tilde{\omega} \in \Omega(\tilde{D}) = \Omega(D + \tilde{D})$, then $\tilde{\omega}$ must be proportional to ω , and $f = \tilde{\omega}/\omega$ is a constant, in particular it belongs to $\mathcal{L}(-D)$. Therefore the map

$$\begin{aligned} \mathcal{L}(-D) &\rightarrow \Omega(\tilde{D}) \\ f &\mapsto f\omega \\ \frac{\tilde{\omega}}{\omega} &\leftarrow \tilde{\omega} \end{aligned} \tag{2-3}$$

is an isomorphism. This implies $r(-D) = i(\tilde{D}) = 1$. The Riemann-Roch theorem is satisfied.

3 Genus 0

On the Riemann sphere we have

$$\mathcal{L}(-D) = \left\{ C \frac{P(z)}{\prod_i (z - p_i)^{\alpha_i}} \mid P \in \mathbb{C}[z] \quad \deg P \leq \deg D \right\} \tag{3-1}$$

$$\implies r(-D) = \max(0, 1 + \deg D) \tag{3-2}$$

$$\Omega(D) = \left\{ C \frac{P(z)}{\prod_i (z - p_i)^{-\alpha_i}} dz \mid P \in \mathbb{C}[z] \quad \deg P \leq -2 - \deg D \right\} \tag{3-3}$$

$$\implies i(D) = \max(0, -1 - \deg D) \tag{3-4}$$

We easily see that the Riemann-Roch theorem is satisfied for all divisors.

4 Genus 1

On a genus 1 curve, there is a unique –up to a scalar factor– holomorphic 1-form ω_0 without poles nor zeros. Choosing a symplectic basis of cycles $\mathcal{A} \cap \mathcal{B} = 1$, we choose ω_0 normalized on \mathcal{A} , so that $\oint_{\mathcal{A}} \omega_0 = 1$, and we define $\tau = \oint_{\mathcal{B}} \omega_0$ and we define the Abel map $p \mapsto u(p)$ such that $du = \omega_0$.

The map $f \mapsto \omega = f\omega_0$ is an isomorphism

$$\mathcal{L}(D) \rightarrow \Omega(D), \tag{4-1}$$

and thus

$$r(D) = i(D). \quad (4-2)$$

If $f \in \mathcal{L}(D)$, we have $(f) = D + \tilde{D}$ with $\tilde{D} \geq 0$, $\deg \tilde{D} = -\deg D$ and $\mathbf{u}(\tilde{D}) = -\mathbf{u}(D)$. Let us denote

$$W(D) = \{\tilde{D} \mid \tilde{D} \geq 0 \text{ and } \deg \tilde{D} = -\deg D \text{ and } \mathbf{u}(\tilde{D}) = -\mathbf{u}(D)\}. \quad (4-3)$$

There is an isomorphism $W(D) \oplus \mathbb{C} \rightarrow \mathcal{L}(D)$, given by

$$(\tilde{D}, C) \mapsto f(p) = C \prod_i \Theta(u(p) - u(p_i) + \frac{1+\tau}{2})^{\alpha_i} \prod_i \Theta(u(p) - u(p'_i) + \frac{1+\tau}{2})^{\alpha'_i}, \quad (4-4)$$

and thus

$$r(D) = \dim W(D) = \max(0, -\deg D). \quad (4-5)$$

We have

$$r(-D) - r(D) = \deg D \quad (4-6)$$

so that the Riemann-Roch theorem is satisfied.

Theorem 4.1 *Every genus 1 Riemann surface is isomorphic to a torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.*

proof: The map $p \mapsto u(p)$ is an isomorphism $\Sigma \rightarrow \mathbb{J}(\Sigma) = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. Indeed it is injective, of maximal rank, and continuous, so it must be surjective. \square

5 Higher genus ≥ 2

Let us first make some observations.

5.1 General remarks

Theorem 5.1

$$i([D]) = r([D] - \mathfrak{K}). \quad (5-1)$$

proof: Let ω an arbitrary meromorphic 1-form. The map

$$\begin{aligned} \Omega(D) &\rightarrow \mathcal{L}(D - (\omega)) \\ \omega' &\mapsto \omega'/\omega \end{aligned} \quad (5-2)$$

is an isomorphism. \square

Write $D = D_+ - D_-$, where $D_+ \geq 0$ and $D_- \geq 0$. We clearly have

$$\Omega(D_+) \subset \Omega(D) \subset \Omega(-D_-) \quad \implies \quad 0 \leq i(D_+) \leq i(D) \leq i(-D_-). \quad (5-3)$$

$$\mathcal{L}(-D_-) \subset \mathcal{L}(-D) \subset \mathcal{L}(-D_+) \quad \implies \quad 0 \leq r(-D_-) \leq r(-D) \leq r(-D_+). \quad (5-4)$$

5.2 Proof of the Riemann–Roch theorem

Negative divisors

Theorem 5.2 *Let D a divisor. If there exists a negative divisor $\tilde{D} \leq 0$ whose divisor class is $[\tilde{D}] = [D]$ or if $[\tilde{D}] = \mathfrak{K} - [D]$, then the Riemann–Roch theorem holds*

proof: The case $\tilde{D} = 0$, and $\tilde{D} < 0$ were already done in section 2.

□

Other cases

The following proposition is very useful:

Proposition 5.1 (Riemann inequality) *Let $D > 0$, we have*

$$r(-D) \geq \deg D - \mathfrak{g} + 1. \quad (5-5)$$

proof: Writing $D = \sum_{i=1}^k \alpha_i p_i$, define $D' = \sum_i (\alpha_i + 1) p_i$.

The map

$$\begin{aligned} \mathcal{L}(-D) &\rightarrow \Omega(-D') \\ f &\mapsto df \end{aligned} \quad (5-6)$$

is a morphism whose kernel consists of constant functions, i.e. whose kernel has dimension 1. Its image consists of exact forms, those for which all cycle integrals and residues vanish. It is a space of codimension at most $2\mathfrak{g} + k - 1$ (because there are at most $2\mathfrak{g} + k$ independent non contractible cycles, and since the sum of all residues is 0, at most $k - 1$ of them are actually independent), therefore

$$r(-D) - 1 \geq i(-D') - 2\mathfrak{g} + 1 - k. \quad (5-7)$$

Moreover, we have already proved Riemann–Roch theorem for negative divisors, and we have $r(D') = 0$ and

$$i(-D') = \deg D' + \mathfrak{g} - 1 = \deg D + k + \mathfrak{g} - 1. \quad (5-8)$$

Therefore:

$$r(-D) - 1 \geq \deg D - \mathfrak{g}. \quad (5-9)$$

□

Proposition 5.2 *If $r(-D) > 0$, then there exists $\tilde{D} \geq 0$ such that $[\tilde{D}] = [D]$.*

proof: Choose $0 \neq f \in \mathcal{L}(-D)$, then $\tilde{D} = (f) + D \geq 0$ and $[\tilde{D}] = [D]$. □

Proposition 5.3 *If $i(D) > 0$, then there exists $\tilde{D} \geq 0$ such that $[\tilde{D}] = \mathfrak{K} - [D]$.*

proof: Use $i([D]) = r(\mathfrak{K} - [D])$. \square

Theorem 5.3 *If D is a divisor such that there exists no positive divisor $\tilde{D} \geq 0$ such that $[D] = [\tilde{D}]$ nor $[\tilde{D}] = \mathfrak{K} - [D]$, then $i(D) = r(-D) = 0$ and*

$$\deg D = \mathfrak{g} - 1. \tag{5-10}$$

So that the Riemann-Roch theorem holds in this case too.

proof: Write $D = D_+ - D_-$ with $D_+ > 0$ and $D_- \geq 0$, with D_+ and D_- having no points in common. We have from the Riemann inequality

$$r(-D_+) \geq \deg D_+ - \mathfrak{g} + 1 = \deg D + \deg D_- - \mathfrak{g} + 1 \tag{5-11}$$

Assume that $\deg D \geq \mathfrak{g}$, this implies that

$$r(-D_+) \geq \deg D_- + 1. \tag{5-12}$$

The subspace of $\{f \mid (f) \geq D_- - D_+\} \subset \mathcal{L}(-D_+)$, is of codimension $\deg D_-$, and thus it is non-vanishing, showing that there exists some $f \neq 0$ such that $(f) + D \geq 0$. This contradicts our hypothesis that D is not equivalent to a positive divisor. Therefore we must have $\deg D \leq \mathfrak{g} - 1$. By the same reasoning we have

$$\mathfrak{g} - 1 \geq \deg(\mathfrak{K} - [D]) = 2\mathfrak{g} - 2 - \deg D \tag{5-13}$$

which implies that $\mathfrak{g} - 1 \leq \deg D \leq \mathfrak{g} - 1$ and thus $\deg D = \mathfrak{g} - 1$. The fact that $i(D) = 0$ and $r(-D) = 0$ follow from prop 5.2 and prop 5.3. The Riemann-Roch theorem thus holds. \square

Chapter 5

Moduli spaces

Throughout this section we shall denote

$$d_{\mathbf{g},n} = 3\mathbf{g} - 3 + n \quad , \quad \chi_{\mathbf{g},n} = 2 - 2\mathbf{g} - n. \quad (0-1)$$

Definition 0.1 Let $(\Sigma, p_1, \dots, p_n)$ and $(\Sigma', p'_1, \dots, p'_n)$ be two compact Riemann surfaces of genus \mathbf{g} , with n distinct and labeled marked points $p_i \in \Sigma$, $p'_i \in \Sigma'$. They are called isomorphic iff there exists a holomorphic map $\phi : \Sigma \rightarrow \Sigma'$, invertible and whose inverse is holomorphic, such that $\phi(p_i) = p'_i$ for all $i = 1, \dots, n$.

Automorphisms of $(\Sigma, p_1, \dots, p_n)$ form a group. We say that $(\Sigma, p_1, \dots, p_n)$ is stable iff its automorphism group is a finite group.

We define the **moduli space**

$$\mathcal{M}_{\mathbf{g},n} = \{(\Sigma, p_1, \dots, p_n) \text{ of genus } \mathbf{g}\} / \text{isomorphisms}. \quad (0-2)$$

1 Genus 0

From corollary II-3.5, every Riemann surface of genus 0 is isomorphic to the Riemann sphere. Automorphisms of the Riemann sphere are Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc = 1$.

Theorem 1.1

$$\mathcal{M}_{0,0} = \{(\mathbb{C}P^1)\} \quad , \quad \text{Aut} = PSL(2, \mathbb{C}). \quad (1-1)$$

$$\mathcal{M}_{0,1} = \{(\mathbb{C}P^1, \infty)\} \quad , \quad \text{Aut} = \{z \mapsto az + b\} \sim \mathbb{C}^* \times \mathbb{C}. \quad (1-2)$$

$$\mathcal{M}_{0,2} = \{(\mathbb{C}P^1, 0, \infty)\} \quad , \quad \text{Aut} = \{z \mapsto az\} \sim \mathbb{C}^*. \quad (1-3)$$

$$\mathcal{M}_{0,3} = \{(\mathbb{C}P^1, 0, 1, \infty)\} \quad , \quad \text{Aut} = \{Id\}. \quad (1-4)$$

And if $n \geq 4$

$$\mathcal{M}_{0,n} = \{(\mathbb{C}P^1, 0, 1, \infty, p_4, \dots, p_n) \mid p_4, \dots, p_n \text{ distinct and } \neq 0, 1, \infty\} \quad , \quad \text{Aut} = \{Id\}, \quad (1-5)$$

$$\mathcal{M}_{0,n} \sim (\mathbb{C}P^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \{\text{coinciding points}\}. \quad (1-6)$$

$\mathcal{M}_{0,n}$ is a smooth complex connected manifold of dimension

$$\dim \mathcal{M}_{0,n} = \max(0, n - 3). \quad (1-7)$$

$\mathcal{M}_{0,n}$ is stable iff $n \geq 3$. If $n \geq 4$, it is not compact.

1.1 $\mathcal{M}_{0,3}$

$\mathcal{M}_{0,3}$ is a single point, with trivial automorphism, this is the simplest possible manifold: the point.

$$\mathcal{M}_{0,3} = \{(\mathbb{C}P^1, 0, 1, \infty)\} \quad , \quad \dim \mathcal{M}_{0,3} = d_{0,3} = 0 \quad , \quad \text{Aut} = \{\text{Id}\}. \quad (1-8)$$

There is a unique topology on it, and it is compact.

Its Euler characteristic is

$$\chi(\mathcal{M}_{0,3}) = 1. \quad (1-9)$$

1.2 $\mathcal{M}_{0,4}$

$\mathcal{M}_{0,4}$ is of dimension 1, it is a sphere with 3 points removed:

$$\mathcal{M}_{0,4} = \{(\mathbb{C}P^1, 0, 1, \infty, p) \mid p \neq 0, 1, \infty\} = \mathbb{C}P^1 - \{0, 1, \infty\} \quad , \quad \dim \mathcal{M}_{0,4} = d_{0,4} = 1 \quad , \quad \text{Aut} = \{\text{Id}\}. \quad (1-10)$$

We put on it the induced topology from $\mathbb{C}P^1$. It is not compact. We also put on it the complex structure inherited from $\mathbb{C}P^1$, it is thus a non compact Riemann surface.

Topologically it is sphere-less-3-points, its Euler characteristic is

$$\chi(\mathcal{M}_{0,4}) = -1. \quad (1-11)$$

Boundary

The boundary is reached when we consider a sequence in $\mathcal{M}_{0,4}$, or equivalently a sequence of points p in $\mathbb{C}P^1 - \{0, 1, \infty\}$, which has no adherence value (and thus no limit). This means a sequence that tends to 0 or 1 or ∞ . In other words, a boundary corresponds to 2 of the marked points colliding.

Consider the limit $p_4 = p \rightarrow 0 = p_1$, and $p_2 = 1, p_3 = \infty$. The chart $\{z \mid |z| > 2|p|\}$ is a neighborhood that contains p_2 and p_3 but not p_1 neither p_4 . In the limit $p \rightarrow 0$, this neighborhood becomes $\mathbb{C}P^1 - \{0\}$. In this chart, a whole basis of neighborhoods of p_1, p_4 becomes contracted to the point $\{0\}$. By using a Möbius transformation and the

coordinate $z' = z/p$, the chart $\{z' \mid |z'| < 2\}$ is a neighborhood that contains p_1 and p_4 but not p_2 neither p_3 . In the limit $p \rightarrow 0$, this neighborhood becomes $\mathbb{C}P^1 - \{\infty\}$, and a whole basis of neighborhoods of p_2, p_3 becomes contracted to the point $\{\infty\}$.

In other words, in the limit $p \rightarrow 0$, we have 2 charts, entirely disconnected, that touch each other only by one point.

This can be described by a notion of **Nodal surface**.

Definition 1.1 (Nodal Riemann surface) *A nodal Riemann surface Σ , is a finite union of compact surfaces Σ_i , together with a set of disjoint nodal points. A nodal point is a pair of distinct points on the union. The nodal surface is*

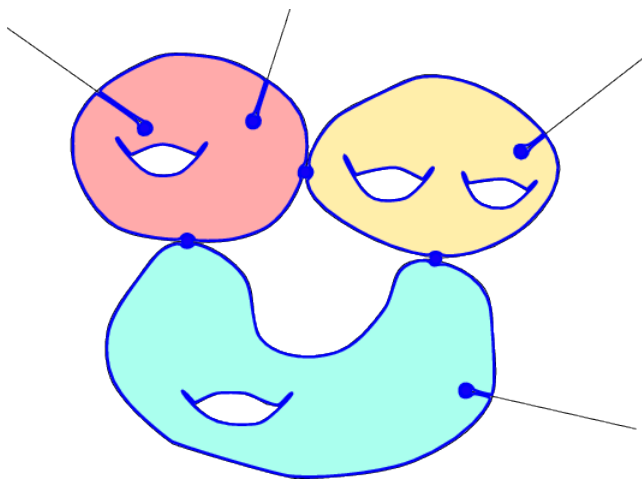
$$\Sigma = \cup_i \Sigma_i / \equiv \tag{1-12}$$

with the quotient by the equivalence relation $p \equiv q$ iff $p = q$ or if (p, q) is a nodal point. The topology of Σ is made of neighborhoods of non-nodal points in the Σ_i s and a neighborhood of a nodal point is the union of 2 neighborhoods of each of the 2 points. With this topology, $\cup_i \Sigma_i$ (before taking the quotient) is not a separated space, and the quotient Σ is separated but is not a manifold because there is no neighborhood of nodal points homeomorphic to Euclidian discs.

Connectivity is well defined, nodal surfaces can be connected or not, also Jordan arcs and their homotopy classes are well defined, and a nodal surface can be simply connected or not.

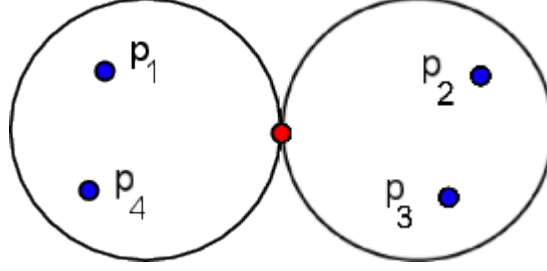
The Euler characteristic is:

$$\chi(\Sigma) = \sum_i \chi(\Sigma_i - \{\text{nodal points}\}). \tag{1-13}$$



We see that the limit $p \rightarrow 0$ in $\mathcal{M}_{0,4}$ is described by a nodal surface, with 2

components, and one nodal point:



- The first component is a Riemann sphere containing $p_2 = 1$ and $p_3 = \infty$ and one side of the nodal point (the point $z = 0$). It is thus $(\mathbb{C}P^1, 1, \infty, 0)$, which is an element of $\mathcal{M}_{0,3}$.
- The second component is a Riemann sphere containing $p_1 = 0$ and $p_4 = 1$ (in the coordinate $z' = z/p$) and one side of the nodal point (the point $z' = \infty$). It is thus $(\mathbb{C}P^1, 0, 1, \infty)$, which is an element of $\mathcal{M}_{0,3}$.

Notice that the Euler characteristic of a nodal surface with 2 sphere components having each 3 points removed is:

$$\begin{aligned} \chi(\Sigma_1 \cup \Sigma_2 - \{p_1, p_2, p_3, p_4, \text{nodal points}\}) &= \chi(\Sigma_1 - \{p_2, p_3, \text{nodal point}\}) \\ &\quad + \chi(\Sigma_2 - \{p_1, p_4, \text{nodal point}\}) \\ &= -1 - 1 = -2, \end{aligned} \quad (1-14)$$

and agrees with the Euler characteristic $2 - 2\mathbf{g} - n = -2$ of a surface of genus $\mathbf{g} = 0$ with $n = 4$ points removed.

Eventually this boundary of $\mathcal{M}_{0,4}$ can be viewed as an element of $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$.

Notice that the boundary $p_4 \rightarrow p_1$, is by a Möbius transform, the same as the boundary $p_2 \rightarrow p_3$.

There are 3 possibilities to choose a pair of colliding points among 4. Therefore there are 3 boundaries, and we have

$$\partial\mathcal{M}_{0,4} \sim \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \cup \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \cup \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}, \quad (1-15)$$

Each of these boundaries is a point, and $\mathcal{M}_{0,4}$ is a sphere with 3 points missing. We can compactify $\mathcal{M}_{0,4}$ by adding its boundary, and then we get the full sphere:

$$\overline{\mathcal{M}_{0,4}} = \mathcal{M}_{0,4} \cup \partial\mathcal{M}_{0,4} \sim \mathbb{C}P^1. \quad (1-16)$$

We equip it with the topology and complex structure of the Riemann sphere $\mathbb{C}P^1$. It is then a manifold, and in fact a complex manifold, a compact Riemann surface. We have

$$\chi(\overline{\mathcal{M}_{0,4}}) = 2. \quad (1-17)$$

Deligne–Mumford compactification

The general case works similarly. Moduli spaces $\mathcal{M}_{g,n}$ are not compact, they have boundaries whenever some marked points collapse, or whenever some cycles get pinched. The limit can always be described by nodal surfaces.

We thus define:

Definition 1.2 (Deligne Mumford compactified moduli space) *Let (g, n) such that $2 - 2g - n < 0$.*

*The Deligne–Mumford compactified moduli space $\overline{\mathcal{M}}_{g,n}$ is defined as the set (modulo isomorphisms) of connected nodal Riemann surfaces with n smooth labelled marked points (smooth means they are distinct from nodal points and from each other), and **stable** (the Euler characteristic of each component with all marked and nodal points removed is < 0), and of total Euler characteristic $\chi = 2 - 2g - n$:*

$$\overline{\mathcal{M}}_{g,n} = \{(\Sigma, p_1, \dots, p_n) \mid \chi = 2 - 2g - n = \sum_i \chi_i, \forall i \chi_i < 0\} / \text{isomorphisms}. \quad (1-18)$$

Isomorphisms are the holomorphic maps whose inverse is analytic, that conserve labeled points and that conserve (up to possible permutations) the nodal points.

We shall not describe here the topology of this moduli space, but just mention that with the appropriate topology it is indeed compact. Also it can be equipped with a differentiable structure, and a complex structure, that we do not describe here. It shall be explained in section 4 below, by providing an explicit atlas of charts and coordinates.

However, it is not a manifold (some neighborhoods are not homeomorphic to Euclidian neighborhoods), it is an orbifold (a manifold quotiented by a group: each neighborhood is homeomorphic to a Euclidian neighborhood quotiented by a group), and it is a **stack**. As we shall see now it has a rather non-trivial topology, in particular, it can have pieces of different dimensions.

Let us first study some simple examples.

- $(g, n) = (0, 5)$

We have

$$\mathcal{M}_{0,5} = \{(\mathbb{C}P^1, 0, 1, \infty, p, q) \mid p \neq 0, 1, \infty, q \neq 0, 1, \infty, p \neq q\}. \quad (1-19)$$

It has no non-trivial automorphisms, because there is a unique Möbius map that fixes $0, 1, \infty$.

Topologically, this is:

$$\mathcal{M}_{0,5} = (\text{sphere-less-3-points}) \times (\text{sphere-less-3-points}) - (\text{sphere-less-3-points}) \quad (1-20)$$

where the last sphere-less-3-points is the diagonal of the product.

We have

$$\dim \mathcal{M}_{0,5} = 2. \quad (1-21)$$

$$\chi(\mathcal{M}_{0,5}) = (-1) \times (-1) - (-1) = 2. \quad (1-22)$$

Naively, one could think that to get a smooth differentiable (and complex) compact manifold, one could add to it the missing pieces to complete the product into 2 full spheres. We would need to add 7 missing sphere-less-3-points, and 9 missing points.

This is wrong, let us study the boundary. There are boundaries of codimension 1 when exactly 2 points collide, and boundaries of codimension 2, when 2 pairs collide.

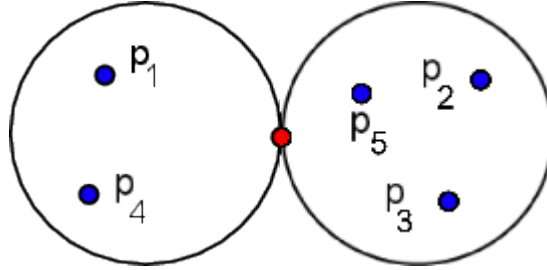
The number of boundaries of codimension 1 is the numbers of pairs of 2 points chosen among 5, i.e.

$$\binom{5}{2} = 10. \quad (1-23)$$

Each such boundary is described by a nodal surface with 1 nodal point and 2 components, one component carrying the 2 colliding points + nodal point, and one component carrying the other 3 points and the nodal point, each component is a sphere. Therefore

$$\partial_{\text{codim } 1} \mathcal{M}_{0,5} = 10 \times (\mathcal{M}_{0,3} \times \mathcal{M}_{0,4}). \quad (1-24)$$

Topologically each boundary of codim 1, is a sphere-less-3-points ($\mathcal{M}_{0,4}$ times a point).



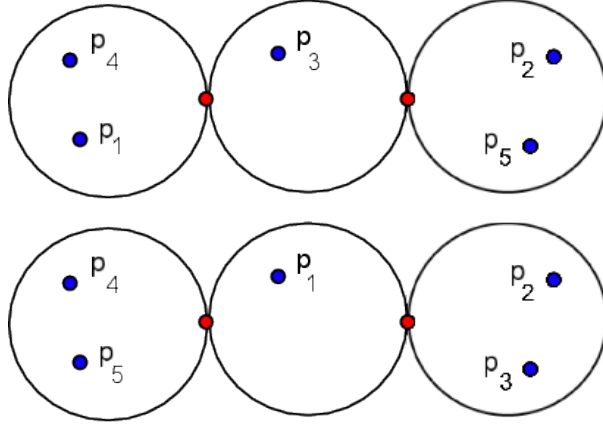
Similarly boundaries of codimension 2 are obtained by choosing 2 pairs of colliding points among 5 points, i.e. choose the non colliding point (5 choices), then split the 4 remainings into 2 pairs (3 choices),

$$5 \times 3 = 15. \quad (1-25)$$

Each such boundary is described by a nodal surface with 2 nodal points and 3 components, two components carrying the 2 pairs of colliding points + 1 nodal point, and one component carrying the 5th marked point and 2 nodal points, each component is a sphere. Therefore

$$\partial_{\text{codim } 2} \mathcal{M}_{0,5} = 15 \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}. \quad (1-26)$$

Topologically each boundary of codim 2, is a point.



Finally we have

$$\overline{\mathcal{M}_{0,5}} = \mathcal{M}_{0,5} \cup 10 \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \cup 15 \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}. \quad (1-27)$$

Topologically, it is a product of 2 spheres, with 7 sphere-less-3-points removed, and 9 points removed, and with 10 sphere-less-3-points added and 15 points added. It is definitely not a smooth manifold.

Some 7 among 10 of the sphere-less-3-points should be glued to the missing ones, namely the colliding $p \rightarrow 0$ should be glued to the missing $p = 0$ in the product, and the same for $p \rightarrow 1$, $p \rightarrow \infty$, $q \rightarrow 0$, $q \rightarrow 1$, $q \rightarrow \infty$, and the colliding $p \rightarrow q$ should be glued to the diagonal $p = q$. The 3 remaining, namely $0 \rightarrow 1$, $0 \rightarrow \infty$ and $1 \rightarrow \infty$ should be glued to just a point respectively to $p = q = \infty$, $p = q = 1$, $p = q = 0$.

9 among 15 of the codimension 2 boundaries, for example the boundary $(p \rightarrow 0, q \rightarrow 1, \infty)$ should be glued to the corresponding missing points in the product. The 6 remaining codimension 2 boundaries, for example $(0 \rightarrow 1, p, q \rightarrow \infty)$ should be glued to the missing point in one of the 3 sphere-less-3-points, the one glued to the point $p = q = \infty$.

In the end we find a union of pieces of different dimensions. Many points have neighborhoods that are not homeomorphic to Euclidian subspaces.

This is a **stack**.

1.3 $n > 5$

The same holds for higher n :

$$\dim \mathcal{M}_{0,n} = n - 3 \quad (1-28)$$

it is a product of $n - 3$ sphere-less-3-points, and to which we remove all submanifolds of coinciding points. One can compute that

$$\chi(\mathcal{M}_{0,n}) = (-1)^{n-1}(n - 3)!. \quad (1-29)$$

Boundaries are nodal surfaces. For $n \geq 5$ there are $n(n-1)/2$ codimension 1 boundaries (choose 2 points among n). All boundaries can be described by a graph whose vertices are the components of the nodal surface, and whose edges are nodal points. Since the genus is 0, we must have

$$2 - 2g - n = 2 - n = \sum_i (2 - 2g_i - n_i - k_i) \quad (1-30)$$

where $\sum_i n_i$ is the number of marked points and $\frac{1}{2} \sum_i k_i$ is the number of nodal points. Observe that the connected components can't exceed 1+ number of nodal points, otherwise the surface would not be connected, which implies that

$$\sum_i (k_i - 2) \geq -2. \quad (1-31)$$

The relationship $2 - n = -n + \sum_i (k_i - 2) - 2 \sum_i g_i$ implies

$$\sum_i g_i = 1 - \frac{1}{2} \sum_i (k_i - 2) \leq 0, \quad (1-32)$$

therefore all connected components must have genus 0, and the number of nodal points (edges in the dual graph) is equal to the number of connected components (vertices in the dual graph)-1, i.e. the graph must be a tree.

2 Genus 1

By the Abel map, every Riemann surface of genus 1 is isomorphic to a standard torus $T_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ for some τ with $\Im\tau > 0$.

Theorem 2.1 T_τ and $T_{\tau'}$ are isomorphic iff

$$\tau = \frac{a\tau' + b}{c\tau' + d}, \quad (a, b, c, d) \in \mathbb{Z}^4, \quad ad - bc = 1. \quad (2-1)$$

proof: Assume that there exists an isomorphism $f : T_\tau \rightarrow T_{\tau'}$. It must satisfy, in each charts:

$$f(z + n + \tau m) = f(z) + n' + \tau' m' \quad (2-2)$$

Its differential df must therefore satisfy

$$df(z + n + \tau m) = df(z) \quad (2-3)$$

showing that it is globally a holomorphic 1-form on T_τ , it must therefore be proportional to dz , i.e.

$$df(z) = \alpha dz. \quad (2-4)$$

This implies that f must be a chart-wise affine function:

$$f(z) = \alpha z + \beta. \quad (2-5)$$

A priori this function is defined on the fundamental domain. It must satisfy (2-2), in particular we must have

$$\alpha = f(1) - f(0) = c\tau' + d \quad , \quad \alpha\tau = f(\tau) - f(0) = a\tau' + b, \quad (2-6)$$

and therefore

$$\tau = \frac{a\tau' + b}{c\tau' + d}. \quad (2-7)$$

Saying that this transformation is invertible in the same form implies that $ad - bc = 1$.

Vice-versa, if τ and τ' are related by such a transformation, the map:

$$f : z \mapsto (c\tau' + d)z \bmod \mathbb{Z} + \tau'\mathbb{Z} \quad (2-8)$$

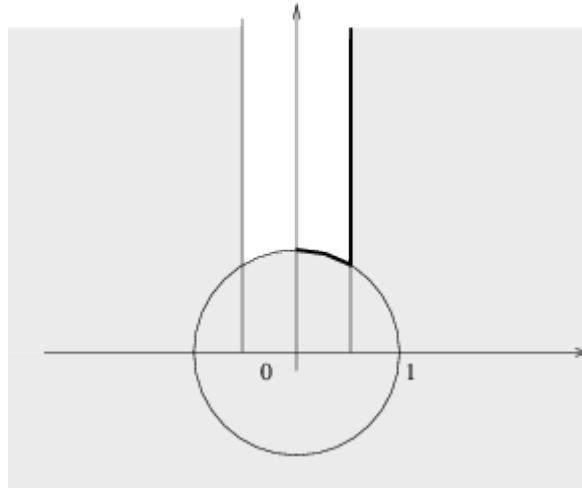
is a holomorphic map $T_\tau \rightarrow T_{\tau'}$ since it satisfies the transition condition that $f(z + n + \tau m) \equiv f(z)$. \square

Corollary 2.1 (Moduli space $\mathcal{M}_{1,0}$)

$$\mathcal{M}_{1,0} = \{T_\tau \mid \tau \in \mathbb{C}_+\} / \text{isomorphisms} = \mathbb{C}_+ / PSL(2, \mathbb{Z}). \quad (2-9)$$

However, each T_τ has an infinite group of automorphisms, indeed every translation $z \mapsto z + \beta$ for $\beta \in \mathbb{C}$ is an automorphism.

$\mathcal{M}_{1,0}$ is unstable.



If we have a marked point p_1 , up to performing a translation, we choose it to be the origin $z = 0$. Therefore:

Theorem 2.2 (Moduli space $\mathcal{M}_{1,1}$)

$$\mathcal{M}_{1,1} = \{(T_\tau, 0) \mid \tau \in \mathbb{C}_+\} / \text{isomorphisms} = \mathbb{C}_+ / PSL(2, \mathbb{Z}). \quad (2-10)$$

Since the modular group $PSL(2, \mathbb{Z})$ is generated by $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$, a fundamental domain is

$$\begin{aligned} \mathcal{M}_{1,1} = & \{z \mid \Im z > 0, \frac{-1}{2} < \Re z < \frac{1}{2}, |z| > 1\} \\ & \cup \{z \mid \Re z = \frac{1}{2}, \Im z > \frac{\sqrt{3}}{2}\} \cup \{z \mid |z| = 1, \frac{\pi}{3} < \text{Arg } z < \frac{\pi}{2}\} \\ & \cup \{i\} \cup \{e^{\frac{2\pi i}{3}}\}. \end{aligned} \quad (2-11)$$

$\mathcal{M}_{1,1}$ is an orbifold of dimension 1.

Automorphisms:

The map $z \mapsto -z$ is always an automorphism.

- For generic $(\Sigma, p) \in \mathcal{M}_{1,1}$, i.e. generic τ we have $\text{Aut} = \mathbb{Z}_2$.
- For $\tau = i$, the map $z \mapsto iz$ is an automorphism, and we have $\text{Aut} = \mathbb{Z}_4$.
- For $\tau = e^{i\pi/3}$, the map $z \mapsto e^{i\pi/3}z$ is an automorphism, and we have $\text{Aut} = \mathbb{Z}_6$.

In all cases the number of automorphisms is finite, $\mathcal{M}_{1,n}$ is stable iff $n \geq 1$.

Euler characteristic of $\mathcal{M}_{1,1}$:

$\mathcal{M}_{1,1}$ is made of a 2-cell (with \mathbb{Z}_2 automorphism), two 1-cells (with \mathbb{Z}_2 automorphism), and 2 points with automorphisms \mathbb{Z}_4 and \mathbb{Z}_6 . The Euler characteristic is thus

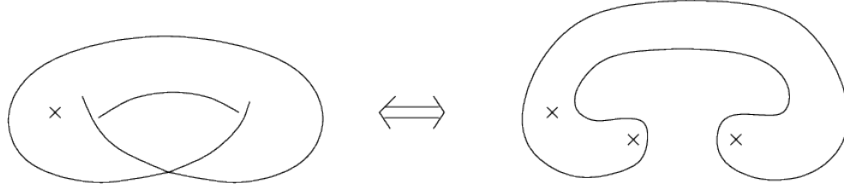
$$\chi(\mathcal{M}_{1,1}) = \frac{1}{2} - 2 \times \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{-1}{12}. \quad (2-12)$$

What is interesting, is to see that for an orbifold, the Euler characteristic is a rational number rather than an integer.

2.1 Boundary of $\mathcal{M}_{1,1}$

The boundary is reached when a cycle gets pinched into a nodal point, and this corresponds to $\tau \rightarrow \infty$ (or in fact $(a\tau + b)/(c\tau + d) \rightarrow \mathbb{Q} \cup \{\infty\}$ at the boundary of the hyperbolic plane \mathbb{C}_+). We can identify a torus with a pinched cycle (a nodal point) and a marked point, with a sphere with 3 marked points, 2 of the marked points when glued together provide a nodal point (and they can be exchanged by a \mathbb{Z}_2 symmetry), and the 3rd marked point is the initial marked point of the torus. In other words

$$\partial\mathcal{M}_{1,1} \sim \mathcal{M}_{0,3}/\mathbb{Z}_2. \quad (2-13)$$



We have

$$\overline{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \cup \mathcal{M}_{0,3}/\mathbb{Z}_2. \quad (2-14)$$

The fundamental domain of $\mathcal{M}_{1,1}$ is a hyperbolic triangle in the upper complex plane \mathbb{C}_+ equipped with the hyperbolic metric. In this metric, geodesics are lines or circles orthogonal to the real axis, so that indeed the boundaries of the fundamental domain are geodesics, it is an hyperbolic triangle. This triangle has 3 vertices: 2 of them ($e^{i\pi/3}$ and $e^{2i\pi/3}$) have angle $\pi/3$ and one of them (∞) has angle 0.

In hyperbolic geometry, the area of a triangle is its deficit angle:

$$\text{Volume}_{\text{Hyperbolic}}(\overline{\mathcal{M}}_{1,1}) = \pi - \frac{\pi}{3} - \frac{\pi}{3} - 0 = \frac{\pi}{3}. \quad (2-15)$$

3 Higher genus

For $\mathfrak{g} \geq 2$, the Abel map embeds the curve into a submanifold of its Jacobian, which is a compact torus of dimension $2\mathfrak{g}$.

Given a symplectic basis of cycles, and an origin point to define the Abel map $p \mapsto \mathbf{u}(p)$, define

$$\begin{aligned} \forall i = 1, \dots, \mathfrak{g} \quad , \quad v_i(p) &= \sum_j (\Im\tau)_{i,j}^{-1} \Im u_j(p) \\ \forall i = 1, \dots, \mathfrak{g} \quad , \quad \tilde{v}_i(p) &= \Re u_i(p) - \sum_j (\Re\tau)_{i,j} v_j(p) \end{aligned} \quad (3-1)$$

By definition we have

$$u_i(p) = v_i(p) + \sum_j \tau_{i,j} \tilde{v}_j(p), \quad (3-2)$$

which we write

$$\mathbf{u}(p) = v(p) + \tau \tilde{v}(p). \quad (3-3)$$

The following sets of points of Σ can be chosen as arcs representing the cycles \mathcal{A}_i or \mathcal{B}_i s:

$$\begin{aligned} \{p \mid \tilde{v}_i(p) = 0\} &\sim \mathcal{A}_i \\ \{p \mid \tilde{v}_i(p) = 1\} &\sim \mathcal{A}_i \\ \{p \mid v_i(p) = 0\} &\sim \mathcal{B}_i \end{aligned}$$

$$\{p \mid v_i(p) = 1\} \sim \mathcal{B}_i \quad (3-4)$$

If we remove them from Σ , we get the fundamental domain $\Sigma_0 \subset \Sigma$. The map

$$\begin{aligned} \Sigma &\rightarrow]0, 1[^{2g} \\ p &\mapsto \{v_i(p), \tilde{v}_i(p)\} \end{aligned} \quad (3-5)$$

embeds the curve into a polygon with $4g$ sides in $]0, 1[^{2g}$. The surface Σ is obtained by gluing corresponding sides together.

It is equipped with a Kähler metric corresponding to a symplectic form (which is nothing but the restriction of the canonical symplectic form of $[0, 1]^{2g} \subset \mathbb{R}^{2g}$ to the image of the curve)

$$\sum_{i=1}^g d\tilde{v}_i \wedge dv_i = \sum_{i,j} (\Im\tau)_{i,j}^{-1} d\Re u_i \wedge d\Im u_i = \frac{i}{2} \sum_{i,j} (\Im\tau)_{i,j}^{-1} du_i \wedge d\bar{u}_i. \quad (3-6)$$

Notice that this metric is positive definite, and is modular invariant.

With this metric, the total area is (this is proved by Riemann bilinear identity)

$$\text{Area}(\Sigma) = g. \quad (3-7)$$

Example: Torus. The torus is not only embedded in the Jacobian, it is isomorphic to its Jacobian. Writing $u = v + \tau\tilde{v}$ where v and \tilde{v} are real, the lines $v = 0$, $v = 1$ are the 2 sides of the cycle \mathcal{B} , the lines $\tilde{v} = 0$ and $\tilde{v} = 1$ are the 2 sides of the cycle \mathcal{A} , and the fundamental domain comprised between them is the parallelogram with summits $0, 1, 1 + \tau, \tau$. This is a parallelogram whose basis has length 1, and whose height is $\Im\tau$, therefore with the metric

$$\frac{1}{\Im\tau} d\Re u \wedge d\Im u = d\tilde{v} \wedge dv, \quad (3-8)$$

its area is 1, which is equal to the genus.

4 Coordinates in the moduli space

We shall find an explicit atlas of the moduli space $\mathcal{M}_{g,n}$. Charts will be homeomorphic to $\mathbb{R}_+^{6g-6+2n}$, and with gluing rules encoded by graphs.

4.1 Strebel graphs

Let (g, n) such that $2g - 2 + n > 0$. Rather than considering $\mathcal{M}_{g,n}$, let us consider the product $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, that we prefer to view as a trivial bundle over $\mathcal{M}_{g,n}$ whose fiber is \mathbb{R}_+^n :

$$\tilde{\mathcal{M}}_{g,n} = \mathcal{M}_{g,n} \times \mathbb{R}_+^n \rightarrow \mathcal{M}_{g,n}. \quad (4-1)$$

Let $\mathcal{S} = (\Sigma, p_1, \dots, p_n, L_1, \dots, L_n) \in \tilde{\mathcal{M}}_{\mathbf{g}, n}$, where Σ is a smooth Riemann surface of genus \mathbf{g} , and p_1, \dots, p_n are n labeled distinct marked points on Σ , and L_1, \dots, L_n are positive real numbers.

Let us consider the set $\Omega_{\mathcal{S}}$ of **quadratic differentials** ω (see def. II-1.3), having double poles at the marked points, and no other poles, and behaving (in some chart of Σ with coordinate ϕ) near the marked point p_i as:

$$\omega(p) \underset{p \rightarrow p_i}{\sim} \frac{-L_i^2}{(\phi(p) - \phi(p_i))^2} (1 + O(\phi(p) - \phi(p_i))) d\phi(p)^2. \quad (4-2)$$

$\Omega_{\mathcal{S}}$ is an affine space, whose underlying linear space is the vector space of quadratic differentials with at most simple poles at the p_i s.

- Example when $(\mathbf{g}, n) = (0, 3)$, and $\Sigma = \mathbb{C}P^1$, we must have

$$\omega(z) = \frac{-L_{\infty}^2 z^2 + (L_{\infty}^2 + L_0^2 - L_1^2)z - L_0^2}{z^2(z-1)^2} dz^2 \quad (4-3)$$

and $\Omega_{\mathcal{S}}$ consists of a unique quadratic differential, $\dim \Omega_{\mathcal{S}} = 0$.

- Example when $(\mathbf{g}, n) = (0, 4)$, and $\mathcal{S} = (\mathbb{C}P^1, 0, 1, \infty, p, L_0, L_1, L_{\infty}, L_p)$, we have

$$\omega(z) = \frac{-dz^2}{z(z-1)(z-p)} \left(L_{\infty}^2 z + \frac{L_0^2}{z} + \frac{L_1^2}{z-1} + \frac{L_p^2}{z-p} + c \right) \quad (4-4)$$

where $c \in \mathbb{C}$ can be any constant, in other words $\dim \Omega_{\mathcal{S}} = 1$.

- Example when $\mathbf{g} = 0$ and $n \geq 4$, and $\mathcal{S} = (\mathbb{C}P^1, p_1, \dots, p_n; L_{\infty}, L_1, \dots, L_n)$,

$$\omega(z) = \frac{-dz}{\prod_{i=1}^n (z - p_i)} \left(\sum_{i=1}^n \frac{L_i^2 \prod_{j \neq i} (p_i - p_j)}{z - p_i} + \sum_{j=0}^{n-4} c_j z^j \right) \quad (4-5)$$

where c_0, c_1, \dots, c_{n-4} are arbitrary complex numbers, in other words $\Omega_{\mathcal{S}} \sim \mathbb{C}^{n-3}$.

- Example when $(\mathbf{g}, n) = (1, 1)$, and $\mathcal{S} = (T_{\tau}, 0, L_0)$, we must have

$$\omega(z) = (-L_0^2 \wp(z; \tau) + c) dz^2 \quad (4-6)$$

where \wp is the Weierstrass function and $c \in \mathbb{C}$, so that $\dim \Omega_{\mathcal{S}} = 1$.

Theorem 4.1

$$\dim \Omega_{\mathcal{S}} = d_{\mathbf{g}, n} = 3\mathbf{g} - 3 + n. \quad (4-7)$$

proof: • If $\mathbf{g} = 0$, on the Riemann sphere, $\Omega_{\mathcal{S}}$ is the set of forms given in (4-5). Indeed ω/dz^2 must be a rational function with n double poles of given leading behavior, and that behaves as $O(1/z^4)$ at ∞ . Therefore $f(z) = \frac{\omega(z)}{dz^2} \prod_{i=1}^n (z - p_i)^2$ must be a

polynomial, of degree at most $2n - 4$, and such that $f(p_i) = -L_i^2$. The space of such polynomials has dimension $n - 3$.

- Now assume $\mathfrak{g} = 1$, and $\Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. Then the following 1-form

$$\tilde{\omega}(z) = \frac{\omega(z)}{dz} + \sum_{i=1}^n L_i^2 \wp(z - p_i) dz \quad (4-8)$$

must be a meromorphic 1-form with at most simple poles at the p_i s. Let $r_i = \text{Res}_{p_i} \tilde{\omega}$. Then the 1-form (where σ is a primitive of \wp , i.e. $d\sigma = \wp$)

$$\tilde{\omega}(z) - \sum_{i=1}^n r_i \sigma(z - p_i) dz \quad (4-9)$$

must be holomorphic, i.e. proportional to dz , therefore there must exist r_1, \dots, r_n , such that $\sum_i r_i = 0$, and C such that

$$\omega(z) = \left(C - \sum_{i=1}^n L_i^2 \wp(z - p_i) - r_i \sigma(z - p_i) \right) dz^2. \quad (4-10)$$

Vice versa, every such quadratic form is in $\Omega_{\mathcal{S}}$, therefore

$$\dim \Omega_{\mathcal{S}} = n. \quad (4-11)$$

- Now assume $\mathfrak{g} \geq 1$. First, $\Omega_{\mathcal{S}}$ is not empty, indeed the quadratic form

$$\omega(z) = - \sum_{i=1}^n L_i^2 B(z, p_i) \frac{\omega_1(z)}{\omega_1(p_i)} \in \Omega_{\mathcal{S}}. \quad (4-12)$$

Any element of $\Omega_{\mathcal{S}}$ is of the form

$$\omega + \omega' \quad (4-13)$$

where ω' is a quadratic form having at most simple poles at the p_i s, i.e. in the linear underlying space of the affine space $\Omega_{\mathcal{S}}$.

Let us choose once for all, a generic (having no zero at the p_i s) holomorphic 1-form $\nu \in \mathcal{O}^1(\Sigma)$ (typically choose $\nu = \omega_1$), and let o a zero of ν , and let

$$r_i = \text{Res}_{p_i} \frac{\omega'}{\nu}. \quad (4-14)$$

The holomorphic quadratic form

$$\tilde{\omega}(z) = \omega'(z) - \sum_{i=1}^n r_i \omega'_{p_i, o}(z) \nu(z) \quad (4-15)$$

has no poles (where $\omega_{p,q}(z)$ is the 3rd kind form introduced in cor.II-3.2 or def. III-5.2). Since the ratio of quadratic forms is a meromorphic function, and since meromorphic

functions have the same number of poles and zeros, this implies that $\tilde{\omega}$ must have the same number of zeros as any quadratic form, in particular as ν^2 , and therefore $\tilde{\omega}$ must have $4\mathbf{g} - 4$ zeros.

Among these zeros, choose $\mathbf{g} - 1$ of them, and choose the unique (up to scalar multiplication) holomorphic 1-form μ that vanishes at those $\mathbf{g} - 1$ points. Therefore $\frac{\tilde{\omega}}{\mu}$ is a meromorphic 1-form with $3\mathbf{g} - 3$ zeros and $\mathbf{g} - 1$ poles (the other $\mathbf{g} - 1$ zeros of μ) that we name $q_1, \dots, q_{\mathbf{g}-1}$. Let $s_i = \text{Res}_{q_i} \frac{\tilde{\omega}}{\mu}$, then

$$\frac{\tilde{\omega}}{\mu} - \sum_{i=1}^{\mathbf{g}-1} s_i \omega_{q_i, o} \quad (4-16)$$

must be a holomorphic 1-form, therefore a linear combination of $\omega_1, \dots, \omega_{\mathbf{g}}$. In the end we may write

$$\omega' = \nu \sum_{i=1}^n r_i \omega_{p_i, o} + \mu \left(\sum_{i=1}^{\mathbf{g}-1} s_i \omega_{q_i, o} + \sum_{i=1}^{\mathbf{g}} \alpha_i \omega_i \right). \quad (4-17)$$

The decomposition is not unique, as we could have chosen any $\mathbf{g} - 1$ zeros of $\tilde{\omega}$, but this is a discrete ambiguity. Vice versa, for any choice of r_i, s_i (subject to $\sum_i s_i = 0$), μ, α_i , we get an element of $\Omega_{\mathcal{S}}$. The only redundancy is that we can multiply μ by a scalar and divide s_i and α_i s by the same scalar. This shows that

$$\dim \Omega_{\mathcal{S}} = \left(\underbrace{n}_{r_i \in \mathbb{C}^n} + \underbrace{\mathbf{g} - 2}_{s_i \in \mathbb{C}^{\mathbf{g}-2}} + \underbrace{\mathbf{g}}_{\mu \in \mathcal{O}^1} + \underbrace{\mathbf{g}}_{\sum_i \alpha_i \omega_i \in \mathcal{O}^1} \right) - \underbrace{1}_{\text{scalar redundancy}} = 3\mathbf{g} - 3 + n. \quad (4-18)$$

□

Definition 4.1 Given $\omega \in \Omega_{\mathcal{S}}$, a **horizontal trajectory** is a maximal connected set $\gamma \subset (\Sigma - \{p_1, \dots, p_n\})^{\text{universal cover}}$, on which the map

$$p \mapsto \mathfrak{S} \left(\int^p \sqrt{\omega(z)} \right) \quad (4-19)$$

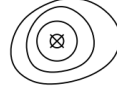
is constant. This is independent of a choice of initial point of integration, and of a choice of a sign of the square root.

Horizontal trajectories have the following properties:

- Locally, in a neighborhood of a point where ω has neither pole nor zero, horizontal trajectories are C^∞ Jordan arcs.
- In a neighborhood of a pole p_i , we have

$$\sqrt{\omega(z)} \sim i L_i \frac{dz}{z - p_i} = i L_i d \log(z - p_i) \quad (4-20)$$

so that horizontal trajectories are circles $|z - p_i| \sim \text{constant}$.



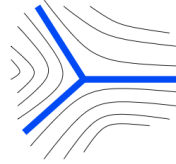
- Horizontal trajectories can not cross except at zeros of ω .
- In a neighborhood of a zero a of ω , of order k_a , we have

$$\omega(z) \sim c_a (z - a)^{k_a} dz^2 \quad (4-21)$$

and thus the horizontal trajectories going through a are locally the rays

$$\sim a + e^{i(-\text{Arg } c_a + \pi j) \frac{2}{k_a + 2}} \mathbb{R}_+ \quad , \quad j = 1, \dots, k_a + 2 \quad (4-22)$$

they form a star with $k_a + 2$ branches. These are called "critical trajectories". Generically, zeros are simple, $k_a = 1$, so that critical trajectories have generically trivalent vertices.



- a critical trajectory starting from a vertex (zero of ω) can either meet another (or the same) vertex, it is then called a finite trajectory, or not, it is then called an infinite trajectory.

A finite critical trajectory starting at a , is a Jordan arc $\gamma : [0, l] \rightarrow \Sigma$ such that $\frac{1}{2\pi} \int_a^{\gamma(t)} |\sqrt{\omega}| = t$.

An infinite critical trajectory starting at a , is a Jordan arc $\gamma : [0, \infty[\rightarrow \Sigma$ such that $\frac{1}{2\pi} \int_a^{\gamma(t)} |\sqrt{\omega}| = t$.

- infinite trajectories have an adherence, which is also a horizontal trajectory. Indeed, let $\gamma : \mathbb{R}_+ \rightarrow \Sigma$ an infinite trajectory, and let $\hat{\gamma} = \{p \in \Sigma \mid \forall \epsilon > 0, \exists q \in \gamma \mid d_{|\sqrt{\omega}|}(q, p) < \epsilon\}$, where the distance is defined by the metric $|\sqrt{\omega}|$. This adherence is non empty, because if we take an infinite sequence t_1, t_2, \dots in \mathbb{R}_+ tending to $+\infty$, then the sequence $\gamma(t_n)$ must have a limit on Σ (because Σ is compact), so $\hat{\gamma}$ is not empty. Moreover, if $\gamma(t_n)$ converges to a point $p \in \hat{\gamma}$, then $\omega(\gamma(t_n))$ converges to $\omega(p)$, and thus $\hat{\gamma}$ has a tangent vector, it must be a compact C^1 curve. Moreover, it must be a horizontal trajectory. We call it a critical horizontal trajectory. It is compact and has finite length, but doesn't need to go through a vertex.

There are at most 3 critical trajectories per vertex, thus the number of compact critical trajectories is finite.

- The set of compact critical trajectories forms a graph Γ embedded in Σ . The graph is not necessarily connected. Its vertices are zeros of ω .
- The connected components of $\Sigma - \Gamma$ are called faces. For each $i = 1, \dots, n$, there is a unique face containing p_i , and it is topologically a disc.
- Faces that do not contain any p_i , have the topology of cylinders.

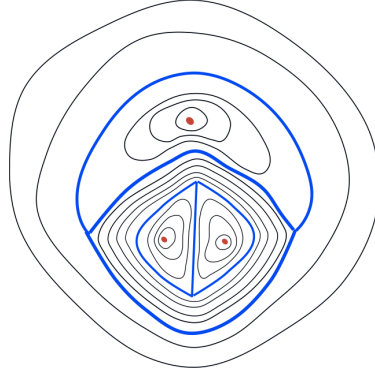
proof: Choose a finite number ≥ 1 of marked points on each edge of the graph. Consider the vertical trajectories emanating from them, and continue them until they reach another edge or a point p_i . Those trajectories can have finite or infinite length. If a trajectory has infinite length, this means that $|\Im \int \sqrt{\omega}|$ becomes larger than the distance (measured with the metric $|\sqrt{\omega}|$) between any 2 edges, and thus an infinite vertical trajectory must necessarily enter one of the discs around one p_i . The vertical trajectories inside the faces not containing the p_i s have finite length. Consider the graph $\tilde{\Gamma}$ of all these finite vertical trajectories.

Consider the connected components of $\Sigma \setminus (\Gamma \cup \tilde{\Gamma})$. Every component f not containing a p_i has at least one marked point on its boundary, call it q_f . The map $g_f : f \rightarrow \mathbb{C}$, $p \mapsto \int_{q_f}^p \sqrt{\omega}$ is bounded, i.e. $\Re g_f$ and $\Im g_f$ have a minimum and maximum in f , defining a rectangle R_f in \mathbb{C} . The map g_f is analytic in f , and thus the image of f is an open set of R_f , whose boundary can be made only of horizontal and vertical lines, i.e. must be a rectangle in \mathbb{C} .

This construction provides an atlas of Σ , whose charts are either discs centered around the p_i s and a finite number of rectangles. The transition functions are translations $x \mapsto \pm x + c$, up to the choice of sign \pm for the square root.

Consider a compact critical trajectory, and the set of all rectangles bordering its left (reps. right) side. Since Σ is orientable, the choice of sign of the square root can be chosen in such a way that all transition maps from a rectangle to its neighbour, are $c \mapsto x + c$ with $c \in \mathbb{R}$. Moreover the other horizontal boundary of the face, must be a horizontal trajectory, i.e. all rectangles must have the same height. The gluing of all rectangles bordering the trajectory is then the gluing of a finite number of rectangles of the same height along their parallel sides, it is a cylinder.

□

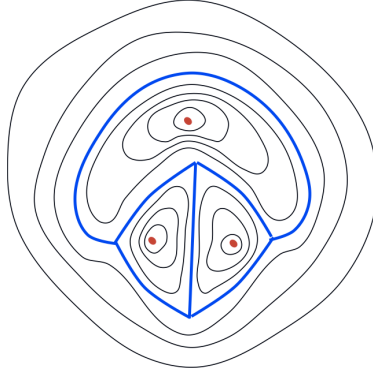


Theorem 4.2 (Strebel) *There exists a unique element $\omega \in \Omega_{\mathcal{S}}$ such that the graph is cellular, i.e. all faces are discs (no cylinder). ω is called the **Strebel differential**, and the graph of its critical horizontal trajectories is called the **Strebel graph**.*

The Strebel graph map:

$$\begin{aligned} \tilde{\mathcal{M}}_{\mathfrak{g},n} &\rightarrow \bigoplus_{\Gamma \in \mathcal{G}_{\mathfrak{g},n}} \mathbb{R}_+^{\{\text{edges}(\Gamma)\}} \\ \mathcal{S} &\mapsto (\Gamma, \{\ell_e\}_{e \in \text{edges}(\Gamma)}) \quad , \quad \ell_e = \left| \int_e \sqrt{\omega} \right| \end{aligned} \quad (4-23)$$

is an isomorphism of orbifolds (it sends $\text{Aut } \mathcal{S} \rightarrow \text{Aut } \Gamma$). Here $\mathcal{G}_{\mathfrak{g},n}$ is the set of trivalent cellular graphs of genus \mathfrak{g} , with n faces. ℓ_e is the length of edge e , measured with the metric $|\sqrt{\omega}|$.



proof: Given $\omega \in \Omega_{\mathcal{S}}$, for each p_i and a given chart, we define the unique primitive of $\sqrt{\omega}$ such that

$$dg_{p_i}(z) = \sqrt{\omega(z)} \quad , \quad g_{p_i}(z) \sim_{z \rightarrow p_i} i L_i \log(\phi(z) - \phi(p_i)) + O((\phi(z) - \phi(p_i))) \quad (4-24)$$

(in other words we have fixed the integration constant so that there is no term of order 0). Now, given some positive real numbers r_1, \dots, r_n , we define:

$$\mathcal{A}(\omega) = \int_{\Sigma \setminus \cup_i \{\Im g_{p_i}(z) < \log r_i\}} |\sqrt{\omega}|^2 + 2\pi \sum_i L_i \log r_i. \quad (4-25)$$

One easily checks (Stokes theorem) that $\mathcal{A}(\omega)$ is actually independent of r_i , provided that r_i is sufficiently small so that the discs $\Im g_{p_i}(z) < \log r_i$ do not intersect Γ .

We have

$$\mathcal{A}(\omega) \geq 2\pi \sum_i L_i \log R_i(\omega) \quad (4-26)$$

where $R_i(\omega)$ is the largest radius such that $\Im g_{p_i}(z) < \log R_i$ does not intersect Γ , so that

$$\mathcal{A}(\omega) - 2\pi \sum_i L_i \log R_i(\omega) = \int_{\cup \text{cylinders}} |\sqrt{\omega}|^2. \quad (4-27)$$

Moreover \mathcal{A} is a convex functional of ω , with second derivative

$$\mathcal{A}''(\delta\omega, \tilde{\delta}\omega) = \int_{\Sigma} |\sqrt{\omega}|^2 \Im \frac{\delta\omega}{\omega} \Im \frac{\tilde{\delta}\omega}{\omega} \quad (4-28)$$

which is a positive definite quadratic form. Therefore \mathcal{A} possesses a minimum, at which the gradient of \mathcal{A} vanishes. The gradient is

$$\mathcal{A}'(\delta\omega) = \int_{\Sigma} |\sqrt{\omega}|^2 \Im \frac{\delta\omega}{\omega}. \quad (4-29)$$

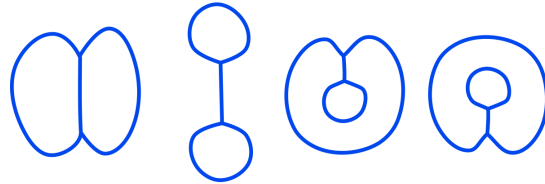
$$\delta \log R_i(\omega) = \frac{1}{2} \Im \int_{p_i}^a \frac{\delta\omega}{\sqrt{\omega}} \quad (4-30)$$

Therefore the minimum is the Strebel differential.

□

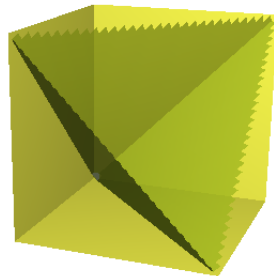
Example

$\tilde{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} \times \mathbb{R}_+^3$ is a sum of 4 graphs times \mathbb{R}_+^3 .



In each graph there are 3 lengths corresponding to the 3 edges.

In the 2nd, 3rd, 4th graph we have a triangular inequality respectively $L_\infty \geq L_0 + L_1$, $L_1 \geq L_0 + L_\infty$, $L_0 \geq L_1 + L_\infty$, whereas in the 1st graph no triangular inequality is satisfied. This cuts \mathbb{R}_+^3 into 4 disjoint regions, each labelled by a graph



In the 1st graph we have $L_0 = \ell_1 + \ell_2$, $L_1 = \ell_2 + \ell_3$ and $L_\infty = \ell_3 + \ell_1$, whereas in the second graph we have $L_0 = \ell_1$, $L_1 = \ell_2$ and $L_\infty = \ell_1 + \ell_2 + 2\ell_3$.

4.2 Topology of the moduli space

An atlas of $\tilde{\mathcal{M}}_{\mathfrak{g},n}$ is thus made of charts labelled by 3-valent graphs of genus \mathfrak{g} with n faces.

The boundary between the domains correspond to one or several edge lengths vanishing. Shrinking an edge amounts to merge 2 trivalent vertices into a 4-valent vertex. The gluing of charts amounts to glue together all graphs whose shrinking edges give the same higher valence graph.

This atlas makes $\tilde{\mathcal{M}}_{\mathfrak{g},n}$ a smooth real manifold of dimension $6\mathfrak{g} - 6 + 3n$ (number of edges), equipped with the topology inherited from $\mathbb{R}_+^{6\mathfrak{g}-6+3n}$. It is connected.

Bounday and compactification

Not all vanishing edge lengths correspond to graphs embedded on a smooth surface, some of them can be embedded only in nodal surfaces, and these correspond to the boundary of $\tilde{\mathcal{M}}_{\mathfrak{g},n}$.

Adding the "nodal graphs" makes $\overline{\mathcal{M}}_{\mathfrak{g},n}$ a Deligne–Mumford compact space, but not a manifold because there are pieces of different dimensions. It is connected.

Complex structure

Instead of the real lengths ℓ_e , we can parametrize the Strebel differential as a point in $\Omega_{\mathcal{S}}$, as in (4-18):

$$\Omega_{\mathcal{S}} \sim \mathbb{C}^{n+\mathfrak{g}-2} \times (\mathcal{O}^1(\Sigma) \times \mathcal{O}^1(\Sigma))/\mathbb{C} \sim \mathbb{C}^{3\mathfrak{g}-3+n}. \quad (4-31)$$

The Strebel differential has complex coordinates in that space, and these complex coordinates can be used as coordinates of $\mathcal{M}_{\mathfrak{g},n}$.

A basis of $\mathcal{O}^1(\Sigma)$ can be defined locally in a chart where a symplectic basis of cycles can be held fixed. Changing charts changes the symplectic basis, and the transition functions to glue coordinates are obtained from the transition functions of the bundle with fiber $\mathcal{O}^1(\Sigma) \rightarrow \mathcal{M}_{\mathfrak{g},n}$, and they are analytic.

This provides a complex structure to $\mathcal{M}_{\mathfrak{g},n}$.

5 Uniformization theorem

Question: Poincaré metric

Let (\mathfrak{g}, n) be such that $2-2\mathfrak{g}-n < 0$. Let $(\Sigma, p_1, \dots, p_n) \in \mathcal{M}_{\mathfrak{g}, n}$ a compact Riemann surface of genus \mathfrak{g} with n marked points. Let $\alpha_1, \dots, \alpha_n$ be n real numbers.

Does there exists a Riemannian metric of constant curvature -1 , that vanishes at order $2\alpha_i$ at marked point p_i ? Is it unique ?

In a chart with coordinate z , the **Poincaré metric** (if it exists) can be written

$$e^{-\phi(z, \bar{z})} |dz| \quad , \quad e^{-\phi(z, \bar{z})} \underset{z \rightarrow p_i}{\sim} C_i |z - p_i|^{2\alpha_i} (1 + o(1)) \quad (5-1)$$

where ϕ is a real valued function which we write as a function of z and \bar{z} instead of $\Re z$ and $\Im z$ in charts $U \subset \mathbb{R}^2$ identified with \mathbb{C} . Under a holomorphic change of chart and coordinates, i.e. under a holomorphic transition function $z \rightarrow \tilde{z} = \psi(z)$, $\phi(z, \bar{z})$ changes as

$$\tilde{\phi}(\psi(z), \bar{\psi}(z)) = \phi(z, \bar{z}) + \log |\psi'(z)| = \phi(z, \bar{z}) + \frac{1}{2} \log \psi'(z) + \frac{1}{2} \log \overline{\psi'(z)}. \quad (5-2)$$

The curvature is

$$-1 = R(z, \bar{z}) = e^{2\phi(z, \bar{z})} \Delta \phi(z, \bar{z}). \quad (5-3)$$

Finding a metric with constant curvature $= -1$ thus amounts to solving **Liouville's equation**

$$\Delta \phi(z, \bar{z}) = 4\partial\bar{\partial}\phi(z, \bar{z}) = -e^{-2\phi(z, \bar{z})}. \quad (5-4)$$

Stress energy tensor and projective connection

From (5-4) we have

$$\begin{aligned} \bar{\partial}(\partial^2\phi + (\partial\phi)^2) &= \bar{\partial}\partial^2\phi + 2\partial\phi\bar{\partial}\partial\phi \\ &= \partial(\bar{\partial}\partial\phi) - \frac{1}{2}e^{-2\phi}\partial\phi \\ &= \frac{-1}{4}(\partial e^{-2\phi} + 2e^{-2\phi}\partial\phi) \\ &= 0, \end{aligned} \quad (5-5)$$

which we rewrite as

$$\bar{\partial}T(z) = 0 \quad \text{where} \quad T(z) = \partial^2\phi(z, \bar{z}) + (\partial\phi(z, \bar{z}))^2. \quad (5-6)$$

$$T(z) \underset{z \rightarrow p_i}{\sim} \frac{-\Delta_i}{(z - p_i)^2} (1 + o(1)) \quad , \quad \Delta_i = \alpha_i(1 - \alpha_i). \quad (5-7)$$

$T(z)$ is called the **stress energy tensor**, and we see that it must be analytic outside of the marked points. We may drop the \bar{z} dependence because $\bar{\partial}T = 0$.

Under a change of chart $z \rightarrow \tilde{z} = \psi(z)$, $T(z)$ changes (using (5-2)) as

$$\tilde{T}(\psi(z)) = \frac{1}{\psi'(z)^2} \left(T(z) + \frac{1}{2}\{\psi, z\} \right) \quad (5-8)$$

where $\{\psi, z\}$ is called the **Schwartzian derivative** of ψ :

$$\{\psi, z\} = \frac{\psi'''}{\psi'} - \frac{3}{2} \left(\frac{\psi''}{\psi} \right)^2 \quad (5-9)$$

A quadratic differential form

$$2T(z)dz^2 \quad (5-10)$$

with transitions given by (5-8) is called a **projective connexion**.

If $B(z_1, z_2)$ is the Bergman kernel, the fundamental second kind differential on Σ normalized on a chosen symplectic basis $\mathcal{A}_i, \mathcal{B}_i$ of cycles, and f a meromorphic function of Σ , then

$$S_f(z) = -6 df(z)^2 \lim_{z' \rightarrow z} \left(\frac{B(z, z')}{df(z)df(z')} - \frac{1}{(f(z) - f(z'))^2} \right) \quad (5-11)$$

is a projective connection. It has poles at the zeros of df .

It follows that, for any choice of a given projective connection S independent of p_i s and α_i s (for instance S_f as above), then

$$\omega(z) = T(z)dz^2 - \frac{1}{2}S(z) \quad (5-12)$$

is a meromorphic quadratic differential on Σ .

It has poles at the poles of S , and it has double poles at the p_i s:

$$\omega(z) \underset{z \rightarrow p_i}{\sim} \frac{-\Delta_i dz^2}{(z - p_i)^2} (1 + o(1)) \quad , \quad \Delta_i = \alpha_i(1 - \alpha_i). \quad (5-13)$$

It belongs to an affine space, whose underlying linear space is the space of quadratic differentials introduced in section 4.1, i.e.

$$\omega \in \omega_S + \Omega'_{(\Sigma, p_1, \dots, p_n; \Delta_1, \dots, \Delta_n)}. \quad (5-14)$$

This is a space of real dimension $6g - 6 + 3n$.

Oper

If $T(z)$ would be known, we would recover $f(z, \bar{z}) = e^{\phi(z, \bar{z})}$ by solving the Schrödinger equation in each chart:

$$\partial^2 f(z, \bar{z}) = -T(z) f(z, \bar{z}). \quad (5-15)$$

The operator:

$$dz^2(\partial^2 + T(z)) \quad (5-16)$$

is in fact independent of a choice of chart and coordinate (we leave to the reader to verify that it transforms well under chart transitions), it is called an **oper**.

Notice that $f(z, \bar{z}) = e^{\phi(z, \bar{z})}$ is not a function, according to (5-2) it transforms as a $(\frac{-1}{2}, \frac{-1}{2})$ spinor form, so the oper does not act in the space of functions but in the space of spinors. It sends a $\frac{-1}{2}$ spinor to a $\frac{3}{2}$ spinor.

Monodromies

This ODE in z (at fixed \bar{z} , since the oper is independent of \bar{z}), is a second order ODE, it has 2 linearly independent solutions, call a choice of basis $f_1(z), f_2(z)$. Doing the same thing for the \bar{z} dependence, since $\bar{\partial}^2 f(z, \bar{z}) = -\bar{T}(\bar{z})f(z, \bar{z})$, we see that there must exist 4 complex constants $c_{i,j}$ such that

$$f(z, \bar{z}) = c_{1,1}f_1(z)\bar{f}_1(\bar{z}) + c_{1,2}f_1(z)\bar{f}_2(\bar{z}) + c_{2,1}f_2(z)\bar{f}_1(\bar{z}) + c_{2,2}f_2(z)\bar{f}_2(\bar{z}) \quad (5-17)$$

They form a 2×2 matrix

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}. \quad (5-18)$$

The choice of constants must be such that $f(z, \bar{z})$ is a real monovalued function. f real implies that the matrix C must be hermitian

$$C^\dagger = C. \quad (5-19)$$

Up to a change of basis we may choose C to be diagonal and real, and in fact we can choose $C = \text{Id}$.

Solutions of ODE usually have monodromies while going around a closed cycle γ : the vector space of solutions remains unchanged, but solutions can be replaced by linear combinations, so that the monodromy around a closed contour γ is encoded by a matrix:

$$\begin{pmatrix} f_1(z + \gamma) \\ f_2(z + \gamma) \end{pmatrix} = M(\gamma) \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}. \quad (5-20)$$

The 2×2 monodromy matrix $M(\gamma)$ is independent of z and actually depends only on the homotopy class of γ . We have $M(-\gamma) = M(\gamma)^{-1}$ and $M(\gamma_1 + \gamma_2) = M(\gamma_2)M(\gamma_1)$, so that monodromies provide a representation of the fundamental group $\pi_1(\Sigma - \{p_1, \dots, p_N\})$ into $Sl_2(\mathbb{C})$:

$$\begin{aligned} \pi_1(\Sigma - \{p_1, \dots, p_N\}) &\rightarrow Sl_2(\mathbb{C}) \\ \gamma &\mapsto M(\gamma). \end{aligned} \quad (5-21)$$

$M(\gamma) \in Sl_2(\mathbb{C})$ rather than $Gl_2(\mathbb{C})$, i.e. $\det M(\gamma) = 1$, thanks to the fact that the Wronskian $f_1'(z)f_2(z) - f_1(z)f_2'(z)$ is constant independent of z , and in particular remains constant after going around a cycle.

Requiring that $f(z, \bar{z})$ is monovalued, i.e. has no monodromy, implies that $\forall \gamma$:

$$M(\gamma)^\dagger C M(\gamma) = C. \quad (5-22)$$

The rank of $\pi_1(\Sigma - \{p_1, \dots, p_n\})$ is $2\mathbf{g} - 2 + n$, and therefore (5-22) impose $3 \times (2\mathbf{g} - 2 + n) = 6\mathbf{g} - 6 + 3n$ real constraints on the choice of $\omega \in \omega_S + \Omega'_{(\Sigma, p_1, \dots, p_n; \Delta_1, \Delta_n)}$, which is precisely of that dimension. We admit that this fixes a unique choice of ω , and thus this determines uniquely the stress energy tensor $T(z)$ and then the function $\phi(z, \bar{z})$.

Mapping class group

Therefore, for every $\alpha_1, \dots, \alpha_n \in \mathbb{R}^n$ and every $(\Sigma, p_1, \dots, p_n) \in \mathcal{M}_{\mathbf{g},n}$, there is a unique Riemannian metric on Σ of constant curvature -1 , which has zeros (or poles) of order α_i at p_i .

This implies that a universal cover of $\Sigma - \{p_1, \dots, p_n\}$ is the hyperbolic plane, i.e. the upper complex plane \mathbb{C}_+ . We recover Σ by quotienting the universal cover, by the fundamental group $\pi_1(\Sigma - \{p_1, \dots, p_n\})$.

If we homotopically move a neighborhood $U \in \Sigma - \{p_1, \dots, p_n\}$ around a closed cycle γ , it should come back to itself in Σ , and to an isometric copy in the universal cover. In other words, to each closed contour γ is associated an isometry in \mathbb{C}_+ , and the fundamental group has a representation into the group of isometries of the hyperbolic plane.

The **Fuchsian group** \mathfrak{K} is the discrete subgroup of the hyperbolic isometries (called $PSL(2, \mathbb{R})$) of \mathbb{C}_+ , generated by $\pi_1(\Sigma - \{p_1, \dots, p_n\})$. We have

$$\Sigma - \{p_1, \dots, p_n\} \sim \mathbb{C}_+/\mathfrak{K}. \quad (5-23)$$

We shall admit that it is possible to find a fundamental domain of \mathbb{C}_+ , bounded by geodesics, i.e. a polygon in the hyperbolic plane. The quotient by \mathfrak{K} then amounts to glue together some sides of the polygon to recover the surface $\Sigma - \{p_1, \dots, p_n\}$. The points p_1, \dots, p_n sit at the boundary of \mathbb{C}_+ (i.e. on $\mathbb{R} \cup \{\infty\}$), and are corners of the polygon, of angles $2\pi\alpha_i$, and all other angles are $\pi/2$.

It is possible to prove that the Fuchsian group is always a torsion free (no finite order element) discrete subgroup of the group $PSL(2, \mathbb{R})$ of hyperbolic isometries. And vice-versa, every such group is the Fuchsian group of a Riemann surface.

Uniformization theorem

This leads to

Theorem 5.1 (Uniformization theorem) *Every compact Riemann surface of genus $\mathbf{g} = 0$ is isomorphic to the Riemann sphere, every compact Riemann surface of genus $\mathbf{g} = 1$ is isomorphic to the standard torus T_τ (its Jacobian), and if $2\mathbf{g} - 2 + n > 0$:*

For every $\alpha_1, \dots, \alpha_n \in \mathbb{R}^n$ and every $(\Sigma, p_1, \dots, p_n) \in \mathcal{M}_{\mathbf{g},n}$, there is a unique Riemannian metric on Σ of constant curvature -1 , which has zeros (or poles) of order $2\alpha_i$ at p_i .

This allows to identify Σ with a polygon in the hyperbolic plane \mathbb{C}_+ , whose sides are glued pairwise, i.e. to $\mathbb{C}_+/\mathfrak{K}$ where \mathfrak{K} is a Fuchsian group, a discrete subgroup of isometries of \mathbb{C}_+

$$\Sigma - \{p_1, \dots, p_n\} \sim \mathbb{C}_+/\mathfrak{K}. \quad (5-24)$$

Remark 5.1 The actual uniformization theorem is slightly stronger than the one we have written here, in particular it also considers surfaces with boundaries, and it characterizes Fuchsian groups in deeper details.

Remark 5.2 An interesting fact is that the uniformization theorem strongly relies on the Liouville equation and the stress energy tensor, in a way very similar to classical conformal field theory.

Remark 5.3 The Stress energy tensor, or more precisely the projective connexion, or more precisely the projective connexion shifted by a fixed projective connection, is found as a unique element of the space of quadratic differentials $\Omega_{(\Sigma, p_1, \dots, p_n; \Delta_1, \Delta_n)}$, like the Strebel differential. It is similar but slightly different, indeed the Strebel differential was found by requiring that closed cycle-integrals $\oint_\gamma \sqrt{\omega}$ had to be real, whereas the stress energy tensor is found by requiring that the monodromy $M(\gamma)$ had to be unitary. In a "heavy limit", where all α_i would be "large", the solutions of Schrödinger equation could be approximated by the WKB approximation, and the monodromies in that approximation would have eigenvalues of the form

$$\text{eigenvalues of } M(\gamma) \sim e^{\pm i \oint_\gamma \sqrt{\omega}} \quad (5-25)$$

and saying that the matrix be unitary implies that the integrals in the exponential are real. In other words, in the heavy limit, the stress energy tensor tends to the Strebel quadratic differential.

Remark 5.4 To each choice of $(\Sigma, p_1, \dots, p_n, \alpha_1, \dots, \alpha_n) \in \mathcal{M}_{g,n} \times \mathbb{R}^n$ corresponds a $SU(2)$ representation (by the monodromies) of the fundamental group:

$$\mathcal{M}_{g,n} \times \mathbb{R}^n \rightarrow \text{Betti} \quad (5-26)$$

where the **Betti space** is the set of representations of the fundamental group into $SU(2)$ (with monodromies of given eigenvalues $e^{\pm 2\pi i \alpha_i}$ on the small cycles \mathcal{C}_{p_i}):

$$\text{Betti} = \text{Hom}(\pi_1(\Sigma - \{p_1, \dots, p_n\}), SU(2)) / (\text{sp}(M(\mathcal{C}_{p_i})) = (e^{2\pi i \alpha_i}, e^{-2\pi i \alpha_i})). \quad (5-27)$$

Remark 5.5 An infinitesimal change of point in the moduli space $\mathcal{M}_{g,n}$, i.e. an infinitesimal change of complex structure, i.e. a cotangent vector to $\mathcal{M}_{g,n}$, corresponds to an infinitesimal change in the uniformization. This can be seen as an infinitesimal change also in the Betti space.

In other words, this shows that the cotangent space to the moduli space, as well as the cotangent space to the Betti space, and the cotangent space to the space of opers, or also to the space of flat $SU(2)$ connections, are all isomorphic to the space of quadratic differentials with double poles at p_i . Their common real dimension is

$$2d_{g,n} + n. \quad (5-28)$$

where the addition of n actually corresponds to the trivial factor by \mathbb{R}^n .

6 Teichmüller space

Definition 6.1 (Teichmüller space) Let S_g a smooth orientable surface of genus g . The Teichmüller space $T(S_g)$ is the set of all complex structures on S_g , modulo diffeomorphism isotopic to identity. An element of $T(S_g)$, i.e. a surface with a class of complex structures, is called a **marked surface**.

Due to the uniformization theorem, for $g \geq 2$, $T(S_g)$ is also the set of complete hyperbolic (curvature $R = -1$) Riemannian metrics on S_g , modulo diffeomorphism isotopic to identity (there is a similar statement for $g = 1$, with parabolic metric $R = 0$, and for $g = 0$ with elliptic metric $R = 1$).

The **mapping class group** $\Gamma(S_g)$ is the quotient of the group of all diffeomorphisms of S_g , by the subgroup of diffeomorphisms isotopic to identity. The moduli space is the quotient

$$\mathcal{M}_{g,0} = T(S_g)/\Gamma(S_g). \quad (6-1)$$

$T(S_g)$ is a universal cover of the moduli space $\mathcal{M}_{g,0}$.

There are many ways of putting a topology on $T(S_g)$.

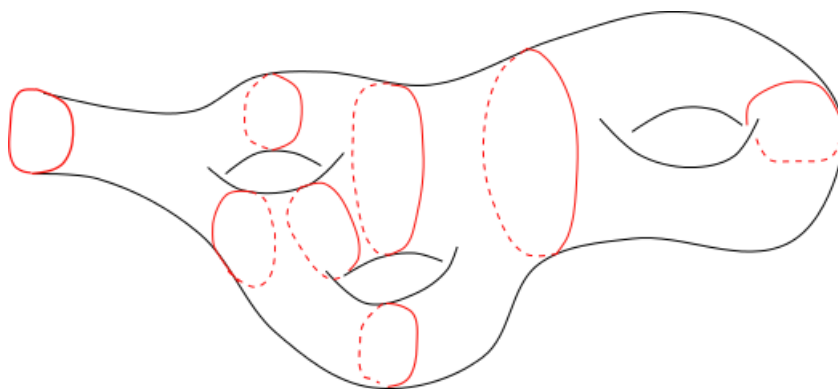
6.1 Fenchel–Nielsen coordinates

Consider

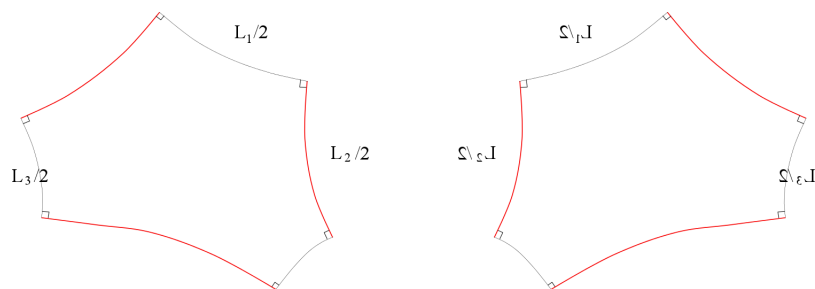
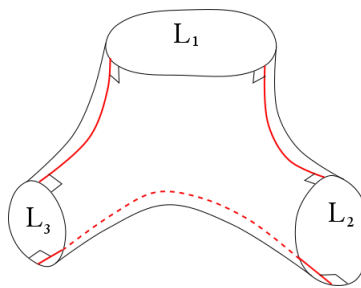
Definition 6.2 $\mathcal{M}_{g,n}(L_1, \dots, L_n)$ be the moduli space of hyperbolic metrics on a connected surface of genus g , with n labelled boundaries, such that the boundaries are geodesic of respective lengths $L_1, \dots, L_n \in \mathbb{R}_+^n$.

We shall admit that it is always possible to find $3g - 3 + n$ non intersecting closed geodesic curves, that cut Σ into $2g - 2 + n$ disjoint pairs of pants.

A pant decomposition is not unique.



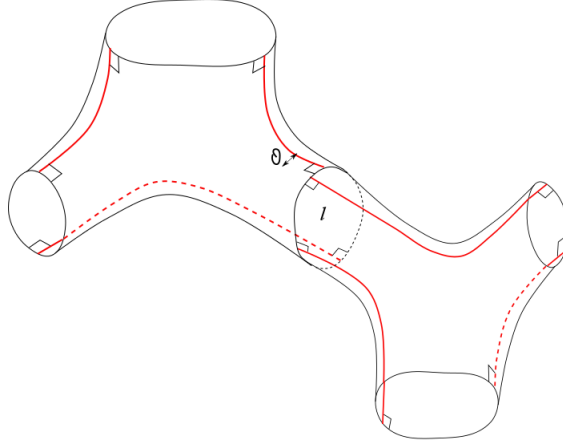
Lemma 6.1 (Pair of pants) *The moduli space $\mathcal{M}_{0,3}(L_1, L_2, L_3)$ contains a single element. In other words, there is a unique (up to isometries) pair of pants, with 3 given boundary lengths L_1, L_2, L_3 (this can be extended if some boundary length is 0, to given cusp angle rather than given length). It is built by gluing the unique hyperbolic right-angles hexagon with 3 edge lengths $L_1/2, L_2/2, L_3/2$ (the other intermediate 3 edge lengths are then uniquely determined as functions of $L_1/2, L_2/2, L_3/2$), and its mirror image, along the 3 other edges (see figure).*



Notice that each pair of pants carries marked points on its boundary (the points at which geodesics orthogonal to 2 boundaries meet the boundary).

Hyperbolic surfaces can be built by gluing pairs of pants along their geodesic boundaries, provided that the glued boundaries have the same lengths, but then the bound-

aries can be glued rotated by an arbitrary twisting angle (angle between marked points).



Every hyperbolic surfaces can be obtained in that way (not uniquely, because of the many ways of cutting the same surface into pairs of pants, but this is a discrete ambiguity). This leads to introduce

Definition 6.3 (Fenchel-Nielsen coordinates) *Every hyperbolic surface $\Sigma \in \mathcal{M}_{g,n}(L_1, \dots, L_n)$, can be built by gluing $2g - 2 + n$ pairs of pants along $3g - 3 + n$ non-intersecting geodesic closed curves. The $3g - 3 + n$ pairs (ℓ_i, θ_i) of geodesic lengths and twisting angles at the cutting geodesics, are the Fenchel-Nielsen coordinates of Σ .*

They are local coordinates in $\mathcal{M}_{g,n}(L_1, \dots, L_n)$ (but not global because of the non-uniqueness of the pant decomposition).

Therefore locally $\mathcal{M}_{g,n}(L_1, \dots, L_n) \sim \mathbb{R}^{6g-6+2n}$, which defines a topology and metric on $\mathcal{M}_{g,n}(L_1, \dots, L_n)$.

It was proved by Weil and Petersson, that the transition maps from a pant decomposition to another, are symplectic transformations in $\mathbb{R}^{6g-6+2n}$ (equipped with the canonical symplectic form), and this allows to define:

Definition 6.4 (Weil-Petersson form) *The following 2-form on $\mathcal{M}_{g,n}(L_1, \dots, L_n)$:*

$$\omega = \sum_{i=1}^{3g-3+n} dl_i \wedge d\theta_i \quad (6-2)$$

*is independent of the pair of pant decomposition, it is a globally defined 2-form on $\mathcal{M}_{g,n}(L_1, \dots, L_n)$. It is called the **Weil-Petersson form**.*

*Notice that ω^{3g-3+n} is a top-dimensional form on $\mathcal{M}_{g,n}(L_1, \dots, L_n)$, and we define the **Weil-Petersson volume** of $\mathcal{M}_{g,n}(L_1, \dots, L_n)$ as*

$$\mathcal{V}_{g,n}(L_1, \dots, L_n) = \frac{1}{(3g-3+n)!} \int_{\mathcal{M}_{g,n}(L_1, \dots, L_n)} \omega^{3g-3+n}. \quad (6-3)$$

It can be proved that the volume is finite, and is a polynomial in the L_i^2 , moreover, the coefficients of the polynomial, are powers of π^2 times rational numbers, i.e.

$$\mathcal{V}_{\mathbf{g},n}(L_1, \dots, L_n) \in \mathbb{Q}[L_1^2, \dots, L_n^2, \pi^2] \quad (6-4)$$

is a homogeneous polynomial of $L_1^2, L_2^2, \dots, L_n^2, \pi^2$ with rational coefficients, of total degree $3\mathbf{g} - 3 + n$. For example

$$\begin{aligned} \mathcal{V}_{0,3}(L_1, L_2, L_3) &= 1 \quad , \quad \mathcal{V}_{1,1}(L_1) = \frac{1}{48} (4\pi^2 + L_1^2) \quad , \\ \mathcal{V}_{0,4}(L_1, L_2, L_3, L_4) &= 2\pi^2 + \frac{1}{2} \sum_{i=1}^4 L_i^2. \end{aligned} \quad (6-5)$$

Mirzakhani's recursion

Maryam Mirzakhani won the Fields medal in 2014 for having found a recursion relation that computes all volumes (recursion on $2\mathbf{g} - 2 + n$).

Let us introduce the Laplace transforms of the volumes:

$$W_{g,n}(z_1, \dots, z_n) = \int_0^\infty \cdots \int_0^\infty \mathcal{V}_{g,n}(L_1, \dots, L_n) \prod_{i=1}^n e^{-z_i L_i} L_i dL_i, \quad (6-6)$$

for example

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) &= \frac{1}{z_1^2 z_2^2 z_3^2} \quad , \quad W_{1,1}(z_1) = \frac{1}{24z_1^2} \left(2\pi^2 + \frac{3}{z_1^2} \right) \quad , \\ W_{0,4}(z_1, z_2, z_3, z_4) &= \frac{1}{\prod_{i=1}^4 z_i^2} \left(2\pi^2 + \sum_{i=1}^4 \frac{3}{z_i^2} \right). \end{aligned} \quad (6-7)$$

Observe that these are polynomials of $1/z_i^2$.

Mirzakhani's theorem, restated in Laplace transform is the following recursion

Theorem 6.1 (Mirzakhani's recursion, Laplace transformed)

$$\begin{aligned} W_{g,n+1}(z_1, \dots, z_n, z_{n+1}) &= \operatorname{Res}_{z \rightarrow 0} \frac{dz}{z_{n+1}^2 - z^2} \frac{\pi}{\sin 2\pi z} \left[W_{g,n-1}(z, -z, z_1, \dots, z_n) \right. \\ &\quad \left. + \sum'_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{z_1, \dots, z_n\}}} W_{g_1,1+|I_1|}(z, I_1) W_{g_2,1+|I_2|}(-z, I_2) \right] \end{aligned} \quad (6-8)$$

where \sum' means that we exclude the terms $(g_1, I_1) = (0, \emptyset)$ and $(g_2, I_2) = (0, \emptyset)$, and where we defined (not the Laplace transform of an hyperbolic volume):

$$W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}. \quad (6-9)$$

This theorem efficiently computes all volumes recursively. In particular it easily proves that the Laplace transforms are indeed polynomials of $1/z_i^2$, and therefore that the volumes are polynomials of L_i^2 .

Chapter 6

Eigenvector bundles and solutions of Lax equations

A good introduction can be found in [1]. Many known integrable systems, can be put in "Lax form", i.e. the Hamilton equations of motions, can be generated by a single matrix equation, called **Lax equation**

$$\frac{\partial}{\partial t} L(x, t) = [M(x, t), L(x, t)] \quad (0-1)$$

where $L(x, t)$ and $M(x, t)$ depend rationally on an auxiliary parameter x , that generates the equations, for instance the Taylor expansion in powers of x generates a sequence of matrix equations for the Taylor coefficients.

We shall see now that such equations can be solved by algebraic geometry methods, their solutions can be expressed in terms of Θ -functions.

Equation (0-1) implies that the eigenvalues of $L(x, t)$ do not depend on t , they are conserved, indeed

$$\begin{aligned} \frac{\partial}{\partial t} \log \det(y - L(x, t)) &= \frac{\partial}{\partial t} \text{Tr} \log(y - L(x, t)) \\ &= -\text{Tr} [M(x, t), L(x, t)] (y - L(x, t))^{-1} \\ &= -\text{Tr} M(x, t) [L(x, t), (y - L(x, t))^{-1}] \\ &= 0. \end{aligned} \quad (0-2)$$

The **conserved quantities** are the Taylor coefficients in the x expansion, of the eigenvalues, or of symmetric polynomials of the eigenvalues, in particular coefficients of the characteristic polynomial:

$$\det(y - L(x, t)) = \sum_{k,l} x^k y^l P_{k,l}(t) \quad \implies \quad \frac{\partial}{\partial t} P_{k,l} = 0. \quad (0-3)$$

The time dependence is thus only in the eigenvectors of $L(x, t)$. As we shall see, the fact that $L(x, t)$ is a rational fraction of x , implies that the eigenvalues are algebraic functions of x , and the eigenvectors are also algebraic functions of x . Algebraic

functions, can be thought of as meromorphic functions on an algebraic curve, i.e. on a Riemann surface. Meromorphic functions are determined by their behavior at their poles, and thus characterized by a small number of parameters, they can also be decomposed on the basis of Θ -functions. This will allow to entirely characterize the eigenvectors, and actually find an explicit formula for eigenvectors using Θ -functions. This is called **Baker-Akhiezer** functions.

1 Eigenvalues and eigenvectors

Let us for the moment work at fixed time t . The question we want to solve is the following: let $L(x)$ an $n \times n$ matrix, rational function of x :

$$L(x) \in M_n(\mathbb{C}(x)). \quad (1-1)$$

The eigenvalues and eigenvectors are functions of x , what can these functions be ?

1.1 The spectral curve

Let $P(x, y) = \det(y - L(x))$ be the characteristic polynomial, and $\tilde{\Sigma} = \{(x, y) \mid P(x, y) = 0\} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$. Let us call Σ its desingularisation, i.e. a compact Riemann surface. Σ has a projection to $\tilde{\Sigma}$, and an immersion into $\mathbb{C}P^1 \times \mathbb{C}P^1$, and two projections x and y to $\mathbb{C}P^1$:

$$\begin{array}{ccc} \Sigma & \rightarrow \tilde{\Sigma} & \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \\ & x \searrow & \searrow \downarrow \\ & & \mathbb{C}P^1 \end{array} \quad (1-2)$$

The eigenvalues of $L(x)$ are thus points $(x, y) \in \tilde{\Sigma}$, and should be thought of as points $z \in \Sigma$.

Locally, in some neighborhood, we may label the preimages of x :

$$z_1(x), \dots, z_n(x), \quad (1-3)$$

and thus locally, we may label the eigenvalues $Y_1(x), \dots, Y_n(x)$, with $Y_i(x) = y(z_i(x))$ and define the diagonal matrix

$$Y(x) = \text{diag}(Y_1(x), \dots, Y_n(x)). \quad (1-4)$$

The eigenvalues are algebraic functions of $x \in \mathbb{C}P^1$, and thus they are meromorphic functions on Σ .

Remark 1.1 The 1-form $y(z)dx(z)$ is a meromorphic 1-form on Σ , it is called the **Liouville form**. In fact the 1-form ydx is defined in the whole $\mathbb{C}P^1 \times \mathbb{C}P^1$, it is called the **tautological form**, its differential is the 2-form $dy \wedge dx$ the canonical symplectic 2-form in $\mathbb{C}P^1 \times \mathbb{C}P^1$. The Liouville form is thus the restriction of the tautological form to the locus of the immersion of the spectral curve. The immersion of the spectral curve is a Lagrangian with respect to the symplectic form $dy \wedge dx$ of $\mathbb{C}P^1 \times \mathbb{C}P^1$.

1.2 Eigenvectors and principal bundle

Let $Y_j(x)$ be an eigenvalue of $L(x)$, and $V_j(x) = \{V_{i,j}(x)\}_{i=1,\dots,n}$ be a non-vanishing eigenvector for that eigenvalue. With $j = 1, \dots, n$ we define a complete set of eigenvectors, and define a matrix $V(x) = \{V_{i,j}(x)\} \in GL_n$, with

$$\det V(x) \neq 0. \quad (1-5)$$

We then have

$$L(x) = V(x)Y(x)V(x)^{-1}. \quad (1-6)$$

However, eigenvectors are not uniquely defined, we may rescale them arbitrarily, and in particular rescale them by a non-vanishing x -dependent factor. This is equivalent to say that we may right-multiply $V(x)$ by an arbitrary x -dependent invertible diagonal matrix.

We say that the eigenvector matrix $V(x)$ is a section of a bundle over $\mathbb{C}P^1$, whose fiber over each point x is the group GL_n . Moreover, when we represent $V(x)$ as a matrix, we assume a choice of basis, and we could change our choice of basis, i.e. conjugate $L(x)$ by an arbitrary matrix, $L(x) \rightarrow UL(x)U^{-1}$, equivalent to $V(x) \rightarrow UV(x)$. In other words we are interested in GL_n only modulo left-multiplication, this is called modulo gauge transformation. Somehow we may fix the identity matrix in GL_n to our will, this is called an affine group. A bundle whose fiber is an affine Lie group, is called a **principal bundle**.

Remark 1.2 [Other Lie groups]

So far we have not assumed that $L(x,t)$ had any particular symmetry, we could also require some symmetries conserved under time evolution. This would imply that eigenvectors matrices would belong to a subgroup of GL_n . We can obtain any Lie group in this way. It is thus possible to consider any principal bundle.

The spectral curve also gets extra symmetries, not all coefficients of the characteristic polynomial are independent. The set of independent coefficients is called the **Hitchin base**. The good notion to describe a spectral curve with those extra symmetries, is the notion of **cameral curve**, beyond the scope of these lectures.

1.3 Monodromies

The labelling of eigenvalues can only be local, in a small neighborhood, and when we move x around a closed cycle γ (which may surround branchpoints), the eigenvalues

get permuted by a permutation σ_γ (we shall identify the permutation group \mathfrak{S}_n with its representation as matrices in Gl_n), and the eigenvectors get right multiplied:

$$\begin{aligned} Y(x + \gamma) &= \sigma_\gamma^{-1} Y(x) \sigma_\gamma, \\ V(x + \gamma) &= V(x) \sigma_\gamma. \end{aligned} \tag{1-7}$$

In other words, the eigenvector bundle has monodromies, and these monodromies are permutations, they are precisely the deck transformations of the spectral curve.

Remark 1.3 In case the group is a Lie subgroup G of Gl_n , the monodromies form a subgroup of \mathfrak{S}_n , in fact they are in the **Weyl group** of G .

1.4 Algebraic eigenvectors

Let us first show that it is possible to choose $V(x)$ as an algebraic function of x .

More precisely, let y an eigenvalue, i.e. $(x, y) = (x(z), y(z))$ a point of $\tilde{\Sigma}$ for $z \in \Sigma$, and $V(z) = (V_1(z), \dots, V_n(z))$ a corresponding eigenvector. Since $V(z) \neq 0$, there must exist at least one i such that $V_i(z) \neq 0$, and let us assume here that, up to relabelling, $i = n$. In a neighborhood, we may choose the normalization $V_n(z) = 1$.

The equation $L(x(z))V(z) = y(z)V(z)$ can then be written as an $(n-1) \times (n-1)$ linear system

$$\forall i = 1, \dots, n-1, \quad \sum_{k=1}^{n-1} L_{i,k}(x(z))V_k(z) - y(z)V_i(z) = -L_{i,n}(x(z))V_n(z) = -L_{i,n}(x(z)). \tag{1-8}$$

This linear system is solved by Kramers formula

$$V_i(z) = (-1)^{n-i} \frac{M_{i,n}(L(x(z)) - y(z)\text{Id})}{M_{n,n}(L(x(z)) - y(z)\text{Id})} \tag{1-9}$$

where $M_{u,v}(A)$ is the minor of the matrix A obtained by removing the u^{th} line and v^{th} column and taking the determinant. This expression is a rational function of $x(z)$ and $y(z)$, it is thus a meromorphic function of z . Therefore we see that there exists some meromorphic functions $\tilde{\psi}_i$ such that

$$V_i(z) = \tilde{\psi}_i(z) \in \mathfrak{M}^1(\Sigma). \tag{1-10}$$

Remark: this formula automatically has the monodromies (1-7) because monodromies are the deck transformations of Σ .

Instead of having to look for an everywhere invertible $n \times n$ matrix of algebraic functions $V_{i,j}(x)$ for $x \in \mathbb{C}$, we have to look for a everywhere non-vanishing vector of meromorphic functions $\tilde{\psi}_i(z)$ on Σ .

This is called the ”**Abelianization procedure**”: we have transformed the problem of finding an algebraic section of a principal bundle of a non-Abelian group over the Riemann sphere $\mathbb{C}P^1$, into the problem of finding a meromorphic section of a projective vector bundle whose fiber is a projective vector space ($\mathbb{C}P^n$), over a compact Riemann surface Σ covering $\mathbb{C}P^1$. Instead of matrices (thus in a non-Abelian space), we have vectors of functions, the problem has somehow become Abelian. The price to pay is to have replaced the Riemann sphere by a higher genus Riemann surface Σ that is a covering of $\mathbb{C}P^1$.

The next idea is that meromorphic functions are entirely determined by their behavior at their poles and zeros, and their periods. The Riemann–Roch theorem says what is the dimension (the number of parameters to choose) to characterize all such functions. The main difficulty is the invertibility of the matrix, or the non-vanishing of the vector. The full solution was explicitly found by Russian mathematicians (see [7, 3, 1]), and we shall now present the final solution, starting from the end. We shall ”reconstruct” the integrable system, the Lax matrix $L(x, t)$ from the spectral curve.

1.5 Geometric reconstruction method

Let us now start from a spectral curve $\tilde{\Sigma} = \{(x, y) \mid P(x, y) = 0\} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$, with Σ its desingularisation. Σ has a projection to $\tilde{\Sigma}$, and an immersion into $\mathbb{C}P^1 \times \mathbb{C}P^1$, and a projection to $\mathbb{C}P^1$ by x :

$$\begin{array}{ccc} \Sigma & \rightarrow \tilde{\Sigma} & \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \\ & x \searrow & \searrow \downarrow \\ & & \mathbb{C}P^1 \end{array} \quad (1-11)$$

Let us assume that Σ has a genus $\mathfrak{g} > 0$, and choose a symplectic basis of cycles, $\mathcal{A}_i, \mathcal{B}_j$, and define the Abel map $z \mapsto \mathbf{u}(z)$. Let c a non-singular odd half characteristic (and thus a zero of Θ), and E the corresponding prime form. Let Ω be a meromorphic 1-form on Σ , of the second kind, having no residues at its poles. Let $\zeta(\Omega) \in \mathbb{C}^{\mathfrak{g}}$ be the vector with coordinates

$$\zeta_i(\Omega) = \oint_{\mathcal{B}_i} \Omega - \sum_j \tau_{i,j} \oint_{\mathcal{A}_j} \Omega, \quad (1-12)$$

which for short we denote

$$\zeta(\Omega) = \oint_{\mathcal{B}-\tau\mathcal{A}} \Omega. \quad (1-13)$$

Let us define:

Definition 1.1 We define the **Szegö kernel**, for z and z' two distinct points of Σ

$$\psi(\Omega; z', z) = \frac{e^{\int_{z'}^z \Omega}}{E(z, z')} \frac{\Theta(\mathbf{u}(z) - \mathbf{u}(z') + \zeta(\Omega) + c)}{\Theta(\zeta(\Omega) + c)}. \quad (1-14)$$

It is a $\frac{1}{2} \otimes \frac{1}{2}$ bi-spinor form on $\Sigma \times \Sigma$.

Definition 1.2 For x and x' two distinct points of $\mathbb{C}P^1$, let us define the $n \times n$ matrix in a neighborhood where is defined an ordering of preimages of x and x'

$$\Psi(\Omega; x', x)_{i,j} = \psi(\Omega; z_i(x'), z_j(x)). \quad (1-15)$$

It is a matrix-valued $\frac{1}{2} \otimes \frac{1}{2}$ bi-spinor form on $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Proposition 1.1 It satisfies

$$\Psi(\Omega; x_1, x)\Psi(\Omega; x, x_2) = \frac{(x_1 - x_2) dx}{(x - x_1)(x - x_2)} \Psi(\Omega; x_1, x_2) \quad (1-16)$$

In particular taking the limit $x_1 \rightarrow x_2 = x'$ this gives

$$\Psi(\Omega; x', x)\Psi(\Omega; x, x') = \frac{dx dx'}{(x - x')^2} Id \quad (1-17)$$

which shows that for $x \neq x'$, $\Psi(\Omega; x', x)$ is invertible.

Proposition 1.2 The matrix

$$L(\Omega; x', x) = \Psi(\Omega; x_1, x)Y(x)\Psi(\Omega; x_1, x)^{-1} \quad (1-18)$$

is a $n \times n$ matrix rational in x . It is algebraic in x' , and it depends on Ω .

Remark that changing the choice of x' amounts to a conjugation by an x -independent matrix:

$$L(\Omega; x'', x) = \Psi(\Omega; x'', x')L(\Omega; x', x)\Psi(\Omega; x'', x')^{-1}, \quad (1-19)$$

in other words x' plays the role of a choice of gauge, i.e. a choice of basis for GL_n .

Proposition 1.3 Let us choose a basis $\{\Omega_i\}$ (or an independent family) of meromorphic 1-forms of the second kind, and let

$$\Omega_t = \sum_i t_i \Omega_i \quad (1-20)$$

where $t = \{t_i\}$ is called the "time" or more precisely the "times".

Let us define

$$L(x'; x, t) = \Psi(\Omega_t; x_1, x)Y(x)\Psi(\Omega_t; x_1, x)^{-1} \quad (1-21)$$

and

$$M_i(x'; x, t) = \left(\frac{\partial}{\partial t_i} \Psi(\Omega_t; x', x) \right) \Psi(\Omega_t; x', x)^{-1}. \quad (1-22)$$

$M_i(x'; x, t)$ is a rational function of x (its poles are the x -projections of points where Ω_i has poles, and of at most the same degrees), and we have the Lax equations

$$\frac{\partial}{\partial t_i} L(x'; x, t) = [M_i(x'; x, t), L(x'; x, t)]. \quad (1-23)$$

In fact every (finite dimension n) solution of Lax equation, can be obtained in this way. What we see, is that the time dependence, is encoded in the choice of $\Omega_t \in \mathfrak{M}^1(\Sigma)$, i.e. times are some linear coordinates in the affine space of meromorphic 1-forms. This means that, under this parametrization the motion, in the space $\mathfrak{M}^1(\Sigma)$ is linear at constant velocity.

The \mathfrak{g} -dimensional vector $\zeta(\Omega_t)$ is called the **angle variables**. It follows a linear motion at constant velocity in $\mathbb{C}^{\mathfrak{g}}$. The velocity is:

$$\nu_i = \frac{\partial}{\partial t_i} \zeta(\Omega_t) = \oint_{\mathcal{B}-\tau\mathcal{A}} \Omega_i. \quad (1-24)$$

The **action variables** parametrize the spectral curve, and it is usual to choose the \mathfrak{g} dimensional vector of \mathcal{A} -cycles periods of the Liouville 1-form ydx :

$$\epsilon_i = \frac{1}{2i\pi} \oint_{\mathcal{A}_i} ydx. \quad (1-25)$$

This \mathfrak{g} dimensional vector parametrizes the spectral curve, i.e. the polynomial $P(x, y) = 0$, or more precisely it parametrizes all the coefficients of P that are interior of the convex envelope of the Newton's polygon. The coefficients that are at the boundary of the convex envelope, are called **Casimirs** of our integrable system. We have

$$ydx = 2\pi i \sum_{i=1}^{\mathfrak{g}} \epsilon_i \omega_i + \sum_{(k,l) \in \partial\mathcal{N}(P)} c_{k,l} \omega_{(k-1,l-1)}. \quad (1-26)$$

1.6 Genus 0 case

The previous section assumed that $P(x, y)$ was generic, with the genus of Σ equal to the number of points inside the Newton's polygon, i.e. no cycle pinched to a nodal point.

When some cycles are pinched into nodal points, Θ functions of genus \mathfrak{g} degenerate and become polynomial combinations of Θ functions of lower genus.

The extreme case is when all non-contractible cycles of Σ have been pinched into nodal points of $\tilde{\Sigma}$, the genus of Σ is then 0.

Let $(p_{i,+}, p_{i,-})_{i=1,\dots,N}$ be the N nodal points ($N = \# \mathring{\mathcal{N}}$ is the genus of the unpinched curve) i.e. all the pairs of distinct points of Σ that have the same projection to $\tilde{\Sigma}$:

$$x(p_{i,+}) = x(p_{i,-}) \quad \text{and} \quad y(p_{i,+}) = y(p_{i,-}). \quad (1-27)$$

The theta functions of a pinched curve degenerate into determinants of rational functions, and the Szegö kernel degenerates into

$$\psi(\Omega; z', z) = \frac{\det_{0 \leq i, j \leq N} \frac{e^{\int_{p_{j,-}}^{p_{i,+}} \Omega}}{p_{i,+} - p_{j,-}}}{\det_{1 \leq i, j \leq N} \frac{e^{\int_{p_{j,-}}^{p_{i,+}} \Omega}}{p_{i,+} - p_{j,-}}} \quad (1-28)$$

where we defined $p_{0,+} = z$ and $p_{0,-} = z'$.

1.7 Tau function, Sato and Hirota relation

Let us come back to the non-degenerate case where there is no pinched cycle.

Definition 1.3 (Tau function)

$$\mathcal{T}(\Omega) = e^{\frac{1}{2} \sum_{i,j} Q_{i,j} t_i t_j} \Theta(\zeta(\Omega) + c) \quad (1-29)$$

where, introducing the generalized cycle $\Omega_i^* \in \mathfrak{M}_1(\Sigma)$ that generates $\Omega_i = \oint_{\Omega_i^*} B$ in theorem III-5.2 (using the meromorphic function $f = x$ in theorem III-5.2), we define the quadratic form as the integral (i.e. the Poincaré pairing):

$$Q_{i,j} = \oint_{\Omega_i^*} \Omega_j = \langle \Omega_i^*, \Omega_j \rangle. \quad (1-30)$$

We thus have

$$\sum_{i,j} Q_{i,j} t_i t_j = \sum_i t_i \oint_{\Omega_i^*} \Omega = \oint_{\sum_i t_i \Omega_i^*} \Omega = \oint_{\Omega^*} \Omega = \oint_{\Omega^*} \oint_{\Omega^*} B. \quad (1-31)$$

Notice that

$$\zeta(\Omega) = \oint_{\mathcal{B}-\tau\mathcal{A}} \Omega = \oint_{\mathcal{B}-\tau\mathcal{A}} \oint_{\Omega^*} B = 2\pi i (\mathcal{B} - \tau\mathcal{A}) \cap \Omega^*. \quad (1-32)$$

so that

$$\mathcal{T}(\Omega) = e^{\frac{1}{2} \oint_{\Omega^*} \oint_{\Omega^*} B} \Theta(c + 2\pi i (\mathcal{B} - \tau\mathcal{A}) \cap \Omega^*) \quad (1-33)$$

Remark 1.4 We see that in fact, using the form-cycle duality, it seems easier and more natural to define the Tau function in the space of cycles Ω^* rather than the space of 1-forms Ω . What is hidden here, is that the map $\hat{B} : \Omega^* \mapsto \Omega = \oint_{\Omega^*} B$ is not invertible, it has a kernel (a huge kernel). The map $\Omega \mapsto \Omega^*$ is ill-defined, it can be defined only by choosing representents of equivalence classes modulo $\text{Ker } \hat{B}$, i.e. as in theorem III-5.2 make an explicit choice of basis of $\mathfrak{M}_1(\Sigma)$. We could change this basis by shifting with elements of $\text{Ker } \hat{B}$. Doing so would change the quadratic form, and would change the Tau function by multiplication by a phase. The choice of basis is in fact a choice of a Lagrangian **polarization** in $\mathfrak{M}_1(\Sigma)$, and thus the Tau function is not unique, it depends on a choice of Lagrangian polarization. Under a –time independent– change of Lagrangian polarization, \mathcal{T} gets multiplied by e^S where S is the generating function of the Lagrangian change of polarization.

Theorem 1.1 (Sato) *The Baker Akhiezer function is a ratio of the Tau function shifted by a 3rd kind form*

$$\psi(\Omega; z', z) = \frac{\mathcal{T}(\Omega + \omega_{z',z})}{\mathcal{T}(\Omega)}. \quad (1-34)$$

proof: It is obvious by explicit computation. \square

Notice that the ratio of \mathcal{T} -functions is independent of a "choice of polarization".

Theorem 1.2 (Fay identities and Plücker relations)

$$\frac{\mathcal{T}(\Omega + \omega_{z_1, z_2} + \omega_{z_3, z_4})}{\mathcal{T}(\Omega)} = \frac{\mathcal{T}(\Omega + \omega_{z_1, z_2})}{\mathcal{T}(\Omega)} \frac{\mathcal{T}(\Omega + \omega_{z_3, z_4})}{\mathcal{T}(\Omega)} - \frac{\mathcal{T}(\Omega + \omega_{z_1, z_4})}{\mathcal{T}(\Omega)} \frac{\mathcal{T}(\Omega + \omega_{z_3, z_2})}{\mathcal{T}(\Omega)}. \quad (1-35)$$

More generally

$$\frac{\mathcal{T}(\Omega + \sum_{i=1}^n \omega_{z_i, z'_i})}{\mathcal{T}(\Omega)} = \det_{1 \leq i, j \leq n} \left(\frac{\mathcal{T}(\Omega + \omega_{z_i, z'_j})}{\mathcal{T}(\Omega)} \right). \quad (1-36)$$

proof: These are **Fay identities** for Theta functions [9]. This can be proved by showing that the ratio of the right and left side, is a well defined meromorphic function on Σ (in particular there is no phase when some z_i goes around a cycle), and has no pole, therefore it must be a constant. The constant is seen to be 1 in a limit $z_i \rightarrow z_j$. \square

Definition 1.4 (Hirota derivative) For a function f on $\mathfrak{M}^1(\Sigma)$, and for any $z \in \Sigma$, choosing a chart and local coordinate ϕ in a neighborhood of z , we defined (in theorem III-5.2 using ϕ) the 1-form $\omega_{\phi, z, 1}$ that has a double pole at z . We define

$$\Delta_z f(\Omega) = d\phi(z) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (f(\Omega + \epsilon \omega_{\phi, z, 1}) - f(\Omega)) \quad (1-37)$$

It is a meromorphic 1-form of z on Σ , it is independent of the choice of chart and coordinate.

Proposition 1.4 Let $p \in \Sigma$ in a chart U , and a coordinate ϕ in U . For z in a neighborhood of p we define the KP times as the negative part coefficients of the Taylor-Laurent expansion

$$\Omega \sim \sum_{k=0}^{-\text{order}_p y dx} t_{p, k} \frac{d\phi(z)}{(\phi(z) - \phi(p))^{k+1}} + \text{analytic at } p. \quad (1-38)$$

Then the Hirota derivative can be locally written as the following series of times derivatives

$$\Delta_z \sim d\phi(z) \sum_{k=1}^{\infty} k (\phi(z) - \phi(p))^{k-1} \frac{\partial}{\partial t_{p, k}} \quad (1-39)$$

(1-39) is the usual way of writing the Hirota operator for KP hierachies, but as we see here, this is just an asymptotic Taylor expansion in a neighborhood of p , whereas

the Hirota operator is globally defined on Σ . In other words, for z in a neighborhood of p

$$\Delta_z - d\phi(z) \sum_{k=1}^n k (\phi(z) - \phi(p))^{k-1} \frac{\partial}{\partial t_{p,k}} = O((\phi(z) - \phi(p))^n) d\phi(z). \quad (1-40)$$

proof: In a local coordinate ϕ , if $|\phi(z) - \phi(p)| < |\phi(q) - \phi(p)|$ we have

$$\begin{aligned} \omega_{\phi,z,1}(q) &\sim \sum_{k=1}^{\infty} k \frac{(\phi(z) - \phi(p))^{k-1}}{(\phi(q) - \phi(p))^{k+1}} d\phi(q) \\ &\sim \sum_{k=1}^{\infty} k (\phi(z) - \phi(p))^{k-1} \omega_{p,k}(q). \end{aligned} \quad (1-41)$$

from which we see that the Hirota derivative acts as (1-39).

□

Theorem 1.3 (Hirota equations)

$$\Delta_z \frac{\mathcal{T}(\Omega + \omega_{z_1, z_2})}{\mathcal{T}(\Omega)} = - \frac{\mathcal{T}(\Omega + \omega_{z_1, z})}{\mathcal{T}(\Omega)} \frac{\mathcal{T}(\Omega + \omega_{z, z_2})}{\mathcal{T}(\Omega)}, \quad (1-42)$$

this can also be written

$$\Delta_z \psi(\Omega; z_1, z_2) = -\psi(\Omega; z_1, z) \psi(\Omega; z, z_2). \quad (1-43)$$

proof: This is the limit $z_3 \rightarrow z_4 = z$ of the Fay identities. □

Proposition 1.5 (Sato formula as a shift of times) *Let $p \in \Sigma$ in a chart U , and a coordinate ϕ in U . For z in a neighborhood of p the Sato formula can be written as the Taylor expansion*

$$\mathcal{T}(\Omega + \omega_{z, z'}) \sim \mathcal{T}(\Omega + \omega_{p, z'} + \sum_{k=1}^{\infty} (\phi(z) - \phi(p))^k \omega_{p,k}) \quad (1-44)$$

in other words, writing $\Omega = \sum_k t_{p,k} \omega_{p,k} + \text{analytic at } p$, the Sato shift is equivalent to

$$t_{p,k} \rightarrow t_{p,k} + (\phi(z) - \phi(p))^k. \quad (1-45)$$

Similarly, if z' is in a neighborhood of p the Sato formula can be written as the Taylor expansion

$$\mathcal{T}(\Omega + \omega_{z, z'}) \sim \mathcal{T}(\Omega - \omega_{p, z} - \sum_{k=1}^{\infty} (\phi(z') - \phi(p))^k \omega_{p,k}) \quad (1-46)$$

in other words, the Sato shift is equivalent to

$$t_{p,k} \rightarrow t_{p,k} - (\phi(z') - \phi(p))^k. \quad (1-47)$$

And if both z and z' are in a neighborhood of p the Sato formula can be written as the Taylor expansion

$$\mathcal{T}(\Omega + \omega_{z,z'}) \sim \mathcal{T}(\Omega + \sum_{k=1}^{\infty} (\phi(z) - \phi(p))^k \omega_{p,k} - \sum_{k=1}^{\infty} (\phi(z') - \phi(p))^k \omega_{p,k}) \quad (1-48)$$

in other words, the Sato shift is equivalent to

$$t_{p,k} \rightarrow t_{p,k} + (\phi(z) - \phi(p))^k - (\phi(z') - \phi(p))^k. \quad (1-49)$$

proof: In a local coordinate ϕ , if $|\phi(z) - \phi(p)| < |\phi(q) - \phi(p)|$ we have

$$\begin{aligned} \omega_{z,z'}(q) &\sim \sum_{k=0}^{\infty} \frac{(\phi(z) - \phi(p))^k}{(\phi(q) - \phi(p))^{k+1}} d\phi(q) \\ &\sim \omega_{p,z'}(q) + \sum_{k=1}^{\infty} (\phi(z) - \phi(p))^k \omega_{p,k}(q). \end{aligned} \quad (1-50)$$

□

To go further

Readers interested in learning more about this way of presenting the algebraic reconstruction method, and in particular in the space of cycles $\mathfrak{M}_1(\Sigma)$ can see [4].

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