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Waves in Plasmas

- Lecture notes -

- 2016-2017 -

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Notations and conventions

In these notes, the following conventions have been employed:

- Vectors and tensors are in bold characters. The latter are overlined with a double line to avoid any confusion.
- The indices “ \parallel ” (resp. “ \perp ”) refer to the parallel (resp. perpendicular) component of a vector with respect to the confining magnetic field \mathbf{B}_0 .
- Following a well established tradition in plasma physics, pulsation ω is often abusively referred to as “frequency”.
- To make the non-sequential reading of these notes easier, the notations have been gathered in the following table. When it appears, the index s refers to the species s of the plasma.

Notations:

\mathbf{E}, \mathbf{D}	Electric field, displacement
\mathbf{H}, \mathbf{B}	Magnetic intensity, induction
ρ_s, ρ	Charge density for species s , total
\mathbf{j}_s, \mathbf{j}	Current density for species s , total
$\overline{\overline{\boldsymbol{\sigma}}}$	Conductivity tensor
$\overline{\overline{\boldsymbol{\epsilon}}}$	Dielectric tensor
$\overline{\overline{\mathbf{K}}} \equiv \overline{\overline{\boldsymbol{\epsilon}}}/\epsilon_0$	Permittivity tensor
$\overline{\overline{\boldsymbol{\epsilon}}}^h, \overline{\overline{\boldsymbol{\epsilon}}}^a$	Hermitian, anti-Hermitian part of $\overline{\overline{\boldsymbol{\epsilon}}}$
W	Electromagnetic energy density
\mathbf{S}, \mathbf{T}	Poynting flux, kinetic flux
P_{abs}	Wave power dissipated in the plasma
\mathbf{B}_0	Confining magnetic field
v_a	Alfvén velocity
φ	Electrostatic potential
λ_D	Debye length
Z	Plasma dispersion function

$\Re(z), \Im(z)$	Real, imaginary part of z
\mathcal{P}	Cauchy principal value
ω	Wave pulsation
λ	Wavelength
\mathbf{k}	Wave vector
$\mathbf{n} \equiv \mathbf{k}c/\omega$	Refraction index
θ	Propagation angle with respect to \mathbf{B}_0
θ_{rc}	Resonance cone angle
\mathbf{v}_φ	Phase velocity
\mathbf{v}_g	Group velocity
ω_r, ω_i	Real, imaginary part of the frequency
$\mathbf{k}_r, \mathbf{k}_i$	Real, imaginary part of the wave vector
q_s	Charge
$\sigma_s \equiv q_s/ q_s $	Charge sign
m_s	Mass
Z_s, A_s	Charge number, mass number
f_s	Distribution function
T_s	Temperature
ε_s	Kinetic energy
ω_{ps}	Plasma frequency
ω_{cs}	Cyclotron frequency
$\Omega_{cs} \equiv \sigma_s \omega_{cs}$	Signed cyclotron frequency
$\overline{\phi}_s$	Fluid stress tensor
$\overline{\mathbf{p}}_s$	Fluid pression tensor
p_s	Fluid pression (scalar)
\mathbf{v}_d	Drift velocity
ν_{cs}	Collision frequency
$v_{th,s} \equiv \sqrt{2k_b T_s/m_s}$	Thermal velocity
c_s	Sound velocity
$x_{n,s} \equiv (\omega - n\Omega_{cs})/(kv_{th,s})$	Deviation from (cold) resonance
ρ_s	Larmor radius
$c \approx 3.00 \times 10^8 \text{m/s}$	Velocity of light
$\epsilon_0 = 1/(36\pi \times 10^9) \text{F/m}$	Vacuum dielectric permittivity
$\mu_0 = 4\pi \times 10^{-7} \text{H/m}$	Vacuum magnetic permeability
$k_b \approx 1.38065 \times 10^{-23} \text{J/K}$	Boltzmann constant

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Chapter 1

Fundamentals

1.1 Introduction

1.1.1 Plasma waves

The physical description of an electromagnetic wave propagating in a given medium necessitates a self-consistent handling of the particles comprising the medium (and their mutual interactions) on one hand, and of the electromagnetic field on the other hand. In the case of a plasma, the problem is summarized on Fig. 1.1.

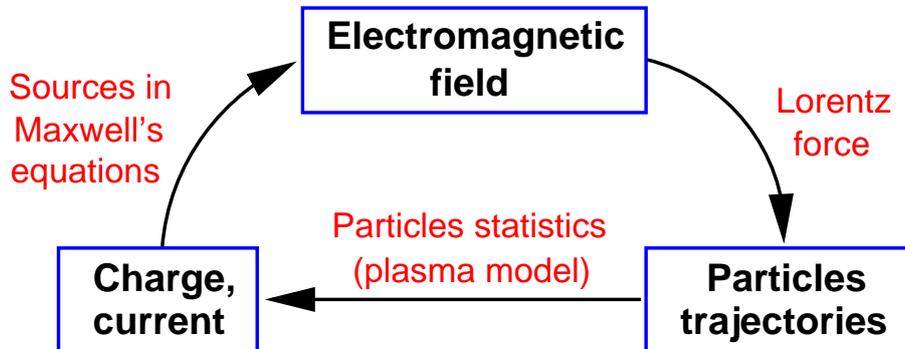


Figure 1.1: Self-consistent description of the electromagnetic field in a plasma.

The electromagnetic field, given by the Maxwell's equations, influences the particles trajectories. Since the handling of all individual particles is largely beyond the computational capabilities of available computers in the present, but also in any foreseeable future, a plasma model is needed to derive statistical quantities, such as the charge and current density. In turn, these quantities enter as sources in the Maxwell's equations, and influence the field. Depending on the problem under study, various approximations are introduced to close this loop. In the present lecture, we

will be developing a linear theory of plasma waves, by introducing a clear separation between the “equilibrium” fields and the wave perturbation.

1.1.2 Maxwell’s equations

The electromagnetic field in the plasma is described by the Maxwell’s equations, which we write in the form:

$$\nabla \cdot \mathbf{D} = \rho_{free} + \rho_{ext}, \quad (1.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (1.3)$$

$$\nabla \times \mathbf{H} = \mathbf{j}_{free} + \mathbf{j}_{ext} + \frac{\partial \mathbf{D}}{\partial t}. \quad (1.4)$$

In these relations, \mathbf{E} is the electric field, \mathbf{D} is the electric displacement, \mathbf{H} is the magnetic intensity, \mathbf{B} is the magnetic induction (which we shall refer to as the magnetic field). \mathbf{j}_{free} is the current carried by the free charges flowing in the medium, and ρ_{free} is the corresponding charge density. \mathbf{j}_{ext} and ρ_{ext} are the current and charge densities from external sources, such as antennas. It is important to notice that in this form, the polarization and magnetization currents are included in \mathbf{D} . Formally, it is possible to solve these equations as long as we are able to describe the medium response to a given electromagnetic excitation. In other words, we need to establish the *constitutive relations of the medium*:

$$\mathbf{D} \stackrel{?}{=} \mathbf{D}(\mathbf{E}), \quad (1.5)$$

and

$$\mathbf{B} \stackrel{?}{=} \mathbf{B}(\mathbf{H}). \quad (1.6)$$

In a classical electromagnetism problem[1], it is usual to introduce a polarization vector \mathbf{P} , and also a magnetization vector \mathbf{M} to write

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (1.7)$$

and

$$\mathbf{B} \equiv \mu_0 \mathbf{H} + \mathbf{M}, \quad (1.8)$$

with $\epsilon_0 = 1/36\pi \times 10^9 \text{F/m}$ the vacuum dielectric permittivity and $\mu_0 = 4\pi \times 10^{-7} \text{H/m}$ the vacuum magnetic permeability. We can then manipulate Eq. 1.4 to obtain the more familiar form

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j}_{free} + \mathbf{j}_{mag} + \mathbf{j}_{pol} + \mathbf{j}_{ext}) + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (1.9)$$

with $\partial_t \mathbf{P} \equiv \mathbf{j}_{pol}$, and $\nabla \times \mathbf{M} \equiv \mu_0 \mathbf{j}_{mag}$. \mathbf{j}_{pol} and \mathbf{j}_{mag} are respectively the polarization and magnetization currents.

So far, we have followed the exact same method that is employed, e.g., in solid state physics. However, in plasma physics, it is impractical to separate the polarization, the magnetization and the free charges currents. Indeed, all charges are free (at least in a fully ionized plasma), yet all do contribute to the polarization of the medium. Therefore, we rewrite Eq. 1.9 in the form

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{j} + \mathbf{j}_{ext}) + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (1.10)$$

where \mathbf{j} is the total current flowing in the plasma in response to the wave perturbation. It is now straightforward to deduce the *wave equation* from Eqs. 1.3 and 1.10

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial (\mathbf{j} + \mathbf{j}_{ext})}{\partial t}. \quad (1.11)$$

Despite its apparent simplicity, this relation is extremely complicated, because of its non-linear nature: \mathbf{j} is a function of \mathbf{E} and the properties of the plasma make this relation far from being trivial, as discussed in the next section. In this lecture, we will always assume that this relation is *linear* in essence, which restricts us to waves of moderate amplitude. This will allow us to retain the self-consistent nature of the problem. Moreover, it has been shown numerous times that the linear theory of waves was well suited to describe a large class of problems, such as high power heating and current drive in magnetic fusion devices.

1.1.3 Properties of a magnetized plasma

In an isotropic “standard” medium, the fact that \mathbf{j} is a linear function of \mathbf{E} can be written as

$$\mathbf{j}(\mathbf{r}, t) = \sigma(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t), \quad (1.12)$$

σ is the linear conductivity. Eq. 1.12 is local, both spatially (i.e., the response at location \mathbf{r} only depends on the excitation at location \mathbf{r}) and temporally (i.e., the response at instant t depends only on the excitation at instant t). Unfortunately, several properties of the plasma make the description more complicated than in this ideal dielectric medium

Anisotropy: In many situations, plasmas are confined by strong magnetic fields (magnetic fusion plasmas, space plasmas...). In this case, the response will obviously be different depending on the direction of the excitation (Fig. 1.2)

The relation between \mathbf{j} and \mathbf{E} thus becomes tensorial in essence, as in a crystal, for instance. Hence, we write

$$\mathbf{j}(\mathbf{r}, t) = \bar{\bar{\sigma}} \cdot \mathbf{E}(\mathbf{r}, t). \quad (1.13)$$

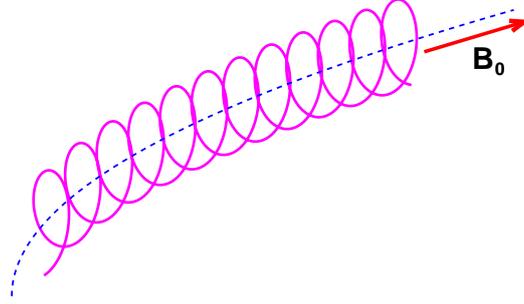


Figure 1.2: Anisotropy in a plasma confined by a magnetic field.

Time dispersion: The plasma is comprised of an assembly of electrons and ions, with various weights. Depending on the wave frequency, due to their inertia, the heavy ions may respond to the excitation with a delay (Fig. 1.3)

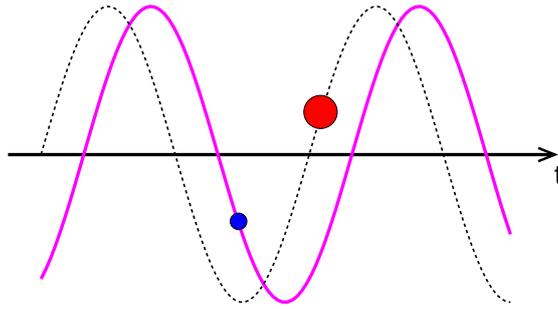


Figure 1.3: Time dispersion in a plasma. The various species respond differently to the wave depending on its frequency and on their respective masses.

In this case, the response of the plasma at instant t is determined by the excitation at all previous instants t' . Taking into account the causality principle which imposes to perform the integral only on times prior to t , we obtain a relation which is non local in time:

$$\mathbf{j}(\mathbf{r}, t) = \int_{-\infty}^t dt' \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r}, t, t') \cdot \mathbf{E}(\mathbf{r}, t'). \quad (1.14)$$

Due to the non-local character of the relation between \mathbf{j} and \mathbf{E} , it is usual to refer $\bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r}, t, t')$ as the *conductivity kernel*.

Space dispersion: In a plasma, the finite temperature of the species induces a thermal agitation, and the particles have erratic motions superimposed to their integrable displacement (if any). This means that the particles at position \mathbf{r}

are influenced by the electromagnetic field in the domain they explore due to this non-deterministic part of their motion. Space dispersion is therefore a consequence of thermal effects (Fig. 1.4). We can thus expect a cold plasma to be non-dispersive in space (but not in time). We will find out later that this is indeed the case.

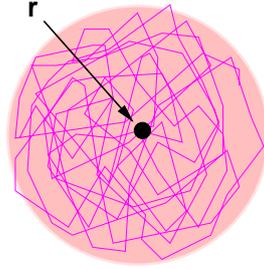


Figure 1.4: Space dispersion in a plasma. The thermal agitation causes the particles located at position r to actually “experience” the field in a region around this position.

The relation between \mathbf{j} and \mathbf{E} must thus be written in a spatially non local form:

$$\mathbf{j}(\mathbf{r}, t) = \int d^3\mathbf{r}' \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r}, \mathbf{r}', t) \cdot \mathbf{E}(\mathbf{r}', t). \quad (1.15)$$

Gathering these three essential properties, it is clear that the functional $\mathbf{j}(\mathbf{E})$ must be written in the form

$$\mathbf{j}(\mathbf{r}, t) = \int_{-\infty}^t dt' \int d^3\mathbf{r}' \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r}, \mathbf{r}', t, t') \cdot \mathbf{E}(\mathbf{r}', t'). \quad (1.16)$$

This relation is linear (assuming $\bar{\bar{\boldsymbol{\sigma}}}$ is independent \mathbf{E}) but retains the fundamental properties of the plasma medium.

1.2 Stationary and homogeneous plasmas

1.2.1 The dielectric tensor

If we inject the relation between \mathbf{j} and \mathbf{E} (Eq. 1.16) in the wave equation (Eq. 1.11), we end up with a integro-differential equation, which is linear, but still terribly complicated. In order to make further progress in the physical understanding of waves in plasmas, it is useful to start by limiting ourselves to homogeneous and stationary plasmas.

If the plasma is stationary, the conductivity can only depend on the time elapsed between the excitation (t') and the response (t). In other words, the conductivity kernel must verify $\bar{\bar{\sigma}}(\mathbf{r}, \mathbf{r}', t, t') = \bar{\bar{\sigma}}(\mathbf{r}, \mathbf{r}', t - t')$. Furthermore, if the plasma is homogeneous, the conductivity can only depend on the difference between the excitation location (\mathbf{r}') and the response location (\mathbf{r}), which simply reflects the fact that the medium is invariant by translation. We obtain $\bar{\bar{\sigma}}(\mathbf{r}, \mathbf{r}', t, t') = \bar{\bar{\sigma}}(\mathbf{r} - \mathbf{r}', t, t')$

When both conditions are fulfilled, Eq. 1.16 then becomes

$$\mathbf{j}(\mathbf{r}, t) = \int_{-\infty}^t dt' \int d^3\mathbf{r}' \bar{\bar{\sigma}}(\mathbf{r} - \mathbf{r}', t - t') \cdot \mathbf{E}(\mathbf{r}', t'). \quad (1.17)$$

It is useful to perform a Fourier analysis of the oscillating quantities and write the electric field and the current density as superpositions of plane waves.

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mathbf{k}, \omega} \mathbf{E}_{\mathbf{k}, \omega} \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)), \quad (1.18)$$

and

$$\mathbf{j}(\mathbf{r}, t) = \sum_{\mathbf{k}, \omega} \mathbf{j}_{\mathbf{k}, \omega} \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)), \quad (1.19)$$

with¹

$$\mathbf{E}_{\mathbf{k}, \omega} = \int_0^\infty dt \int d^3\mathbf{r} \mathbf{E}(\mathbf{r}, t) \exp(-i(\mathbf{k} \cdot \mathbf{r} - \omega t)), \quad (1.20)$$

and

$$\mathbf{j}_{\mathbf{k}, \omega} = \int_0^\infty dt \int d^3\mathbf{r} \mathbf{j}(\mathbf{r}, t) \exp(-i(\mathbf{k} \cdot \mathbf{r} - \omega t)), \quad (1.21)$$

It is readily seen that in Fourier space, Eq. 1.17 has a local character:

$$\mathbf{j}_{\mathbf{k}, \omega} = \bar{\bar{\sigma}}_{\mathbf{k}, \omega} \cdot \mathbf{E}_{\mathbf{k}, \omega}, \quad (1.22)$$

where $\bar{\bar{\sigma}}_{\mathbf{k}, \omega}$ is the Fourier transform of the conductivity kernel. Using the latter relation in Eq. 1.11, we can rewrite the wave equation for the Fourier components of \mathbf{E} as

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E}_{\mathbf{k}, \omega} + \frac{\omega^2}{c^2} \bar{\bar{\mathbf{K}}}_{\mathbf{k}, \omega} \cdot \mathbf{E}_{\mathbf{k}, \omega} = i\omega\mu_0 (\mathbf{j}_{ext})_{\mathbf{k}, \omega}, \quad (1.23)$$

where ∇ has been replaced by $i\mathbf{k}$ and ∂_t by $-i\omega$. We also have introduced the *dielectric permittivity tensor* as

$$\bar{\bar{\mathbf{K}}}_{\mathbf{k}, \omega} = \bar{\mathbf{1}} + \frac{i}{\omega\epsilon_0} \bar{\bar{\sigma}}_{\mathbf{k}, \omega}. \quad (1.24)$$

¹We will generally omit the normalization factors appearing in the Fourier transforms.

The *dielectric tensor* $\bar{\bar{\epsilon}}_{\mathbf{k},\omega}$ is then defined as²

$$\bar{\bar{\epsilon}}_{\mathbf{k},\omega} \equiv \epsilon_0 \bar{\bar{\mathbf{K}}}_{\mathbf{k},\omega}. \quad (1.25)$$

The wave equation 1.23 is now linear for the electric field and is therefore referred to as the *linearized wave equation*³.

The dielectric tensor presents some mathematical properties which will be important in the subsequent sections.

Firstly, ϵ can be decomposed as

$$\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}^h + i\bar{\bar{\epsilon}}^a, \quad (1.26)$$

where the Hermitian part $\bar{\bar{\epsilon}}^h$ is defined as

$$\bar{\bar{\epsilon}}^h \equiv \frac{\bar{\bar{\epsilon}} + \bar{\bar{\epsilon}}^\dagger}{2}, \quad (1.27)$$

and the anti-Hermitian part as

$$\bar{\bar{\epsilon}}^a \equiv \frac{\bar{\bar{\epsilon}} - \bar{\bar{\epsilon}}^\dagger}{2i}. \quad (1.28)$$

The \dagger symbol means ‘‘adjoint’’, i.e. $(\bar{\bar{\epsilon}}^\dagger)_{ij} = (\bar{\bar{\epsilon}})_{ji}^*$.

Interesting mathematical relations exist between the Hermitian and anti-Hermitian parts of the dielectric tensor as fundamental consequences of the physical properties of the system. For instance, the principle of causality has strong mathematical implications. As underlined above (see Eq. 1.22), if the plasma is assumed to be stationary and uniform, the relation between \mathbf{j} and \mathbf{E} has a local character in Fourier space:

$$\mathbf{j}_{\mathbf{k},\omega} = \bar{\bar{\sigma}}_{\mathbf{k},\omega} \cdot \mathbf{E}_{\mathbf{k},\omega}, \quad (1.29)$$

The causality principle imposes the plasma response (\mathbf{j}) to follow the cause (\mathbf{E}), which implies

$$\bar{\bar{\sigma}}(t - t') = 0, \quad t < t', \quad (1.30)$$

so that the Fourier transform is written as

$$\bar{\bar{\sigma}}(\omega) = \int_{-\infty}^{\infty} dt \bar{\bar{\sigma}}(t) e^{i\omega t}, \quad (1.31)$$

²From now on and unless otherwise stated, we shall omit the \mathbf{k} and ω indices and implicitly make use the Fourier-Laplace components of the oscillating quantities (electromagnetic field, dielectric tensor...).

³It is worth mentioning that the wave equation could have been obtained without assuming that the medium be homogeneous, thus keeping the space dispersion property. In this case, the dielectric tensor depends on the positions \mathbf{r} and \mathbf{r}' . In this situation, the $\nabla \times \nabla \times \cdot$ operator in Eq. 1.11 should not be replaced by $-\mathbf{k} \times \mathbf{k} \times \cdot$. Despite its linear character, the obtained wave equation is difficult to handle and hardly lends itself to analytical treatments.

where the integral is performed from $t \rightarrow -\infty$ to $+\infty$. Actually, the expression “Fourier transform” is improper. What we really have is a Laplace transform, with the usual variable s purely imaginary: $s \equiv -i\omega$. The properties of the Laplace transform impose[2] that the elements of $\bar{\bar{\sigma}}$, and thus the elements of $\bar{\bar{\epsilon}}$ must be analytical functions for $\Im(\omega) > 0$.

In order to illustrate this idea, let us consider an instantaneous excitation at $t = 0$: $\mathbf{E} = \mathbf{E}_0\delta(t)$. In this case, we simply have

$$\mathbf{j}_\omega = \bar{\bar{\sigma}}(\omega) \cdot \mathbf{E}_0, \quad (1.32)$$

so that

$$\mathbf{j}(t) = \int_{-\infty}^{\infty} d\omega \bar{\bar{\sigma}}(\omega) \cdot \mathbf{E}_0 e^{-i\omega t}. \quad (1.33)$$

Decomposing $\omega \equiv \omega_r + i\omega_i$, it can be seen that for $t < 0$, $\exp(\omega_i t) \rightarrow 0$ when $\omega_i \rightarrow \infty$. The integral can be performed along a circular contour in the region $\Im(\omega) > 0$ of the complex plan. By definition, if $\bar{\bar{\sigma}}$ is analytical in this region and vanishes for $\omega \rightarrow \infty$ (which must be the case, because of the finite inertia of the plasma particles), the integral is zero and $\mathbf{j}(t < 0) = 0$, which is in conformity with the causality principle. A necessary condition for our solution to be correct is therefore that $\bar{\bar{\epsilon}}$ be an analytical function in the superior half of the complex plan. This has consequences on the calculation of integrals such as

$$I(\omega) \equiv \int_{\mathcal{C}} d\omega' \frac{K_{ij}(\omega') - \delta_{ij}}{\omega' - \omega}, \quad (1.34)$$

with \mathcal{C} the contour shown on Fig. 1.5 and $\bar{\bar{\mathbf{K}}} = \bar{\bar{\epsilon}}/\epsilon_0$. As demonstrated above, $\bar{\bar{\epsilon}}$ is an analytical function in the region delimited by \mathcal{C} and thus $I(\omega) = 0$. This integral can be written as the sum of an integral along the real axis, $I_1(\omega)$, and another integral along a half-circle lying in the superior half of the plan, $I_2(\omega)$, which is extended to $\Im(\omega) \rightarrow +\infty$. Therefore, we have $I_2 \rightarrow 0$. From the theorem of residues, we deduce

$$I_1(\omega) = -i\pi(K_{ij} - \delta_{ij}) + \mathcal{P}\left(\frac{K_{ij} - \delta_{ij}}{\omega' - \omega}\right) = 0, \quad (1.35)$$

where \mathcal{P} designates the Cauchy principal value of the integral[2].

Adding and subtracting the complex conjugate of this expression, we obtain

$$\bar{\bar{\epsilon}}^h = \epsilon_0 \bar{\mathbf{1}} + \frac{1}{\pi} \mathcal{P} \int d\omega' \frac{\bar{\bar{\epsilon}}^a(\omega')}{\omega' - \omega}, \quad (1.36)$$

and also

$$\bar{\bar{\epsilon}}^a = -\frac{1}{\pi} \mathcal{P} \int d\omega' \frac{\bar{\bar{\epsilon}}^h(\omega')}{\omega' - \omega}. \quad (1.37)$$

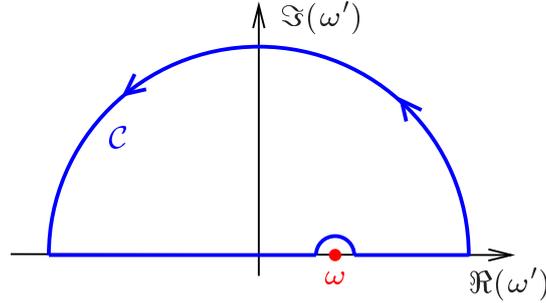


Figure 1.5: Integration contour in the complex plan in the presence of a singularity at $\omega' = \omega$.

The two previous expressions are called the *Kramers-Kronig relations*. As we shall see later in the course of this lecture, $\bar{\epsilon}^h$ describes the wave propagation, whereas $\bar{\epsilon}^a$ describes the absorption. The causality principle therefore implies that both parts are intrinsically linked. Practically, this means that in principle, the Hermitian part can be evaluated assuming that ω is real in the conductivity tensor, and deduce the anti-Hermitian part by appending an imaginary part to the frequency.

1.2.2 Wave equation, dispersion equation and polarization

As discussed above, if the plasma is stationary and homogeneous, it is convenient to handle the problem directly in Fourier-Laplace space. Maxwell's equations (1.3) and (1.10) can be written as

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}, \quad (1.38)$$

and

$$\mathbf{k} \times \mathbf{B} = -\omega \mu_0 \bar{\epsilon} \cdot \mathbf{E} - i \mu_0 \mathbf{j}_{ext}. \quad (1.39)$$

In this case, Eqs. 1.1 yields

$$\mathbf{k} \cdot \mathbf{B} = 0, \quad (1.40)$$

which is a mere consequence of Eq.1.38. On the other hand, Eq. 1.2 gives

$$\mathbf{k} \cdot \bar{\epsilon} \cdot \mathbf{E} = -i \rho_{ext}, \quad (1.41)$$

which is identical to Eq. 1.39 as long as the current continuity is verified in the exciting structure, *i.e.*

$$\frac{\partial \rho_{ext}}{\partial t} + \nabla \cdot \mathbf{j}_{ext} = 0. \quad (1.42)$$

As shown before, Eqs. 1.38 and 1.39 can be combined to lead to the wave equation

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} + \frac{\omega^2}{c^2} \bar{\mathbf{K}} \cdot \mathbf{E} = -i \omega \mu_0 \mathbf{j}_{ext}. \quad (1.43)$$

As a consequence of the locality of this equation, and of the fact that we are interested in modes that can propagate in the plasma as a response to an excitation typically located outside the ionized medium, we can assume that the sources are infinitely remote and it is convenient to work with the homogeneous version⁴ of the wave equation:

$$\overline{\mathbf{M}}_{\mathbf{k},\omega} \cdot \mathbf{E} \equiv \mathbf{k} \times \mathbf{k} \times \mathbf{E} + \frac{\omega^2}{c^2} \overline{\mathbf{K}} \cdot \mathbf{E} = 0, \quad (1.44)$$

where $\overline{\mathbf{M}}_{\mathbf{k},\omega}$ is a matrix representing the operator $\mathbf{k} \times \mathbf{k} \times \cdot + \omega^2/c^2 \overline{\mathbf{K}}$. Using the relation $\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = (\mathbf{k} \cdot \mathbf{E})\mathbf{k} - k^2\mathbf{E}$, it is readily shown that its terms are written as

$$(\overline{\mathbf{M}}_{\mathbf{k},\omega})_{ij} = k_i k_j - k^2 \delta_{ij} + \frac{\omega^2}{c^2} (\overline{\mathbf{K}})_{ij} \quad (1.45)$$

The non-trivial solutions of Eq. 1.44 are obtained from

$$\det(\overline{\mathbf{M}}_{\mathbf{k},\omega}) = 0. \quad (1.46)$$

which can be formally inverted to yield $\mathbf{k} = \mathbf{k}(\omega)$. In other words, (1.46) is the *dispersion relation* of the medium. Introducing the *refraction index* $\mathbf{n} \equiv \mathbf{k}c/\omega$, we can still write

$$\det \left(n_i n_j - n^2 \delta_{ij} + (\overline{\mathbf{K}})_{ij} \right) = 0, \quad (1.47)$$

which can be expanded in powers of n as

$$A_{ij} n^4 + B_{ij} n^2 + \det(\overline{\mathbf{K}}) = 0, \quad (1.48)$$

where the sum over i and j is implicit

If $\overline{\mathbf{K}}$ does not depend on the refraction index (i.e. in the absence of space dispersion), Eq. 1.48 is a simple quadratic equation for n^2 which gives at most two waves propagating in the plasma. The introduction of thermal effects in the description generally results in a dependence of $\overline{\mathbf{K}}$ on the wave vector: $\overline{\mathbf{K}} = \overline{\mathbf{K}}(\mathbf{n})$. In the case, we obtain new modes propagating which are usually referred to as *hot plasma modes*.

After having obtained $\mathbf{n}_\alpha = \mathbf{n}_\alpha(\mathbf{k}/k, \omega)$ (α denotes a given mode) from the dispersion relation, injecting this eigenvalue in (1.44), we obtain the non trivial solution for \mathbf{E}_α or more precisely the ratios of its components two by two (as eigenvectors). The amplitude remains arbitrary. By doing this, we obtain the mode *polarization*.

1.2.3 Cut-off - Resonance - Evanescence

Useful information can be obtained by solving the dispersion relation (1.48). Generally, this is made assuming that the frequency is fixed, and seeking solutions for

⁴In a mathematical sense.

n. Fundamental wave features for given plasma parameters can thus be deduced. For instance, a *cut-off* is determined by $n = 0$: the refraction index goes to zero (see Fig. 1.6).

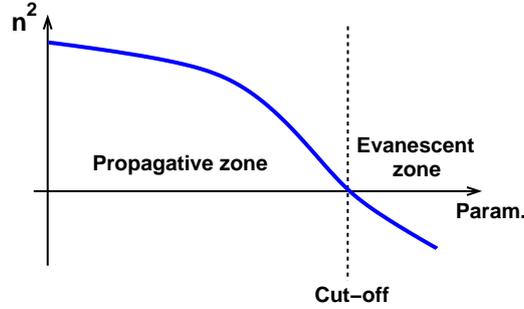


Figure 1.6: Cut-off.

From relation (1.48), this clearly occurs when

$$\det(\bar{\mathbf{K}}) = 0. \quad (1.49)$$

However, it is important to notice that depending on the configuration of the plasma under study, the cut-off can be defined in a different fashion. For instance, in a tokamak, because of the geometry of the system, the antenna on the edge of the plasma imposes the parallel index $n_{\parallel} = \mathbf{n} \cdot \mathbf{B}_0/B_0$, with \mathbf{B}_0 the confining magnetic field⁵. In this case, the dispersion relation is more conveniently expressed as

$$A'_{ij}n_{\perp}^4 + B'_{ij}n_{\perp}^2 + C'_{ij} = 0, \quad (1.50)$$

with $n_{\perp} \equiv (n^2 - n_{\parallel}^2)^{1/2}$. In this situation, it makes more sense to look for locations where $n_{\perp} \rightarrow 0$. Although they should be called *perpendicular cut-offs*, for brevity, they are often simply referred to as *cut-offs*.

On the contrary, in order to study longitudinal waves propagating in a plasma column, a better choice is to introduce *parallel cut-offs* as the locations where, for fixed n_{\perp} , $n_{\parallel} = 0$. In this case, the dispersion relation can be written as

$$A''_{ij}n_{\parallel}^4 + B''_{ij}n_{\parallel}^2 + C''_{ij} = 0. \quad (1.51)$$

A *resonance* occurs when $n \rightarrow \infty$ (see Fig. 1.7). We can deduce from Eq. 1.48 that a necessary (but not sufficient) condition is $A_{ij} = 0$. At this point and without further assumptions on the dielectric tensor, however, it is difficult to be more specific about the occurrence of resonances.

⁵“Parallel” (resp. “perpendicular”) refers to the projection along (resp. normal to) the magnetic field direction.

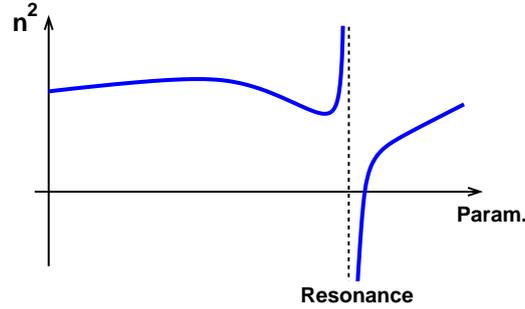


Figure 1.7: Resonance. In the case shown above, a resonance and a cut-off are simultaneously present in the plasma.

Up to now, our analysis has been conducted under the assumption that both \mathbf{k} and ω were real, in order to describe our wave as a superposition of plane waves. In reality, however, the plasma can either dissipate the wave power or emit a wave on its own in the process of relaxing towards thermodynamical equilibrium. The description of this kind of situation necessitates the refraction index to be complex. In this case, the solutions of (1.48) such as $n^2 > 0$ will give *propagative mode*. On the other hand, $n^2 < 0$ describes an *evanescent wave*, whose propagation is limited, either in time if $\Im(\omega) < 0$ or in the space direction x for $\Im(k_x) < 0$ (for $\Re(k_x) > 0$). It should be mentioned that evanescent modes play a crucial role in the antenna region: even if the field penetration in the plasma is limited to a narrow region on the edge of the medium, it determines the quality of the wave coupling.

1.3 Wave in inhomogeneous plasmas

1.3.1 Stationarity and homogeneity

In order to perform plane wave expansions, we have assumed so far that the plasma was stationary and homogeneous. The former hypothesis requires that the timescale Δt of the phenomena under study be slow compared to the wave period:

$$\Delta t \gg \frac{2\pi}{\omega}. \quad (1.52)$$

If we consider, for instance, a low frequency heating wave in a tokamak plasma, $f \approx 10\text{MHz}$, this condition implies $\Delta t \gg 10^{-7}\text{s}$. Practically, we are not really interested in processes taking place on such a short timescale and it is safe to assume that the stationarity of the plasma with respect to the wave dynamics is a legitimate assumption.

The homogeneity hypothesis is equivalent to assume that the observed phenom-

ena take place on a space-scale Δr large compared to the wavelength:

$$\Delta r \gg \lambda = 2\pi \frac{c}{n\omega}. \quad (1.53)$$

Considering the vacuum wavelength of a wave with frequency $f \approx 10\text{MHz}$ propagating in a fusion device, this condition gives $\Delta r \gg 30\text{m}$! This time, we have a problem because the observed phenomena obviously have scales which generally are much smaller⁶. The consequence is that our description should be refined to reflect, at least partly, the inhomogeneous nature of the plasma.

From the point of view of the energy transfers, an issue also arises: the energy of a homogeneous and stationary is infinite. This can be seen by writing the electromagnetic field as

$$(\mathbf{E}, \mathbf{B}) \equiv (\mathbf{E}_0, \mathbf{B}_0) \times \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)). \quad (1.54)$$

In order to describe the wave absorption, we append an imaginary part to the frequency and to the wave vector: $\omega \equiv \omega_r + i\gamma$ with $\gamma < 0$, and $\mathbf{k} \equiv \mathbf{k}_r + i\mathbf{k}_i$. The phase term can be written as

$$e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = e^{i(\mathbf{k}_r \cdot \mathbf{r} - \omega_r t)} \times e^{-\mathbf{k}_i \cdot \mathbf{r}} \times e^{\gamma t}. \quad (1.55)$$

When computing the total energy, the volume integral will at least encompass an integration to infinity with the integrand containing $\exp(-\mathbf{k}_i \cdot \mathbf{r})$ obviously diverging. Therefore, the homogeneity condition will most likely be a problem if we are interested in describing energy transfers between the wave and the plasma.

1.3.2 Propagation in an inhomogeneous medium

In order to cure the shortcomings arising from the homogeneity condition discussed in the previous section, we will now try to relax the latter but nevertheless place ourselves in an idealized physical situation: we assume that the medium consists of a stack of infinitely thin layer, homogeneous in the directions $\hat{\mathbf{u}}_x$ and $\hat{\mathbf{u}}_y$. We also assume that the plasma is an isotropic, non-dispersive, dielectric medium whose refraction index is n , i.e. $\mathbf{D} = n^2 \mathbf{E}$. For a wave with frequency ω , the wave equation can be written as

$$\nabla \times \nabla \times \mathbf{E} - \frac{\omega^2}{c^2} n^2 \mathbf{E} = \mathbf{0}. \quad (1.56)$$

The problem configuration is such as $\nabla \equiv d/dz \hat{\mathbf{u}}_z$ and we thus obtain two independent equations:

$$\frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2} n^2 E_x = 0, \quad (1.57)$$

⁶It should be remarked that the wavelength of a low frequency wave propagating in a plasma is often very different from the vacuum value. Nevertheless, the homogeneity assumption remains doubtful for such waves.

and

$$\frac{d^2 E_y}{dz^2} + \frac{\omega^2}{c^2} n^2 E_y = 0. \quad (1.58)$$

Since the E_x and E_y components are uncoupled, we can assume, without affecting the generality of the problem, that the wave is linearly polarized along $\hat{\mathbf{u}}_y$. For the wave magnetic field, we have from Maxwell's equations

$$\frac{dB_x}{dz} = -i\omega n^2 E_y. \quad (1.59)$$

We are looking for solutions of Eq. 1.58 in the form

$$E_y = A e^{i\phi}. \quad (1.60)$$

If the plasma is homogeneous, Eq. 1.58 simply yields $\phi = \pm n k_0 z$, with $k_0 \equiv \omega/c$, and the general solution can be written as

$$E_y(z) = A_+ e^{i n k_0 z} + A_- e^{-i n k_0 z}. \quad (1.61)$$

We obtain two modes propagating in opposite directions, with the same module for the phase velocity $v_\phi = c/n$.

If the plasma is not homogeneous along z , on the other hand, (1.61) does not constitute a solution for Eq. 1.58. In this case, we must instead use the more general form (1.60). Substituting in Eq. 1.58, we obtain the following relation, assuming that $\phi = \phi(z)$:

$$\left(\frac{d\phi}{dz} \right)^2 = i \frac{d^2 \phi}{dz^2} + n^2 k_0^2, \quad (1.62)$$

which is a non-linear differential equation, complicated to solve in its general form. However, we remember that in the homogeneous case, we had $\phi = \pm n k_0 z$, or $d^2 \phi / dz^2 = 0$. Therefore, it is interesting to consider the situation in which the medium is *slowly variable* as a starting point. We do this by using the *ansatz* $\phi(z) \equiv \pm n(z) k_0 z$ (note that the refraction index now depends on z) which gives

$$\frac{d\phi}{dz} = \pm n k_0 \pm \frac{dn}{dz} k_0 z. \quad (1.63)$$

Considering that the medium is slowly variable means that we assume

$$\frac{z}{n} \frac{dn}{dz} \ll 1, \quad (1.64)$$

so that in Eq. 1.63, we neglect the second term on the right hand side to obtain

$$\frac{d\phi}{dz} \approx \pm n(z) k_0, \quad (1.65)$$

or yet

$$\frac{d^2\phi}{dz^2} = \pm \frac{dn}{dz} k_0. \quad (1.66)$$

Substituting this approximate expression for $d^2\phi/dz^2$ in 1.62, we obtain

$$\frac{d\phi}{dz} \approx \pm n k_0 \left(1 \pm i \frac{c}{\omega} \frac{1}{n^2} \frac{dn}{dz} \right)^{1/2} \approx \pm n k_0 + \frac{i}{2n} \frac{dn}{dz}, \quad (1.67)$$

which can be directly integrated to yield

$$\phi(z) = \pm k_0 \int dz n(z) + i \ln(n^{1/2}). \quad (1.68)$$

Replacing in the general expression for E_y , this leads to

$$E_y = A_{\pm} n^{-1/2} \exp \left(\pm i k_0 \int dz n(z) \right) \quad (1.69)$$

The solution obtained in the inhomogeneous case clearly presents significant qualitative differences with respect to the homogeneous solution. $\phi(z)$, as defined by Eq. 1.68, is called the *eikonal function* and its expression features an integral over time: the wave keeps a memory of the medium properties along its propagation. This kind of expression is traditionally called a *WKB* solution, in reference to its inventors (in quantum mechanics), Wentzel, Kramers and Brillouin. We thus obtain an oscillating electromagnetic field, modulated by a slowly variable amplitude, as illustrated on Fig. 1.8

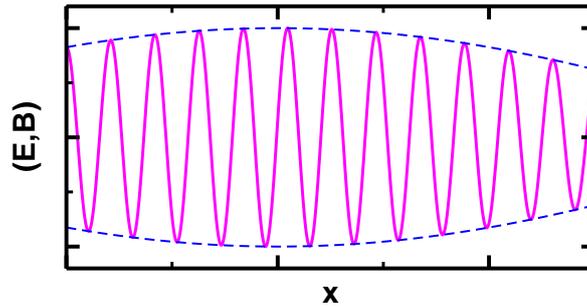


Figure 1.8: Slow space variation of the electromagnetic field envelope.

In order to assess the validity of the obtained solution, it is natural to replace E_y in Eq. 1.58 by its expression (1.69). This yields

$$\frac{d^2 E_y}{dz^2} + n^2 k_0^2 E_y = A_{\pm} \left[\frac{3}{4} \frac{1}{n^{5/2}} \left(\frac{dn}{dz} \right)^2 - \frac{1}{2n^{3/2}} \frac{d^2 n}{dz^2} \right] \exp \left(\pm i k_0 \int dz n(z) \right). \quad (1.70)$$

By comparing to Eq. 1.58 and since our solution was constructed for a slowly varying medium, we see that the magnitude of the right-hand side of this relation must be small with respect to the terms on the left-hand side. This allows to write the sought condition as

$$\left| \frac{3}{4} \left(\frac{1}{n^2} \frac{dn}{dz} \right)^2 - \frac{1}{2n^3} \frac{d^2n}{dz^2} \right| \ll \frac{\omega^2}{c^2}, \quad (1.71)$$

which demonstrates that the WKB solution can be used provided that dn/dz and d^2n/dz^2 remain small, and that n does not vanish, which excludes the cut-offs. It is also incorrect to describe the solution in the vicinity of resonances by a WKB solution because expression (1.69) shows that in this case, $E_y \rightarrow \infty$, which is unphysical (and in contradiction with the slow-varying amplitude condition). Finally, we see that the WKB solution becomes more valid as the frequency increases. In a magnetic fusion device, it is commonly used to describe the high-frequency radiofrequency waves (Lower Hybrid and Electron Cyclotron) but it generally should not be applied to lower frequency wave (Ion Cyclotron and Alfvén). It is still extremely useful because it extends the concepts of dispersion relation and mode polarization in inhomogeneous plasmas

1.3.3 Propagation in the presence of a cut-off

In situations such as the WKB method can clearly not be used, it is necessary to directly solve the Maxwell's equations and compute the electromagnetic field. The latter approach is known as a *full-wave description* and it often requires a numerical treatment. In this section, we study the propagation of a wave in the vicinity of a cut-off, which forbids the use of the WKB approximation, as shown in the previous section. Away from this zone, however, it can still be safely used and we will therefore connect the full-wave solution to the WKB solution.

We assume that the plasma density varies along z , so that the refraction index vanishes when $z = z_c$ (see Fig. 1.9)

In order to be able to handle this problem analytically, we further assume that the refraction index varies as

$$n^2 = a(z_c - z), \quad (1.72)$$

where a is a positive constant. The wave equation (1.58) then yields

$$\frac{d^2 E_y}{dz^2} + a(z_c - z) \frac{\omega^2}{c^2} E_y = 0. \quad (1.73)$$

Letting

$$\bar{z} \equiv \left(\frac{\omega^2}{c^2} a \right)^{1/3} (z - z_c), \quad (1.74)$$

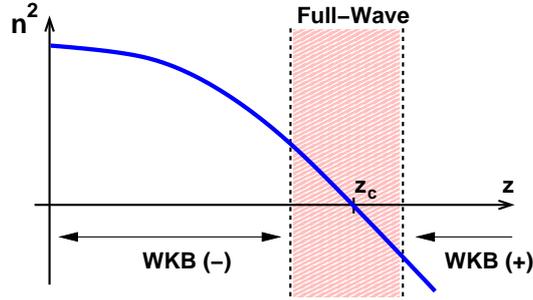


Figure 1.9: In the vicinity of a cut-off, the WKB theory does not apply (*shaded region*) and a full-wave treatment must be performed.

we obtain

$$\frac{d^2 E_y}{d\bar{z}^2} - \bar{z} E_y = 0, \quad (1.75)$$

which is known as the *Airy equation*[3]. It has two classes of solutions, denoted A_i and B_i . The behavior of the latter, which is such as $\lim_{z \rightarrow \pm\infty} |B_i(z)| = +\infty$ imposes

$$E_y(\bar{z}) = C A_i(\bar{z}), \quad (1.76)$$

with C a constant to be determined.

Assuming that the wave is excited by an antenna located at $z \rightarrow -\infty$, we can use a WKB solution between the antenna and the vicinity of the cut-off (region denoted “WKB (-)” on Fig. 1.9). According to section 1.3.2, it can be written in the general form

$$E_y(z) = I n^{-1/2} \exp\left(ik_0 \int dz n(z)\right) + R n^{-1/2} \exp\left(-ik_0 \int dz n(z)\right). \quad (1.77)$$

The first term on the right hand side describes the incident wave (I is therefore the amplitude of the wave excited by the antenna). The second one describes the reflected wave and R is the reflection coefficient. Using the prescribed density profile (1.72), we can write

$$E_y(\bar{z}) = I n^{-1/2} \exp\left(-\frac{2i}{3} |\bar{z}|^{3/2}\right) + R n^{-1/2} \exp\left(\frac{2i}{3} |\bar{z}|^{3/2}\right). \quad (1.78)$$

In order to connect the latter expression with the full-wave solution (1.76), we must use the asymptotic expansion of A_i for large negative arguments[3]

$$A_i(\bar{z}) \sim \frac{1}{\sqrt{\pi}} |\bar{z}|^{-1/4} \sin\left(\frac{2}{3} |\bar{z}|^{3/2} + \frac{\pi}{4}\right), \quad (1.79)$$

so that Eq. 1.76 gives for $\bar{z} \rightarrow -\infty$:

$$E_y(\bar{z}) \sim \frac{C}{2^{3/2}\sqrt{\pi}} |\bar{z}|^{-1/4} (1+i) \left[\exp\left(-\frac{2i}{3} |\bar{z}|^{3/2}\right) - i \exp\left(\frac{2i}{3} |\bar{z}|^{3/2}\right) \right]. \quad (1.80)$$

Comparing Eqs. 1.78 and (1.80), we see that

$$R = -iI, \quad (1.81)$$

which means that the entire wave power is reflected at the cut-off and that the phase undergoes a $\pi/2$ shift in the process.

In the region located on the right off the cut-off, the WKB solution can be written in the form

$$E_y(z) = Tn^{-1/2} \exp\left(ik_0 \int dz n(z)\right) + Mn^{-1/2} \exp\left(-ik_0 \int dz n(z)\right). \quad (1.82)$$

T is the wave transmission coefficient and the second term describes a wave propagating towards the left on Fig. 1.9. In reality, this could be a wave excited in a mode conversion process[4].

Using 1.72, we have

$$E_y(\bar{z}) = Tn^{-1/2} \exp\left(-\frac{2}{3} |\bar{z}|^{3/2}\right) + Mn^{-1/2} \exp\left(\frac{2}{3} |\bar{z}|^{3/2}\right). \quad (1.83)$$

In the vicinity of the cut-off, expression (1.76) must be employed. Using the expansion of A_i for large positive arguments[3]:

$$A_i(\bar{z}) \sim \frac{1}{2\sqrt{\pi}} |\bar{z}|^{-1/4} \exp\left(-\frac{2}{3} |\bar{z}|^{3/2}\right), \quad (1.84)$$

yields

$$E_y(\bar{z}) \sim \frac{C}{2\sqrt{\pi}} |\bar{z}|^{-1/4} \exp\left(-\frac{2}{3} |\bar{z}|^{3/2}\right). \quad (1.85)$$

The comparison of expressions (1.83) and (1.85) shows that we necessarily have $M = 0$: in our model, there can only be a transmitted wave, but the exponential factor shows that it is evanescent.

On Fig. 1.10, the solution (1.76) is shown, along with the approximated expressions employed for $z \rightarrow \pm\infty$ to connect to the WKB solution. It can be noted that on the right of the cut-off, the evanescent wave only propagates on a very short distance.

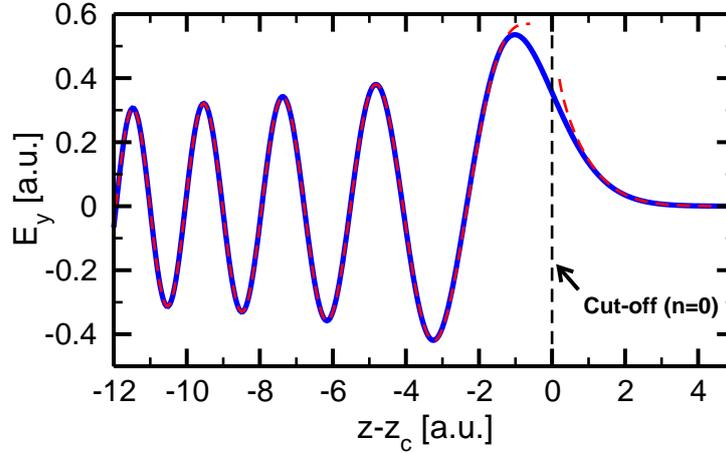


Figure 1.10: Full-Wave solution for E_y in the vicinity for the cut-off, located here at $z = 0$ (solid). The asymptotic expansions used in the derivation are also shown as dashed curves.

1.4 Energy transfers

1.4.1 Poynting theorem

In electromagnetism problems, the energy transfers between the medium and the wave, as the wave-field itself, are described by the Maxwell's equations. Dotting Eq. 1.3 with \mathbf{H} , Eq. 1.4 with \mathbf{E} and using the vector relation $\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}$, the *Poynting theorem* can be expressed as a conservation relation:

$$\nabla \cdot \mathbf{S} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = -(\mathbf{j}_{free} + \mathbf{j}_{ext}) \cdot \mathbf{E}, \quad (1.86)$$

with the *Poynting vector* defined as

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}. \quad (1.87)$$

The Poynting theorem can take several forms. The latter one has the advantage of clearly showing the energy flow of the electromagnetic wave (\mathbf{S}), the power required to establish the electric field in the medium; $\mathbf{E} \cdot \partial_t \mathbf{D}$, the power required to establish the magnetic field in the medium: $\mathbf{H} \cdot \partial_t \mathbf{B}$. On the right hand side is the dissipated power, caused by the Joule effect.

Another possibility is to use Eqs. 1.3 and 1.9 to write

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} + (\mathbf{j}_{pol} + \mathbf{j}_{mag}) \cdot \mathbf{E} = -(\mathbf{j}_{free} + \mathbf{j}_{ext}) \cdot \mathbf{E}, \quad (1.88)$$

with the vacuum energy density

$$W \equiv \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2, \quad (1.89)$$

and the Poynting vector defined as

$$\mathbf{S} \equiv \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}. \quad (1.90)$$

Both relations (1.86) and (1.88), although written differently, features the reversible processes on the left hand side, and the irreversible phenomena (dissipation) on the right hand side. In a solid, for instance, the two first terms of Eq. 1.88 can be identified respectively as the electromagnetic energy variation and the power flowing in the electromagnetic wave. The third and fourth term represent the work exerted by the polarization and magnetization current, caused by the displacement of atoms or nuclei around their equilibrium positions as a response to the applied electromagnetic field. This energy exchange is reversible: when the excitation is switched off, the power is immediately transferred back to the wave. On the right hand side appears the power dissipated by Joule effect. This energy transfer is irreversible and contributes to the heating of the medium. Often, in a solid, it is safe to assume that the mean free path is much smaller than the wavelength⁷. This means that the equilibrium between the electromagnetic field and the medium is practically instantaneous.

1.4.2 Energy balance in a plasma

Again, in a plasma, the situation is more complicated: there is no clear separation between the free carriers current, the polarization and magnetization currents. All charges are free, but also contribute to the polarization. Also, the mean free path is often much longer than the wave period. In other words, there is no equilibrium between the electromagnetic field and the medium on a single wave period. Therefore, we will need to work with quantities averaged over a number of wave periods to describe the exchanges of energy between the plasma and the wave: an analysis on the timescale of the wave period does not yield much useful information. The instantaneous Poynting theorem will nevertheless be our starting point. It can be written by combining Eqs. 1.3 and 1.10 as

$$\nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) + \frac{\partial}{\partial t} \left[\frac{|\mathbf{B}|^2}{2\mu_0} + \frac{\epsilon_0 |\mathbf{E}|^2}{2} \right] + \mathbf{j} \cdot \mathbf{E} = -\mathbf{j}_{ext} \cdot \mathbf{E}. \quad (1.91)$$

As shown in section 1.3.1, a homogeneous plasma has infinite energy. Therefore, we will make use of a WKB form for all oscillating quantities, instead of a simple plane wave decomposition. For instance, the electric field is written as

$$\mathbf{E}(t) = \frac{1}{2} [\mathbf{E}_0 e^{i\phi} + \mathbf{E}_0^* e^{-i\phi}]. \quad (1.92)$$

⁷If we assume that the distance between two atoms is 1nm, and that this constitutes the average mean free path of a current carrier, we see that the wave frequency needed to have a wavelength comparable to this mean free path is $f \gtrsim 10^{17}$ Hz, which is in the X-rays range of frequencies.

Assuming that the wave properties vary slowly both spatially and temporally (see section 1.3.1 for the meaning of this requirement), the eikonal function can be approximated as⁸

$$\phi(t) = \mathbf{k} \cdot \mathbf{r} - \omega t. \quad (1.93)$$

We define the time average of a given quantity A as

$$\langle A \rangle \equiv \frac{1}{\tau} \int_0^\tau dt A(t), \quad (1.94)$$

with $\tau\omega \gg 2\pi$, where $\omega/2\pi$ is the wave frequency. If we consider the product of two quantities \mathbf{A} and \mathbf{B} written in the form (1.92), it is readily shown that the time-average operation yields

$$\langle \mathbf{A} \cdot \mathbf{B} \rangle = \frac{1}{2} \Re(\mathbf{A}_0 \cdot \mathbf{B}_0^*). \quad (1.95)$$

The time-average of the first term of (1.91) leads to

$$\left\langle \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) \right\rangle = \frac{1}{2\mu_0} \nabla \cdot \Re(\mathbf{E}_0 \times \mathbf{B}_0^*). \quad (1.96)$$

For the second term, we obtain

$$\left\langle \frac{\partial}{\partial t} \left[\frac{|\mathbf{B}|^2}{2\mu_0} + \frac{\epsilon_0 |\mathbf{E}|^2}{2} \right] \right\rangle = \frac{1}{2\mu_0} \frac{\partial}{\partial t} |\mathbf{B}_0|^2 + \frac{\epsilon_0}{2} \frac{\partial}{\partial t} |\mathbf{E}_0|^2. \quad (1.97)$$

Obviously, the complexity of our problem lies in the third term on the left hand side of Eq. 1.91, because of the dispersive nature of the plasma. In its non-local form, the current \mathbf{j} is expressed as a function of the electric field as

$$\mathbf{j}(\mathbf{r}, t) = \int_{-\infty}^t dt' \int d^3\mathbf{r}' \bar{\bar{\sigma}}(\mathbf{r}, \mathbf{r}', t, t') \cdot \mathbf{E}(\mathbf{r}', t'). \quad (1.98)$$

Since we assume that the plasma is stationary, and also *locally* homogeneous⁹, we have

$$\bar{\bar{\sigma}}(\mathbf{r}, \mathbf{r}', t, t') \approx \bar{\bar{\sigma}}(\mathbf{r} - \mathbf{r}', t - t'). \quad (1.99)$$

The current can thus be written as

$$\mathbf{j}(\mathbf{r}, t) = \frac{1}{2} \left[e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \int_{-\infty}^t dt' \int d^3\mathbf{r}' \bar{\bar{\sigma}}(\mathbf{r} - \mathbf{r}', t - t') \cdot \mathbf{E}_0(\mathbf{r}', t') e^{i(\mathbf{k}(\mathbf{r}' - \mathbf{r}) - \omega(t' - t))} + \text{c.c.} \right], \quad (1.100)$$

⁸Owing to the validity conditions of the WKB approximations, it must be noted that this derivation is not well adapted to describe abrupt space or time changes in the medium properties, such as cut-offs or resonances. The description of the wave-medium energy transfer in the context of a full-wave modeling accounting for arbitrary space and time dispersion is a formidable problem.

⁹In section 1.3, we have shown that “locally homogeneous” essentially means homogeneous on the scale of the wavelength.

where c.c. means ‘‘complex conjugate’’. We can use the fact that the envelope of the electromagnetic field varies slowly to perform the Taylor expansion:

$$\mathbf{E}_0(\mathbf{r}', t') \approx \mathbf{E}_0(\mathbf{r}, t) + (\mathbf{r}' - \mathbf{r}) \cdot \left. \frac{\partial \mathbf{E}_0}{\partial \mathbf{r}'} \right|_{\mathbf{r}'=\mathbf{r}} + (t' - t) \cdot \left. \frac{\partial \mathbf{E}_0}{\partial t'} \right|_{t'=t}. \quad (1.101)$$

In the obtained expression, the Fourier transform of $\bar{\bar{\boldsymbol{\sigma}}}$ and its derivatives with respect to \mathbf{k} and ω can be readily identified. This gives

$$\mathbf{j}(\mathbf{r}, t) = \frac{1}{2} \left\{ e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \left[\bar{\bar{\boldsymbol{\sigma}}}_{\mathbf{k}, \omega} \cdot \mathbf{E}_0(\mathbf{r}, t) - i \frac{\partial \bar{\bar{\boldsymbol{\sigma}}}_{\mathbf{k}, \omega}}{\partial \mathbf{k}} \cdot \frac{\partial \mathbf{E}_0}{\partial \mathbf{r}} + i \frac{\partial \bar{\bar{\boldsymbol{\sigma}}}_{\mathbf{k}, \omega}}{\partial \omega} \cdot \frac{\partial \mathbf{E}_0}{\partial t} \right] + \text{c.c.} \right\}. \quad (1.102)$$

Dotting the latter expression with \mathbf{E} and averaging over time yields

$$\langle \mathbf{j} \cdot \mathbf{E} \rangle = \frac{1}{4} \left\{ \mathbf{E}_0^* \cdot \left[\bar{\bar{\boldsymbol{\sigma}}}_{\mathbf{k}, \omega} \cdot \mathbf{E}_0 - i \frac{\partial \bar{\bar{\boldsymbol{\sigma}}}_{\mathbf{k}, \omega}}{\partial \mathbf{k}} \cdot \frac{\partial \mathbf{E}_0}{\partial \mathbf{r}} + i \frac{\partial \bar{\bar{\boldsymbol{\sigma}}}_{\mathbf{k}, \omega}}{\partial \omega} \cdot \frac{\partial \mathbf{E}_0}{\partial t} \right] + \text{c.c.} \right\} \quad (1.103)$$

Recalling the definition

$$\bar{\bar{\boldsymbol{\epsilon}}} = \epsilon_0 \left(\bar{\bar{\mathbf{1}}} + \frac{i}{\omega \epsilon_0} \bar{\bar{\boldsymbol{\sigma}}}_{\mathbf{k}, \omega} \right), \quad (1.104)$$

where the subscript \mathbf{k}, ω has been omitted when writing the dielectric tensor and using Defs. 1.27 and 1.28, one obtains

$$\langle \mathbf{j} \cdot \mathbf{E} \rangle = \frac{\omega}{2} \mathbf{E}_0^* \cdot \bar{\bar{\boldsymbol{\epsilon}}}^a \cdot \mathbf{E}_0 - \frac{\omega}{2} \nabla \cdot \left(\mathbf{E}_0^* \cdot \frac{\partial \bar{\bar{\boldsymbol{\epsilon}}}^h}{\partial \mathbf{k}} \cdot \mathbf{E}_0 \right) + \frac{1}{2} \frac{\partial}{\partial t} \left(\mathbf{E}_0^* \cdot \frac{\partial}{\partial \omega} (\omega \bar{\bar{\boldsymbol{\epsilon}}}^h) \cdot \mathbf{E}_0 - \epsilon_0 |\mathbf{E}_0|^2 \right), \quad (1.105)$$

Finally, the time-average of the right-hand side of Eq. 1.91 corresponds to the power dissipated on the wave-exciting device. In the context of an antenna aimed at heating a plasma, it makes sense to define the power coupled by the antenna to the ionized medium as

$$P_{ant} \equiv -\mathbf{j}_{ext} \cdot \mathbf{E}. \quad (1.106)$$

Eventually, the energy conservation can be written as

$$\frac{\partial W}{\partial t} + \nabla \cdot (\mathbf{S} + \mathbf{T}) = P_{ant} - P_{abs}, \quad (1.107)$$

where W represents the electromagnetic energy density:

$$W \equiv \frac{1}{2} \left[\frac{|\mathbf{B}_0|^2}{\mu_0} + \mathbf{E}_0^* \cdot \frac{\partial}{\partial \omega} (\omega \bar{\bar{\boldsymbol{\epsilon}}}^h) \cdot \mathbf{E}_0 \right]. \quad (1.108)$$

\mathbf{S} , the Poynting vector, describes the power transported by the wave field:

$$\mathbf{S} \equiv \frac{1}{\mu_0} \Re(\mathbf{E}_0^* \times \mathbf{B}_0). \quad (1.109)$$

\mathbf{T} is called the *kinetic flux*:

$$\mathbf{T} \equiv -\frac{\omega}{2} \mathbf{E}_0^* \cdot \frac{\partial \bar{\epsilon}}{\partial \mathbf{k}} \cdot \mathbf{E}_0, \quad (1.110)$$

It corresponds to the electromagnetic power transport caused by the coherent motion of the particles. It is clear from the latter expression that in the case of a cold plasma, where the absence of space dispersion implies $\partial \bar{\epsilon} / \partial \mathbf{k} = 0$, this term is zero.

Finally, the dissipated power can be written as

$$P_{abs} \equiv \omega \mathbf{E}_0^* \cdot \bar{\epsilon}^a \cdot \mathbf{E}_0, \quad (1.111)$$

and directly involves the anti-Hermitian part of the dielectric tensor. It should be kept in mind that $\nabla \cdot \mathbf{T}$ and P_{abs} have different natures: the first one corresponds to power which is reversibly exchanged between the electromagnetic wave and the plasma and therefore does not contribute to its heating. The latter, on the other hand, corresponds to power irreversibly transferred to the plasma. In general, the separation of these two terms is difficult to achieve unambiguously.

Chapter 2

Fluid theory of plasma waves

2.1 Fluid equations

The idea of a fluid description of the plasma is to deal with averaged quantities rather than with the distribution functions themselves. One such quantity is the *fluid density* for species s :

$$n_s(\mathbf{r}) = \int d^3\mathbf{v} f_s(\mathbf{r}, \mathbf{v}), \quad (2.1)$$

where f_s is the distribution function.

The associated flux density is defined as

$$n_s(\mathbf{r})\mathbf{v}_s(\mathbf{r}) = \int d^3\mathbf{v} \mathbf{v} f_s(\mathbf{r}, \mathbf{v}), \quad (2.2)$$

with \mathbf{v}_s the *fluid velocity*. The total current can be written as

$$\mathbf{j} = \sum_s \mathbf{j}_s = \sum_s n_s q_s \mathbf{v}_s, \quad (2.3)$$

where q_s (resp. m_s) is the charge (resp. mass) of the elements of species s .

The second order moment defines the *fluid stress tensor*

$$\bar{\bar{\Phi}}_s(\mathbf{r}) \equiv \int d^3\mathbf{v} m_s \mathbf{v} \mathbf{v} f_s(\mathbf{r}, \mathbf{v}). \quad (2.4)$$

Often, the *fluid pressure tensor* is preferred. It is given by

$$\bar{\bar{\mathbf{p}}}_s(\mathbf{r}) \equiv \int d^3\mathbf{v} m_s (\mathbf{v} - \mathbf{v}_s)(\mathbf{v} - \mathbf{v}_s) f_s(\mathbf{r}, \mathbf{v}). \quad (2.5)$$

The scalar pressure is deduced from the trace of $\bar{\bar{\mathbf{p}}}_s$

$$p_s = \frac{1}{3} \text{Tr}(\bar{\bar{\mathbf{p}}}_s). \quad (2.6)$$

and the following relation holds:

$$\overline{\overline{\Phi}}_s = \overline{\overline{\mathbf{p}}}_s + m_s n_s \mathbf{v}_s \mathbf{v}_s. \quad (2.7)$$

In order to obtain the dielectric tensor, we recall that we need to find the relation between \mathbf{j} and \mathbf{E} (see section 1.2.1). In the context of this fluid description, we must therefore express \mathbf{v}_s as a function of \mathbf{E} and \mathbf{B} (see Eq. 2.3). To obtain this relation, the starting point is the Vlasov equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0. \quad (2.8)$$

Integrating this equation over \mathbf{v} , the continuity equation for species s is obtained as

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0. \quad (2.9)$$

Likewise, multiplying (2.8) by \mathbf{v} and integrating over velocity, we obtain the momentum conservation relation:

$$n_s m_s \left[\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s \right] = n_s q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \nabla \cdot \overline{\overline{\Phi}}_s \quad (2.10)$$

In order to obtain the fluid stress tensor, the recipe is the same, namely multiply (2.8) by the dyad $\mathbf{v}\mathbf{v}$ and integrate over velocity. By doing this, we will obtain a new equation for the energy that will require the knowledge of the third order moment. In other words, in a purely fluid description, the system of equations is virtually infinite. Obviously, at some point, we will need a *closure relation*, whose role is to express the higher moment in a given equation (for instance the pressure in the momentum continuity equation) as a function of lower moments and known quantities. Here, we will limit ourselves to two physical situations for which Eqs. 2.9 and 2.10 are sufficient: cold plasmas, and plasmas with a finite temperature and a given pressure relation.

In the framework of our linear analysis (see chapter 1), we linearize the various quantities appearing in the fluid equations assuming that

- The fluid is immobile in the absence of a wave: $\mathbf{v}_s = \delta \mathbf{v}_s$, where $\delta \mathbf{v}_s$ is the fluid displacement caused by the wave (The specific case of a moving fluid is discussed in section 2.6)
- There is no static electric field in the plasma: $\mathbf{E} = \delta \mathbf{E}$,
- The total magnetic field is the superposition of the confining and wave magnetic fields: $\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}$.

2.2 Cold plasma

2.2.1 Dielectric tensor

The fluid description of a *cold plasma* is obtained by imposing $\overline{\Phi}_s = \overline{\mathbf{0}}$ in Eq. 2.10. Practically, it can be applied provided that $\omega \gg \omega_{cs}$ (no resonant interaction), $k_\perp v_{th,s}/\omega_{cs} \gg 1$ (weak magnetization: the wave does not “feel” the cyclotron motion) and $v_\varphi \gg v_{th,s}$ (the wave does not “feel” the thermal motion of the particles). Actually, its range of application is surprisingly wide and it often yields useful information on waves propagating even in hot plasmas.

The momentum conservation equation (2.10) yields

$$\frac{\partial \mathbf{v}_s}{\partial t} = \frac{d\mathbf{v}_s}{dt} = \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}), \quad (2.11)$$

which is easily identified as a simple equation of motion for the fluids elements comprising species s . To first order and only considering the linear harmonic response of the plasma to a wave of frequency ω , we obtain

$$-i\omega m_s \mathbf{v}_s = q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}_0). \quad (2.12)$$

The latter relation describes the fluid elements as some kind of a “jelly” oscillating around a fixed position in space, the motion being forced by the wave. At this point, to continue the calculation, a spatial frame of reference has to be fixed. We define the direction $\hat{\mathbf{u}}_z$ along the confining magnetic field \mathbf{B}_0 : $\mathbf{B}_0 = B_0 \hat{\mathbf{u}}_z$. Elementary algebraic manipulations of Eq. 2.12 yield for the components of the fluid velocity:

$$\begin{cases} -i(v_x + iv_y) \cdot (\omega - \Omega_{cs}) = q_s(E_x + iE_y)/m_s, \\ -i(v_x - iv_y) \cdot (\omega + \Omega_{cs}) = q_s(E_x - iE_y)/m_s, \\ -iv_z\omega = q_s E_z/m_s. \end{cases} \quad (2.13)$$

where we have defined the *cyclotron frequency*, $\omega_{cs} \equiv |q_s|B_0/m_s$, and its signed version, $\Omega_{cs} \equiv \omega_{cs}q_s/|q_s|$. The first two equations of this system hint at the introduction of the rotating electric field and velocity

$$E_\pm \equiv \frac{1}{\sqrt{2}}(E_x \pm iE_y), \quad (2.14)$$

$$v_\pm \equiv \frac{1}{\sqrt{2}}(v_x \pm iv_y), \quad (2.15)$$

to write

$$\begin{cases} -iv_\pm \cdot (\omega \mp \Omega_{cs}) = \frac{q_s}{m_s} E_\pm, \\ -iv_z\omega = \frac{q_s}{m_s} E_z, \end{cases} \quad (2.16)$$

from which the rotating components of the current can be deduced as

$$\begin{cases} j_{\pm} = i \sum_s \frac{n_s q_s^2}{m_s} \cdot \frac{E_{\pm}}{\omega \mp \Omega_{cs}}, \\ j_z = i \sum_s \frac{n_s q_s^2}{m_s} \cdot \frac{E_z}{\omega}. \end{cases} \quad (2.17)$$

In the rotating basis $(\hat{\mathbf{u}}_+, \hat{\mathbf{u}}_-, \hat{\mathbf{u}}_z)$, where $\hat{\mathbf{u}}_{\pm} \equiv (\hat{\mathbf{u}}_x \pm \hat{\mathbf{u}}_y)/\sqrt{2}$, we obtain the elements of the conductivity tensor, defined as $\mathbf{j} \equiv \bar{\bar{\boldsymbol{\sigma}}} \cdot \mathbf{E}$, in the form

$$\bar{\bar{\boldsymbol{\sigma}}} = i\epsilon_0 \sum_s \omega_{ps}^2 \begin{pmatrix} 1/(\omega - \Omega_{cs}) & 0 & 0 \\ 0 & 1/(\omega + \Omega_{cs}) & 0 \\ 0 & 0 & 1/\omega \end{pmatrix}. \quad (2.18)$$

ω_{ps} is the plasma frequency, defined as

$$\omega_{ps}^2 \equiv \frac{n_s q_s^2}{m_s \epsilon_0}. \quad (2.19)$$

The dielectric tensor is readily deduced from relation (1.24). In the basis $(\hat{\mathbf{u}}_+, \hat{\mathbf{u}}_-, \hat{\mathbf{u}}_z)$, it takes the diagonal form

$$\bar{\bar{\boldsymbol{\epsilon}}} = \epsilon_0 \begin{pmatrix} L & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & P \end{pmatrix}, \quad (2.20)$$

where

$$R \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \Omega_{cs})}, \quad (2.21)$$

$$L \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \Omega_{cs})}, \quad (2.22)$$

and

$$P \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}. \quad (2.23)$$

Going back to the lab basis $(\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z)$, we deduce the classical ‘‘Stix’’ form of the dielectric tensor which can be written as[5]

$$\bar{\bar{\boldsymbol{\epsilon}}} = \epsilon_0 \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}, \quad (2.24)$$

having defined

$$S \equiv \frac{1}{2}(R + L) = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_{cs}^2}, \quad (2.25)$$

and

$$D \equiv \frac{1}{2}(R - L) = \sum_s \frac{\omega_{ps}^2}{\omega} \frac{\Omega_{cs}}{\omega^2 - \Omega_{cs}^2}, \quad (2.26)$$

It should be noted that the cold dielectric tensor does not depend on the wave vector. This is the mathematical consequence of the fact that the cold plasma is not spatially dispersive. This feature had been anticipated from physical considerations in section 1.1.3.

2.2.2 Dispersion relation

The general form of the wave equation is (see section 1.2.2):

$$\mathbf{n} \times \mathbf{n} \times \mathbf{E} + \overline{\overline{\mathbf{K}}} \cdot \mathbf{E} = \mathbf{0}. \quad (2.27)$$

The only constraint we have imposed, in the derivation of $\overline{\overline{\boldsymbol{\epsilon}}}$ (and thus $\overline{\overline{\mathbf{K}}}$), is that $\hat{\mathbf{u}}_z$ is determined by the confining magnetic field direction. Since we are in a homogeneous plasma, and that the only source of anisotropy is precisely \mathbf{B}_0 , there is nothing preventing us from assuming that the propagation occurs in the (x, z) plane (see Fig. 2.1)

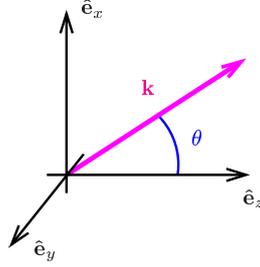


Figure 2.1: Wave propagation in the (x, z) plane.

The refraction index can therefore be decomposed as $\mathbf{n} = n_x \hat{\mathbf{u}}_x + n_z \hat{\mathbf{u}}_z$, and we define the angle θ such as

$$\begin{cases} n_x \equiv n \sin(\theta), \\ n_z \equiv n \cos(\theta). \end{cases} \quad (2.28)$$

Using the terms of the cold dielectric tensor derived above, Eq. 2.27 can thus be written in the form

$$\overline{\overline{\mathbf{M}}}_{\mathbf{k}, \omega} \cdot \mathbf{E} = \begin{pmatrix} S - n^2 \cos^2(\theta) & -iD & n^2 \cos(\theta) \sin(\theta) \\ iD & S - n^2 & 0 \\ n^2 \cos(\theta) \sin(\theta) & 0 & P - n^2 \sin^2(\theta) \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}. \quad (2.29)$$

The associated dispersion relation is by given by the condition $\det(\overline{\mathbf{M}}_{\mathbf{k},\omega}) = 0$, and can readily be cast in the form

$$An^4 - Bn^2 + C = 0, \quad (2.30)$$

with

$$\begin{cases} A = S \sin^2(\theta) + P \cos^2(\theta), \\ B = RL \sin^2(\theta) + PS(1 + \cos^2(\theta)), \\ C = PRL. \end{cases} \quad (2.31)$$

Since a cold plasma is not dispersive (in other words, A , B and C do not depend on \mathbf{n}), (2.30) is a simple quadratic equation for n^2 , whose discriminant

$$B^2 - 4AC = (RL - PS)^2 \sin^4(\theta) + 4P^2 D^2 \cos^2(\theta), \quad (2.32)$$

is always positive. This means that in a cold plasma, we always have two modes, and that these two modes are either purely propagative ($n^2 > 0$) or purely evanescent ($n^2 < 0$). The transition between those two regimes occurs at cut-offs or resonances.

As discussed in section 1.2.3, in many situations, it is useful to write the dispersion relation as a quadratic in n_{\perp}^2 , considering that n_{\parallel} is known. We obtain

$$A'n_{\perp}^4 - B'n_{\perp}^2 + C' = 0, \quad (2.33)$$

with

$$\begin{cases} A' = S, \\ B' = RL + PS - n_{\parallel}^2(P + S), \\ C' = P(n_{\parallel}^2 - R)(n_{\parallel}^2 - L). \end{cases} \quad (2.34)$$

Finally, if we consider that n_{\perp} is fixed, it is natural to write the dispersion relation as a polynomial for n_{\parallel} :

$$A''n_{\parallel}^4 - B''n_{\parallel}^2 + C'' = 0, \quad (2.35)$$

where

$$\begin{cases} A'' = P, \\ B'' = 2PS - n_{\perp}^2(P + S), \\ C'' = (Sn_{\perp}^2 - RL)(n_{\perp}^2 - P). \end{cases} \quad (2.36)$$

2.2.3 Surface of indices

In the plane $(n_{\parallel}^2, n_{\perp}^2)$, where $n_{\parallel} \equiv \mathbf{n} \cdot \mathbf{B}_0/B_0$ and $n_{\perp}^2 \equiv n^2 - n_{\parallel}^2$, it is straightforward to demonstrate that Eq. 2.30 constitutes the equation of an hyperbole[2], called the *dispersion hyperbola*. Its asymptotes are given by

$$n_{\perp}^2 + n_{\parallel}^2 = S + \frac{D^2}{P - S}, \quad (2.37)$$

and

$$n_{\perp}^2 + \frac{P}{S}n_{\parallel}^2 = \frac{P}{S} \left(S - \frac{D^2}{P-S} \right). \quad (2.38)$$

The *surface of indices* is the polar representation of n as a function of the propagation angle θ . We thus obtain a surface of indices for every propagative mode. It can be immediately seen that the first asymptote of the dispersion hyperbola leads to $n^2(\theta) = \text{const}$. If $n^2 > 0$, the corresponding surface of indices is a spheroid and the mode is propagative for any value of the propagation angle θ .

The second asymptote is more complex. To study it, it is useful to rewrite Eq. 2.38 as

$$n^2 \left(\sin^2(\theta) + \frac{P}{S} \cos^2(\theta) \right) = \frac{P}{S} \left(S - \frac{D^2}{P-S} \right) \quad (2.39)$$

If $P/S > 0$, we obtain again a spheroid. On the other hand, if $P/S < 0$, two situations occur:

The mode is propagative in parallel propagation: In this case, $S - D^2/(P - S) > 0$ and the phase velocity is real if and only if

$$\tan^2(\theta) \leq -\frac{P}{S}. \quad (2.40)$$

The surface of indices is a dumbbell-shaped lemniscoid.

The mode is propagative in perpendicular propagation: In this case, $S - D^2/(P - S) < 0$ and the phase velocity is real if and only if

$$\tan^2(\theta) \geq -\frac{P}{S}. \quad (2.41)$$

The corresponding surface of indices is a wheel-shaped lemniscoid.

The three types of surface are shown on Fig. 2.2, where the axis shows the direction of the magnetic field.

It is important to notice that in the two cases with $P/S < 0$, a limiting angle for the propagation direction with respect to the magnetic field, θ_{rc} , appears. Locally, the dispersion surface becomes a cone, and the corresponding wave is called a *resonance cone wave*. This angle is given by the relation

$$\tan(\theta_{rc}) = \sqrt{-\frac{P}{S}}. \quad (2.42)$$

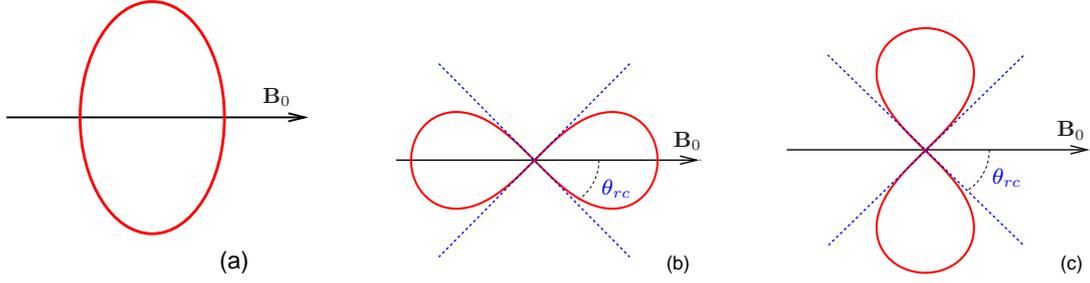


Figure 2.2: Typical surfaces of indices: (a) spheroid; (b) dumbbell lemniscoid; (c) wheel lemniscoid. θ_{rc} is the resonance cone angle.

2.2.4 Group velocity

In a cold plasma, the calculation of the group velocity is generally simple. The easiest way is to reconsider the dispersion relation written in the form:

$$\mathcal{D}(\omega, \mathbf{k}) \equiv An^4 - Bn^2 + C = 0, \quad (2.43)$$

which can be differentiated to obtain

$$d\mathcal{D} = \frac{\partial \mathcal{D}}{\partial \omega} d\omega + \frac{\partial \mathcal{D}}{\partial \mathbf{k}} d\mathbf{k} = 0. \quad (2.44)$$

Recalling the definition of the group velocity $\mathbf{v}_g \equiv \partial\omega/\partial\mathbf{k}$, this yields

$$\mathbf{v}_g = -\frac{\partial \mathcal{D}/\partial \mathbf{k}}{\partial \mathcal{D}/\partial \omega}. \quad (2.45)$$

The angle of propagation of the wave energy (not to be confused with the wave propagation angle), θ_g , is thus given by

$$\tan(\theta_g) = \frac{\partial \mathcal{D}/\partial k_{\perp}}{\partial \mathcal{D}/\partial k_{\parallel}}. \quad (2.46)$$

The propagation angle (direction of propagation of the wave fronts) is

$$\tan(\theta) = \frac{k_{\perp}}{k_{\parallel}}. \quad (2.47)$$

It is therefore possible to write, by using Eqs. 2.33 and 2.35,

$$\frac{\tan(\theta_g)}{\tan(\theta)} = \frac{2Sn_{\perp}^2 - [RL + PS - n_{\parallel}^2(P + S)]}{2Pn_{\parallel}^2 - [2PS - n_{\perp}^2(P + S)]}. \quad (2.48)$$

When the right hand side of this equation is finite, for $\theta = 0$ or $\theta = \pi/2$, we see that $\theta_g = \theta$: the energy propagates in the same direction as the wave fronts. The

case of the resonance cone is interesting: approaching the asymptote, we have (see Eq. 2.42)

$$\frac{n_{\perp}^2}{n_{\parallel}^2} \rightarrow -\frac{P}{S}. \quad (2.49)$$

When the wave approaches a resonance, by definition, we have $n \rightarrow \infty$. This gives $n_{\parallel}^2 = n^2 S / (S - P) \rightarrow \infty$. Performing the substitution $n_{\perp}^2 = -n_{\parallel}^2 P / S$ in Eq. 2.48 and letting $n_{\parallel}^2 \rightarrow \infty$ yields

$$\frac{\tan(\theta_g)}{\tan(\theta)} \rightarrow \frac{S}{P} = -\frac{1}{\tan^2(\theta_{rc})}, \quad (2.50)$$

which shows $\theta_g \rightarrow \theta_{rc} - \pi/2$ when $\theta \rightarrow \theta_{rc}$: the group and phase velocity are perpendicular. In the vicinity of the resonance, the energy of a resonance cone wave tends to propagate perpendicularly to the wave fronts. This case applies to the lower hybrid wave, discussed in further details in section 2.3.3.

2.3 Cold plasma waves

2.3.1 Parallel propagation

When $\theta = 0$, the wave equation (2.29) takes the form

$$\bar{\mathbf{M}}_{\mathbf{k},\omega} \cdot \mathbf{E} = \begin{pmatrix} S - n^2 & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}. \quad (2.51)$$

It is clear from the structure of $\bar{\mathbf{M}}_{\mathbf{k},\omega}$ that the two perpendicular and the parallel components of the electric field are uncoupled.

Langmuir wave

Starting with the parallel component, the wave equation is $PE_z = 0$. The corresponding dispersion relation is simply

$$P = 0 \quad (2.52)$$

Using expression (2.23) yields

$$\omega = \sqrt{\sum_s \omega_{ps}^2}. \quad (2.53)$$

Assuming that the plasma is comprised of only one species of ions (i) and electrons (e), we have $\omega = \sqrt{\omega_{pe}^2 + \omega_{pi}^2}$. This shows that the obtained dispersion relation

is very peculiar: it directly imposes the wave frequency. The phase velocity $v_\phi \equiv \omega/k$ does not have any meaning, in this context (k , the magnitude of the wave vector, is not defined). The group velocity is $v_g \equiv \partial\omega/\partial k = 0$. According to Eq. 2.51, the polarization is such as $E_x = E_y = 0$ and $E_z \neq 0$ (see Fig. 2.3). Therefore, we are in the presence of an oscillation parallel to the confining magnetic field \mathbf{B}_0 . The wave magnetic field is given by Maxwell's equation

$$\mathbf{k} \times \mathbf{E} = \omega \delta \mathbf{B}. \quad (2.54)$$

Since \mathbf{k} and \mathbf{E} are parallel, $\delta \mathbf{B}$ is zero, which characterizes an *electrostatic wave*. The obtained oscillation, called the Langmuir wave is not a propagative phenomenon. It merely corresponds to the response of the charged particles (essentially the electrons, owing to their much larger mobility) to any deviation of the plasma with respect to quasi-neutrality.

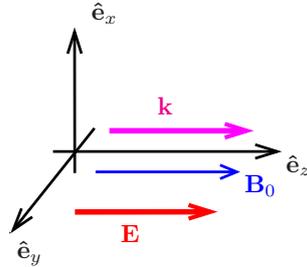


Figure 2.3: Langmuir wave polarization.

Whistlers

Using the definition of the rotating coordinates for the electric field (Eq. 2.14), the two remaining equations of (2.51) can be written in the form

$$\begin{cases} (L - n^2)E_+ = 0 \\ (R - n^2)E_- = 0 \end{cases} \quad (2.55)$$

We obtain two modes, called *whistlers*.

Ion whistler: The dispersion relation is $n^2 = L$. In the presence of a single ion species and of electrons, we have

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega - \omega_{ci})}. \quad (2.56)$$

The cut-off condition $n^2 = 0$ is fulfilled when

$$\omega \equiv \omega_l = \frac{\omega_{ci} - \omega_{ce}}{2} + \sqrt{\left(\frac{\omega_{ci} + \omega_{ce}}{2}\right)^2 + \omega_p^2}, \quad (2.57)$$

with $\omega_p^2 \equiv \omega_{pe}^2 + \omega_{pi}^2$. This cut-off is called the *left cut-off*. We also have a resonance when $\omega = \omega_{ci}$. When $\omega \rightarrow \infty$, $n^2 \rightarrow 1$, which reflects the fact that, as the frequency increases, the plasma becomes progressively transparent to the wave. It propagates as in vacuum, because its frequency is such that the plasma particles are not able to respond to the oscillating field. Finally, for $\omega \rightarrow 0$, we obtain $n^2 \rightarrow c^2/v_a^2$, where the *Alfvén velocity* is defined as

$$v_a \equiv \frac{B_0}{(\mu_0 \sum_s n_s m_s)^{1/2}} \approx \frac{B_0}{(\mu_0 n_i m_i)^{1/2}}. \quad (2.58)$$

These features appear on Fig. 2.4(a), which shows the dispersion relation and the polarization of the ion whistler

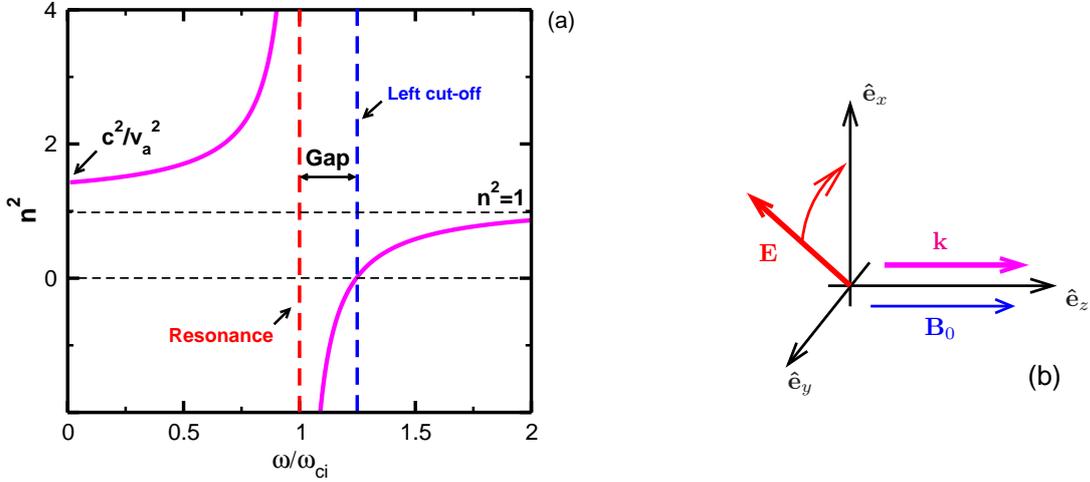


Figure 2.4: Ion whistler: (a) dispersion relation; (b) polarization.

The left cut-off and the resonance for $\omega = \omega_{ci}$ are easily recognized. Between these two layers exists a zone in which the wave is evanescent (denoted “gap” on the figure).

The dispersion relation being $n^2 = L$, the only possibility to verify the wave equation (2.55) is to have $E_- = 0$, or $E_x = iE_y$ and $E_+ \neq 0$. This means that the polarization of the ion whistler is circular, and that the electric field rotates in the same direction as the ions, hence the name. In literature¹, this mode is sometimes called “left whistler” or “L-whistler”. The polarization is shown on Fig. 2.4(b).

The fact that the field rotates in the same direction as the ions is responsible for a very potent interaction occurring between the ion whistlers and the plasma

¹Some authors refer to this wave as a “whistler” only for $\omega \leq \omega_{ci}$. Above the ion cyclotron frequency, they prefer to name it *electromagnetic wave*.

ions around $\omega_{ci} \approx \omega$, and also for the lack of any sensible effect at $\omega \sim \omega_{ce}$. For a detailed study of the wave-particle interaction at the resonance, however, the cold approximation is not sufficient.

Electron whistler: The dispersion relation for this mode is $n^2 = R$, which translates to

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega + \omega_{ci})}. \quad (2.59)$$

We see that $n^2 \rightarrow 0$ when

$$\omega \equiv \omega_r = \frac{\omega_{ce} - \omega_{ci}}{2} + \sqrt{\left(\frac{\omega_{ci} + \omega_{ce}}{2}\right)^2 + \omega_p^2}, \quad (2.60)$$

constituting the *right cut-off*. A resonance occurs at $\omega = \omega_{ce}$. At high frequency, $n^2 \rightarrow 1$ and at low frequency $n^2 \rightarrow c^2/v_a^2$. In fact the features of the electron whistler could have been directly obtained from those of the ion whistler by operating the substitution $\Omega_{ci} \rightarrow \Omega_{ce}$, and vice versa. The obtained dispersion relation is shown on Fig. 2.5(a).

The polarization is deduced by examining the dispersion relation and the wave equation. From Eq. 2.55, it appears that $n^2 = R$ imposes $E_+ = 0$ and $E_- \neq 0$. This mode is thus circularly polarized, but the electric field rotates in the electron direction (see Fig. 2.5).

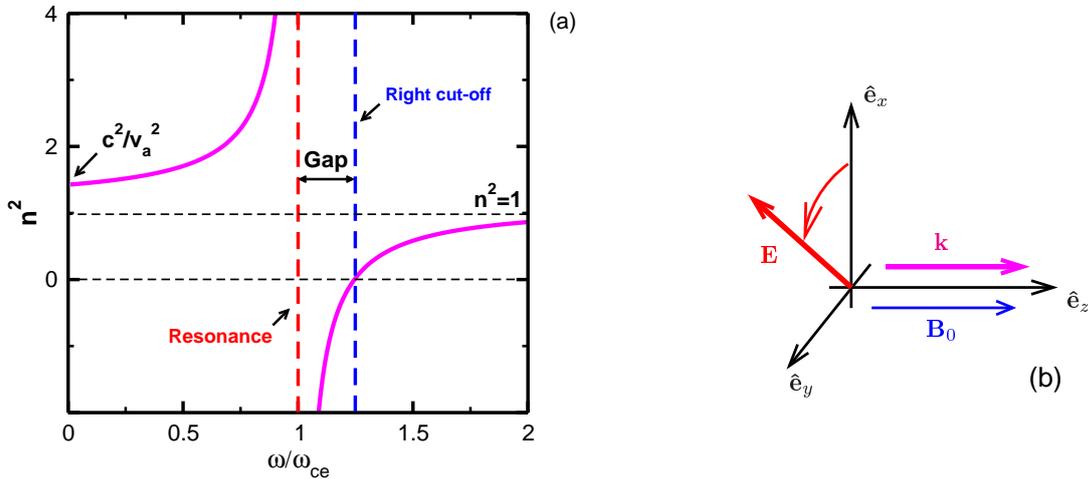


Figure 2.5: Electron whistler: (a) dispersion relation; (b) polarization.

For the same reasons as in the ion whistler case, the electron whistler strongly interacts with the electrons when $\omega_{ce} \approx \omega$.

An interesting feature of this mode can be deduced from its phase and group velocities. In order to simplify the analysis, we assume $\omega_{ci}, \omega_{pi} \ll \omega \ll \omega_{ce} \sim \omega_{pe}$. In this case, Eq. (2.59) can be approximated as

$$n^2 \approx 1 + \frac{\omega_{pe}^2}{\omega\omega_{ce}} - \frac{\omega_{pi}^2}{\omega^2} \approx \frac{\omega_{pe}^2}{\omega\omega_{ce}}. \quad (2.61)$$

The phase velocity can be written as

$$v_\varphi \equiv \frac{\omega}{k} = \frac{c}{n} \approx c \sqrt{\frac{\omega\omega_{ce}}{\omega_{pe}^2}}, \quad (2.62)$$

and the group velocity:

$$v_g \equiv \frac{\partial\omega}{\partial k} \approx 2c \sqrt{\frac{\omega\omega_{ce}}{\omega_{pe}^2}} = 2v_\varphi. \quad (2.63)$$

We see that $v_\varphi, v_g \propto \omega^{1/2}$, which means that the *dispersivity* of this wave is large. Practically, this means that the high frequency components of the wave propagate significantly faster than the low frequency components.

The electron whistlers are plentiful in the earth magnetosphere[6]. They are generally the result of magnetic storms and are guided² by the Van Allen belts magnetic field lines. Placing a radio receiver at the earth magnetic poles, their manifestation can be heard as a “whistling” sound caused by the large dispersion. Furthermore, they are largely responsible for the high electron loss rate observed in the radiation belts: their interaction with the electrons precipitate the latter in the loss cones, where they end up lost in collisions with the earth atmosphere species (see Fig. 2.6)

2.3.2 Perpendicular propagation

When the wave propagates perpendicularly to the magnetic field ($\theta = \pi/2$), the wave equation (2.29) simplifies to give:

$$\begin{cases} SE_x - iDE_y = 0, \\ iDE_x + (S - n^2)E_y = 0, \\ (P - n^2)E_z = 0. \end{cases} \quad (2.64)$$

Again, in this particular case, there is a complete decoupling between the perpendicular and parallel components of \mathbf{E} .

²The commonly employed term is “ducted”.

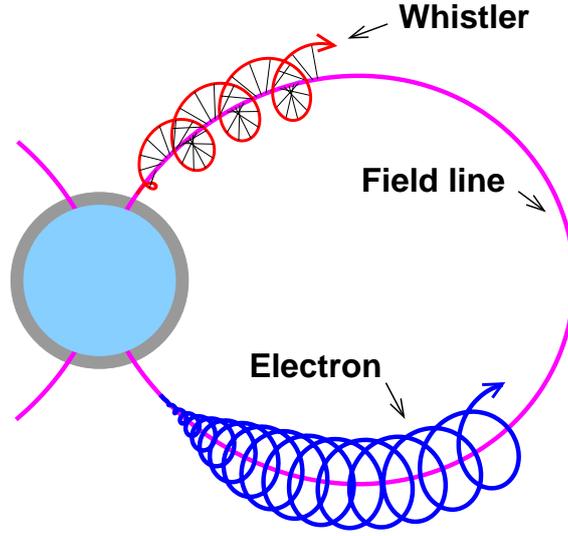


Figure 2.6: Interaction between guided electron whistlers and electrons trapped in the Van Allen belts.

Ordinary wave

The wave equation corresponding to the *ordinary mode* (O-mode) is

$$(P - n^2)E_z = 0, \quad (2.65)$$

with associated dispersion relation

$$n^2 = P = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2} \quad (2.66)$$

It appears that for this mode, $n^2 > 0$ if $\omega^2 > \omega_{pe}^2 + \omega_{pi}^2$: the wave is propagative only if its frequency is larger than the plasma frequency $\omega_p \equiv \sqrt{\omega_{pe}^2 + \omega_{pi}^2}$, which therefore defines a cut-off, named the *plasma cut-off*. For the same reasons as in previous section, we have $n^2 \rightarrow 1$ when $\omega \rightarrow \infty$. The obtained dispersion relation is shown on Fig. 2.7(a)

The polarization is obtained by re-injecting the dispersion relation in the wave equation, which yields $E_x = E_y = 0$, $E_z \neq 0$ (see Fig. 2.7(b)).

The electric field is aligned with the confining magnetic field, and is purely transverse, i.e. $\mathbf{E} \cdot \mathbf{k} = 0$. Physically, it is clear that since the particles oscillate along the direction defined by \mathbf{B}_0 , their motion is not influenced by the static magnetic field.

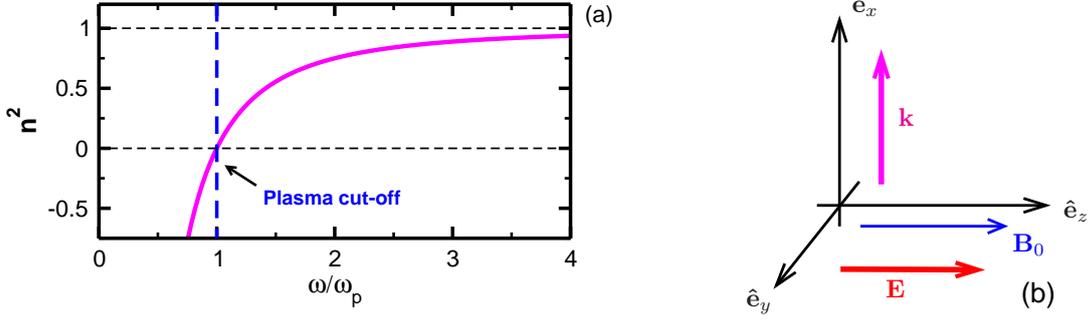


Figure 2.7: Ordinary wave: (a) dispersion relation; (b) polarization.

Extraordinary mode

The wave equation corresponding to the *extraordinary* mode (X-mode) is

$$\begin{cases} SE_x - iDE_y = 0 \\ iDE_x + (S - n^2)E_y = 0. \end{cases} \quad (2.67)$$

The dispersion relation can be written in the form

$$n^2 = \frac{RL}{S}. \quad (2.68)$$

Its analysis is slightly more complicated than for the other modes studied so far. Two cut-offs are obtained, defined by the relations

$$\omega = \pm \frac{\omega_{ce} - \omega_{ci}}{2} + \sqrt{\left(\frac{\omega_{ci} + \omega_{ce}}{2}\right)^2 + \omega_p^2}. \quad (2.69)$$

Two cold resonances also exist, given by

$$\omega^2 = \frac{\omega_e + \omega_i}{2} \pm \sqrt{\left(\frac{\omega_e^2 - \omega_i^2}{2}\right)^2 + \omega_{pe}^2 \omega_{pi}^2}, \quad (2.70)$$

where

$$\omega_{e,i}^2 \equiv \omega_{pe,pi}^2 + \omega_{ce,ci}^2. \quad (2.71)$$

The resonance with the lowest frequency is called *lower hybrid resonance* (ω_{lh}), the other one is known as the *upper hybrid resonance* (ω_{uh}). At low frequency, we obtain, as for the whistlers, $n^2 \rightarrow c^2/v_a^2$ and when $\omega \rightarrow \infty$, $n \rightarrow 1$. The field polarization is given by $E_z = 0$ and

$$\frac{E_x}{E_y} = i \frac{R - L}{R + L}. \quad (2.72)$$

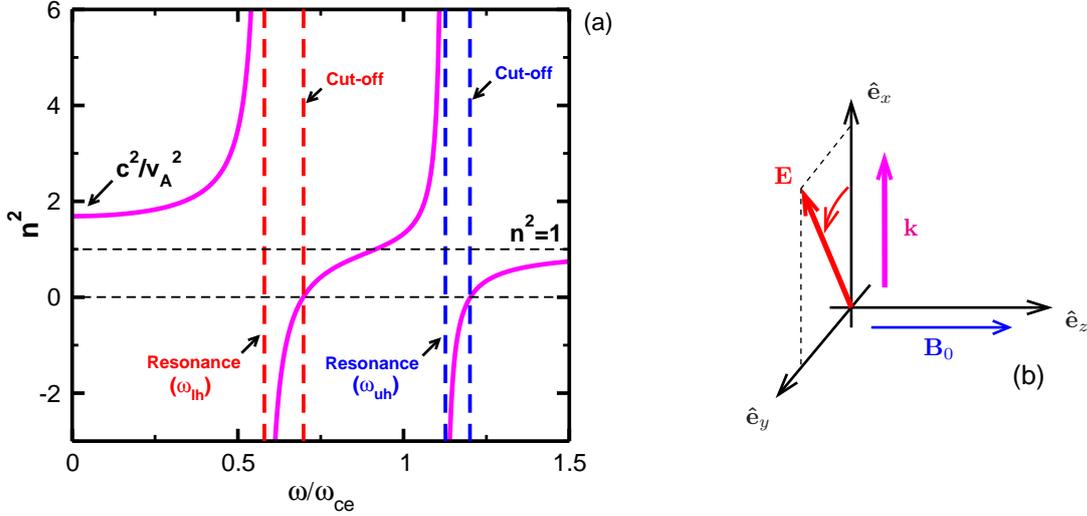


Figure 2.8: Extraordinary wave: (a) dispersion relation; (b) polarization.

Therefore, this mode has an elliptic polarization such as $\mathbf{E} \perp \mathbf{B}_0$. The dispersion relation and the mode polarization are shown on Fig. 2.8

A thorough analysis of the X-mode polarization reveals an interesting property. Indeed, when $\omega \rightarrow \omega_{ci}$, we observe that $L \rightarrow \infty$, and therefore $E_x/E_y \rightarrow -i$. If we consider the rotating coordinates of the field, this corresponds to $E_+ \rightarrow 0$: the electric field component rotating in the same direction as the ions vanishes at the ion cyclotron resonance. Likewise, when $\omega \rightarrow \omega_{ce}$, we obtain $R \rightarrow \infty$, which corresponds to $E_- \rightarrow 0$. At the electron cyclotron resonance, the field component rotating in the electron direction also vanishes. This phenomenon is called *screening effect* and results in a significantly reduced interaction between the wave and the particles at resonances. This feature is crucial in the context of magnetic fusion plasma heating by ion cyclotron waves. It is worthwhile mentioning that the only wave which are not subject to the screening effect are the whistlers.

2.3.3 Lower hybrid wave

In the previous section, the particular cases of parallel and perpendicular propagation have been examined. This choice was motivated by the fact that in both cases, the two modes were naturally separated. The case with θ arbitrary does not pose any conceptual difficulty, but the obtained expressions are significantly less convenient. The general conclusions of our analysis remain valid: there are always two distinct modes in a cold plasma. Actually, a refined analysis[4] shows that when the propagation angle varies from perpendicular to parallel, the X-mode transforms in one of the two whistlers (depending on the parameters, this can be either the L- or the R-whistler) and that the O-mode (or the resonance cone mode) becomes

the other whistler. Therefore, in fact, depending on the plasma parameters and the propagation angle, the modes are called “L-whistler” and “R-whistler”, “ordinary wave” and “extraordinary wave”, “slow wave”, “fast wave”... It should be kept in mind that these terms actually merely refer to the two branches of the cold dispersion relation.

An interesting case, which lends itself to an analytical treatment even when the propagation angle is arbitrary, is encountered in the intermediate range of frequencies:

$$\omega_{ci}, \omega_{pi} \ll \omega \ll \omega_{ce}, \omega_{pe}. \quad (2.73)$$

The elements of the dielectric tensor can be approximated as

$$R \approx 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} + \frac{\omega_{pe}^2}{\omega_{ce}\omega} - \frac{\omega_{pi}^2}{\omega(\omega + \omega_{ci})}, \quad (2.74)$$

$$L \approx 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{\omega_{pe}^2}{\omega_{ce}\omega} - \frac{\omega_{pi}^2}{\omega(\omega - \omega_{ci})}, \quad (2.75)$$

$$P \approx 1 - \frac{\omega_{pe}^2}{\omega^2}, \quad (2.76)$$

and

$$S = \frac{1}{2}(R + L) \approx 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2}, \quad (2.77)$$

$$D = \frac{1}{2}(R - L) \approx \frac{\omega_{pe}^2}{\omega_{ce}\omega} - \frac{\omega_{ci}}{\omega} \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2}. \quad (2.78)$$

On Fig. 2.9(a) appears the dispersion hyperbola obtained with $\omega_{pe}^2/\omega_{ce}^2 = 0.01$ and $\omega/\omega_{ce} = 0.02$, in a hydrogen plasma. Also shown are the asymptotes, whose expressions are given by Eqs. 2.37 and 2.38.

This allows to identify the whistlers (for $n_{\perp} = 0$), already discussed in section 2.3.1. There is a supplementary interesting feature: one of the branches follows the asymptote corresponding to the resonance cone, since we are in a situation where $-P/S > 0$. Indeed, in this frequency domain, it is readily seen that $R \sim -L$, $S \sim 1$, $|D| \sim |R|, |L|$ and $-P \gg |R|, |L|, |S|, |D|$. The dispersion relation 2.33 can then be factorized in the approximate form

$$(n_{\perp}^2 - n_{\perp,F}^2)(n_{\perp}^2 - n_{\perp,S}^2) = 0, \quad (2.79)$$

with

$$n_{\perp,F}^2 \equiv -\frac{(n_{\parallel}^2 - R)(n_{\parallel}^2 - L)}{n_{\parallel}^2 - S}, \quad (2.80)$$

and

$$n_{\perp,S}^2 \equiv -\frac{P}{S}(n_{\parallel}^2 - S). \quad (2.81)$$

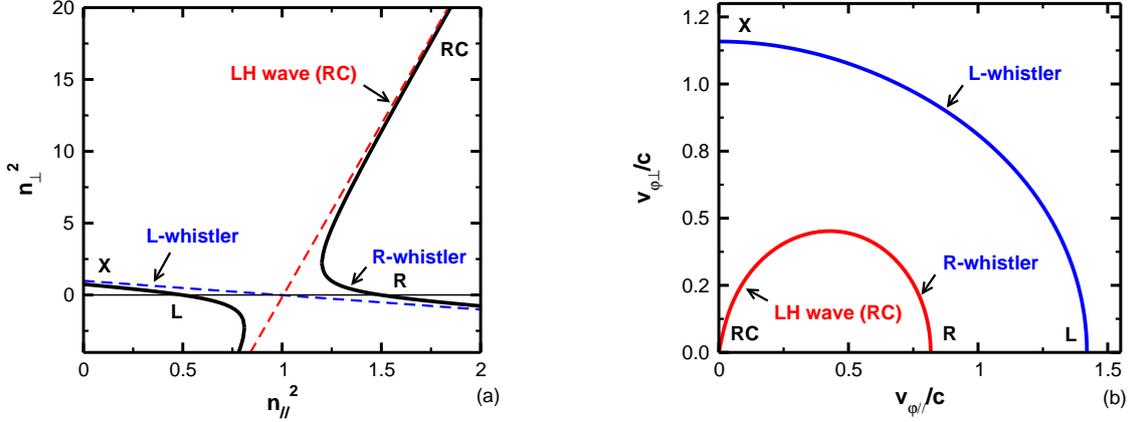


Figure 2.9: (a) Dispersion hyperbola; (b) Phase velocity surfaces for a hydrogen plasma with $\omega_{pe}^2/\omega_{ce}^2 = 0.01$ and $\omega/\omega_{ce} = 0.02$. “R” and “L” refer to the right and left whistler, respectively. “RC” denotes the cone resonance mode and “X” is the extraordinary wave.

S and F mean “slow” and “fast”. Generally, $n_{\perp,S} \gg n_{\parallel,F}$, and the corresponding phase velocity $v_\varphi = c/n$ of the *slow wave* is thus smaller than the phase velocity for the *fast wave*, hence their names. This can be seen on Fig. 2.9(b) where the corresponding surface of indices appears. Two branches can be seen: one corresponds to the L-whistler and X-mode, the other one to the R-whistler and the resonance cone. We also observe that $n_{\perp,S} \rightarrow \infty$ when $S \rightarrow 0$. This is the lower hybrid resonance, already discussed in section 2.3.2. In the vicinity of the resonance, it is reasonable to assume that $n_{\parallel}^2 \gg S$ and we obtain the simplified dispersion relation

$$n_{\perp}^2 \approx -n_{\parallel}^2 \frac{P}{S}. \quad (2.82)$$

Using the fact that $\omega_{pi} \ll \omega$, we deduce

$$n_{\perp}^2 \approx n_{\parallel}^2 \frac{m_i}{Z_i m_e} \frac{\omega_{lh}^2}{\omega^2 - \omega_{lh}^2}, \quad (2.83)$$

where Z_i is the ion charge number and m_i the ion mass. The *lower hybrid frequency*, defined by $S(\omega_{lh}) = 0$ is approximately given by

$$\omega_{lh}^2 \approx \frac{\omega_{pi}^2}{1 + \omega_{pe}^2/\omega_{ce}^2}. \quad (2.84)$$

An interesting feature of the lower hybrid wave is its polarization. Injecting the approximate dispersion relation in the wave equation gives

$$\frac{E_x}{E_z} \approx \frac{n_{\perp}}{n_{\parallel}} = \sqrt{\frac{-P}{S}}, \quad (2.85)$$

Since, by definition, $n_{\perp} = \mathbf{n} \cdot \hat{\mathbf{u}}_x$, this shows that \mathbf{E} is parallel to \mathbf{k} , and the wave thus becomes electrostatic as it approaches the resonance. This is actually a general feature of waves near cold resonances[7]. We can see that in this case of arbitrary propagation angle, we recover the L- and R-whistlers, and the resonance cone. By analogy with the results obtained in parallel propagation, it appears that the latter belongs to the same dispersion branch as the Langmuir wave, which is also electrostatic. Unlike the Langmuir oscillation, however, the wave is here propagative.

2.3.4 Alfvén waves

Another case with θ arbitrary lending itself to an analytical treatment is the MHD limit, which consists of assuming $\omega \ll \omega_{ci}$. In this situation, the cold dielectric tensor terms can be simplified as:

$$L \approx 1 + \frac{\omega_{pi}^2}{\omega_{ci}} \frac{1}{(\omega + \omega_{ci})} \approx 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} = 1 + \frac{c^2}{v_a^2}, \quad (2.86)$$

$$R \approx 1 - \frac{\omega_{pi}^2}{\omega_{ci}} \frac{1}{(\omega - \omega_{ci})} \approx 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} = 1 + \frac{c^2}{v_a^2} \quad (2.87)$$

where use has been made of the plasma quasi-neutrality, which can be written as $\omega_{pe}^2/\omega_{ce} = \omega_{pi}^2/\omega_{ci}$. Also, we have

$$S \approx 1 - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} \approx 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} = 1 + \frac{c^2}{v_a^2}, \quad (2.88)$$

$$D \approx \frac{\omega_{pi}^2}{\omega_{ci}} \frac{\omega}{\omega^2 - \omega_{ce}^2} \approx -\frac{\omega}{\omega_{ci}} \frac{c^2}{v_a^2} \quad (2.89)$$

and

$$P \approx -\frac{\omega_{pe}^2}{\omega^2} \gg 1. \quad (2.90)$$

The third line of the wave equation (2.29) gives to lowest order

$$E_z = 0. \quad (2.91)$$

Using the expressions derived above, which show that $|D| \ll S$, we obtain the approximate wave equation:

$$\begin{cases} \left(-n^2 \cos^2(\theta) + 1 + \frac{c^2}{v_a^2} \right) E_x = 0, \\ \left(-n^2 + 1 + \frac{c^2}{v_a^2} \right) E_y = 0. \end{cases} \quad (2.92)$$

Torsional Alfvén wave

The first equation of (2.92) leads to the dispersion relation

$$n^2 \cos^2(\theta) = 1 + \frac{c^2}{v_a^2}, \quad (2.93)$$

and the associated polarization is such as $E_x \neq 0$, and $E_y = E_z = 0$. It is interesting to notice that the associated magnetic perturbation is

$$\delta \mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}}{\omega} = k \cos(\theta) E_x \hat{\mathbf{u}}_y, \quad (2.94)$$

so that this perturbation acts perpendicularly to the confining magnetic field. Therefore, it induces a *torsion* of the magnetic field lines and is called *torsional Alfvén wave* (see Fig. 2.10(a)).

The phase velocity is

$$v_\varphi = \frac{\omega}{k} = \frac{c}{\sqrt{1 + c^2/v_a^2}} \cos(\theta) \approx v_a \cos(\theta), \quad (2.95)$$

if we assume that $v_a \ll c$, which, in most cases of practical interest, constitutes a reasonable approximation.

The group velocity is

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \frac{c}{\sqrt{1 + c^2/v_a^2}} \hat{\mathbf{u}}_z \approx v_a \hat{\mathbf{u}}_z, \quad (2.96)$$

showing that the energy flows along the field lines at the Alfvén velocity. It is also of interest to examine the motion of the particles influenced by the wave. The linearized equation of motion for ions

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{q_i}{m_i} (\mathbf{v} \times \mathbf{B}_0 + \mathbf{E}), \quad (2.97)$$

can be written as

$$i\omega \mathbf{v} + \frac{q_i}{m_i} \mathbf{v} \times \mathbf{B}_0 + \frac{q_i}{m_i} \mathbf{E} = 0, \quad (2.98)$$

The wave frequency is such as $\omega \ll \Omega_{ci}$, meaning that the magnitude of the first term on the left hand side of this equation, $\omega|\mathbf{v}|$, is much smaller than the magnitude of the second one, $\Omega_{ci}|\mathbf{v}|$. Therefore, the ion drift velocity is given by

$$\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B}_0}{B_0^2} \approx -\frac{E_x}{B_0} \hat{\mathbf{u}}_y. \quad (2.99)$$

Both the ions and electrons follow the magnetic field perturbation (no charge separation occurs, since the latter relation does not depend on the charge sign). Actually, this is not a surprise, since in ideal MHD, the particles are known to be “frozen” in the magnetic field lines. The polarization, the phase velocity, the group velocity and the particles velocity directions are shown on Fig. 2.10(b)

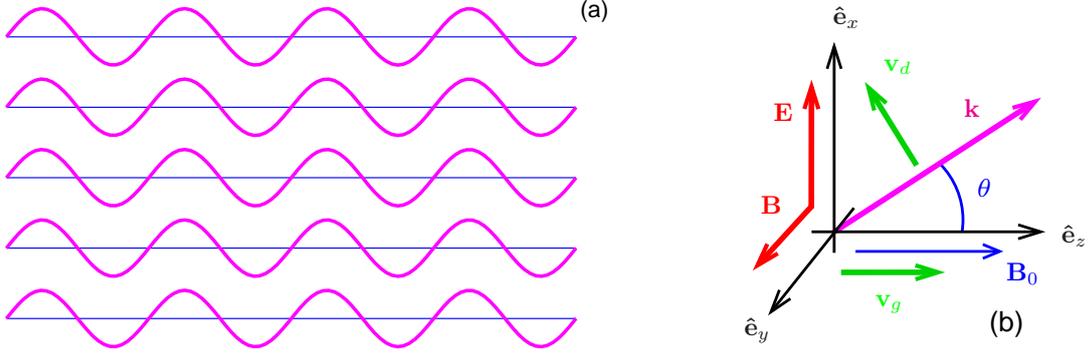


Figure 2.10: (a) Torsion of the magnetic field lines under the effect of the torsional Alfvén wave. (b) Corresponding polarization.

Compressional Alfvén wave

The second equation of (2.92) leads to the following dispersion relation:

$$n^2 = 1 + \frac{c^2}{v_a^2}, \quad (2.100)$$

with associated polarization $E_x = E_z = 0$, $E_y \neq 0$. The magnetic perturbation is given by

$$\delta\mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}}{\omega}, \quad (2.101)$$

which shows that $\delta\mathbf{B}$ lies in the plane $(\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_z)$. For a normal propagation angle, $\theta = \pi/2$, we obtain $\delta\mathbf{B} \parallel \mathbf{B}_0$. This time, the perturbation thus tends to compress the magnetic field lines (Fig. 2.11(b)), and this mode is therefore commonly referred to as the *compressional Alfvén wave*.

The phase velocity is

$$v_\varphi = \frac{c}{\sqrt{1 + c^2/v_a^2}} \approx v_a, \quad (2.102)$$

and the group velocity

$$\mathbf{v}_g = \frac{c}{\sqrt{1 + c^2/v_a^2}} \frac{\mathbf{k}}{k} \approx v_a \frac{\mathbf{k}}{k}. \quad (2.103)$$

The ion drift velocity is given by

$$\mathbf{v}_d \approx \frac{E_y}{B_0} \hat{\mathbf{u}}_x. \quad (2.104)$$

The associated polarization and velocities are shown on Fig. 2.11(b).

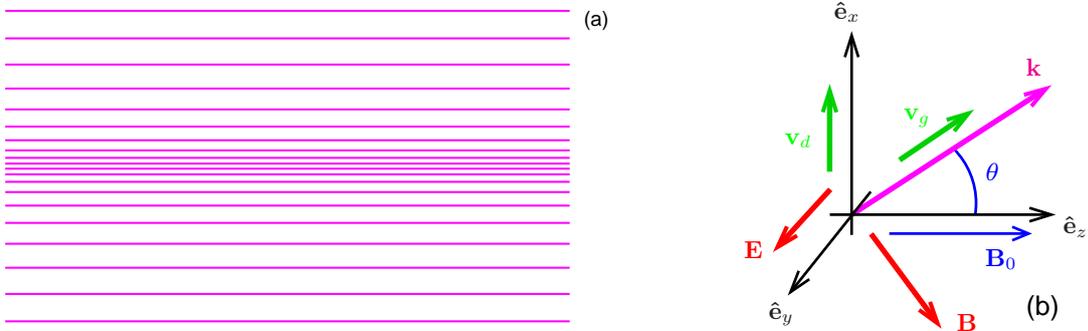


Figure 2.11: (a) Compression of the magnetic field lines under the effect of the compressional Alfvén wave. (b) Corresponding polarization.

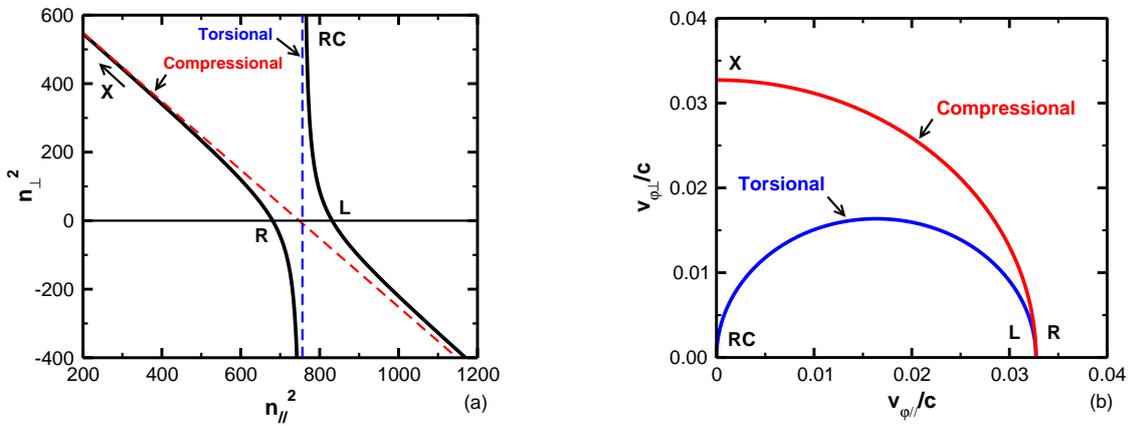


Figure 2.12: Hydrogen plasma with $\omega/\omega_{ci} = 0.1$ and $\omega_{pe}^2/\omega_{ce}^2 = 0.4$. (a) Dispersion hyperbola; (b) Phase velocity surface. “R” and “L” respectively refer to the right and left whistlers. “RC” is the resonance cone mode and “X” is the extraordinary mode.

In order to summarize the features of the two cold Alfvén waves, the corresponding dispersion hyperbolas and phase velocity surfaces are shown on Figs. 2.12(a) and (b)

It can be observed that the torsional Alfvén wave belongs to the same branch as the L-whistler in parallel propagation, and as the resonance cone in perpendicular propagation. The compressional wave belongs to the R-X branch. It can be noted that the resonance cone presents a vertical asymptote, which is associated to the fact that the dispersion relation of the torsional wave does not constrain n_{\perp} , but does impose a finite value to n_{\parallel} .

2.4 Thermal corrections

2.4.1 Finite pressure model

If we are interested in refining our plasma model to include finite temperature effects, we obviously need to consider a non-zero pressure tensor in Eq. 2.10. The simplest model we can employ is an isotropic pressure, which translates into $\bar{\mathbf{p}}_s = p_s \bar{\mathbf{1}}$. Alternatively, we may be willing to consider a pressure exerting only along the field lines, whose direction is $\hat{\mathbf{u}}_z$. In this case, we will have $\bar{\mathbf{p}}_s = p_s \hat{\mathbf{u}}_z \hat{\mathbf{u}}_z$. But no matter which of these options we consider, an expression for p_s is still required. More specifically, in the context of our linear problem, we write $p_s \equiv p_{s,0} + \delta p_s$, with $p_{s,0} = n_{s,0} k_b T_s$. $k_b \approx 1.38065 \times 10^{-23} \text{J/K}$ is the Boltzmann constant. T_s is the temperature of species s , which is assumed to be given. In this lecture, we will limit ourselves to two possibilities:

Isothermal model: This model applies when the density “follows” the fluid, whereas the temperature remains constant along the magnetic field lines. This situation occurs, for instance when the ions motion determine the fluid motion, while the electrons travel back and forth along the field line in order to ensure the plasma quasi-neutrality, i.e. are not “trapped” in the fluid ensemble motion. From this argument, we expect this law to be valid when the electrons velocity is large. More precisely, from the wave point of view, it will hold if the electrons, assumed to be traveling at thermal speed $v_{th,e}$, have a transit time on one wavelength $\tau_e \sim \lambda/v_{th,e}$ much smaller than the wave period $2\pi/\omega$. This condition can be written $v_{\phi} \ll v_{th,e}$, and places an upper limit on the wave frequency. In this case, the total pressure is given by:

$$p_s = n_s k_b T_s, \quad (2.105)$$

which implies for the pressure perturbation

$$\delta p_s = \delta n_s k_b T_s. \quad (2.106)$$

Adiabatic model: This model consists of considering that both the density and temperature accompany the fluid motion. The range of application of this model will encompass the situation in which the particles do not diffuse significantly on the wave period. If we consider again the case of electrons, this means that the adiabatic condition will hold true provided that $v_\varphi \gg v_{th,e}$. In this case, the total pressure verifies the condition

$$\frac{p_s}{n_s^{\gamma_s}} = \text{const.} \quad (2.107)$$

with γ_s the ratio of specific heats. Linearizing the pressure, we can readily obtain the expression for the first order perturbation as

$$\delta p_s = \gamma_s p_{s,0} \frac{\delta n_s}{n_{s,0}} \quad (2.108)$$

The choice of either model will essentially be determined by the wave frequency, wavenumber, and the fluid temperature.

2.4.2 Electrostatic waves

In this section, we will only be considering electrostatic waves, i.e. the wave electric field is given by the relation $\mathbf{E} = -\nabla\varphi$ (φ is the electrostatic potential). In Fourier space, this can be written as $\mathbf{E}_{\omega,\mathbf{k}} = -i\mathbf{k}\varphi_{\omega,\mathbf{k}}$: the wave electric field is parallel to the wave vector. Maxwell's equation $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ shows that the magnetic field perturbation is zero for an electrostatic wave. We will also assume that the plasma pressure induces a force in the parallel direction: $\overline{\mathbf{p}}_s = p_s \hat{\mathbf{u}}_z \hat{\mathbf{u}}_z$.

Adiabatic electrons and cold ions

We start by considering that only the electrons are warm (but not hot), and that the ions are cold. In other words, we place ourselves in a regime in which $v_{th,e} \ll \omega/k$. From the section 2.4.1, it appears that this corresponds to the range of applicability of the adiabatic model for the electrons.

From the density continuity equation (2.9) applied to electrons and ions, we can write the corresponding perturbations as

$$\delta n_e = n_{e,0} \frac{\mathbf{k} \cdot \delta \mathbf{v}_e}{\omega}, \quad \delta n_i = n_{i,0} \frac{\mathbf{k} \cdot \delta \mathbf{v}_i}{\omega}. \quad (2.109)$$

To first order, the momentum equation (2.9) applied to electrons yields

$$-in_{e,0}m_e\omega\delta\mathbf{v}_e = -en_{e,0}(\mathbf{E} + \delta\mathbf{v}_e \times \mathbf{B}_0) - i\delta p_e(\mathbf{k} \cdot \hat{\mathbf{u}}_z)\hat{\mathbf{u}}_z. \quad (2.110)$$

Dotting this equation with \mathbf{k} gives

$$-in_{e,0}m_e\omega\mathbf{k}\cdot\delta\mathbf{v}_e = -en_{e,0}\mathbf{E}\cdot\mathbf{k} - en_{e,0}\mathbf{k}\cdot(\delta\mathbf{v}_e\times\mathbf{B}_0) - i\delta p_e(\mathbf{k}\cdot\hat{\mathbf{u}}_z)^2. \quad (2.111)$$

In order to simplify the calculation, let us restrict ourselves to the case of a parallel propagation: $\mathbf{k}\equiv k_{\parallel}\hat{\mathbf{u}}_z$. We obtain

$$-in_{e,0}m_e\omega k_{\parallel}\delta v_{e,z} = -en_{e,0}k_{\parallel}E - i\delta p_e k_{\parallel}^2. \quad (2.112)$$

Using the expression for δp_e for adiabatic electrons (2.108) and expressions (2.109), we obtain for the velocity perturbation

$$\delta v_{e,z} = -ie\frac{\omega}{m_e}\frac{E}{\omega^2 - \gamma k_{\parallel}^2 k_b T_e/m_e}. \quad (2.113)$$

For the cold ions with charge number Z_i , Eq. 2.9 with $\bar{\mathbf{p}}_i = \bar{\mathbf{0}}$ yields

$$-im_i\omega\delta\mathbf{v}_i = Z_i e(\mathbf{E} + \delta\mathbf{v}_i\times\mathbf{B}_0). \quad (2.114)$$

Dotting this relation with \mathbf{k} leads to

$$\delta v_{i,z} = iZ_i e\frac{E}{\omega m_i}. \quad (2.115)$$

Rather than following the usual procedure and deduce the dielectric tensor, it is more convenient to use Maxwell's equations directly to obtain another relation linking the electric field to the wave perturbation. In electrostatic problems, it is usual to use Poisson's equation

$$\nabla\cdot\mathbf{E} = \sum_s \frac{\rho_s}{\epsilon_0}. \quad (2.116)$$

Written to first order and in parallel propagation, we obtain

$$ik_{\parallel}E = \frac{e}{\epsilon_0}(Z_i\delta n_i - \delta n_e). \quad (2.117)$$

Hence, using (2.109)

$$ik_{\parallel}E = \frac{en_{e,0}}{\epsilon_0\omega}k_{\parallel}(\delta v_{i,z} - \delta v_{e,z}). \quad (2.118)$$

Injecting Eqs. 2.113 and 2.115 in the latter expression yields

$$E\left[1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2 - \gamma k_{\parallel}^2 k_b T_e/m_e}\right] = 0, \quad (2.119)$$

and the dispersion relation is

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2 - \gamma k_{\parallel}^2 k_b T_e / m_e} = 0. \quad (2.120)$$

It is known as the *Bohm-Gross* dispersion relation. In order to investigate the features of this wave, we notice that the adiabatic model requires $(\omega/k_{\parallel})^2 \gg k_b T_e / m_e$. We can thus approximate (2.120) as

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2} \left(1 + \gamma \frac{k_{\parallel}^2}{\omega^2} \frac{k_b T_e}{m_e} \right) = 0. \quad (2.121)$$

Setting $T_e = 0$ in this expression, we obtain the corresponding cold plasma wave:

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2, \quad (2.122)$$

which is the Langmuir wave already discussed before (see, e.g., section 2.3.1). Again, the adiabatic model imposes $\gamma k_{\parallel}^2 / \omega^2 k_b T_e / m_e \ll 1$. We can therefore obtain an approximate solution for (2.121)

$$\omega^2 \approx \omega_{pe}^2 + \frac{3}{2} k_{\parallel}^2 v_{th,e}^2, \quad (2.123)$$

where we have used $\omega_{pi} \ll \omega_{pe}$ to neglect the ion contribution and $v_{th,e}^2 = 2k_b T_e / m_e$. Also, by assuming that the electron pressure was along $\hat{\mathbf{u}}_z$, we have restricted ourselves to one degree of freedom ($n = 1$), which gives $\gamma = (n + 2)/n = 3$. This dispersion relation is reminiscent of the cold Langmuir wave dispersion, with a thermal correction. The most important feature of this correction is that unlike in the cold case, the resulting wave is now propagative: v_{ϕ} is defined and the group velocity is small, but non zero: $v_g \ll v_{th,e}$.

Isothermal electrons and adiabatic ions

A second interesting case is to consider that the ions are warm and that the electrons are hot. This means that we consider a regime of frequencies such as $v_{th,i} \ll \omega/k \ll v_{th,e}$. From section 2.4.1, the adequate models are isothermal for electrons and adiabatic for ions.

The density perturbations δn_e and δn_i are still given by Eqs. 2.109. The electron momentum equation is given by

$$-in_{e,0} m_e \omega \delta \mathbf{v}_e = -en_{e,0} (\mathbf{E} + \delta \mathbf{v}_e \times \mathbf{B}_0) - i\delta p_e (\mathbf{k} \cdot \hat{\mathbf{u}}_z) \hat{\mathbf{u}}_z, \quad (2.124)$$

but now, we must use $\delta p_e = \delta n_e k_b T_e$. Following the exact same steps as in the adiabatic electrons case, and assuming that the propagation is parallel, we obtain

$$\delta v_{e,z} = -ie \frac{\omega}{m_e} \frac{E}{\omega^2 - k_{\parallel}^2 k_b T_e / m_e}. \quad (2.125)$$

For the ions, the linearized momentum equation is

$$-in_{i,0}m_i\omega\delta\mathbf{v}_i = Z_i en_{i,0}(\mathbf{E} + \delta\mathbf{v}_i \times \mathbf{B}_0) - i\delta p_i(\mathbf{k} \cdot \hat{\mathbf{u}}_z)\hat{\mathbf{u}}_z. \quad (2.126)$$

We use $\delta p_i = \gamma p_{i,0}\delta n_i/n_{i,0}$ to obtain

$$\delta v_{i,z} = -ie\frac{\omega}{m_e}\frac{E}{\omega^2 - \gamma k_{\parallel}^2 k_b T_i/m_i}, \quad (2.127)$$

with T_i the ion temperature. Using Eqs. 2.109, 2.125 and 2.113 in Poisson's equation (2.116) leads to the wave equation

$$E \left[1 - \frac{\omega_{pi}^2}{\omega^2 - \gamma k_{\parallel}^2 k_b T_i/m_i} - \frac{\omega_{pe}^2}{\omega^2 - k_{\parallel}^2 k_b T_e/m_e} \right] = 0, \quad (2.128)$$

and the corresponding dispersion relation is

$$1 - \frac{\omega_{pi}^2}{\omega^2 - \gamma k_{\parallel}^2 k_b T_i/m_i} - \frac{\omega_{pe}^2}{\omega^2 - k_{\parallel}^2 k_b T_e/m_e} = 0. \quad (2.129)$$

Once again, if we set $T_i = T_e = 0$ to recover the cold plasma case, we obtain the dispersion relation for the Langmuir wave. This is not surprising since in parallel propagation, this is the only electrostatic wave. Also, if we neglect the ions contributions, we obtain a dispersion relation describing Langmuir waves with thermal corrections

$$\omega^2 = \omega_{pe}^2 + k_{\parallel}^2 \frac{k_b T_e}{m_e}. \quad (2.130)$$

The isothermal we have used for the electrons implies that $\omega^2 \ll k_{\parallel}^2 k_b T_e/m_e$. On the other hand, the adiabatic ions verify the condition $\omega^2 \gg k_{\parallel}^2 k_b T_i/m_i$. Using these two conditions in Eq. 2.129, we can obtain the approximate form

$$\omega^2 \approx \frac{k_{\parallel}^2}{\omega_{pe}^2} \left(\omega_{pi}^2 \frac{k_b T_e}{m_e} + \omega_{pe}^2 \gamma \frac{k_b T_i}{m_i} \right). \quad (2.131)$$

Noting that $\omega_{pi}^2/\omega_{pe}^2 = Z_i m_e/m_i$, we can also write

$$\omega^2 \approx k_{\parallel}^2 \left(\frac{Z_i k_b T_e + \gamma k_b T_i}{m_i} \right). \quad (2.132)$$

The quantity between parentheses is the square of the *sound velocity*

$$c_s^2 \equiv \frac{Z_i k_b T_e + \gamma k_b T_i}{m_i}, \quad (2.133)$$

which shows that the dispersion relation can approximately be written as $\omega = k_{\parallel} c_s$. It is the dispersion relation for the *ion acoustic wave*. This wave has a behavior

which presents similarities with sound waves propagating in a gas. In the latter case, the kinetic pressure is the force opposing the longitudinal compression. In our case, this role is played by the ion and electron pressure (the latter is usually dominant). The inertia is provided by the ion mass. An interesting feature of this case is that the finite temperature effects have introduced a new mode, which was absent in a cold plasma. The appearance of one or several new modes in addition to the (possibly modified) cold plasma waves is caused by the space dispersion related to thermal corrections.

2.4.3 High frequency electromagnetic waves

From previous discussions, it is clear that only electrons are able to react to a wave in the range of frequency $\omega \gg \omega_{ci}, \omega_{pi}$. The linearized momentum continuity equation appears as

$$\mathbf{v}_e = -i \frac{e}{\omega m_e} (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}_0) + \frac{v_{th,e}^2}{\omega^2} (\mathbf{k} \cdot \mathbf{v}_e) \cdot \mathbf{k}. \quad (2.134)$$

For electromagnetic waves, all three components of the electric field are needed. It is therefore necessary to express the electron fluid velocity components as functions of the electric field components, deduce the conductivity tensor $\overline{\boldsymbol{\sigma}}$ and the dielectric tensor $\overline{\boldsymbol{\epsilon}}$. This calculation, which is not detailed here but does not pose any particular difficulty, yields

$$\epsilon_{xx}/\epsilon_0 = 1 - \frac{\omega_{pe}^2 (\omega^2 - k^2 v_{th,e}^2 \cos^2(\theta))}{\omega^2 (\omega^2 - k^2 v_{th,e}^2) - \omega_{ce}^2 (\omega^2 - k^2 v_{th,e}^2 \cos^2(\theta))}, \quad (2.135)$$

$$\epsilon_{xy}/\epsilon_0 = i \frac{\omega_{ce}}{\omega} \frac{\omega_{pe}^2 (\omega^2 - k^2 v_{th,e}^2 \cos^2(\theta))}{\omega^2 (\omega^2 - k^2 v_{th,e}^2) - \omega_{ce}^2 (\omega^2 - k^2 v_{th,e}^2 \cos^2(\theta))}, \quad (2.136)$$

$$\epsilon_{xz}/\epsilon_0 = -i \frac{\omega_{pe}^2 k^2 v_{th,e}^2 \cos(\theta) \sin(\theta)}{\omega^2 (\omega^2 - k^2 v_{th,e}^2) - \omega_{ce}^2 (\omega^2 - k^2 v_{th,e}^2 \cos^2(\theta))}, \quad (2.137)$$

$$\epsilon_{yy}/\epsilon_0 = 1 - \frac{\omega_{pe}^2 (\omega^2 - k^2 v_{th,e}^2)}{\omega^2 (\omega^2 - k^2 v_{th,e}^2) - \omega_{ce}^2 (\omega^2 - k^2 v_{th,e}^2 \cos^2(\theta))}, \quad (2.138)$$

$$\epsilon_{yz}/\epsilon_0 = -i \frac{\omega_{ce}}{\omega} \frac{\omega_{pe}^2 k^2 v_{th,e}^2 \cos(\theta) \sin(\theta)}{\omega [\omega^2 (\omega^2 - k^2 v_{th,e}^2) - \omega_{ce}^2 (\omega^2 - k^2 v_{th,e}^2 \cos^2(\theta))]}, \quad (2.139)$$

and

$$\epsilon_{zz}/\epsilon_0 = 1 - \frac{\omega_{pe}^2 (\omega^2 - \omega_{ce}^2 - k^2 v_{th,e}^2 \sin^2(\theta))}{\omega^2 (\omega^2 - k^2 v_{th,e}^2) - \omega_{ce}^2 (\omega^2 - k^2 v_{th,e}^2 \cos^2(\theta))}. \quad (2.140)$$

Also, the hermiticity properties of the dielectric tensor ensure

$$\epsilon_{yx} = -\epsilon_{xy}, \quad \epsilon_{zx} = \epsilon_{xz} \quad \text{and} \quad \epsilon_{zy} = -\epsilon_{yz}. \quad (2.141)$$

Perpendicular propagation

For $\theta = \pi/2$, we obtain for the various terms of $\bar{\epsilon}$

$$\epsilon_{xx}/\epsilon_0 = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2 - k^2 v_{th,e}^2}, \quad (2.142)$$

$$\epsilon_{xy}/\epsilon_0 = i \frac{\omega_{pe}^2 \omega_{ce}}{\omega(\omega^2 - \omega_{ce}^2 - k^2 v_{th,e}^2)}, \quad (2.143)$$

$$\epsilon_{yy}/\epsilon_0 = 1 - \frac{\omega_{pe}^2 (\omega^2 - k^2 v_{th,e}^2)}{\omega^2 (\omega^2 - \omega_{ce}^2 - k^2 v_{th,e}^2)}, \quad (2.144)$$

$$\epsilon_{zz}/\epsilon_0 = 1 - \frac{\omega_{pe}^2}{\omega^2} = P, \quad (2.145)$$

and

$$\epsilon_{xz} = \epsilon_{yz} = 0. \quad (2.146)$$

In a cold plasma and for perpendicular propagation, two modes were obtained: the ordinary and extraordinary modes (see section 2.3.2). Since the zz term of the tensor has the same expression than in the cold plasma case, the ordinary mode is not influenced by thermal effects. For the extraordinary mode, the dispersion relation takes the form

$$n_X^2 = \frac{\epsilon_{xx}\epsilon_{yy} + \epsilon_{xy}^2}{\epsilon_{xx}}. \quad (2.147)$$

Hence

$$n_X^2 = \frac{(\omega^2 - \omega_e^2 - k^2 v_{th,e}^2)(\omega^2 - \omega_{pe}^2) - \omega_{pe}^2 \omega_{ce}^2}{\omega^2 (\omega^2 - \omega_e^2 - k^2 v_{th,e}^2)}, \quad (2.148)$$

with $\omega_e^2 \equiv \omega_{ce}^2 + \omega_{pe}^2$. In the cold plasma case, the corresponding expression was (section 2.3.2):

$$n_{X,c}^2 = \frac{RL}{S} = \frac{(\omega^2 - \omega_e^2)(\omega^2 - \omega_{pe}^2) - \omega_{pe}^2 \omega_{ce}^2}{\omega^2 (\omega^2 - \omega_e^2)}, \quad (2.149)$$

which is precisely what we obtain by setting $T_e = 0$ in (2.148). In order to quantify thermal effects in the plasma, it is usual to introduce the quantity β , which is defined as the ratio between kinetic and magnetic pressures. Thus:

$$\beta \equiv \beta_e + \beta_i = \frac{2\mu_0 k_b (n_e T_e + n_i T_i)}{B_0^2}. \quad (2.150)$$

A simple algebraic manipulation of this expression shows that β_s can be expressed as

$$\beta_s = \frac{\omega_{ps}^2 v_{th,s}^2}{\omega_{cs}^2 c^2} = \frac{\omega_{ps}^2}{\omega_{cs}^2} \frac{2T_s}{m_s c^2} \quad (2.151)$$

On Fig. 2.13(a) is shown the cold dispersion relation, with the singularity corresponding to the upper hybrid resonance, with $\omega_{pe}^2/\omega_{ce}^2 = 0.5$. Also shown is the dispersion relation obtained for $T_e = 1\text{keV}$, or $\beta_e \approx 0.001$. On Fig. 2.13(b) appears the dispersion relation for various temperatures.

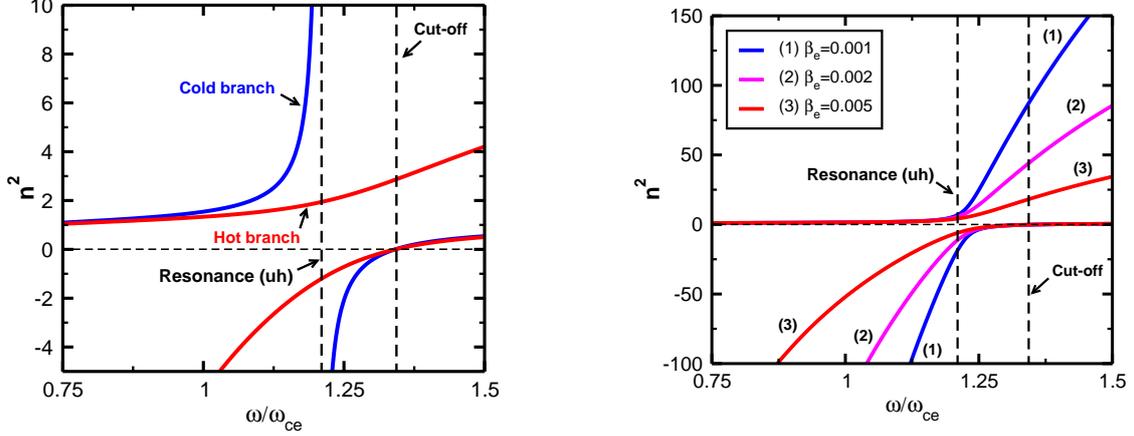


Figure 2.13: Influence of finite temperature effects on the dispersion relation for the X-mode. On this figure, $\omega_{pe}^2/\omega_{ce}^2 = 0.5$. (a) Comparison between cold and hot plasma for $T_e = 1\text{keV}$; (b) influence of the electron temperature.

The most striking manifestation of thermal effects is the disappearance of the singularity at the upper hybrid resonance. Mathematically, this is caused by the fact that the denominator in Eq. 2.148 does not correspond to a singularity (since it depends on k). Furthermore, a new branch is observed. For $\omega \lesssim \omega_{uh}$, this new branch corresponds to a strongly evanescent wave. On the other hand, we obtain a propagative wave for large values of the parallel refraction index when $\omega \gg \omega_{uh}$. This corresponds to the *electron pressure wave*. At this point, it is worth mentioning that the thermal correction which has been introduced here is too simplistic. Indeed, it results in the crossing of the cold resonance by the dispersion relation branch. In a kinetic description, more precise, we would obtain a very different behavior for this acoustic wave. The problem is caused by the incapacity of a fluid description to provide a correct description of the wave-plasma interaction in the vicinity of the cold resonance. This issue is addressed by Ginzburg[8], p. 182.

Parallel propagation

Substituting $\theta = 0$ in the dielectric tensor terms yields

$$\epsilon_{xx}/\epsilon_0 = \epsilon_{yy}/\epsilon_0 = S, \quad (2.152)$$

$$\epsilon_{xy}/\epsilon_0 = -iD, \quad (2.153)$$

$$\epsilon_{xz} = \epsilon_{yz} = 0, \quad (2.154)$$

and

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2 - k^2 v_{th,e}^2}. \quad (2.155)$$

This shows that the xx , xy and yy terms of the dielectric tensor are identical to the corresponding cold plasma terms: the propagation of the R and L-whistlers is not affected by finite temperature effects. On the other hand, ϵ_{zz} is different: we recall that in a cold plasma, the dispersion relation for Langmuir oscillations is $P \approx 1 - \omega_{pe}^2/\omega^2 = 0$, and that this solution corresponds to a non-propagative phenomenon. Here, $\epsilon_{zz} = 0$ gives

$$\omega^2 = \omega_{pe}^2 + k^2 v_{th,e}^2. \quad (2.156)$$

This time, we obtain a real dispersion branch, since the k -dependence ensures that the phase velocity is finite.

2.4.4 MHD waves

In the MHD domain, ($\omega \ll \omega_{ci}$), we find the Alfvén waves, which have already been discussed in the case of a cold plasma (see 2.3.4). It is also of interest to study how finite temperature effects influence the waves propagating in the range of frequency. However, the general treatment of this problem is rather cumbersome[4] and we will make use of the MHD approximation in order to obtain the dispersion relation. As discussed before, in this domain, the role of the ions is obviously preponderant. But the electron contribution can never be neglected since they respond to MHD waves quasi-instantaneously and ensure the plasma quasi-neutrality. It is therefore necessary to write the fluid equations for the two species. To simplify the calculation, we will consider $Z_i = 1$. Also, we will assume that $k\lambda_d \gg 1$, with λ_d the Debye length. This means that we consider that the plasma quasi-neutrality is automatically verified ($n_e = n_i \equiv n$), without needing to resort to the Poisson's equation. We introduce $\rho \equiv \rho_e + \rho_i$ with $\rho_e \equiv m_e n$ and $\rho_i \equiv m_i n$. Since $m_e/m_i \ll 1$, we have $\rho \approx nm_i$: the fluid mass is essentially carried by the ions. We define the macroscopic velocity

$$\mathbf{v} \equiv \frac{\rho_e \mathbf{v}_e + \rho_i \mathbf{v}_i}{\rho}. \quad (2.157)$$

We see that

$$\mathbf{v}_i \approx \mathbf{v} + O(m_e/m_i). \quad (2.158)$$

The current flowing in the plasma is given by

$$\mathbf{j} = en(\mathbf{v}_i - \mathbf{v}_e). \quad (2.159)$$

This equation can be rewritten as

$$\mathbf{v}_e \approx \mathbf{v} - \frac{\mathbf{j}}{en}. \quad (2.160)$$

\mathbf{j} being a perturbation, the electron velocity is of the same order as the ion velocity, and as the fluid velocity. This is not surprising: ions respond to the wave, while the electrons follow them to ensure quasi-neutrality. This corresponds to an ensemble motion of the fluid. Summing the density continuity equations for ions and electrons yields

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\rho \mathbf{v}) = 0. \quad (2.161)$$

Letting $p = p_e + p_i$, the combination of the momentum conservation equations gives

$$\rho \frac{d\mathbf{v}}{dt} = en(\mathbf{v}_i - \mathbf{v}_e) \times \mathbf{B} - \nabla p. \quad (2.162)$$

From the same equations, we can deduce Ohm's law, which is taken here in its ideal form

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} \approx 0 \quad (2.163)$$

Finally, we consider that the the fluid closure is an adiabatic law for electrons and ions (which is compatible with the considered frequency range, see section 2.4.1). If we assume that $\gamma_e = \gamma_i \equiv \gamma$, we can write

$$\frac{d}{dt} \left(\frac{p}{n^\gamma} \right) = 0. \quad (2.164)$$

We obtain a set of four equations, known as the *MHD equations*

$$\frac{d\rho}{dt} + \nabla(\rho \mathbf{v}) = 0, \quad (2.165)$$

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p - \mathbf{j} \times \mathbf{B} = 0, \quad (2.166)$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \quad (2.167)$$

$$\frac{d}{dt} \left(\frac{p}{n^\gamma} \right) = 0. \quad (2.168)$$

The Maxwell's equations also give

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (2.169)$$

where the displacement current has been neglected, as is usual in MHD. Also

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.170)$$

and Ohm's law can be rewritten in the form

$$-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B}) = 0. \quad (2.171)$$

We linearize Eqs. 2.165-2.168 to obtain to first order

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2.172)$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla \delta p - \frac{1}{\mu_0} (\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0 = \mathbf{0}, \quad (2.173)$$

$$-\frac{\partial \delta \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B}_0) = \mathbf{0}, \quad (2.174)$$

$$\frac{\partial}{\partial t} \left(\frac{\delta p}{\rho_0} - \gamma \frac{\delta \rho}{\rho_0} \right) = 0. \quad (2.175)$$

In the case of plane waves

$$-\omega \delta \rho + \rho_0 \mathbf{k} \cdot \mathbf{v} = 0, \quad (2.176)$$

$$-\omega \rho_0 \mathbf{v} + \mathbf{k} \delta p - \frac{1}{\mu_0} (\mathbf{k} \times \delta \mathbf{B}) \times \mathbf{B}_0 = \mathbf{0}, \quad (2.177)$$

$$\omega \delta \mathbf{B} + \mathbf{k} \times (\mathbf{v} \times \mathbf{B}_0) = \mathbf{0}, \quad (2.178)$$

$$\frac{\delta p}{\rho_0} - \gamma \frac{\delta \rho}{\rho_0} = 0, \quad (2.179)$$

which yields

$$\delta \rho = \rho_0 \frac{(\mathbf{k} \cdot \mathbf{v})}{\omega}, \quad (2.180)$$

$$\delta p = \gamma \frac{(\mathbf{k} \cdot \mathbf{v})}{\omega} p_0, \quad (2.181)$$

and

$$\delta \mathbf{B} = -\frac{\mathbf{k} \times (\mathbf{v} \times \mathbf{B}_0)}{\omega}. \quad (2.182)$$

Injecting the three latter quantities in (2.177) and after a few algebraic operations, we get

$$\mathbf{v} - \gamma \frac{p_0}{\omega^2 \rho_0} \mathbf{k} (\mathbf{k} \cdot \mathbf{v}) - \frac{1}{\omega^2 \mu_0 \rho_0} \left[B_0^2 \mathbf{k} (\mathbf{k} \cdot \mathbf{v}) - (\mathbf{B}_0 \cdot \mathbf{k}) (\mathbf{k} \cdot \mathbf{v}) \mathbf{B}_0 - \right. \\ \left. (\mathbf{k} \cdot \mathbf{B}_0) (\mathbf{v} \cdot \mathbf{B}_0) \mathbf{k} + (\mathbf{B}_0 \cdot \mathbf{k})^2 \mathbf{v} \right] = 0. \quad (2.183)$$

We set $\hat{\mathbf{u}}_z \equiv \mathbf{B}_0/|\mathbf{B}_0|$ and also $\mathbf{k} = k \sin(\theta)\hat{\mathbf{u}}_x + k \cos(\theta)\hat{\mathbf{u}}_z$ to write the latter equation in the following matrix form:

$$\overline{\mathbf{N}}_{\mathbf{k},\omega} \cdot \mathbf{v} = 0, \quad (2.184)$$

with

$$\overline{\mathbf{N}}_{\mathbf{k},\omega} = \begin{pmatrix} \omega^2 - k^2 v_a^2 - k^2 c_s^2 \sin^2(\theta) & 0 & -k^2 c_s^2 \sin(\theta) \cos(\theta) \\ 0 & \omega^2 - k^2 v_a^2 \cos^2(\theta) & 0 \\ -k^2 c_s^2 \sin(\theta) \cos(\theta) & 0 & \omega^2 - k^2 c_s^2 \cos^2(\theta) \end{pmatrix}. \quad (2.185)$$

where the Alfvén velocity has been introduced (see section 2.3.1) and is approximately given by

$$v_a^2 \approx \frac{B_0^2}{\mu_0 n m_i}. \quad (2.186)$$

The sound velocity is defined as

$$c_s = \left(\gamma \frac{p_0}{\rho_0} \right)^{1/2}. \quad (2.187)$$

The non-trivial solutions of (2.184) are obtained by setting $\det(\overline{\mathbf{N}}_{\omega,\mathbf{k}}) = 0$. Hence,

$$(\omega^2 - k^2 v_a^2 \cos^2(\theta)) [\omega^4 - \omega^2 k^2 (c_s^2 + v_a^2) + k^4 v_a^2 c_s^2 \cos^2(\theta)] = 0. \quad (2.188)$$

The first factor on the left hand side leads to

$$\omega = k v_a \cos(\theta), \quad (2.189)$$

which is no other than the torsional Alfvén wave (see 2.3.4), unaffected by the thermal corrections. The corresponding velocity vector is $\mathbf{v}_t = (0, v_y, 0)$. From Eqs. 2.180 and 2.181, it also appears that $\delta p = \delta \rho = 0$: this wave is not associated with any density (and thus pressure) perturbation.

The second term in (2.188) yields two solutions:

$$\omega_{\pm} = \frac{k}{\sqrt{2}} [(c_s^2 + v_a^2) \pm \sqrt{(c_s^2 + v_a^2)^2 - 4v_a^2 c_s^2 \cos^2(\theta)}]^{1/2}. \quad (2.190)$$

In order to analyze these solutions, we first consider a zero-pressure, which implies $c_s = 0$. The latter solution gives $\omega_+ = k v_a$, which shows that this wave is a modified version of the compressional Alfvén wave. The other solution is $\omega_- = 0$. It is a slow wave (since $\omega_- < \omega_+$), strictly associated to finite pressure effects.

To further characterize this solution, it is useful to notice

$$\frac{c_s^2}{v_a^2} = \frac{\gamma}{2} \beta, \quad (2.191)$$

with $\beta \equiv 2\mu_0 p_0/B_0^2$. Considering the limit $\beta \ll 1$ (realistic, e.g., in a magnetic fusion device), we have $c_s \ll v_a$. This means that the magnetic stress is such that the waves associated to a magnetic perturbation propagate much faster than pressure waves. In this case, we have

$$\omega_- \approx kc_s \cos(\theta) = k_z c_s, \quad (2.192)$$

which demonstrates that this wave propagates along the field lines, with $v_\varphi = c_s$. It is clear that we recover the *ion acoustic wave*, already discussed in section 2.4.2. On Fig. 2.14 are shown the phase velocity surfaces for each of these three waves, obtained with³ $\beta = 0.4$.

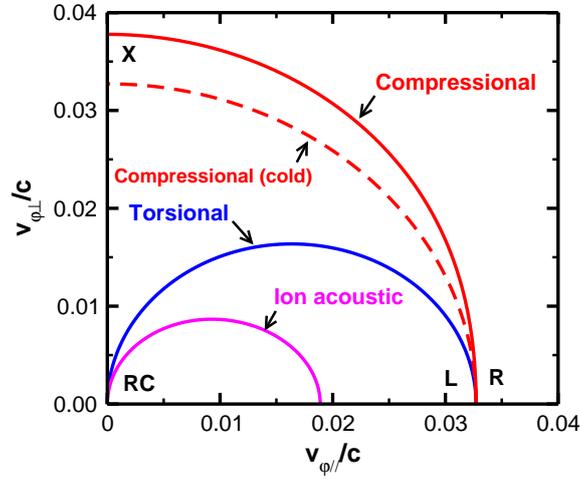


Figure 2.14: Phase velocity surface obtained with $\beta = 0.4$ and $\omega_{pe}^2/\omega_{ce}^2 = 0.5$. Also shown is the compressional wave obtained in the cold plasma limit.

It is possible to find in the literature treatments with less approximations than what has been developed here[4]. In this case, however, the dispersion relation is more complex:

$$\left(1 - \frac{\omega^2}{k^2 v_a^2} - \frac{\omega^2}{\omega_{ce}\omega_{ci}} + \frac{k^2 c_s^2 \sin^2(\theta)}{\omega^2 - k^2 c_s^2}\right) \cdot \left(\cos^2(\theta) - \frac{\omega^2}{k^2 v_a^2} - \frac{\omega^2}{\omega_{ce}\omega_{ci}}\right) = \frac{\omega^2}{\omega_{ci}^2} \cos^2(\theta), \quad (2.193)$$

which is a third order polynomial for ω^2 . In this case, the two Alfvén waves are recovered, as well as the ion acoustic wave. These solutions are shown on Fig. 2.15, which is known as a *Stringer diagram*. The interested reader can refer to the detailed study presented in Swanson’s textbook[4].

³This is much larger than the values encountered in magnetic fusion plasmas, but this allows to have the three waves appearing on the same figure.

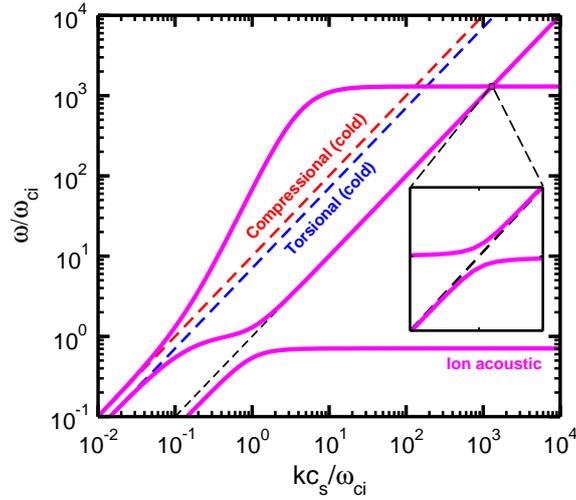


Figure 2.15: Stringer diagram: effects of a finite temperature on the propagation of Alfvén and ion acoustic waves for $\theta = 45^\circ$ and $c_s/v_a = 0.1$. Also shown are the solutions obtained in the cold plasma case.

2.5 Collisional damping

The cold plasma described so far can not absorb any wave power. Mathematically, this is caused by the fact that the anti-Hermitian part of the dielectric tensor is zero. Physically, the reason lies in the fact that the unperturbed particle trajectories are perfectly deterministic (integrable), as is clear from Eq. 2.11, which is merely the equation of motion for the particles of species s . At this point, it is important to realize that the concept of absorption of a wave by a medium means that the information contained in the wave is destroyed, so that its power is irreversibly transferred to the plasma⁴. Clearly, the cold plasma does not have any power “sink” and therefore, no means to “store” the wave energy. In order to add this capability to our model, we need an irreversible mechanism. The most natural idea is to include collisions. With collisions, we have a n -body problem with n very large, and therefore, a phenomenon which is irreversible by essence.

2.5.1 Dielectric tensor

The most straightforward way to include collisions in the cold plasma model described in section 2.2 is to add a *Crook term* to the momentum conservation equation

⁴As discussed in section 1.4, one of the difficulties arising when attempting to establish the energy balance between the wave and the plasma is to distinguish between the reversible and irreversible power transfer.

(2.11) which becomes

$$\frac{d\mathbf{v}_s}{dt} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \nu_s \mathbf{v}_s, \quad (2.194)$$

with ν_s the collision frequency. Hence

$$\mathbf{v}_s(-i\omega + \nu_s) = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}). \quad (2.195)$$

At this point, we could proceed along the exact same lines as in section 2.2.1 and recalculate the dielectric tensor. However, the previous expression shows that the new dielectric tensor can be simply obtained by replacing ω by $\omega + i\nu_s$ in the conductivity tensor (but not in the dielectric tensor as a whole, since the ω appearing in Eq. 1.24 must not be replaced). The dielectric tensor still has the form (2.24), with new terms:

$$R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \Omega_{cs} + i\nu_s)}, \quad (2.196)$$

$$L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \Omega_{cs} + i\nu_s)}, \quad (2.197)$$

and

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + i\nu_s)}. \quad (2.198)$$

Collisions induce a major modification of the dielectric tensor: its terms now have a imaginary part. If, moreover, we assume $\nu_s \ll \omega$ (the opposite would mean that the wave is absorbed before having propagated on even a single period) we obtain

$$R \approx 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \Omega_{cs})} + i \sum_s \frac{\nu_s}{\omega} \frac{\omega_{ps}^2}{(\omega + \Omega_{cs})^2}, \quad (2.199)$$

$$L \approx 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \Omega_{cs})} + i \sum_s \frac{\nu_s}{\omega} \frac{\omega_{ps}^2}{(\omega - \Omega_{cs})^2}, \quad (2.200)$$

and

$$P \approx 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} + i \sum_s \frac{\nu_s}{\omega} \frac{\omega_{ps}^2}{\omega^2}. \quad (2.201)$$

To lowest order, the Hermitian part (Eq. 1.27) is merely the collisionless dielectric tensor. Also, according to Eq. 1.111, we obtain

$$P_{abs} = \sum_s \nu_s \frac{\omega_{ps}^2}{\omega^2} \left[\frac{\omega^2}{(\omega - \Omega_{cs})^2} |E_+|^2 + \frac{\omega^2}{(\omega + \Omega_{cs})^2} |E_-|^2 + |E_z|^2 \right]. \quad (2.202)$$

It should be mentioned that the model which has been described here is applicable only to specific physical problems. A first shortcoming is that P_{abs} is directly proportional to the friction coefficient ν_s , which is difficult to obtain. Even more importantly, a simple friction coefficient such as the one which has been introduced above is not adapted to the description of collisions between the species comprising the plasma. Its application is limited to collisions with external species, such as neutrals, for instance.

2.5.2 Langmuir wave damping

As an example, we can study how the Langmuir wave (see section 2.3.1) is damped by collisions. Since the dielectric tensor has the exact same form as its non-collisional counterpart, the dispersion relation of Langmuir oscillations is still given by $P = 0$. To simplify this problem, we consider an oscillation such as $\omega \sim \omega_{pe} \gg \omega_{pi}$, which yields the approximated dispersion relation (where the ion contribution has been neglected)

$$P = 1 - \frac{\omega_{pe}^2}{\omega(\omega + i\nu_e)} = 0, \quad (2.203)$$

or

$$\omega(\omega + i\nu_e) = \omega_{pe}^2. \quad (2.204)$$

The discriminant of this expression is $\Delta = 4\omega_{pe}^2/2 - \nu_e^2$. If $\nu_e < 2\omega_{pe}$, we obtain

$$\omega = \frac{-i\nu_e + \sqrt{4\omega_{pe}^2 - \nu_e^2}}{2}, \quad (2.205)$$

and if we further assume $\nu_e^2 \ll 4\omega_{pe}^2$, we can write

$$\omega \approx \omega_{pe} - i\frac{\nu_e}{2} \quad (2.206)$$

Since the electromagnetic field varies as $\exp(-i\omega t)$, we see immediately

$$(|\mathbf{E}|, |\mathbf{B}|) \propto \exp(-i\omega_{pe}t) \exp(-\nu_e t/2), \quad (2.207)$$

which demonstrates that we recover the Langmuir oscillations, with an amplitude decaying in time because of collisions.

2.5.3 Electron whistler damping

A more interesting case is the collisional damping of the electron whistler. As was discussed in section 1.3.1, we can not describe energy transfers within the framework of a homogeneous plasma model. We will therefore use a model similar to the one used in section 1.3.2. This model consists of assuming that the medium varies along

$\hat{\mathbf{u}}_z$, but it homogeneous along $\hat{\mathbf{u}}_x$ and $\hat{\mathbf{u}}_y$ (plane stratified geometry). Furthermore, since we want to study the R-whistler, which propagates along the magnetic field (i.e. with $k_x = k_y = 0$), we have $\nabla \equiv d/dz \hat{\mathbf{u}}_z$. Using the collisional cold dielectric tensor, we obtain the following coupled equations for E_x and E_y (the third line of the wave equation corresponds to the Langmuir wave and gives, in our case, $E_z = 0$):

$$\frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2} (S E_x - i D E_y) = 0, \quad (2.208)$$

and

$$\frac{d^2 E_y}{dz^2} + \frac{\omega^2}{c^2} (i D E_x + S E_y) = 0. \quad (2.209)$$

Making use of the rotating components (2.14) of the electric field, we can rewrite (2.208) and (2.209) as

$$\frac{d^2 E_+}{dz^2} + \frac{\omega^2}{c^2} L E_+ = 0, \quad (2.210)$$

and

$$\frac{d^2 E_-}{dz^2} + \frac{\omega^2}{c^2} R E_- = 0. \quad (2.211)$$

As seen before, these two equations respectively describe the L and R -whistler. They are uncoupled, which means that if we choose an initial polarization, it will remain the same during the propagation. Here, we will restrict our study to the R -whistler, with $E_+ = E_x + i E_y = 0$. We readily obtain from Eq. 2.211

$$\frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2} R E_x = 0, \quad \frac{d^2 E_y}{dz^2} + \frac{\omega^2}{c^2} R E_y = 0, \quad (2.212)$$

Away from sources, Poynting's theorem can written as

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E}, \quad (2.213)$$

with the electromagnetic energy density

$$W \equiv \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2, \quad (2.214)$$

and the Poynting vector defined as

$$\mathbf{S} \equiv \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}. \quad (2.215)$$

At steady-state, we have $\partial_t W = 0$. Averaging Eq. 2.213 over time, if we assume that the dissipated power is given by⁵ $P_{abs} \equiv -\langle \mathbf{j} \cdot \mathbf{E} \rangle$, we obtain

$$P_{abs} = \langle \nabla \cdot \mathbf{S} \rangle \quad (2.216)$$

⁵For the sake of simplicity, we disregard the issue of reversible/irreversible energy transfer discussed in section 1.4.

From Maxwell's equations, in the geometry considered here, we have

$$i\omega B_x = -\partial_z E_y, \quad i\omega B_y = \partial_z E_x, \quad (2.217)$$

so that

$$\mathbf{S} \equiv S_z \hat{\mathbf{u}}_z = \frac{1}{\mu_0} (E_x B_y - E_y B_x) \hat{\mathbf{u}}_z. \quad (2.218)$$

In order to perform the time-average of $\nabla \cdot \mathbf{S}$, we need to express $\mathbf{S} = \mathbf{S}(t)$. This is done by writing, for E_x

$$E_x(t) = \frac{1}{2} (E_x e^{i\omega t} + E_x^* e^{-i\omega t}), \quad (2.219)$$

where E_x and E_x^* appearing on the right hand side actually refer to the Fourier components of $E_x(t)$. The same must be done to obtain $E_y(t)$, $B_x(t)$ and $B_y(t)$. Also, we readily show

$$\langle A(t)B(t) \rangle = \frac{AB^*}{2}. \quad (2.220)$$

Hence

$$\langle S_z(t) \rangle = \frac{1}{4\mu_0} (E_x B_y^* + E_x^* B_y - E_y^* B_x - E_y B_x^*). \quad (2.221)$$

Using (2.217), we obtain

$$\langle S_z(t) \rangle = \frac{i}{2\mu_0\omega} (E_y \partial_z E_y^* - E_y^* \partial_z E_y), \quad (2.222)$$

which yields

$$\langle \nabla \cdot \mathbf{S} \rangle = \langle \partial_z S_z(t) \rangle = \frac{i}{2\mu_0\omega} (E_y \partial_z^2 E_y^* - E_y^* \partial_z^2 E_y). \quad (2.223)$$

Using Eqs. 2.212, the latter expression can be written as

$$\langle \nabla \cdot \mathbf{S} \rangle = \frac{i}{2\mu_0\omega} \frac{\omega^2}{c^2} (-R^* |E_y|^2 + R |E_y|^2). \quad (2.224)$$

Finally, the absorbed power is

$$P_{abs} = i\omega \frac{\epsilon_0 |E_y|^2}{2} (R - R^*). \quad (2.225)$$

We have $P_{abs} = 0$ in a collisionless cold plasma, where $R = R^*$. In a collisional plasma, on the other hand, using (2.199) yields

$$P_{abs} = \omega_{pe}^2 \frac{\epsilon_0 |E_y|^2}{2} \frac{2\nu_e}{(\omega - \omega_{ce})^2 + \nu_e^2}, \quad (2.226)$$

where we have used $\omega_{ce} = -\Omega_{ce}$.

In order to better understand the physical nature of this result, it is useful to consider that the resonance is located at $z = z_0$ and perform the following expansion in the vicinity of z_0 :

$$\omega_{ce}(z) = \omega_{ce}(z_0) + (z - z_0)\partial_z\omega_{ce}(z_0). \quad (2.227)$$

Introducing the distance $\delta \equiv \nu_e/\partial_z\omega_{ce}(z_0)$, we can write

$$P_{abs} = \frac{\omega_{pe}^2}{\partial_z\omega_{ce}(z_0)} \frac{\epsilon_0|E_y|^2}{2} \frac{2\delta}{(z - z_0)^2 + \delta^2}, \quad (2.228)$$

which shows that the collision frequency actually determines the width of the absorption layer around the resonance. However, it should be noted that it does not determine the total amount of wave power actually absorbed by the plasma. Actually, in all cases, the whole wave energy will be transferred to the plasma, which is a direct consequence of the simplicity of our dissipation model.

2.6 Waves and beams

2.6.1 Dispersion relation

Up to this point, the derivation of the wave equation and dispersion relation has been done assuming that the plasma was immobile in the absence of wave: $\mathbf{v}_s = \delta\mathbf{v}_s$. In reality, this is obviously not always the case. The condition $\mathbf{v}_{s,0} \neq \mathbf{0}$ has a large variety of interesting physical consequences. It also has the advantage of providing insight into concepts related to the wave absorption or emission by the plasma.

We will still use a fluid description. The derivation of the equations presented in section 2.1 remains valid. The difference lies in their linearization. For instance, the continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0, \quad (2.229)$$

yields to first order

$$\frac{\partial \delta n_s}{\partial t} + \nabla \cdot (\delta n_s \mathbf{v}_{s,0} + n_{s,0} \delta \mathbf{v}_s) = 0. \quad (2.230)$$

By performing the harmonic and local analysis, we obtain

$$\delta n_s = n_{s,0} \frac{\delta \mathbf{v}_s \cdot \mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}_{s,0}}. \quad (2.231)$$

Likewise, the momentum conservation equation

$$n_s m_s \left[\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s \right] = n_s q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \nabla \cdot \bar{\bar{\Phi}}_s, \quad (2.232)$$

yields to first order and in the case of a cold plasma ($\bar{\Phi}_s = \bar{\mathbf{0}}$)

$$\left[\frac{\partial \delta \mathbf{v}_s}{\partial t} + (\mathbf{v}_{s,0} \cdot \nabla) \delta \mathbf{v}_s \right] + (\delta \mathbf{v}_s \cdot \nabla) \mathbf{v}_{s,0} = \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v}_{s,0} \times \delta \mathbf{B} + \delta \mathbf{v}_s \times \mathbf{B}_0). \quad (2.233)$$

In a homogeneous plasma, the term $(\delta \mathbf{v}_s \cdot \nabla) \mathbf{v}_{s,0}$ on the left hand side of this equation is zero. Thus, we obtain

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}_{s,0}) \delta \mathbf{v}_s - \Omega_{cs} (\delta \mathbf{v}_s \times \hat{\mathbf{u}}_z) = \frac{q_s}{m_s} \mathbf{E} + \frac{q_s}{m_s} \mathbf{v}_{s,0} \times \delta \mathbf{B} \quad (2.234)$$

with $\Omega_{cs} \equiv q_s B_0 / m_s$. Using Maxwell's equations to write $\mathbf{k} \times \mathbf{E} = \omega \delta \mathbf{B}$ leads to

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}_{s,0}) \delta \mathbf{v}_s - \Omega_{cs} (\delta \mathbf{v}_s \times \hat{\mathbf{u}}_z) = \frac{q_s}{m_s} \left[\mathbf{E} + \frac{\mathbf{v}_{s,0}}{\omega} \times (\mathbf{k} \times \mathbf{E}) \right]. \quad (2.235)$$

The linearized current appears as

$$\mathbf{j} = \sum_s n_s q_s \mathbf{v}_s \equiv \mathbf{j}_0 + \delta \mathbf{j}, \quad (2.236)$$

with

$$\mathbf{j}_0 = \sum_s q_s n_{s,0} \mathbf{v}_{s,0}, \quad (2.237)$$

and

$$\delta \mathbf{j} \equiv \sum_s q_s (\delta n_s \mathbf{v}_{s,0} + n_{s,0} \delta \mathbf{v}_s). \quad (2.238)$$

Using Eq. 2.231, this expression can be rewritten as

$$\delta \mathbf{j} = \sum_s q_s n_{s,0} \left[\delta \mathbf{v}_s + \frac{(\delta \mathbf{v}_s \cdot \mathbf{k})}{\omega - \mathbf{k} \cdot \mathbf{v}_{s,0}} \mathbf{v}_{s,0} \right], \quad (2.239)$$

At this point, the dielectric tensor can be deduced by using the same procedure as detailed in section 2.2. However, the calculation is more complicated: unlike the case with $\mathbf{v}_{s,0} = \mathbf{0}$, it is readily seen from Eqs. 2.239 and 2.235 that the plasma is spatially dispersive. In other words, the dielectric tensor depends on the wavevector. In order to simplify our study, we will therefore consider the special case of a parallel beam and a parallel propagating wave. We note $\mathbf{k} \equiv k_z \hat{\mathbf{u}}_z$, $\mathbf{v}_{0,s} \equiv v_{0,s} \hat{\mathbf{u}}_z$. For this system, (2.235) simplifies considerably and by introducing the usual rotating components of both the electric field $E_{\pm} \equiv E_x \pm i E_y$ and the velocity perturbation $\delta v_{\pm} \equiv \delta v_x \pm i \delta v_y$, we obtain three uncoupled equations

$$-i(\omega - \Omega_{cs} - k_z v_{s,0}) \delta v_{+} = \frac{q_s}{m_s} \frac{\omega - k_z v_{s,0}}{\omega} E_{+}, \quad (2.240)$$

$$-i(\omega + \Omega_{cs} - k_z v_{s,0})\delta v_- = \frac{q_s}{m_s} \frac{\omega - k_z v_{s,0}}{\omega} E_-, \quad (2.241)$$

and

$$-i(\omega - k_z v_{s,0})\delta v_z = q_s E_z / m_s. \quad (2.242)$$

From (2.239), we can obtain the expressions for the terms of the conductivity and deduce the dielectric tensor. This gives

$$\bar{\boldsymbol{\epsilon}} = \epsilon_0 \begin{bmatrix} S' & -iD' & 0 \\ iD' & S' & 0 \\ 0 & 0 & P' \end{bmatrix} \quad (2.243)$$

Following the lines of section (2.2), the following definitions have been introduced:

$$P' \equiv 1 - \sum_s \frac{\omega_{ps}^2}{(\omega - k_z v_{s,0})^2}, \quad (2.244)$$

$$R' \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega - k_z v_{s,0}}{(\omega + \Omega_{cs} - k_z v_{s,0})}, \quad (2.245)$$

$$L' \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega - k_z v_{s,0}}{(\omega - \Omega_{cs} - k_z v_{s,0})}. \quad (2.246)$$

and S' and D' are now given by

$$S' \equiv \frac{1}{2}(R' + L') = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{(\omega - k_z v_{s,0})^2}{(\omega - k_z v_{s,0})^2 - \Omega_{cs}^2}, \quad (2.247)$$

$$D' \equiv \frac{1}{2}(R' - L') = \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{(\omega - k_z v_{s,0})\Omega_{cs}}{(\omega - k_z v_{s,0})^2 - \Omega_{cs}^2}. \quad (2.248)$$

These expressions should be compared to the dielectric tensor terms of section 2.2. As in section 2.3.1, (2.240) and (2.241) describe the two whistlers. The plasma motion essentially manifests itself by a Doppler shift of the resonance. Another interesting observation is that in the reference frame moving at the wave phase velocity, both S' and D' cancel. This is related to the transverse polarization of the whistlers. Indeed, we have $\delta \mathbf{B} = \mathbf{k} \times \mathbf{E} / \omega$, and it is readily deduced that the Lorentz force felt by particles having an ensemble motion at velocity $v_{s,0}$ along z by the whistler electric field is

$$\mathbf{f}_{L,s} \equiv q_s(\mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) = q_s \left(1 - \frac{k_z v_{s,0}}{\omega} \right) \mathbf{E}, \quad (2.249)$$

which cancels when $v_{s,0} = v_\varphi$. Eq. 2.242, on the other hand, describes the Langmuir wave, which we will now study in more details.

2.6.2 Langmuir oscillations

The dielectric tensor structure ensures that the whistlers and Langmuir wave are completely decoupled and therefore can be studied separately, as in section 2.3.1. If we consider waves at frequency $\omega \gg \omega_{pi}$, we have

$$P' \approx 1 - \frac{\omega_{pe}^2}{(\omega - k_z v_{e,0})^2}. \quad (2.250)$$

The associated dispersion relation can be written as

$$\omega - k v_{e,0} = \pm \omega_{pe}. \quad (2.251)$$

We now have two waves with the following properties:

Fast wave: Its phase velocity is $v_\varphi = v_{e,0} + \omega_{pe}/k$, showing that the wave fronts are faster than the beam. The associated density perturbation, obtained by combining (2.231) and (2.251) is

$$\delta n_e = n_{e,0} \frac{k_z}{\omega_{pe}} \delta v_e. \quad (2.252)$$

Slow wave: At phase velocity $v_\varphi = v_{e,0} - \omega_{pe}/k$, the wave fronts are slower than the beam, and the density perturbation is

$$\delta n_e = -n_{e,0} \frac{k}{\omega_{pe}} \delta v_e \quad (2.253)$$

Both waves have the same group velocity $v_g = v_{e,0}$. In the absence of wave, the beam kinetic energy is

$$\bar{\varepsilon} = \frac{1}{2} m_e n_{e,0} v_{e,0}^2 \quad (2.254)$$

On the other hand, when one of the waves is present, the time evolution of the electron density can be deduced from the Fourier coefficients of the perturbed quantities as

$$n_e(t) = n_{e,0} + \frac{1}{2} (\delta n_e e^{i\omega t} + \delta n_e^* e^{-i\omega t}). \quad (2.255)$$

Likewise, for the velocity perturbation,

$$v_e(t) = v_{e,0} + \frac{1}{2} (\delta v_e e^{i\omega t} + \delta v_e^* e^{-i\omega t}), \quad (2.256)$$

whose square gives to first order

$$v_e^2(t) \approx v_{e,0}^2 + v_{e,0} (\delta v_e e^{i\omega t} + \delta v_e^* e^{-i\omega t}). \quad (2.257)$$

Therefore, we obtain for the kinetic energy

$$\varepsilon(t) = \frac{1}{2}m_e n_e v_e^2 = \bar{\varepsilon} + \frac{1}{2}v_{e,0}(\delta n_e \delta v_e e^{2i\omega t} + \delta n_e^* \delta v_e^* e^{-2i\omega t} + \delta n_e \delta v_e^* + \delta n_e^* \delta v_e). \quad (2.258)$$

In order to draw conclusions regarding the wave influence on the beam energy, the relevant quantity is not the instantaneous kinetic energy $\varepsilon(t)$ but its time average over the wave period (see section 1.4). We have

$$\langle \varepsilon \rangle \equiv \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \varepsilon(t), \quad (2.259)$$

or

$$\langle \varepsilon \rangle = \bar{\varepsilon} + \frac{\omega}{4\pi} \int_0^{2\pi/\omega} dt (\delta n_e \delta v_e^* + \delta n_e^* \delta v_e) = \bar{\varepsilon} + \frac{1}{2}(\delta n_e \delta v_e^* + \delta n_e^* \delta v_e), \quad (2.260)$$

where the terms oscillating at twice the wave frequency have disappeared as a result of the time-averaging. Using the expression for the density perturbation caused by the fast wave (2.252) gives

$$\langle \varepsilon \rangle_f = \bar{\varepsilon} + n_{e,0} \frac{k}{\omega_{pe}} |\delta v_e|^2 > \bar{\varepsilon}, \quad (2.261)$$

The perturbed beam kinetic energy is larger than the equilibrium beam energy. Physically, this is caused by the fact that the density perturbation is in phase with the velocity perturbation. In other words, in regions where particles are numerous, they are also faster. This also means that this wave can also be excited when some energy is brought to the system from the outside, for instance by a suitable antenna[9].

For the slow wave, using (2.252) in (2.260) yields

$$\langle \varepsilon \rangle_s = \bar{\varepsilon} - n_{e,0} \frac{k}{\omega_{pe}} |\delta v_e|^2 < \bar{\varepsilon}. \quad (2.262)$$

The kinetic energy is smaller than the unperturbed beam energy, because the density perturbation is in phase opposition with the velocity perturbation. The slow wave thus corresponds to an instability through which the plasma can channel some energy to the outside world[9].

2.6.3 Beam dissipation

As in section 2.5.1, the easiest way to introduce an absorption mechanism is to add a collision term to the momentum equation. Eq. 2.235 appears with a supplemental term:

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}_{s,0})\delta \mathbf{v}_s + \nu_s \delta \mathbf{v}_s - \Omega_{cs}(\delta \mathbf{v}_s \times \hat{\mathbf{u}}_z) = \frac{q_s}{m_s} \left[\mathbf{E} + \frac{\mathbf{v}_{s,0}}{\omega} \times (\mathbf{k} \times \mathbf{E}) \right]. \quad (2.263)$$

with ν_s the collision frequency. If we consider the same system as in previous sections (i.e. parallel beam and parallel propagation), we obtain for the zz term of the dielectric tensor

$$P' = 1 - \sum_s \frac{\omega_{ps}^2}{(\omega + i\nu_s - k_z v_{s,0})(\omega - k_z v_{s,0})}. \quad (2.264)$$

Assuming that $\nu_e \ll \omega_{pe}$, the high frequency Langmuir wave dispersion relation can be approximated as

$$\omega - k_z v_{e,0} = \pm \omega_{pe}^2 - i \frac{\nu_e}{2}. \quad (2.265)$$

Denoting $\omega \equiv \omega_r + i\omega_i$, we obtain $\omega_i = -\nu_e/2$, showing that both the slow and fast wave experience the same collisional damping.

A more interesting situation arises when the beam propagates in a dissipative (or resistive) medium, characterized by a given isotropic conductivity $\sigma_d > 0$. Physically, this implies that the plasma has the capability of exchanging power with the outside through currents flowing as a response to the wave. Therefore, we write the current perturbation as

$$\delta \mathbf{j} = (\bar{\bar{\sigma}} + \sigma_d \mathbf{1}) \cdot \mathbf{E}. \quad (2.266)$$

With this modification and following the exact same procedure as before leads to

$$P' = 1 + \frac{\omega_d}{\omega} - \sum_s \frac{\omega_{ps}^2}{(\omega + k_z v_{s,0})^2}, \quad (2.267)$$

where we have introduced a frequency characterizing the dielectric dissipation $\omega_d \equiv \sigma_d/\epsilon_0$. The associated dispersion relation of the high frequency Langmuir wave is

$$1 - \frac{\omega_{pe}^2}{(\omega - k_z v_{e,0})^2} + i \frac{\omega_d}{\omega} = 0, \quad (2.268)$$

If we further assume $\omega_d \ll \omega$, we obtain

$$\omega - k_z v_{e,0} \approx \pm \omega_{p,e} \left(1 - i \frac{\omega_d}{2\omega} \right) \quad (2.269)$$

This time, the situation is more complicated because either the imaginary part of ω or the imaginary part of k_z can be positive or negative, depending on the situation under consideration. In principle, the wave damping or wave instabilities are examined assuming that ω and k_z both have an imaginary part, leading to absolute, convective instabilities[4]... , whose study is often complicated. For illustrative purpose, we will simplify the study by separating explicitly the time stability from the space stability.

Time stability: We assume that k_z is real and $\omega \equiv \omega_r + i\omega_i$. Thus

$$(\omega_r - k_z v_{e,0}) + i\omega_i = \pm\omega_{pe} \mp i\frac{\omega_{pe}}{2}\frac{\omega_d}{\omega}, \quad (2.270)$$

so that

$$\omega_i = \mp\frac{\omega_{pe}}{2}\frac{\omega_d}{\omega}. \quad (2.271)$$

This shows that for the fast wave, $\omega_i < 0$: we are in the presence of a decaying process. The slow wave, on the other hand, is unstable since $\omega_i > 0$.

Space stability: Denoting $k_z \equiv k_r + ik_i$, this study is generally more complicated. Indeed, the fields are assumed to be proportional to $\exp(ik_r z)\exp(-k_i z)$ which either increases or decreases depending on direction of z . We will consider the situation $z \rightarrow \infty$, meaning that $k_i > 0$ will correspond to a wave damping, while $k_i < 0$ characterizes an instability.

For the fast and slow waves, we obtain

$$\omega - (k_r + ik_i)v_{e,0} = \pm\omega_{pe} \mp i\frac{\omega_{pe}}{2}\frac{\omega_d}{\omega}, \quad (2.272)$$

so that

$$k_i v_{e,0} = \mp\frac{\omega_{pe}}{2}\frac{\omega_d}{\omega}. \quad (2.273)$$

This shows that $k_i > 0$ for the fast wave, which is stable in space. For the slow wave, we obtain $k_i < 0$, showing that the slow wave corresponds to an instability.

2.6.4 Beam instabilities

We now consider the case of a beam propagating with velocity \mathbf{v}_0 in a plasma at rest. We assume that this beam motion is directed along the confining magnetic field, i.e. $\mathbf{v}_0 = v_0 \hat{\mathbf{u}}_z$. This is actually a very common situation: plasma accelerators are based on this concept. We assume that the plasma is composed of electrons and ions at rest, and the unperturbed beam density is denoted n_b . For a parallel propagating wave and assuming the frequency is large enough to consider only the electron response, the zz term of the dielectric tensor still leads to the dispersion relation

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_b^2}{(\omega - k_z v_0)^2} = 0, \quad (2.274)$$

with $\omega_b^2 \equiv n_b e^2 / (m_e \epsilon_0)$. The quasi-neutrality is obtained by imposing $n_e + n_b = Z_i n_i$. The dispersion relation can be rewritten as

$$\left(1 - \frac{\omega^2}{\omega_{pe}^2}\right)(\omega - k_z v_0)^2 - \eta_b \omega^2 = 0, \quad (2.275)$$

with $\eta_b \equiv \omega_b^2/\omega_{pe}^2 \approx n_b/(n_b+n_e)$, representing the ratio of the number of particles in the beam and in the plasma. As in section 2.6.3, we perform separate studies of the case with ω complex (time stability) and k_z complex (space stability). In the first situation we note $Y \equiv \omega/\omega_{pe}$, and $X \equiv k_z v_0/\omega_{pe}$. The dispersion relation is then given by

$$(Y^2 - 1)(Y - X)^2 - \eta_b Y^2 = 0. \quad (2.276)$$

The result, obtained with $\eta_b = 0.1$, is shown in Fig. 2.16(a).

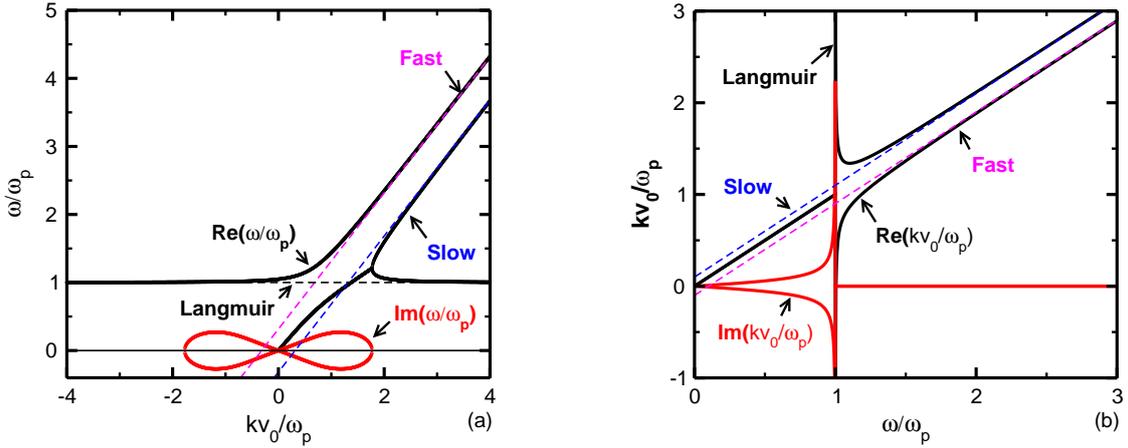


Figure 2.16: Dispersion relation for a beam propagating in a plasma at rest when $\eta_b = 10\%$. (a) ω complex; (b) \mathbf{k} complex.

The Langmuir oscillations in the background plasma can be identified as the line $Y = 1$ on the figure. The fast and slow waves related to the beam are given by $Y = \pm\sqrt{\eta_b} + X$. It can be seen that the Langmuir wave actually couples to the fast and slow waves in a region characterized by $\Im(\omega) \neq 0$, corresponding to a wave damping and wave instability, respectively.

If we now assume that ω is real and k_z is complex, we prefer to consider $Y \equiv kv_0/\omega_{pe}$ versus $X \equiv \omega/\omega_{pe}$. Doing this leads to

$$Y = X \pm \eta_b \frac{X^2}{(X^2 - 1)^{1/2}}. \quad (2.277)$$

This dispersion relation is shown on Fig. 2.16(b). the Langmuir oscillations are this time characterized by the vertical line $X = 1$. The situation $X < 1$ gives an evanescent wave ($k_i > 0$) or a space instability ($k_i < 0$). For $X \gtrsim 1$, the fast and slow wave can be readily recognized[9].

Chapter 3

Kinetic theory of plasma waves

So far, the properties of the various waves propagating in the plasma have been obtained by solving the Maxwell's equations in which the anisotropic, time and space dispersive relation between the current and the electromagnetic field has been injected. This relation has been established in the framework of the fluid theory. However, for the problem at hand, this approach has limitations, some of them having been encountered in the previous chapter. For instance, the individuality of particles is not taken into account, which has the consequence that the plasma fluid can exchange energy with the wave only if a resonance relation such as $\omega = \mathbf{k} \cdot \mathbf{v}_{th,s}$ is verified. Since $\mathbf{v}_{th,s}$ is an averaged velocity, the latter relation clearly does not allow the selection of a class of particles with a given velocity. Realistically, however, we expect that something will happen for particles having velocity \mathbf{v} such as $\omega = \mathbf{k} \cdot \mathbf{v}$, regardless of the details of the distribution function. Another striking illustration of the shortcomings of our simple fluid description has been discussed in section 2.4.3, where the dispersion relation obtained in the presence of thermal effects is clearly flawed. In this chapter, we will show how the substitution of the fluid description with the tools of the kinetic theory allows to cure these limitations.

3.1 Elements of kinetic theory

3.1.1 The Maxwell-Vlasov system

For the particles of species s , when collisions are neglected, it is possible to write the kinetic equation in the form

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \quad (3.1)$$

which is called the Vlasov equation. It can also be written as

$$\frac{df_s}{dt} = 0, \quad (3.2)$$

where d/dt is the convective (Lagrangian) derivative. The Vlasov equation can be interpreted as the fact that the distribution function is constant along its characteristics, which are simply given by

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad (3.3)$$

and

$$\frac{d\mathbf{v}}{dt} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (3.4)$$

The equations governing the motion of the charged particles in the electromagnetic field (\mathbf{E}, \mathbf{B}) are clearly recognized. This field itself is solution of the Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (3.5)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.6)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3.7)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \quad (3.8)$$

The system comprised of the Vlasov equation (3.1) and Maxwell's equations (3.5-3.8) is closed: in principle, the electromagnetic field and the associated particle response can be obtained in a self-consistent fashion.

3.1.2 Linearization

Following the same lines as in the previous chapters, we separate the equilibrium quantities on one hand, and the quantities fluctuating under the effect of the wave on the other hand, by writing the distribution function in the form

$$f_s(\mathbf{r}, \mathbf{v}, t) = f_{s,0}(\mathbf{r}, \mathbf{v}) + \delta f_s(\mathbf{r}, \mathbf{v}, t), \quad (3.9)$$

where the equilibrium distribution function is assumed to be stationary, and thus verifies (we still assume that there is no applied static electric field)

$$\mathbf{v} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = 0. \quad (3.10)$$

If, furthermore, it is assumed that $f_{s,0}$ is homogeneous, the latter equation yields

$$(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = 0. \quad (3.11)$$

In the presence of an equilibrium magnetic field, it is natural to use a system of cylindrical coordinates, $(v_\perp, \phi, v_\parallel)$, with

$$\begin{cases} v_x = v_\perp \cos(\phi), \\ v_y = v_\perp \sin(\phi), \\ v_z = v_\parallel. \end{cases} \quad (3.12)$$

Eq. 3.11 shows that $\partial f_{s,0}/\partial\phi = 0$, which is a consequence of the symmetry properties of the considered system. Therefore, we have $f_{s,0} \equiv f_{s,0}(v_\perp, v_\parallel)$. On the other hand, the fluctuating part of the distribution function is solution of

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} (\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} = -\frac{q_s}{m_s} (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}}. \quad (3.13)$$

where second order terms are neglected. This expression may be rewritten in the form

$$\frac{d \delta f_s}{dt} = -\frac{q_s}{m_s} (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}}, \quad (3.14)$$

where d/dt is the convective derivative. Formally, the solution is given by

$$\delta f_s(\mathbf{r}, \mathbf{v}, t) = -\frac{q_s}{m_s} \int_{-\infty}^t dt' \left\{ [\delta \mathbf{E}(\mathbf{r}', t') + \mathbf{v}' \times \delta \mathbf{B}(\mathbf{r}', t')] \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}'} \right\}. \quad (3.15)$$

This time, the integration is performed along the characteristics

$$\frac{d\mathbf{r}'}{dt'} = \mathbf{v}', \quad (3.16)$$

and

$$\frac{d\mathbf{v}'}{dt'} = \frac{q_s}{m_s} \mathbf{v}' \times \mathbf{B}_0, \quad (3.17)$$

associated to $\mathbf{r}'(t' = t) = \mathbf{r}$ and $\mathbf{v}'(t' = t) = \mathbf{v}$. Since these relations clearly represent the equations of motion of a particle reaching point (\mathbf{r}, \mathbf{v}) à $t' = t$ in the absence of wave, they are called the *unperturbed trajectories (or orbits)*. It should be stressed that the fact that the integral must be performed along the particle orbits obtained in the absence of wave is a direct consequence of the linear hypothesis.

3.2 Electrostatic waves in an unmagnetized plasma

3.2.1 Dispersion relation

In order to start our study of kinetic effects, we first impose the condition $\mathbf{B}_0 = \mathbf{0}$ (unmagnetized plasma). In this case, the linearized Vlasov equation is given by

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = 0. \quad (3.18)$$

Furthermore, we consider electrostatic waves, which are such that $\delta\mathbf{B} = 0$, so that

$$\frac{\partial\delta f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial\delta f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} \delta\mathbf{E} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = 0. \quad (3.19)$$

In this situation, rather than the electric field itself, it is natural to use the electrostatic potential φ , such as $\delta\mathbf{E} \equiv -\nabla\varphi$.

$\varphi \equiv \varphi(\mathbf{r}, t)$ must satisfy the *Poisson equation*, which is obtained by rewriting the linearized version of Eq. 3.5 in the form

$$\nabla^2\varphi = -\sum_s \frac{q_s}{\epsilon_0} \int d^3\mathbf{v} \delta f_s(\mathbf{r}, \mathbf{v}, t). \quad (3.20)$$

Assuming the plasma is locally homogeneous and restricting ourselves to frequency ω , we have in Fourier space

$$-i(\omega - \mathbf{k} \cdot \mathbf{v})\delta f_s - i\frac{q_s}{m_s}\varphi\mathbf{k} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = 0, \quad (3.21)$$

and from Eq. 3.20

$$k^2\varphi(\mathbf{k}, t) = \sum_s \frac{q_s}{\epsilon_0} \int d^3\mathbf{v} \delta f_s(\mathbf{k}, \mathbf{v}, t). \quad (3.22)$$

Solving Eq. 3.21 for δf_s and injecting the obtained result in Eq. 3.22, gives (for $\varphi \neq 0$)

$$1 + \sum_s \frac{q_s^2}{m_s\epsilon_0 k^2} \int d^3\mathbf{v} \frac{\mathbf{k} \cdot \partial f_{s,0}/\partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} = 0. \quad (3.23)$$

This is the dispersion relation for electrostatic waves in an unmagnetized plasma. To obtain $\mathbf{k} = \mathbf{k}(\omega)$, however, there is a serious problem: the integral presents a singularity in velocity space for $\omega = \mathbf{k} \cdot \mathbf{v}$. Historically, Vlasov proposed to solve this issue by considering only the principal value of the integral, which is equivalent to consider two integration paths: one passing above the pole, the other one below the pole, and perform the average. In addition to being rather delicate to justify, this solution leads in fact to incoherent results: for instance, in the case of electrostatic oscillations, if one postulates $\Im(\omega) > 0$, the calculation irremediably leads to $\Im(\omega) < 0$ and vice versa[4]. This problem has been solved by Landau who, in a very influential paper, has shown that it was the result of the substitution $\partial/\partial t \rightarrow -i\omega$. Rather than assuming that the distribution function varies as $\exp(-i\omega t)$, it should be considered that it is given at $t = 0$, and deduce its evolution. We start over with Vlasov equation, keeping the local nature only in space

$$\frac{\partial\delta f_s}{\partial t} + i\mathbf{k} \cdot \mathbf{v}\delta f_s - i\frac{q_s}{m_s}\varphi\mathbf{k} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = 0, \quad (3.24)$$

To solve this initial value problem, the Laplace transform is introduced

$$\delta \tilde{f}_s(\mathbf{r}, \mathbf{v}, p) \equiv \int_0^\infty dt e^{-pt} \delta f_s(\mathbf{r}, \mathbf{v}, t). \quad (3.25)$$

If δf_s does not grow faster than an exponential, i.e. $|\delta f_s| < |M| \exp(\gamma t)$, this integral converges and defines $\delta \tilde{f}_s$ as an analytical function of p , provided that [2] $\Re(p) > |\gamma|$. The inverse Laplace transform is defined as

$$\delta f_s(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} dp e^{pt} \delta \tilde{f}_s(\mathbf{r}, \mathbf{v}, p), \quad (3.26)$$

where σ is such as the integration is performed on the right of all singularities of the integrand. This contour is known as the *Bromwich contour* (see Fig. 3.1(a)).

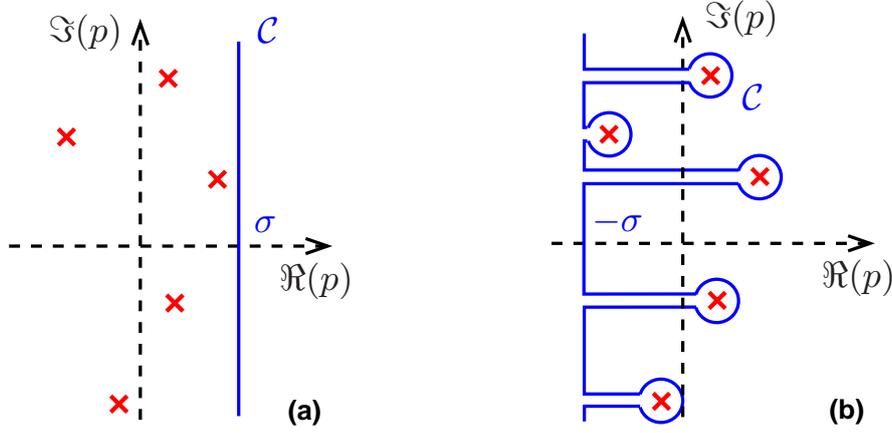


Figure 3.1: (a) Bromwich contour. (b) Deformation of the integration contour following Landau's method. The poles are denoted by crosses.

The Laplace transform of Eq. 3.24 can be written as

$$(p + i\mathbf{k} \cdot \mathbf{v}) \delta \tilde{f}_s - i \frac{q_s}{m_s} \tilde{\varphi} \mathbf{k} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = \delta f_s(t=0), \quad (3.27)$$

with $\tilde{\varphi}$ the Laplace transform of the electrostatic potential, which verifies

$$\tilde{\varphi}(\mathbf{k}, p) = \sum_s \frac{q_s}{\epsilon_0} \frac{1}{k^2} \int d^3\mathbf{v} \delta \tilde{f}_s(\mathbf{k}, \mathbf{v}, p). \quad (3.28)$$

Using Eq. 3.27 in the previous expression yields

$$\tilde{\varphi} \left[1 + \sum_s \frac{q_s^2}{m_s \epsilon_0 k^2} \int d^3\mathbf{v} \frac{\mathbf{k} \cdot \partial f_{s,0} / \partial \mathbf{v}}{ip - \mathbf{k} \cdot \mathbf{v}} \right] = i \sum_s \frac{q_s}{\epsilon_0} \frac{1}{k^2} \int d^3\mathbf{v} \frac{\delta f_s(t=0)}{ip - \mathbf{k} \cdot \mathbf{v}}. \quad (3.29)$$

To deduce φ from this expression, an inverse Laplace transform must be applied. If the number of poles is assumed to be finite, rather than directly considering the Bromwich contour, we will deform it in order to have the vertical line at $\Re(p) = -\sigma$, while staying on the right of all singularities (see Fig. 3.1(b)). This allows to write

$$\varphi(\mathbf{k}, t) = \frac{1}{2i\pi} \int_C dp e^{pt} \tilde{\varphi}(\mathbf{k}, p) = \frac{1}{2i\pi} \int_{-\sigma-i\infty}^{-\sigma+i\infty} dp e^{pt} \tilde{\varphi}(\mathbf{k}, p) + \sum_n e^{p_n t} \left\{ (p - p_n) \tilde{\varphi} \right\} \Big|_{p=p_n}, \quad (3.30)$$

where (p_n) represent the various poles.

The first term on the right hand side comes from the vertical portion of the contour and if we let $p' \equiv \sigma + p$, we see that

$$\frac{1}{2i\pi} \int_{-\sigma-i\infty}^{-\sigma+i\infty} dp e^{pt} \tilde{\varphi}(\mathbf{k}, p) = \frac{\exp(-\sigma t)}{2i\pi} \int_{-i\infty}^{i\infty} dp' e^{p't} \tilde{\varphi}(\mathbf{k}, p), \quad (3.31)$$

which becomes negligible when t increases: this constitutes the justification for the deformation of the Bromwich contour. In other words, rather than looking for a completely general expression for φ , we focus on its behavior after a sufficiently long time (asymptotic behavior).

The second term corresponds to the contribution of the various poles. The presence of the exponential $\exp(p_n t)$ implies that the rightmost term (p_0) will dominate all others for t sufficiently large: the sum can thus be restricted to a single term

$$\varphi(\mathbf{k}, t) \sim \exp(p_0 t) \left[(p - p_0) \tilde{\varphi}(\mathbf{k}, p) \right] \Big|_{p=p_0} \quad (3.32)$$

At this point, another difficulty arises: we have assumed that $\tilde{\varphi}$ was an analytical function for $\Re(p) > |\gamma|$, which is in contradiction with our choice of contour. It is thus necessary to perform the analytical continuation of $\tilde{\varphi}$ and study its behavior as the integration contour is deformed. To do this, it is useful to notice that both the numerator and denominator of $\tilde{\varphi}$ contain integrals of the type

$$I = \int_{-\infty}^{\infty} du \frac{G(u)}{u - ip/k}, \quad (3.33)$$

having let $u \equiv \mathbf{k} \cdot \mathbf{v}/k$. Clearly, $G(u)$ is always analytical and the behavior of these integrals is thus determined by the resonant denominator.

To calculate I when $\Re(p) > 0$, the contour shown on Fig. 3.2(a) can be employed. It is subsequently deformed in order to stay *below* the pole as $\Re(p)$ becomes negative. This is shown on Fig. 3.2(b).

By doing this, the analytical continuation of I has been rigorously obtained. Therefore, I remains an analytical function as the contour is deformed and the behavior of φ is determined by the zeros of the denominator of $\tilde{\varphi}$ (see Eq. 3.30).

$\delta \tilde{f}_s$ is given by

$$\delta \tilde{f}_s(\mathbf{k}, \mathbf{v}, p) = -\frac{q_s}{m_s} \tilde{\varphi} \frac{\mathbf{k} \cdot \partial f_{s,0} / \partial \mathbf{v}}{ip - \mathbf{k} \cdot \mathbf{v}} + i \frac{\delta f_s(t=0)}{ip - \mathbf{k} \cdot \mathbf{v}}. \quad (3.34)$$

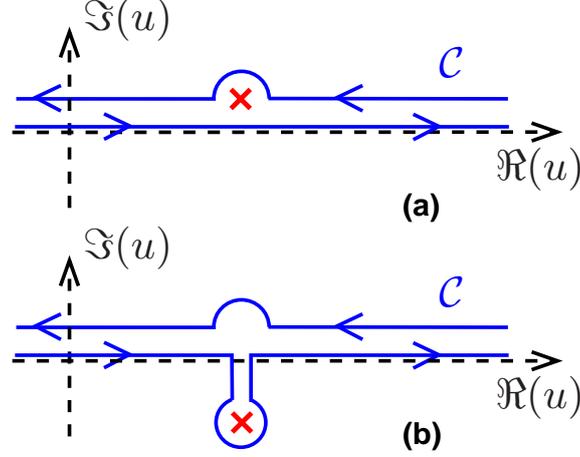


Figure 3.2: Integration contour of I for (a) $\Re(p) > 0$, (b) $\Re(p) < 0$.

The arguments developed above still apply: the behavior of the first term is fixed by the zeros in the denominator of $\tilde{\varphi}$. The Laplace transform of the second term yields

$$\delta f_{s,2}(\mathbf{k}, \mathbf{v}, t) = \frac{1}{2\pi} \int_{\mathcal{C}} dp \frac{\delta f_s(t=0)}{ip - \mathbf{k} \cdot \mathbf{v}} \exp(pt) \quad (3.35)$$

The contribution of the vertical part of the contour can be disregarded, to focus on the pole $p_0 = -i\mathbf{k} \cdot \mathbf{v}$. This pole clearly gives a complex exponential $\exp\{-i(\mathbf{k} \cdot \mathbf{v})t\}$, which varies very rapidly. Its contribution to the charge density, obtained by integration over \mathbf{v} is negligible. The discussion concerning φ is thus directly applicable to the case of δf_s and its behavior is determined by the zeros of the denominator of $\tilde{\varphi}$, given by

$$1 + \sum_s \frac{q_s^2}{m_s \epsilon_0 k^2} \int d^3\mathbf{v} \frac{\mathbf{k} \cdot \partial f_{s,0} / \partial \mathbf{v}}{ip - \mathbf{k} \cdot \mathbf{v}} = 0. \quad (3.36)$$

It is worth mentioning that substituting p with $-i\omega$ in this expression leads to the dispersion relation given by Eq. 3.23, which was obtained directly from a Fourier analysis in time. However, we now have a clear prescription on the way the integral must be performed, since p has a finite real part. Actually, our initial analysis was almost correct. It would have been completely correct, had we started by letting ω have a positive imaginary part, before performing an analytical continuation for $\Im(\omega) < 0$, which actually describes the wave damping by the plasma particles.

3.2.2 Langmuir wave

To illustrate the consequences of the results obtained in the previous section, we restrict ourselves to high frequency oscillations, i.e. we assume that only the electrons

respond to the wave. The dispersion relation can be written as

$$1 + \frac{e^2}{m_e \epsilon_0 k^2} \int d^3 \mathbf{v} \frac{\mathbf{k} \cdot \partial f_{e,0} / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} = 0, \quad (3.37)$$

where we use the ‘‘usual’’ variable ω , rather than p , keeping in mind that ω has a small positive imaginary part. Letting $v_z \equiv \mathbf{k} \cdot \mathbf{v} / k$ and denoting v_x and v_y the two components of the velocity vector perpendicular to \mathbf{k} , we have $d^3 \mathbf{v} = dv_x dv_y dv_z$ and we let

$$F_{e,0}(v_z) \equiv \int dv_x dv_y f_{e,0}(\mathbf{v}). \quad (3.38)$$

The dispersion relation (3.37) becomes

$$1 + \frac{e^2}{m_e \epsilon_0 k} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{e,0} / \partial v_z}{\omega - kv_z} = 0. \quad (3.39)$$

Since ω has a positive imaginary part, the integral is performed by passing below the pole, yielding

$$1 + \frac{e^2}{m_e \epsilon_0 k} \left[\mathcal{P} \left(\int_{-\infty}^{\infty} dv_z \frac{\partial F_{e,0} / \partial v_z}{\omega - kv_z} \right) - i \frac{\pi}{|k|} \frac{\partial F_{e,0}}{\partial v_z} \Big|_{v_z=\omega/k} \right] = 0, \quad (3.40)$$

where \mathcal{P} designates the Cauchy principal value. We assume that the electron distribution function is Maxwellian along \mathbf{k} , or

$$F_{e,0}(v_z) = \frac{n_{e,0}}{v_{th,e} \sqrt{\pi}} \exp \left(- \frac{v_z^2}{v_{th,e}^2} \right), \quad (3.41)$$

where the thermal velocity is defined as $v_{th,e}^2 = 2k_B T_e / m_e$, with T_e the electron temperature and k_B Boltzmann’s constant.

To proceed further with this analytical treatment, we assume $\omega \gg kv_{th,e}$ and thus perform the approximation

$$\frac{1}{\omega - kv_z} \approx \frac{1}{\omega} \left[1 + \frac{kv_z}{\omega} + \frac{(kv_z)^2}{\omega^2} + \frac{(kv_z)^3}{\omega^3} + o\left(\frac{(kv_z)^3}{\omega^3}\right) \right]. \quad (3.42)$$

Noticing that the Maxwellian (3.41) is even for v_z , and thus that its derivative is odd yields

$$\mathcal{P} \left(\int_{-\infty}^{\infty} dv_z \frac{\partial F_{e,0} / \partial v_z}{\omega - kv_z} \right) \approx - \frac{n_{e,0}}{\omega} \left(\frac{k}{\omega} + \frac{3k^3}{2\omega^3} v_{th,e}^2 \right), \quad (3.43)$$

so that the dispersion relation takes the form

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{3k^2 v_{th,e}^2 \omega_{pe}^2}{2\omega^2} - i \frac{\pi}{k|k|} \frac{e^2}{m_e \epsilon_0} \frac{\partial F_{e,0}}{\partial v_z} \Big|_{v_z=\omega/k} = 0, \quad (3.44)$$

with $\omega_{pe}^2 = n_{e,0}e^2/(m_e\epsilon_0)$.

To deduce the propagation and absorption properties, we let $\omega \equiv \omega_r + i\omega_i$ with $|\omega_i| \ll \omega_r$. By doing this, we place ourselves in a regime of weak damping, meaning that the wave can propagate over at least a few periods before being absorbed by the plasma. The advantage of this assumption is that the dispersion relation $\mathcal{D}(\mathbf{k}, \omega) = 0$ can be split into a real and an imaginary part, i.e.,

$$\mathcal{D}(\mathbf{k}, \omega) \equiv \mathcal{D}_r(\mathbf{k}, \omega) + i\mathcal{D}_i(\mathbf{k}, \omega). \quad (3.45)$$

It is readily shown that if $|\mathcal{D}_i| \ll |\mathcal{D}_r|$, ω_r is solution of

$$\mathcal{D}_r(\mathbf{k}, \omega_r) = 0, \quad (3.46)$$

and ω_i is given by

$$\omega_i = -\frac{\mathcal{D}_i(\mathbf{k}, \omega_r)}{\partial\mathcal{D}_r/\partial\omega|_{\omega=\omega_r}}. \quad (3.47)$$

Coming back to Eq. 3.44, we obtain for the real part of ω

$$1 - \frac{\omega_{pe}^2}{\omega_r^2} \left(1 + \frac{3}{2} \frac{k^2 v_{th,e}^2}{\omega_r^2} \right) = 0. \quad (3.48)$$

Since $k^2 v_{th,e}^2/\omega_r^2 \ll 1$, this gives the approximate expression

$$\omega_r^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{th,e}^2, \quad (3.49)$$

which should be compared with the result obtained in section 2.4.2.

The imaginary part is given by

$$\omega_i = \frac{\pi}{2} \frac{\omega_{pe}}{k|k|} \frac{e^2}{m_e\epsilon_0} \left. \frac{\partial F_{e,0}}{\partial v_z} \right|_{v_z=\omega/k}. \quad (3.50)$$

We see that, for $k \geq 0$, if the distribution function decreases with velocity ($\partial F_{e,0}/\partial u \leq 0$), the imaginary part is negative, describing the absorption of the wave by the plasma. This phenomenon is known as *Landau damping* and will be discussed in further details in the next section. Its magnitude is a function of the slope of the distribution in velocity space. The stronger the dependence on u , the more efficient the absorption. On the other hand, if the distribution function is locally flat, no damping occurs. Finally, if F has a positive slope, the plasma will emit a wave in the process of relaxing towards thermodynamical equilibrium.

3.2.3 Landau damping

It is instructive to express the dispersion relation in terms of Debye length. Indeed, we have $\lambda_D^2 \equiv v_{th,e}^2/(2\omega_{pe}^2)$, which means that the approximation employed above, $k^2 v_{th,e}^2/\omega^2 \ll 1$, is equivalent to $k\lambda_D \ll 1$. In physical terms, we assume that the wavelength is much larger than the electron Debye length, which allows to rewrite Eq. 3.49 as

$$\omega^2 = \omega_{pe}^2(1 + 3k^2\lambda_D^2), \quad (3.51)$$

and Eq. 3.50 as¹

$$\omega_i = -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_{pe}}{(k\lambda_D)^3} \exp\left(-\frac{1}{2(k\lambda_D)^2}\right), \quad (3.52)$$

This shows that if $k\lambda_D \ll 1$, the wave is only weakly absorbed by the plasma. In the opposite case, the damping is very strong: a gross interpretation of this phenomenon is that when the wavelength becomes comparable to the Debye length, the wave “explores” the fine structure of the plasma². In the process, it directly exchanges energy with the electrons, and the part of this energy which gets transferred from the wave to the particles is instantaneously dissipated in the irreversible thermal motion. In fact, it is possible to show that this conclusion remains correct in the general case, even though the hypotheses that have been employed to derive Eq. 3.52 impose that $|\omega_i| \ll \omega_r$.

In order to clarify this interpretation, we can build a simple one-dimensional model. We consider a wave whose electric field is given by $E(x, t) = E_0 \cos(kx - \omega t)$ and a particle at initial position x_0 and initial velocity v_0 (see Fig. 3.3).

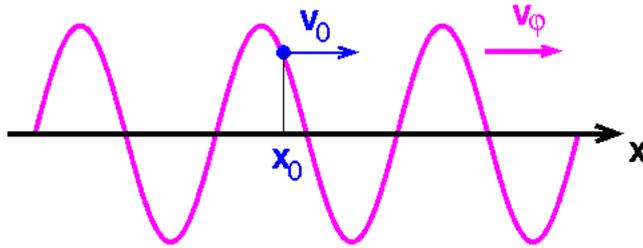


Figure 3.3: Particle in an oscillating field.

The equation of motion of the particle in the wave field is

$$\frac{dv}{dt} = \frac{q}{m} E_0 \cos(kx(t) - \omega t). \quad (3.53)$$

¹Except otherwise noted, we assume $k \geq 0$.

²This really is a crude explanation, because the Vlasov equation itself only holds for space scalelengths large compared to the Debye length.

In the framework of a linear theory, in which the wave acts as a perturbation, it is legitimate to consider that for small values of t , the variation of $x(t)$ in the phase term is determined by the unperturbed motion, i.e. $x(t) \approx x_0 + v_0 t$. In this case, Eq. 3.53 can be rewritten as

$$\frac{dv}{dt} = \frac{q}{m} E_0 \cos[(kv_0 - \omega)t + kx_0], \quad (3.54)$$

which is integrated to give

$$v(t) = \frac{q}{m} \frac{E_0}{kv_0 - \omega} \left\{ \sin[(kv_0 - \omega)t] \cos(kx_0) + (\cos[(kv_0 - \omega)t] - 1) \sin(kx_0) \right\} + v_0, \quad (3.55)$$

where the initial condition $v(t = 0) = v_0$ has been used.

We now consider a particle with initial velocity very close to the wave phase velocity: $v_0 \approx v_\varphi = \omega/k$, thus $kv_0 - \omega \rightarrow 0$. We obtain

$$\Delta v \equiv v(t) - v_0 \approx \frac{q}{m} E_0 t \cos(kx_0), \quad (3.56)$$

which shows that at small times, resonant particles have an energy varying linearly with time. That they gain or lose energy depends on their initial phase, as shown by the $\cos(kx_0)$ terms. The non-resonant particles have an oscillatory motion.

In order to understand how the wave gets eventually absorbed or emitted by the plasma, we firstly consider a population of particles with velocities $v_0 + \delta v$ slightly larger than the wave phase velocity. If the wave gives energy to the particles, their velocity will increase and they will become progressively less resonant, since they leave the “interaction zone” in velocity space. On the other hand, particles which yield energy to the wave will slow down, and will be even more resonant. On average, the particles losing energy will therefore have an efficient interaction with the wave and will tend to reinforce the electric field. In the opposite case, the interaction of a population of particles with initial velocities $v_0 - \delta v$ slightly smaller than the wave phase velocity will be the most efficient when the wave provides them energy. Thus, they tend to absorb the wave. Now, in a plasma in thermodynamical equilibrium, the distribution function decreases with energy and the particles with velocity $v_0 - \delta v$ are more numerous than particles with velocity $v_0 + \delta v$. Therefore, on average, the wave will be absorbed. On the other hand, a plasma with a distribution function increasing with energy will relax by emitting a wave. This explains why the absorption strength is directly determined by the slope of the distribution function. Incidentally, we can see that the Landau damping phenomenon can never be explained by models considering a single particle in the wave field: it is necessary to consider an assembly with a given distribution function. This process is therefore intrinsically of kinetic nature.

3.2.4 Ion acoustic wave

In section 2.4.2, we have found a wave propagating in the range of frequencies $v_{th,i} \ll \omega/k \ll v_{th,e}$. For these parameters, it is not possible to neglect the ion contribution and the appropriate electrostatic dispersion relation is therefore

$$1 + \frac{e^2}{m_e \epsilon_0 k} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{e,0}/\partial v_z}{\omega - kv_z} + \frac{Z_i^2 e^2}{m_i \epsilon_0 k} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{i,0}/\partial v_z}{\omega - kv_z} = 0, \quad (3.57)$$

written for ions with charge number Z_i and mass m_i . As in the previous section, the two distribution functions are assumed to be Maxwellian in the direction determined by \mathbf{k} , respectively characterized by the thermal velocities $v_{th,e}$ and $v_{th,i}$ (see Eq. 3.41). For the electrons, the fact that $\omega \ll kv_{th,e}$ and that the distribution function is Maxwellian allows to approximate the resonant denominator as

$$\frac{\partial F_{e,0}/\partial v_z}{\omega - kv_z} \approx -\frac{\partial F_{e,0}}{\partial v_z} \frac{1}{kv_z} \left(1 + \frac{\omega}{kv_z} + \frac{\omega^2}{k^2 v_z^2}\right). \quad (3.58)$$

On the other hand, for the ions, the appropriate expansion is

$$\frac{\partial F_{i,0}/\partial v_z}{\omega - kv_z} \approx \frac{\partial F_{i,0}}{\partial v_z} \frac{1}{\omega} \left(1 + \frac{kv_z}{\omega} + \frac{k^2 v_z^2}{\omega^2}\right). \quad (3.59)$$

Retaining only the lowest order non-zero contributions, following the same method as in section 3.2.2, we obtain

$$\mathcal{P} \left(\int_{-\infty}^{\infty} dv_z \frac{\partial F_{e,0}/\partial v_z}{\omega - kv_z} \right) \approx \frac{2n_{e,0}}{k_z v_{th,e}^2}, \quad (3.60)$$

and also

$$\mathcal{P} \left(\int_{-\infty}^{\infty} dv_z \frac{\partial F_{i,0}/\partial v_z}{\omega - kv_z} \right) \approx -\frac{kn_{i,0}}{\omega^2}. \quad (3.61)$$

The real part of the dispersion relation can now be obtained in the form

$$\mathcal{D}_r(\omega, k) = 1 + 2 \frac{\omega_{pe}^2}{k^2 v_{th,e}^2} - \frac{\omega_{pi}^2}{\omega^2}. \quad (3.62)$$

From Eq. 3.46, we deduce ω_r , the real part of the frequency:

$$\omega_r^2 = \omega_{pi}^2 \frac{\lambda_D^2 k^2}{1 + \lambda_D^2 k^2}. \quad (3.63)$$

As discussed in the previous section, the regime of weak absorption is valid if $\lambda_D^2 k^2 \ll 1$. On the other hand, we note that

$$\omega_{pi}^2 \lambda_D^2 = Z_i \frac{k_B T_e}{m_i} \equiv c_s^2, \quad (3.64)$$

with c_s the sound velocity, already introduced in section 2.4.2. (3.62) can be approximated by the relation

$$\omega_r \approx kc_s. \quad (3.65)$$

Using Eq. 3.57 gives, for the imaginary part of the dispersion relation

$$\mathcal{D}_i(\omega, k) = -i\pi \frac{e^2}{m_e \epsilon_0 k} \frac{\partial F_{e,0}}{\partial v_z} \Big|_{v_z=\omega/k} - i\pi \frac{Z_i^2 e^2}{m_i \epsilon_0 k} \frac{\partial F_{i,0}}{\partial v_z} \Big|_{v_z=\omega/k}, \quad (3.66)$$

yielding for the imaginary part of the frequency, ω_i , using Eqs. 3.47 and 3.62

$$\omega_i = -\frac{\sqrt{\pi}}{2} \left(\frac{c_s}{v_{th,i}} \right)^3 kc_s \left[e^{-\omega_{pi}^2 \lambda_D^2 / v_{th,i}^2} + \frac{1}{Z_i} \left(\frac{m_e}{m_i} \right)^{1/2} \left(\frac{T_i}{T_e} \right)^{3/2} e^{-\omega_{pi}^2 \lambda_D^2 / v_{th,e}^2} \right]. \quad (3.67)$$

The second term in the bracket appears to be significant only if $T_i \gtrsim 12T_e$, which would characterize a very peculiar plasma. This means that in general, the electron damping is negligible compared to the ion damping, so that

$$\omega_i \approx -\frac{\sqrt{\pi}}{2} \left(\frac{c_s}{v_{th,i}} \right)^3 kc_s e^{-\omega_{pi}^2 \lambda_D^2 / v_{th,i}^2}. \quad (3.68)$$

The latter expression shows that ion damping will be small only when $\omega_{pi}^2 \lambda_D^2 \gg v_{th,i}^2$, which translates into

$$\frac{Z_i T_e}{2 T_i} \gg 1, \quad (3.69)$$

demonstrating that the ion acoustic wave is heavily damped in plasmas with $T_e \sim T_i$. It is usually considered that the ion acoustic wave can propagate only if $T_e \gtrsim 5T_i$. Therefore, it cannot propagate in magnetic fusion device. Nevertheless, it can still have a significant influence on non-linear phenomena[7], for instance by mediating interactions between other propagating waves.

On Fig. 3.4, the dispersion relations for the Langmuir wave (3.49) and the ion acoustic wave (3.63) are shown versus k .

3.2.5 The plasma dispersion function

In this section, we consider the general dispersion relation for electrostatic waves

$$1 + \sum_s \frac{q_s^2}{m_s \epsilon_0 k} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s,0} / \partial v_z}{\omega - kv_z} = 0. \quad (3.70)$$

Assuming that all species are Maxwellian along z with thermal velocity $v_{th,s}$, we obtain

$$1 + 2 \sum_s \frac{\omega_{ps}^2}{kv_{th,s}^3} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv_z \frac{v_z \exp(-v_z^2 / v_{th,s}^2)}{kv_z - \omega} = 0. \quad (3.71)$$

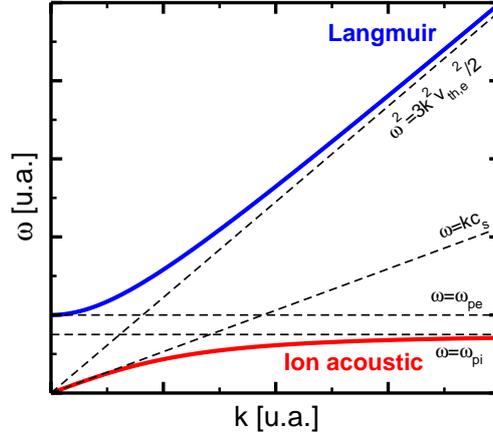


Figure 3.4: Dispersion relation for electrostatic waves in parallel propagation: Langmuir wave and ion acoustic wave.

Letting $\xi \equiv v_z/v_{th,s}$ and $\eta \equiv \omega/(k_z v_{th,s})$, we have

$$1 + 2 \sum_s \frac{\omega_{ps}^2}{k^2 v_{th,s}^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \frac{\xi \exp(-\xi^2)}{\xi - \eta} = 0. \quad (3.72)$$

At this point, we can introduce the *plasma dispersion function*

$$Z(\eta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \frac{\exp(-\xi^2)}{\xi - \eta}, \quad (3.73)$$

for $\Im(\eta) > 0$. The analytical continuation for negative values of $\Im(\eta)$ is obtained by writing

$$Z(\eta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \frac{\exp(-\xi^2)}{\xi - \eta} + i\sqrt{\pi} \exp(-\eta^2). \quad (3.74)$$

Rather than having to deal with two different expressions, it is possible to show that a general expression for Z , valid in the whole complex plane for η is

$$Z(\eta) = i\sqrt{\pi} \exp(-\eta^2) \operatorname{erfc}(-i\eta), \quad (3.75)$$

with erfc the complementary error function[3].

At this point, it is useful to notice that

$$\frac{\partial}{\partial \eta} \int_{-\infty}^{\infty} d\xi \frac{\exp(-\xi^2)}{\xi - \eta} = 2 \int_{-\infty}^{\infty} d\xi \frac{\xi \exp(-\xi^2)}{\xi - \eta}, \quad (3.76)$$

where an integration by parts has been performed on ξ . This proves that

$$Z'(\eta) \equiv -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \frac{\xi \exp(-\xi^2)}{\xi - \eta} = -2[1 + \eta Z(\eta)], \quad (3.77)$$

and the dispersion relation (3.70) becomes

$$1 - \sum_s \frac{\omega_{ps}^2}{k^2 v_{th,s}^2} Z' \left(\frac{\omega}{k v_{th,s}} \right) = 0. \quad (3.78)$$

The Z function is very frequently encountered in kinetic theory, when the considered distribution function are Maxwellians. The advantage is that its properties have been thoroughly studied and tabulated[10]. It also has the advantage of embedding the issue of Landau's analytical continuation.

For small arguments, Z can be approximated as

$$Z(\eta) = i\sqrt{\pi} \exp(-\eta^2) - 2\eta \left[1 - \frac{2}{3}\eta^2 + \frac{4}{15}\eta^4 \right] + o(\eta^6). \quad (3.79)$$

The asymptotic expansion is also very useful:

$$Z(\eta) = i\sqrt{\pi}\sigma \exp(-\eta^2) - \frac{1}{\eta} \left[1 + \frac{1}{2\eta^2} + \frac{3}{4\eta^4} \right] + o(1/\eta^6), \quad (3.80)$$

with

$$\begin{cases} \sigma = 0 & \text{if } \Im(\eta) > 1/\Re(\eta), \\ \sigma = 1 & \text{if } |\Im(\eta)| < 1/\Re(\eta), \\ \sigma = 2 & \text{if } -\Im(\eta) > 1/\Re(\eta). \end{cases} \quad (3.81)$$

A straightforward consequence, immediately apparent from relations (3.79) and (3.80), but valid in all cases, is that the imaginary part of Z , linked to the wave damping (or emission) becomes significant only when $\eta \lesssim 1$. The physical reason is that in this case, we have $\omega/k \lesssim v_{th,s}$, meaning that the wave phase velocity is lower than or in the range of the particle thermal velocity (which is representative of the various particles velocities, for a Maxwellian distribution), allowing for an efficient wave/particle interaction. All results obtain in sections 3.2.2 and 3.2.4 can be easily recovered by using the plasma dispersion function rather than perform expansions of the resonant denominator.

3.3 Waves in a magnetized plasma

3.3.1 Unperturbed orbits

Unfortunately, in the presence of a finite confining magnetic field \mathbf{B}_0 and as discussed in section 3.1.2, there is no way to circumvent the integral over unperturbed orbits in the expression for the distribution function perturbation

$$\delta f_s(\mathbf{r}, \mathbf{v}, t) = -\frac{q_s}{m_s} \int_{-\infty}^t dt' \left\{ [\delta \mathbf{E}(\mathbf{r}', t') + \mathbf{v}' \times \delta \mathbf{B}(\mathbf{r}', t')] \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}'} \right\}. \quad (3.82)$$

The procedure of analytical continuation which has been followed in section 3.2.1 to obtain the dispersion relation for $\Im(\omega) < 0$ is still applicable. If we assume that the perturbed quantities vary as $\exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t))$, the integral can be rewritten in Fourier space as

$$\delta f_{\mathbf{k},\omega} = -\frac{q_s}{m_s} \int_{-\infty}^t dt' \left\{ \left[\delta \mathbf{E}_{\mathbf{k},\omega} + \mathbf{v}' \times \delta \mathbf{B}_{\mathbf{k},\omega} \right] \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}'} \right\} e^{i(\mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}) + \omega \tau)}, \quad (3.83)$$

with $\tau \equiv t - t'$. Maxwell's equations give

$$\delta \mathbf{B}_{\mathbf{k},\omega} = \frac{\mathbf{k}}{\omega} \times \delta \mathbf{E}_{\mathbf{k},\omega}, \quad (3.84)$$

so that

$$\delta f_{\mathbf{k},\omega} = -\frac{q_s}{m_s} \int_0^\infty d\tau \left\{ \left[\delta \mathbf{E}_{\mathbf{k},\omega} + \frac{\mathbf{v}' \times (\mathbf{k} \times \delta \mathbf{E}_{\mathbf{k},\omega})}{\omega} \right] \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}'} \right\} e^{i(\mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}) + \omega \tau)}, \quad (3.85)$$

which can be cast in the form

$$\delta f_{\mathbf{k},\omega} = -\frac{q_s}{m_s} \int_0^\infty d\tau \left\{ \left[\left(1 - \frac{\mathbf{k} \cdot \mathbf{v}'}{\omega} \right) \bar{\mathbf{1}} + \frac{\mathbf{k} \mathbf{v}'}{\omega} \right] \delta \mathbf{E}_{\mathbf{k},\omega} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}'} \right\} e^{i(\mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}) + \omega \tau)}, \quad (3.86)$$

where the $\mathbf{k} \mathbf{v}'$ term appearing in the bracket is a dyad³.

The differential equation for the unperturbed velocity is given by

$$\frac{d\mathbf{v}'}{dt} = \frac{q_s}{m_s} \mathbf{v}' \times \mathbf{B}_0. \quad (3.87)$$

In the cylindrical system of coordinates $(v_\perp, \phi, v_\parallel)$, this gives

$$\begin{cases} dv'_x/dt = \Omega_{cs} v'_y, \\ dv'_y/dt = -\Omega_{cs} v'_x, \\ dv'_x/dt = 0, \end{cases} \quad (3.88)$$

The previous equations can be integrated to give

$$\begin{cases} v'_x = v_x \cos(\Omega_{cs} \tau) - v_y \sin(\Omega_{cs} \tau) \\ v'_y = v_x \sin(\Omega_{cs} \tau) + v_y \cos(\Omega_{cs} \tau) \\ v'_z = v_\parallel, \end{cases} \quad (3.89)$$

where the ‘‘final’’ condition $\mathbf{v}'(t' = t) = \mathbf{v}$ has been used. It is actually more convenient to express v'_x and v'_y as functions of v_\perp rather than v_x and v_y :

$$\begin{cases} v'_x = v_\perp \cos(\Omega_{cs} \tau + \phi) \\ v'_y = v_\perp \sin(\Omega_{cs} \tau + \phi) \\ v'_z = v_\parallel, \end{cases} \quad (3.90)$$

³This dyad is obtained by performing the outer product of \mathbf{k} and \mathbf{v}' , which yields a second rank tensor, whose terms are given by $(\mathbf{k} \mathbf{v}')_{ij} = k_i v'_j$.

These three equations can again be integrated once to obtain the position, using the final condition $\mathbf{r}'(t' = t) = \mathbf{r}$

$$\begin{cases} x' &= x - v_{\perp}/\Omega_{cs} [\sin(\Omega_{cs}\tau + \phi) - \sin(\phi)] \\ y' &= y + v_{\perp}/\Omega_{cs} [\cos(\Omega_{cs}\tau + \phi) - \cos(\phi)] \\ z' &= z - v_{\parallel}\tau, \end{cases} \quad (3.91)$$

In a homogeneous plasma (and uniform magnetic field), we know that $v_{\perp} \equiv (v_x^2 + v_y^2)^{1/2}$ and v_{\parallel} are constants of motion, which implies $f_{s,0}(v'_{\perp}, v'_z) = f_{s,0}(v_{\perp}, v_{\parallel})$. On the other hand, we have

$$\frac{\partial f_{s,0}}{\partial v'_x} = \frac{\partial v_{\perp}}{\partial v_x} \frac{\partial f_{s,0}}{\partial v_{\perp}} = \cos(\Omega_{cs}\tau + \phi) \frac{\partial f_0}{\partial v_{\perp}}, \quad (3.92)$$

$$\frac{\partial f_{s,0}}{\partial v'_y} = \frac{\partial v_{\perp}}{\partial v_y} \frac{\partial f_{s,0}}{\partial v_{\perp}} = \sin(\Omega_{cs}\tau + \phi) \frac{\partial f_0}{\partial v_{\perp}}, \quad (3.93)$$

and also

$$\frac{\partial f_{s,0}}{\partial v'_z} = \frac{\partial f_0}{\partial v_{\parallel}}. \quad (3.94)$$

3.3.2 Dielectric tensor

Owing to the axial symmetry around $\hat{\mathbf{u}}_z$, it is possible to assume that \mathbf{k} lies in the plane $(\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_z)$, and write $\mathbf{k} \equiv k_{\perp}\hat{\mathbf{u}}_x + k_z\hat{\mathbf{u}}_z$. Using relations 3.91, the phase term in (3.86) can be rewritten as

$$\begin{aligned} \mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}) + \omega\tau &= -\frac{k_{\perp}v_{\perp}}{\Omega_{cs}} [\sin(\Omega_{cs}\tau + \phi) - \sin(\phi)] \\ &\quad + (\omega - k_{\parallel}v_{\parallel})\tau. \end{aligned} \quad (3.95)$$

At this point, a useful relation is

$$\exp(ia \sin(x)) = \sum_{n=-\infty}^{\infty} J_n(a) \exp(inx), \quad (3.96)$$

with J_n the Bessel function of the first kind and of order n . This gives

$$e^{i(\mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}) + \omega\tau)} = \sum_{m,n=-\infty}^{\infty} J_m\left(\frac{k_{\perp}v_{\perp}}{\Omega_{cs}}\right) J_n\left(\frac{k_{\perp}v_{\perp}}{\Omega_{cs}}\right) e^{i(\omega - n\Omega_{cs} - k_{\parallel}v_{\parallel})\tau} e^{i(m-n)\phi}. \quad (3.97)$$

Using the unperturbed particle trajectories and Eqs. 3.92, 3.93 and 3.94, we can expand the integrand of (3.86) as

$$\begin{aligned} \left[\left(1 - \frac{\mathbf{k} \cdot \mathbf{v}'}{\omega}\right) \bar{\mathbf{1}} + \frac{\mathbf{k}\mathbf{v}'}{\omega} \right] \delta \mathbf{E}_{\mathbf{k},\omega} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}'} &= \hat{U} f_{s,0} \cos(\Omega_{cs}\tau + \phi) \delta E_x \\ &\quad + \hat{U} f_{s,0} \sin(\Omega_{cs}\tau + \phi) \delta E_y + \left(\frac{\partial f_{s,0}}{\partial v_{\parallel}} + \hat{V} f_{s,0} \right) \cos(\Omega_{cs}\tau + \phi), \end{aligned} \quad (3.98)$$

having defined

$$\hat{U}f_{s,0} \equiv \frac{\partial f_{s,0}}{\partial v_{\perp}} + \frac{k_{\parallel}}{\omega} \left(v_{\perp} \frac{\partial f_{s,0}}{\partial v_{\parallel}} - v_{\parallel} \frac{\partial f_{s,0}}{\partial v_{\perp}} \right), \quad (3.99)$$

and

$$\hat{V}f_{s,0} \equiv -\frac{k_{\perp}}{\omega} \left(v_{\perp} \frac{\partial f_{s,0}}{\partial v_{\parallel}} - v_{\parallel} \frac{\partial f_{s,0}}{\partial v_{\perp}} \right). \quad (3.100)$$

We recall that the dielectric tensor is deduced from the conductivity. To obtain the latter, we write

$$\mathbf{j} \equiv \sum_s \mathbf{j}_s = \sum_s q_s \int d^3\mathbf{v} \mathbf{v} \delta f_s \equiv \bar{\bar{\boldsymbol{\sigma}}} \cdot \delta \mathbf{E}. \quad (3.101)$$

Using relations 3.90 to obtain explicit expressions for the velocity, we can write

$$\sum_{i=1}^3 \sigma_{xi} \delta E_i = \sum_s q_s \int d^2\mathbf{v} v_{\perp} \int_0^{2\pi} d\phi \cos(\Omega_{cs}\tau + \phi) \delta f_{\mathbf{k},\omega}, \quad (3.102)$$

$$\sum_{i=1}^3 \sigma_{yi} \delta E_i = \sum_s q_s \int d^2\mathbf{v} v_{\perp} \int_0^{2\pi} d\phi \sin(\Omega_{cs}\tau + \phi) \delta f_{\mathbf{k},\omega}, \quad (3.103)$$

and

$$\sum_{i=1}^3 \sigma_{zi} \delta E_i = \sum_s q_s \int d^2\mathbf{v} v_{\parallel} \int_0^{2\pi} d\phi \delta f_s. \quad (3.104)$$

We have defined

$$\int d^2\mathbf{v} \equiv \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} dv_{\perp} v_{\perp}. \quad (3.105)$$

As an example, only the calculation of σ_{xx} will be presented here. The calculation of the various terms of $\bar{\bar{\boldsymbol{\sigma}}}$ can be performed by following the exact same lines and is left as an exercise to the reader. From Eqs. 3.86 and 3.102, we have

$$\begin{aligned} \sigma_{xx} = - \sum_s \frac{q_s^2}{m_s} \int d^2\mathbf{v} v_{\perp} \int_0^{\infty} d\tau \sum_{m,n} e^{i(\omega - n\Omega_{cs} - k_{\parallel}v_{\parallel})\tau} J_m(\xi_s) J_n(\xi_s) \times \\ \hat{U}f_{s,0} \int_0^{2\pi} d\phi \cos(\Omega_{cs}\tau + \phi) \cos(\phi) e^{i(m-n)\phi} \end{aligned} \quad (3.106)$$

The integration on the variable ϕ yields

$$\int_0^{2\pi} d\phi \cos(\Omega_{cs}\tau + \phi) \cos(\phi) e^{i(m-n)\phi} = \frac{\pi}{2} \left[e^{i\Omega_{cs}\tau} (\delta_{m,n-2} + \delta_{m,n}) + e^{-i\Omega_{cs}\tau} (\delta_{m,n} + \delta_{m,n+2}) \right]. \quad (3.107)$$

Injecting in (3.106), we obtain

$$\sigma_{xx} = -\frac{1}{4} \sum_s \frac{q_s^2}{m_s} \int d^3\mathbf{v} v_\perp \int_0^\infty d\tau \hat{U} f_{s,0} \sum_{m,n} J_m(\xi_s) J_n(\xi_s) \times \left[e^{i(\omega - (n-1)\Omega_{cs} - k_\parallel v_\parallel)\tau} (\delta_{m,n-2} + \delta_{m,n}) + e^{i(\omega - (n+1)\Omega_{cs} - k_\parallel v_\parallel)\tau} (\delta_{m,n-2} + \delta_{m,n}) \right]. \quad (3.108)$$

In the bracket on the right hand side, it is convenient to perform the substitution $n \pm 1 \rightarrow n$, using the fact that the sum over m and n is infinite. We can write

$$\sigma_{xx} = -\frac{1}{4} \sum_s \frac{q_s^2}{m_s} \int d^3\mathbf{v} v_\perp \int_0^\infty d\tau e^{i(\omega - n\Omega_{cs} - k_\parallel v_\parallel)\tau} \hat{U} f_{s,0} \sum_{m,n} (\delta_{m,n-1} + \delta_{m,n+1}) \left[J_m(\xi_s) J_{n+1}(\xi_s) + J_m(\xi_s) J_{n-1}(\xi_s) \right]. \quad (3.109)$$

Letting $\xi_s \equiv k_\perp v_\perp / \Omega_s$, we obtain

$$\sigma_{xx} = -i \sum_s \frac{q_s^2}{m_s} \sum_{n=-\infty}^\infty \int d^3\mathbf{v} \frac{v_\perp \hat{U} f_{s,0}}{\omega - n\Omega_{cs} - k_\parallel v_\parallel} \left(\frac{n J_n(\xi_s)}{\xi_s} \right)^2, \quad (3.110)$$

where the relation

$$J_{l+1}(z) + J_{l-1}(z) = \frac{2l}{z} J_l(z), \quad (3.111)$$

has been used. The time integral has been performed by recalling that ω has a small positive imaginary part, so that the integrand is zero when $\tau \rightarrow \infty$ (i.e. $t' \rightarrow -\infty$). The corresponding term of the dielectric tensor can be written as

$$K_{xx} \equiv \frac{\epsilon_{xx}}{\epsilon_0} = 1 + \sum_s \frac{q_s^2}{\omega \epsilon_0 m_s} \sum_{n=-\infty}^\infty \int d^3\mathbf{v} \frac{v_\perp \hat{U} f_{s,0}}{\omega - n\Omega_{cs} - k_\parallel v_\parallel} \left(\frac{n J_n(\xi_s)}{\xi_s} \right)^2. \quad (3.112)$$

As mentioned above, the calculation of the other terms does not pose any essential difficulty. In order to obtain a final expression as compact as possible, the following relation is useful:

$$J_{l+1}(z) - J_{l-1}(z) = -2J'_l(z). \quad (3.113)$$

Finally, the dielectric tensor terms can be written as

$$K_{ij} = \delta_{ij} + \sum_s \frac{q_s^2}{\omega^2 \epsilon_0 m_s} \sum_{n=-\infty}^\infty \int d^3\mathbf{v} \frac{S_{n,ij}}{\omega - n\Omega_{cs} - k_\parallel v_\parallel}, \quad (3.114)$$

where the *susceptibility tensor* is given by

$$\bar{\mathbf{S}}_n = \begin{pmatrix} v_{\perp}(nJ_n/\xi_s)^2 U & iv_{\perp}(n/\xi_s)J_n J'_n U & v_{\perp}(n/\xi_s)J_n^2 W \\ -iv_{\perp}(n/\xi_s)J_n J'_n U & v_{\perp}(J'_n)^2 U & -iv_{\perp}J_n J'_n W \\ v_{\parallel}(n/\xi_s)J_n^2 U & iv_{\parallel}J_n J'_n U & v_{\parallel}J_n^2 W \end{pmatrix}, \quad (3.115)$$

having defined

$$U \equiv \omega \hat{U} f_{s,0} = (\omega - k_{\parallel} v_{\parallel}) \frac{\partial f_{s,0}}{v_{\perp}} + v_{\perp} k_{\parallel} \frac{\partial f_{s,0}}{\partial v_{\parallel}}, \quad (3.116)$$

and

$$W \equiv n\Omega_{cs} \frac{v_{\parallel}}{v_{\perp}} \frac{\partial f_{s,0}}{\partial v_{\perp}} + (\omega - n\Omega_{cs}) \frac{\partial f_{s,0}}{\partial v_{\parallel}}. \quad (3.117)$$

The argument in all Bessel functions is $k_{\perp} v_{\perp} / \Omega_{cs}$.

3.3.3 Maxwellian distribution functions

In many situations, the considered distribution functions can be taken as Maxwellians, which allows to express the various terms of the dielectric tensor (3.114) as functions of the plasma dispersion function (see section 3.2.5). To be specific, we consider

$$f_{s,0}(\mathbf{r}, v_{\parallel}, v_{\perp}) = \frac{n_{s,0}(\mathbf{r})}{\pi^{3/2} v_{th,s}^3} \exp\left(-\frac{v^2}{v_{th,s}^2}\right), \quad (3.118)$$

with $v_{th,s}$ the thermal velocity for species s . In this case, U (Eq. 3.116) is simply given by

$$U = -2 \frac{\omega v_{\perp}}{v_{th,s}^2} f_{s,0}, \quad (3.119)$$

which yields the following expression for the xx term of the dielectric tensor:

$$K_{xx} = 1 - 4 \sum_s \frac{\omega_{ps}^2}{\omega} \frac{\Omega_{cs}^2}{v_{th,s}^5 k_{\perp}^2} \sum_{n=-\infty}^{\infty} n^2 \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv_{\parallel} \frac{\exp(-v_{\parallel}^2/v_{th,s}^2)}{\omega - n\Omega_{cs} - k_{\parallel} v_{\parallel}} \times \int_0^{\infty} dv_{\perp} v_{\perp} J_n^2(k_{\perp} v_{\perp} / \Omega_{cs}) e^{-v_{\perp}^2/v_{th,s}^2}. \quad (3.120)$$

In order to handle the integral over perpendicular velocity, the following formula[11] can be employed

$$\int_0^{\infty} dt t J_{\nu}(at) J_{\nu}(bt) \exp(-p^2 t^2) = \frac{1}{2p^2} \exp\left(-\frac{a^2 + b^2}{4p^2}\right) I_{\nu}\left(\frac{ab}{2p^2}\right). \quad (3.121)$$

It is valid for $\Re(\nu) > -1$, $|\arg p| < \pi/4$ and $a, b > 0$. I_ν is the modified Bessel function of order ν . It is then possible to recognize the Z function in the velocity integral and we finally obtain

$$K_{xx} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{k_{\parallel} v_{th,s}} \sum_n n^2 \frac{I_n(\lambda_s) e^{-\lambda_s}}{\lambda_s} Z_n \left(\frac{\omega - n\Omega_{cs}}{k_{\parallel} v_{th,s}} \right), \quad (3.122)$$

with

$$\lambda_s \equiv \frac{k_{\perp}^2 v_{th,s}^2}{2\Omega_{cs}^2}. \quad (3.123)$$

The calculation of the other terms is based on the same principle and will not be detailed here. Eventually, the dielectric tensor terms can be expressed as

$$K_{ij} = \delta_{ij} + \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{k_{\parallel} v_{th,s}} \exp(-\lambda_s) \sum_{n=-\infty}^{\infty} T_{n,ij}, \quad (3.124)$$

with

$$\bar{\mathbf{T}}_n = \begin{pmatrix} n^2 I_n Z / \lambda_s & -in(I_n - I'_n)Z & -nI_n Z' / (2\lambda_s)^{1/2} \\ in(I_n - I'_n)Z & [n^2 I_n / \lambda_s + 2\lambda_s(I_n - I'_n)]Z & -i(2\lambda_s)^{1/2}(I_n - I'_n)Z' / 2^{1/2} \\ -nI_n Z' / (2\lambda_s)^{1/2} & i(2\lambda_s^{1/2})(I_n - I'_n)Z' & -x_{n,s} I_n Z' \end{pmatrix}, \quad (3.125)$$

where

$$x_{n,s} \equiv \frac{\omega - n\Omega_{cs}}{k_{\parallel} v_{\parallel}}. \quad (3.126)$$

In Eq. 3.125, the abbreviations $I_n \equiv I_n(\lambda_s)$ and $Z \equiv Z(x_{n,s})$ have been employed.

It is possible to show[7] that in the limit $T_s \rightarrow 0$, the terms of the cold dielectric tensor (section 2.2.1) are recovered. It is worthwhile noting that in this case, the fluid and kinetic descriptions lead to the same result. Rather than verifying this for every term of $\bar{\bar{\epsilon}}$, we will again consider two particular cases: parallel and perpendicular propagation.

3.3.4 Parallel propagation

When the wave propagates along the confining magnetic field, we have $k_{\perp} \rightarrow 0$, so that $\lambda_s \rightarrow 0$. In this case, it is legitimate to employ the small argument expansion of I_n :

$$I_n(x) \sim \left(\frac{x}{2} \right)^{|n|} + o(x^{|n|}). \quad (3.127)$$

We thus obtain, denoting $\bar{\bar{\mathbf{T}}} \equiv \sum_n \bar{\bar{\mathbf{T}}}_n$

$$\bar{\bar{\mathbf{T}}} = \frac{1}{2} \begin{pmatrix} Z(x_{1,s}) + Z(x_{-1,s}) & i[Z(x_{1,s}) - Z(x_{-1,s})] & 0 \\ -i[Z(x_{1,s}) - Z(x_{-1,s})] & Z(x_{1,s}) + Z(x_{-1,s}) & 0 \\ 0 & 0 & -2x_{0,s} Z'(x_{0,s}) \end{pmatrix}, \quad (3.128)$$

and

$$K_{ij} = \delta_{ij} + \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{k_{\parallel} v_{th,s}} T_{ij}, \quad (3.129)$$

In section 1.2.2, we have seen that in all generality, the wave equation could be written as

$$\overline{\mathbf{M}}_{\mathbf{k},\omega} \cdot \mathbf{E} \equiv \mathbf{n} \times \mathbf{n} \times \mathbf{E} + \overline{\mathbf{K}} \cdot \mathbf{E} = 0, \quad (3.130)$$

with $\mathbf{n} \equiv \mathbf{k}c/\omega$. The terms of $\overline{\mathbf{M}}$ are given by

$$M_{ij} = n_i n_j - n^2 \delta_{ij} + K_{ij}, \quad (3.131)$$

and we thus obtain

$$M_{11} = M_{22} = 1 - \frac{k_{\parallel}^2 c^2}{\omega^2} + \frac{1}{2} \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{k_{\parallel} v_{th,s}} [Z(x_{1,s}) + Z(x_{-1,s})], \quad (3.132)$$

$$M_{12} = -M_{21} = \frac{i}{2} \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{k_{\parallel} v_{th,s}} [Z(x_{1,s}) - Z(x_{-1,s})], \quad (3.133)$$

and

$$M_{33} = 1 - \sum_s \frac{\omega_{ps}^2}{(k_{\parallel} v_{th,s})^2} Z(x_{0,s}). \quad (3.134)$$

All other terms are zero. It is straightforward to verify that the dispersion relation $\det(\overline{\mathbf{M}}) = 0$ appears as the product of two factors.

Langmuir wave

Nullifying the first of these factors leads to

$$1 - \sum_s \frac{\omega_{ps}^2}{(k_{\parallel} v_{th,s})^2} Z'(x_{0,s}) = 0. \quad (3.135)$$

In the cold limit, obtained by setting $T_s \rightarrow 0$, we have $v_{th,s} = 0$, and thus $x_{0,s} \rightarrow \infty$. In this case, the asymptotic expansion of Z' (Eq. 3.80) can be employed to obtain

$$Z'(x_{0,s}) \sim \frac{1}{x_{0,s}^2}, \quad (3.136)$$

leading directly to

$$1 - \frac{\omega_{ps}^2}{\omega^2} = 0. \quad (3.137)$$

The dispersion relation of Langmuir waves in a cold plasma is recognized. By extension, Eq. 3.135 thus represents more generally the kinetic Langmuir wave. The polarization is given by $\delta E_x = \delta E_y = 0$ and $\delta E_z \neq 0$.

Using relation (3.77), it is also possible to write Eq. 3.135 as

$$1 + 2 \sum_s \frac{\omega_{ps}^2}{(k_{\parallel} v_{th,s})^2} \left[1 + x_{0,s} Z(x_{0,s}) \right] = 0. \quad (3.138)$$

As described in section 3.2.5, significant wave damping occurs if $x_{0,s} \lesssim 1$, or $\omega \lesssim k_{\parallel} v_{th,s}$. The absorption mechanism is Landau damping. Except in the rather pathological cases in which $T_i \gg T_e$, we have $v_{th,e}/v_{th,i} \propto (m_i/m_e)^{1/2} \gg 1$, showing that practically, the electron absorption always dominates the ion absorption. In magnetic fusion devices, where usually $T_e \sim T_i$, the Landau damped waves are usually completely absorbed by the electrons. It is however worthwhile mentioning that in a future reactor, a significant fraction of the lower hybrid wave power, whose “normal” damping mechanism is Landau damping by the electrons, could be absorbed by the very energetic fusion born alpha particles if the plasma and wave parameters were not carefully chosen so as to minimize this effect.

Whistlers

The second solution of the dispersion relation is given by

$$M_{11}M_{22} - M_{12}M_{21} = M_{11}^2 + M_{12}^2 = 0, \quad (3.139)$$

leading to two further possibilities.

The first one is

$$M_{11} - iM_{12} = 0, \quad (3.140)$$

or

$$1 - \frac{k_{\parallel}^2 c^2}{\omega^2} + \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{k_{\parallel} v_{th,s}} Z(x_{1,s}) = 0, \quad (3.141)$$

In order to identify unambiguously this wave, it is appropriate to try and recover the cold limit results, by using the asymptotic expansion

$$Z(x_{0,s}) \sim -\frac{1}{x_{0,s}}, \quad (3.142)$$

to obtain

$$1 - n_{\parallel}^2 - \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{\omega - \Omega_{cs}} = 0. \quad (3.143)$$

If we only consider electrons and one type of ions

$$n_{\parallel}^2 = 1 - \frac{\omega_{pe}^2}{\omega} \frac{1}{\omega + \omega_{ce}} - \frac{\omega_{pi}^2}{\omega} \frac{1}{\omega - \omega_{ci}} = L, \quad (3.144)$$

with L given by Eq. 2.22. This mode is thus recognized as the L-whistler (ion whistler), extensively discussed in section 2.3.1.

The polarization is given by $\delta\mathbf{E} = (\delta E_x, -i\delta E_x, 0)$. Significant wave damping occurs when

$$x_{1,s} = \frac{\omega - \Omega_{cs}}{k_{\parallel} v_{th,s}} \lesssim 1. \quad (3.145)$$

When the wave frequency approaches the ion cyclotron frequency, i.e. $\omega \sim \omega_{ci}$, the inequality 3.145 is easily verified. The ion whistler is essentially absorbed by the plasma ions. Electron damping is also possible, but is less easy, since $x_{1,e} = (\omega + \omega_{ce})/(k_{\parallel} v_{th,e})$ becomes small enough only if the electron temperature becomes very large. Generally, however, unless $T_i \ll T_e$, the ion absorption will dominate.

The second possibility corresponds to

$$M_{11} - iM_{12} = 0, \quad (3.146)$$

which can be rewritten as

$$1 - \frac{k_{\parallel}^2 c^2}{\omega^2} + \sum_s \frac{\omega_{ps}^2}{\omega} \frac{1}{k_{\parallel} v_{th,s}} Z(x_{-1,s}) = 0. \quad (3.147)$$

In the cold plasma limit, we obtain

$$n_{\parallel}^2 = 1 - \frac{\omega_{pe}^2}{\omega} \frac{1}{\omega - \omega_{ce}} - \frac{\omega_{pi}^2}{\omega} \frac{1}{\omega + \omega_{ci}} = R, \quad (3.148)$$

which corresponds to the R-whistler (electron whistler). The polarization is such as $\delta\mathbf{E} = (\delta E_x, i\delta E_x, 0)$ and the condition for the absorption to be significant is

$$x_{-1,s} = \frac{\omega + \Omega_{cs}}{k_{\parallel} v_{th,s}} \lesssim 1, \quad (3.149)$$

which is most easily verified for electrons if $\omega \sim \omega_{ce}$, since $x_{-1,e} = (\omega - \omega_{ce})/(k_{\parallel} v_{th,e})$. The same discussion as in the L-whistler case can be applied, exchanging the respective roles of ions and electrons. Unless the condition $T_i \gg T_e$ holds, the plasma electrons will absorb the wave power.

In both cases (L- and R-whistler), the absorption mechanism is different from the familiar Landau damping. It is referred to as *cyclotron absorption*. This discussion of the Landau effect (see section 3.2.3) can be generalized, but the interaction now takes place between the cyclotron rotation and the wave electromagnetic field. On average, the particles rotating slightly faster than the wave field will have a tendency to lose energy to the wave, whereas those that rotate slightly slower will gain energy from the wave. For a Maxwellian distribution function, and in most practical cases, the former are more numerous than the latter, which implies a net damping of the wave.

3.3.5 Perpendicular propagation

In perpendicular propagation, we have $k_{\parallel} \rightarrow 0$ and therefore, expect in the vicinity of resonances ($\omega = n\Omega_{cs}$), we have $x_{n,s} \rightarrow \infty$. The appropriate tool is therefore the asymptotic expansions of Z .

In this case, it is possible to show that the dielectric tensor terms can be simplified to yield

$$K_{11} = 1 - \sum_s \frac{\omega_{ps}^2 e^{-\lambda_s}}{\omega \lambda_s} \sum_{n=-\infty}^{\infty} n^2 \frac{I_n(\lambda_s)}{\omega - n\Omega_{cs}}, \quad (3.150)$$

$$K_{12} = -K_{21} = i \sum_s \frac{\omega_{ps}^2 e^{-\lambda_s}}{\omega} \sum_{n=-\infty}^{\infty} n \frac{I_n - I'_n}{\omega - n\Omega_{cs}}, \quad (3.151)$$

$$K_{22} = 1 - \sum_s \frac{\omega_{ps}^2 e^{-\lambda_s}}{\omega \lambda_s} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(\lambda_s) + 2\lambda_s^2 (I_n - I'_n)}{\omega - n\Omega_{cs}}, \quad (3.152)$$

and

$$K_{33} = 1 - \sum_s \frac{\omega_{ps}^2 e^{-\lambda_s}}{\omega} \sum_{n=-\infty}^{\infty} \frac{I_n(\lambda_s)}{\omega - n\Omega_{cs}}. \quad (3.153)$$

All other terms are zero:

$$K_{13} = K_{23} = K_{31} = K_{32} = 0. \quad (3.154)$$

The non-zero terms in the wave equation matrix $\bar{\mathbf{M}}_{\mathbf{k},\omega}$ are

$$M_{11} = K_{11}, \quad (3.155)$$

$$M_{12} = K_{12}, \quad (3.156)$$

$$M_{22} = K_{22} - n_{\perp}^2, \quad (3.157)$$

and

$$M_{33} = K_{33} - n_{\perp}^2. \quad (3.158)$$

Once again, the dispersion relation can be expressed as the product of two factors.

Ordinary wave

The zero condition for the first one of these factors gives $M_{33} = 0$, or

$$n_{\perp}^2 = 1 - \sum_s \frac{\omega_{ps}^2 e^{-\lambda_s}}{\omega} \sum_{n=-\infty}^{\infty} \frac{I_n(\lambda_s)}{\omega - n\Omega_{cs}}. \quad (3.159)$$

In a cold plasma, $\lambda_s \rightarrow 0$, leading to $I_n(\lambda_s) \approx (\lambda_s/2)^{|n|}$. The only non zero term in the sum over harmonics corresponds to $n = 0$ and thus

$$n_{\perp}^2 = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}, \quad (3.160)$$

which should be compared to Eq. 2.66 to identify the ordinary wave. It is however worth noticing that Eq. 3.159 is radically different from its cold counterpart, because of the influence of the cyclotron resonance harmonics $\omega = n\Omega_{cs}$, with $n \neq 0$. The process at hand is called a *finite Larmor radius effect*, and is the result of the fact that the wave phase varies along the particle orbit. The ratio between the Larmor radius of the particle

$$\rho_s = \frac{v_{\perp}}{\omega_{cs}}, \quad (3.161)$$

and the perpendicular wavelength $\lambda_{\perp} \equiv 1/k_{\perp}$ is determined by $\lambda_s = k_{\perp}\rho_s/2$, allowing to identify several regimes:

1. $\lambda_s = 0$, as is the case in a cold plasma. The Larmor circle degenerates to a point and there is no influence from the resonances.
2. $\lambda_s \ll 1$, or $\rho_s \ll \lambda_{\perp}$. The phase variation of the wave is weak along the particle trajectory. Taking $I_n(\lambda_s) \sim (\lambda_s/2)^{|n|}$ in Eq. 3.159, it is possible to see that the influence of the various cyclotron harmonics decreases rapidly with n , so that only the lowest order harmonics have a significant effect.
3. $\lambda_s \gtrsim 1$. In this case, the high harmonic have a large influence. In the limit $\lambda_s \gg 1$, they actually all have a quantitatively equivalent effect.

Extraordinary wave - Bernstein wave

The other solution of the dispersion relation verifies

$$M_{11}M_{22} + M_{12}^2 = 0, \quad (3.162)$$

which can be expanded as

$$\left[\left(1 - \sum_s \frac{\omega_{ps}^2}{\omega} \frac{e^{-\lambda_s}}{\lambda_s} \sum_{n=-\infty}^{\infty} n^2 \frac{I_n(\lambda_s)}{\omega - n\Omega_{cs}} \right) \times \left(1 - n_{\perp}^2 - \sum_s \frac{\omega_{ps}^2}{\omega} \frac{e^{-\lambda_s}}{\lambda_s} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(\lambda_s) + 2\lambda_s^2 (I_n(\lambda_s) - I'_n(\lambda_s))}{\omega - n\Omega_{cs}} \right) \right] - \left(\sum_s \frac{\omega_{ps}^2}{\omega} e^{-\lambda_s} \sum_{n=-\infty}^{\infty} n \frac{I_n(\lambda_s) - I'_n(\lambda_s)}{\omega - n\Omega_{cs}} \right)^2 = 0. \quad (3.163)$$

In order to identify the obtained waves, we set $\lambda_s \rightarrow 0$ to place ourselves in the cold plasma limit. This yields

$$\left[1 - \frac{1}{2} \sum_s \frac{\omega_{ps}^2}{\omega} \left(\frac{1}{\omega - \Omega_{cs}} + \frac{1}{\omega + \Omega_{cs}} \right) \right] \left[1 - n_{\perp}^2 - \frac{1}{2} \sum_s \frac{\omega_{ps}^2}{\omega} \left(\frac{1}{\omega - \Omega_{cs}} + \frac{1}{\omega + \Omega_{cs}} \right) \right] - \left(\frac{1}{2} \sum_s \frac{\omega_{ps}^2}{\omega} \left(\frac{1}{\omega - n\Omega_{cs}} - \frac{1}{\omega + n\Omega_{cs}} \right) \right)^2 = 0. \quad (3.164)$$

Recalling the definitions of S (Eq. 2.25) and D (Eq. 2.26), we can also write

$$n_{\perp}^2 = \frac{S^2 - D^2}{S} = \frac{RL}{S}, \quad (3.165)$$

which is the dispersion relation for the extraordinary wave in a cold plasma, obtained here in the general case. Recalling that the wave equation is given by

$$\mathbf{n} \times \mathbf{n} \times \delta \mathbf{E} + \overline{\mathbf{K}} \cdot \delta \mathbf{E} = 0, \quad (3.166)$$

it can be realized that the electrostatic modes, for which $\mathbf{n} \parallel \delta \mathbf{E}$ verify the particular dispersion relation

$$\delta \mathbf{E} \cdot \overline{\mathbf{K}} \cdot \delta \mathbf{E} = 0. \quad (3.167)$$

Since we have considered $k_{\perp} \equiv k_x$, and since we are looking for solutions such as $\delta \mathbf{E} = (\delta E, 0, 0)$, we obtain $M_{11} = 0$. This yields

$$1 - \sum_s \frac{\omega_{ps}^2}{\omega} \frac{e^{-\lambda_s}}{\lambda_s} \sum_{n=-\infty}^{\infty} n^2 \frac{I_n(\lambda_s)}{\omega - n\Omega_{cs}} = 0. \quad (3.168)$$

This constitutes the dispersion relation of *Bernstein waves*. At every harmonic of the cyclotron frequency, a wave-particle resonance occurs but once again, the Bessel function $I_n(\lambda_s)$ limits the influence of high order harmonics unless $\lambda_s \gg 1$.

3.4 Quasilinear theory

Up to this point, the equilibrium distribution function $f_{s,0}$ was assumed to be given, and constant in time. These assumptions provide a convenient framework to describe the wave-plasma interaction on the time scale of the order of the wave period. However, after a sufficiently long time, we expect some energy to be transferred from the wave to the plasma, resulting in significant heating of the latter. This modification of the macroscopic properties of the plasma manifests itself through non harmonic (secular) changes in the equilibrium distribution functions, which needs to be included in our modeling.

Historically, this issue has been investigated to properly describe plasma instabilities, where the absence of evolution of the equilibrium distribution functions is particularly striking. To illustrate this, we have found in section 3.2 that the imaginary part of the frequency ω (often denoted γ in the study of instabilities, and referred to as the *growth rate*) for the case of the Langmuir oscillations in an unmagnetized plasma (Eq. 3.50) was given by

$$\gamma \equiv \omega_i = \frac{\pi \omega_{pe}}{2} \frac{e^2}{k^2 m_e \epsilon_0} \left. \frac{\partial F_{e,0}}{\partial v_z} \right|_{v_z=\omega/k}. \quad (3.169)$$

where k is the magnitude of the wave vector. It is clear from this expression that $\gamma > 0$ if $\partial F_{e,0}/\partial v_z > 0$. The instability energy, which is proportional to square of the wave field, is given by

$$A_i \propto |\mathbf{E}|^2 \propto \exp(2\gamma t). \quad (3.170)$$

This immediately shows that the instability increases exponentially without any limitation. Evidently, this does not reflect the reality and it is therefore necessary to identify a mechanism responsible for the *saturation* of this instability. In order to add such a capability to our model, the first step is evidently to let $F_{e,0}$ vary slowly (compared to the wave period) over time. The *quasilinear theory* has been precisely introduced to describe such an evolution. Although the quasilinear theory was initially proposed to describe linear instabilities, it is also the adapted tool to deal with wave damping.

3.4.1 Quasilinear equation

Once again, the starting point is the Boltzmann equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \left(\frac{\partial f_s}{\partial t} \right)_{coll}. \quad (3.171)$$

So far, we have only considered weakly collisional plasmas, i.e. such as the wave-particle interaction takes place on a timescale much smaller than the collisional timescale. For magnetic fusion plasmas and many space plasmas, this assumption is very well justified. Here, however, we have to consider that the time evolution of the distribution function induced by the wave is slow enough to be influenced by collisions, which explains the fact that we do not neglect this term.

We now introduce a time-averaging operation over τ , the wave period. By definition, the obtained quantities will vary on a timescale which will be slow compared to τ

$$\langle A \rangle_\tau(\bar{t}) \equiv \frac{1}{\tau} \int_{\bar{t}}^{\bar{t}+\tau} dt A(\mathbf{r}, t). \quad (3.172)$$

\bar{t} was employed to distinguish the slow, quasilinear variation, from the general time t which contains the effects both the secular variation and the effects of the

fast oscillating wave. If we consider a given quantity having a slow variation superimposed to the fast wave response, the effect of this operation will be to extract the slow variation, as illustrated in Fig. 3.5

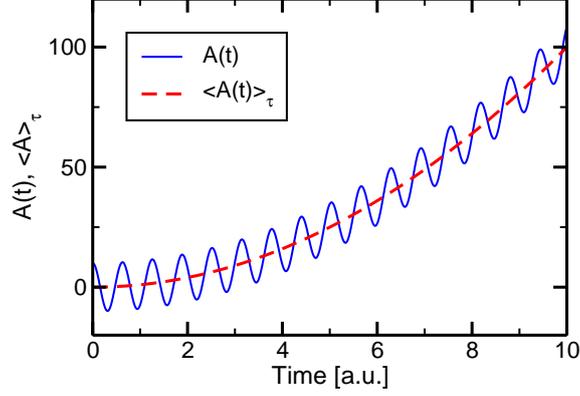


Figure 3.5: Fast varying quantity $A(t)$, and time average, $\langle A(t) \rangle_\tau$. The slow, quasi-linear variation is extracted.

We now introduce the following linearization:

$$f_s(\mathbf{r}, \mathbf{v}, t) \equiv f_{s,0}(\mathbf{r}, \mathbf{v}, \bar{t}) + \delta f_s(\mathbf{r}, \mathbf{v}, t), \quad (3.173)$$

which differs from the linearizations employed previously (see, e.g., Eq. 3.9) in that $f_{s,0}$ may evolve slowly over time. For the perturbation

$$\delta f_s(t) \equiv \frac{1}{2} \{ \delta f_\omega e^{-i\omega t} + \text{c.c.} \}. \quad (3.174)$$

Likewise, for the magnetic field, we have

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}, \bar{t}) + \delta \mathbf{B}(\mathbf{r}, t), \quad (3.175)$$

where

$$\delta \mathbf{B}(\mathbf{r}, t) \equiv \frac{1}{2} \{ \delta \mathbf{B}_\omega e^{-i\omega t} + \text{c.c.} \}, \quad (3.176)$$

and in the absence of a static electric field

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \{ \delta \mathbf{E}_\omega e^{-i\omega t} + \text{c.c.} \}. \quad (3.177)$$

It is readily seen that

$$\langle f_s(\mathbf{r}, \mathbf{v}, t) \rangle_\tau = f_{s,0}(\mathbf{r}, \mathbf{v}, \bar{t}), \quad (3.178)$$

$$\langle \mathbf{B}(\mathbf{r}, t) \rangle_\tau = \mathbf{B}_0(\mathbf{r}, \bar{t}), \quad (3.179)$$

and also

$$\langle \mathbf{E}(\mathbf{r}, t) \rangle_\tau = 0. \quad (3.180)$$

If we apply the averaging operation to the Boltzmann equation as a whole, we obtain

$$\frac{\partial f_{s,0}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{r}} + \frac{q_s}{m_s} (\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = \hat{C}(f_{s,0}) + \hat{Q}(f_{s,0}), \quad (3.181)$$

with $\hat{C}(f_{s,0})$ is the *Fokker-Planck collision operator*, which will not be discussed in this lecture [7] and \hat{Q} is the *wave quasilinear operator*, defined by

$$\hat{Q}(f_{s,0})(\bar{t}) = -\frac{q_s}{m_s} \left\langle (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle_\tau. \quad (3.182)$$

If we consider two quantities, $A(\mathbf{r}, t)$ and $B(\mathbf{r}, t)$ oscillating at frequency ω in terms of their Fourier components,

$$A(\mathbf{r}, t) = \frac{1}{2} \{A_\omega(\mathbf{r})e^{-i\omega t} + \text{c.c.}\}, \quad (3.183)$$

and

$$B(\mathbf{r}, t) = \frac{1}{2} \{B_\omega(\mathbf{r})e^{-i\omega t} + \text{c.c.}\}, \quad (3.184)$$

To account for the field attenuation caused by the wave damping, the frequency must be considered a complex number, i.e. $\omega \equiv \omega_r + i\omega_i$, we can thus write the average of the product of A and B as

$$\langle AB \rangle_\tau(\bar{t}) = \frac{1}{4\tau} \int_{\bar{t}}^{\bar{t}+\tau} dt \{A_\omega B_\omega e^{-2i\omega_r t} + A_\omega^* B_\omega^* e^{2i\omega_r t} + A_\omega B_\omega^* + A_\omega^* B_\omega\} e^{2\omega_i t} \quad (3.185)$$

We place ourselves in a regime of weak dissipation, which imposes $|\omega_i| \ll \omega_r$. By performing the integral over a wave period, τ , the first two terms in the curly braces vanish, and we obtain

$$\langle AB \rangle_\tau(\bar{t}) \approx \frac{1}{4} [A_\omega B_\omega^* + A_\omega^* B_\omega] \frac{1}{\tau} \int_{\bar{t}}^{\bar{t}+\tau} dt e^{2\omega_i t}. \quad (3.186)$$

The weak damping regime allows us to perform the approximation

$$\frac{1}{\tau} \int_{\bar{t}}^{\bar{t}+\tau} dt e^{2\omega_i t} = \frac{e^{2\omega_i \bar{t}}}{2\omega_i \tau} [e^{2\omega_i \tau} - 1] \approx \exp(2\omega_i \bar{t}), \quad (3.187)$$

and we obtain

$$\langle AB \rangle_\tau(\bar{t}) = \frac{1}{4} [A_\omega B_\omega^* + A_\omega^* B_\omega] e^{2\omega_i \bar{t}}, \quad (3.188)$$

where the last exponential factor accounts for the wave damping (or excitation if $\omega_i > 0$).

At this point, it is possible to evaluate \hat{Q} , but it is very complicated and generally requires a numerical treatment. Therefore, to further simplify our calculation, we will neglect any phenomenon occurring on a space scale comparable to the wavelength. This means that we place ourselves in the context of a quasi-local analysis (i.e. local from the wave standpoint, but with macroscopic quantities such as density, temperature possibly varying in space. . .). In this case, we can still decompose the field as a sum of plane waves:

$$\delta\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \sum_{\mathbf{k}} \{ \delta\mathbf{E}_{\omega, \mathbf{k}} \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)) + \text{c.c.} \}, \quad (3.189)$$

and

$$\delta\mathbf{B}(\mathbf{r}, t) = \frac{1}{2} \sum_{\mathbf{k}} \{ \delta\mathbf{B}_{\omega, \mathbf{k}} \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)) + \text{c.c.} \}, \quad (3.190)$$

It is crucial to note that ω and \mathbf{k} are not independent variables, here. Indeed, we should rather write $\omega(\mathbf{k})$ since we must satisfy the local dispersion relation for the considered waves.

Likewise, for the distribution function

$$\delta f_s(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2} \sum_{\mathbf{k}} \{ \delta f_{\omega, \mathbf{k}} \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)) + \text{c.c.} \}. \quad (3.191)$$

Since we are not interested in space scales of the order of the wavelength λ , it is natural to supplement the time averaging operation by a space averaging operation

$$\langle x \rangle_{\mathbf{r}} \equiv \frac{1}{V} \int d^3\mathbf{r} x(\mathbf{r}, t). \quad (3.192)$$

The integration volume V is chosen to be much larger than a sphere of radius λ , but small compared to the macroscopic variations of the plasma. By application of this averaging operator to the Boltzmann equation, we obtain⁴.

$$\frac{\partial F_{s,0}}{\partial t} + \mathbf{v} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{r}} + \frac{q_s}{m_s} (\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}} = \hat{C}(F_{s,0}) + \hat{Q}(F_{s,0}), \quad (3.193)$$

The quasilinear operator is given by

$$\hat{Q}(F_{s,0}) = -\frac{q_s}{m_s} \left\langle (\delta\mathbf{E} + \mathbf{v} \times \delta\mathbf{B}) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle_{\mathbf{r}, \tau}. \quad (3.194)$$

⁴In order to obtain this result, it is necessary to also assume that $f_{s,0}$ has a negligible variation in the integration volume. If this is not verified, supplemental drift terms appear, which will not be addressed in this lecture.

Using Eqs. 3.189, 3.190 and 3.191, and performing directly the time averaging, we are left with

$$\hat{Q}(F_{s,0})(\bar{t}) = - \sum_{\mathbf{k}, \mathbf{k}'} \frac{q_s}{4m_s} \left\langle (\delta \mathbf{E}_{\omega, \mathbf{k}} + \mathbf{v} \times \delta \mathbf{B}_{\omega, \mathbf{k}}) \cdot \frac{\partial \delta f_{\omega, \mathbf{k}'}}{\partial \mathbf{v}} e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} + \text{c.c.} \right\rangle_{\mathbf{r}} e^{2\omega_i \bar{t}}. \quad (3.195)$$

The space averaging has the consequence of selecting the wave number $\mathbf{k} = \mathbf{k}'$, which leads to

$$\hat{Q}(F_{s,0}) = - \sum_{\mathbf{k}} \frac{q_s}{2m_s} \Re \left\{ (\delta \mathbf{E}_{\omega, \mathbf{k}} + \mathbf{v} \times \delta \mathbf{B}_{\omega, \mathbf{k}})^* \cdot \frac{\partial \delta f_{\omega, \mathbf{k}}}{\partial \mathbf{v}} \right\} e^{2\omega_i \bar{t}}. \quad (3.196)$$

To construct the quasilinear operator, the necessary ingredients are the electromagnetic field $\delta \mathbf{E}$ and $\delta \mathbf{B}$, and the perturbation to the distribution function δf_s . By construction, these quantities are to be computed in the framework of a linear theory, i.e. neglecting second order term, following the same procedure as, e.g., in section 3.2. Since we obtain a first non-linear term by the combination of linear results, this theory is called a *quasilinear theory*. Generally and with an ad-hoc collision operator, Eq. 3.193 is known as the *Fokker-Planck equation*.

3.4.2 Electrostatic waves

For illustrative purposes, rather than calculate the quasilinear operator in general, we will assume that the considered wave is electrostatic and propagates in an unmagnetized plasma. We can therefore directly use the results obtained in section 3.2.

Since $\delta \mathbf{B} = \mathbf{0}$, we have

$$\hat{Q}(F_{s,0}) = - \sum_{\mathbf{k}} \frac{q_s}{2m_s} \Re \left(\delta \mathbf{E}_{\omega, \mathbf{k}}^* \cdot \frac{\partial \delta f_{\omega, \mathbf{k}}}{\partial \mathbf{v}} \right) e^{2\omega_i \bar{t}}. \quad (3.197)$$

The linear response of the plasma was obtained in section 3.2.1 (Eq. 3.34) as

$$\delta f_{\omega, \mathbf{k}} = - \frac{q_s}{m_s} \varphi_{\omega, \mathbf{k}} \frac{\mathbf{k} \cdot \partial F_{s,0} / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} + i \frac{\delta F_s(t=0)}{\omega - \mathbf{k} \cdot \mathbf{v}}. \quad (3.198)$$

Hence

$$\frac{\partial \delta f_{\omega, \mathbf{k}}}{\partial \mathbf{v}} = - \frac{q_s}{m_s} \varphi_{\omega, \mathbf{k}} \frac{\partial}{\partial \mathbf{v}} \left\{ \frac{\mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}} \right\} + i \frac{\partial}{\partial \mathbf{v}} \left\{ \frac{\delta F_s(t=0)}{\omega - \mathbf{k} \cdot \mathbf{v}} \right\}, \quad (3.199)$$

yielding

$$\delta \mathbf{E}_{\omega, \mathbf{k}}^* \cdot \frac{\partial \delta f_{\omega, \mathbf{k}}}{\partial \mathbf{v}} = i \varphi_{\omega, \mathbf{k}}^* \mathbf{k} \cdot \frac{\partial \delta f_{\omega, \mathbf{k}}}{\partial \mathbf{v}} \approx -i \frac{q_s}{m_s} |\varphi_{\omega, \mathbf{k}}|^2 \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\mathbf{k} \mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}}, \quad (3.200)$$

where $(\mathbf{k}\mathbf{k})_{ij} \equiv k_i k_j$ is a dyad. In the previous expression, we have neglected the influence of the initial distribution function on the solution. In usual situations, this is a legitimate assumption (the system quickly loses memory of its initial state). Therefore, we can write

$$\hat{Q}(F_{s,0}) = \frac{\partial}{\partial \mathbf{v}} \cdot \bar{\mathbf{D}}_{ql} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}}, \quad (3.201)$$

with

$$\bar{\mathbf{D}}_{ql} \equiv \frac{q_s^2}{2m_s^2} \Re \left(\sum_{\mathbf{k}} i \frac{\mathbf{k}\mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}} |\varphi_{\omega, \mathbf{k}}|^2 \right) e^{2\omega_i \bar{t}}, \quad (3.202)$$

or equivalently, in components notation

$$(D_{ql})_{ij} \equiv \frac{q_s^2}{2m_s^2} \Re \left(\sum_{\mathbf{k}} i \frac{k_i k_j}{\omega - \mathbf{k} \cdot \mathbf{v}} |\varphi_{\omega, \mathbf{k}}|^2 \right) e^{2\omega_i \bar{t}}, \quad (3.203)$$

The Fokker-Planck equation (3.193) can now be written as

$$\frac{dF_{s,0}}{dt} = \frac{\partial F_{s,0}}{\partial t} + \mathbf{v} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{r}} = \hat{C}(F_{s,0}) + \frac{\partial}{\partial \mathbf{v}} \cdot \bar{\mathbf{D}}_{ql} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}}. \quad (3.204)$$

It is clear that when collisions are neglected, we are in the presence of diffusion equation in velocity space. In fact, in its Fokker-Planck form, the collision operator is also a diffusive term[7]. The resulting phenomenon is known as the *quasilinear diffusion*: under the conjugate effects of the wave on one hand and collisions on the other hand, the variation of the distribution function has a diffusive nature. It is worthwhile mentioning that even if we consider the more complex case of an electromagnetic wave in a magnetized plasma, this equation is still a diffusion equation. The form of the quasilinear diffusion coefficient is significantly more complicated, but the physics remains the same.

In order to gain insight into this mechanism, it is useful to treat the resonant denominator appearing in Eq. 3.202 by using the Plemelj formula and write

$$\frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} = \mathcal{P} \left(\frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \right) - i\pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}). \quad (3.205)$$

We recall that this formula is employed to perform the integration over velocity considering that the frequency has a small positive imaginary part, i.e. $\omega \equiv \omega_r + i\omega_i$. If we assume $\omega_i \ll \omega_r$ (weak dissipation), the principal value can be rewritten as

$$\mathcal{P} \left(\frac{1}{\omega_r - \mathbf{k} \cdot \mathbf{v} + i\omega_i} \right) \approx \mathcal{P} \left(\frac{1}{\omega_r - \mathbf{k} \cdot \mathbf{v}} \right) - i\omega_i \mathcal{P} \left(\frac{1}{(\omega_r - \mathbf{k} \cdot \mathbf{v})^2} \right), \quad (3.206)$$

and we obtain

$$\Re \left(\delta \mathbf{E}_{\omega, \mathbf{k}}^* \cdot \frac{\partial \delta f_{\omega, \mathbf{k}}}{\partial \mathbf{v}} \right) = \frac{q_s}{m_s} |\varphi_{\omega, \mathbf{k}}|^2 \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ \mathbf{k}\mathbf{k} \left[\omega_i \mathcal{P} \left(\frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v})^2} \right) + \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}) \right] \right\} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}}. \quad (3.207)$$

The diffusion coefficient (3.202) can be decomposed as

$$\bar{\bar{\mathbf{D}}}_{ql} \equiv \bar{\bar{\mathbf{D}}}_{ql,nr} + \bar{\bar{\mathbf{D}}}_{ql,r}, \quad (3.208)$$

with

$$\bar{\bar{\mathbf{D}}}_{ql,nr} \equiv -\frac{q_s^2}{2m_s^2} \sum_{\mathbf{k}} \omega_i |\varphi_{\omega,\mathbf{k}}|^2 \mathcal{P} \left(\frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v})^2} \right) \mathbf{k} \mathbf{k} e^{2\omega_i \bar{t}}, \quad (3.209)$$

and

$$\bar{\bar{\mathbf{D}}}_{ql,r} \equiv -\frac{\pi}{2} \frac{q_s^2}{m_s^2} \sum_{\mathbf{k}} |\varphi_{\omega,\mathbf{k}}|^2 \delta(\omega - \mathbf{k} \cdot \mathbf{v}) \mathbf{k} \mathbf{k} e^{2\omega_i \bar{t}}, \quad (3.210)$$

We see that $\bar{\bar{\mathbf{D}}}_{ql}$ is the sum of a non-resonant ($\bar{\bar{\mathbf{D}}}_{ql,nr}$) and a resonant ($\bar{\bar{\mathbf{D}}}_{ql,r}$) contribution. As apparent from the Dirac function, the latter is clearly the result of resonant plasma particle interactions. Generally, the non-resonant contribution is small⁵ and we can approximate $\bar{\bar{\mathbf{D}}}_{ql}$ as

$$\bar{\bar{\mathbf{D}}}_{ql} \approx -\bar{\bar{\mathbf{D}}}_{ql,r} \equiv \frac{\pi}{2} \frac{q_s^2}{m_s^2} \sum_{\omega,\mathbf{k}} |\varphi_{\omega,\mathbf{k}}|^2 \delta(\omega - \mathbf{k} \cdot \mathbf{v}) \mathbf{k} \mathbf{k} e^{2\omega_i \bar{t}}. \quad (3.211)$$

3.4.3 Quasilinear diffusion

In order to get insight into the physics of the quasilinear diffusion phenomenon, it is useful to reconsider the model described in section 3.2.3: a particle initially at position x_0 , with velocity v_0 , and a wave with an electric field given by $E(x, t) = E_0 \cos(kx - \omega t)$. The variation of the particle velocity (Eq. 3.55) is given by

$$\Delta v = \frac{q}{m} \frac{E_0}{kv_0 - \omega} \left\{ \sin[(kv_0 - \omega)t] \cos(kx_0) + (\cos[(kv_0 - \omega)t] - 1) \sin(kx_0) \right\}, \quad (3.212)$$

Since we are studying a collective phenomenon, it is natural to define the average over initial positions of a given quantity A as

$$\langle A \rangle_{x_0} \equiv \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L dx_0 A(x_0), \quad (3.213)$$

where the integral is performed over a domain such as $L \gg 1/k$.

On average, the velocity variation of particles is clearly zero. If, on the other hand, we consider the average of $(\Delta v)^2$, we obtain

$$\langle (\Delta v)^2 \rangle_{x_0} = 2 \frac{q^2}{m^2} E_0^2 \frac{\sin^2[(\omega - kv_0)t/2]}{(\omega - kv_0)^2}. \quad (3.214)$$

⁵Also, this is a reversible process, uninteresting in the context of plasma heating.

If we let $X \equiv (\omega - kv_0)t/2$, we can rewrite the previous expression as

$$\langle (\Delta v)^2 \rangle_{x_0} = \frac{q^2}{2m^2} E_0^2 \frac{\sin^2(X)}{X^2} t^2. \quad (3.215)$$

The function $f(X) \equiv (\sin(X)/X)^2$ is peaked around $X = 0$. In the framework of our simple model, we approximate it as a Dirac function centered around $X = 0$ and write

$$f(X) \approx \pi \delta(X). \quad (3.216)$$

The π factor appears as a consequence of

$$\int_{-\infty}^{\infty} dX \frac{\sin^2(X)}{X^2} = \pi, \quad (3.217)$$

so that

$$\langle (\Delta v)^2 \rangle_{x_0} \approx \pi \frac{q^2}{2m^2} E_0^2 t^2 \delta\{(\omega - kv_0)t/2\} = \pi \frac{q^2}{m^2} E_0^2 t \delta(\omega - kv_0). \quad (3.218)$$

We can deduce a phenomenological diffusion coefficient

$$D_{qt} \equiv \frac{\langle \Delta v^2 \rangle_{x_0}}{2t} = \frac{\pi}{2} \frac{q^2}{m^2} E_0^2 \delta(\omega - kv_0), \quad (3.219)$$

which clearly corresponds to the resonant part of the diffusion coefficient which has been derived previously (see Eq. 3.210). It is worth noting that the quasilinear theory is applicable only when certain hypotheses are verified [7, 5, 4]. In particular, if the particles happen to be trapped in the wave field, it is not possible to use this formalism, because of the intrinsic non-linearity of the problem. In the general framework of magnetic fusion plasma heating and current drive by externally excited wave, the fact that the wave spectrum is generally wide makes the quasilinear theory very well suited to the description of the processes at hand, even at high levels of wave power.

The result of the quasilinear diffusion on the distribution function is a local flattening of the distribution function. This can be seen by writing the quasilinear diffusion equation, accounting only for the wave, in the form

$$\frac{\partial F_{s,0}}{\partial t} = \frac{\partial}{\partial v} D_{qt} \frac{\partial F_{s,0}}{\partial v}. \quad (3.220)$$

It is clear that if D_{qt} is non-zero, when the stationary state is attained, ie. when $\partial F_{s,0}/\partial t = 0$, we have

$$D_{qt} \frac{\partial F_{s,0}}{\partial v} = \alpha, \quad (3.221)$$

with α a constant. In the regions of velocity space where D_{qt} is large, we have

$$\frac{\partial F_{s,0}}{\partial v} = \frac{\alpha}{D_{qt}} \rightarrow 0, \quad (3.222)$$

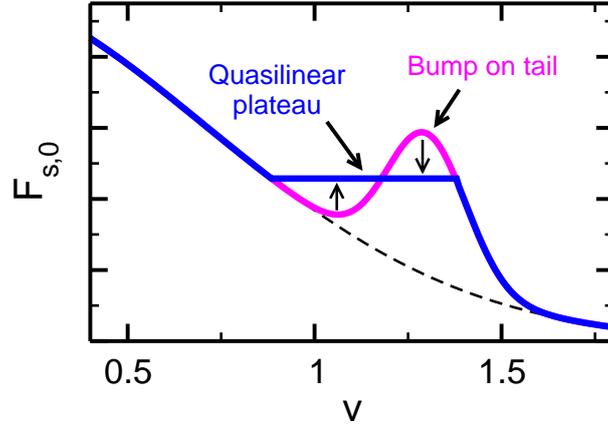


Figure 3.6: Quasilinear relaxation of an unstable distribution function.

so that the distribution function becomes locally flat. An example of distribution function evolution is provided by the classical “bump on tail” problem. In this case, $F_{s,0}$ increases locally with the velocity, i.e. $\partial F_{s,0}/\partial v > 0$. As was discussed previously, this is an unstable situation. Physically, the distribution function will evolve until an equilibrium is reached, and the process will progressively slow down as the distribution function flattens (see Fig. 3.6).

In many situations, the quasilinear diffusion time is comparable to the collision time, meaning that the influence of collisions must be taken into account⁶. When this process is included into the model, the quasilinear equation takes the form

$$\frac{\partial F_{s,0}}{\partial t} = \left(\frac{\partial F_{s,0}}{\partial t} \right)_{coll} + \frac{\partial}{\partial \mathbf{v}} \cdot \bar{\mathbf{D}}_w \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}} \quad (3.223)$$

where the first term on the right hand side represents the quasilinear diffusion induced by collisions, whereas the second term contains the effect of the wave. In the absence of wave, the solution to this equation is the familiar Maxwellian distribution function. By adding the effect of the wave, we will obtain a local flattening of $F_{s,0}$. Therefore, in situation where a wave transfers energy from an outside source to the plasma, these two phenomena oppose each other and the detailed shape of the distribution function will depend on their respective strengths (see Fig. 3.7).

3.4.4 Energy conservation

It is instructive to examine the problem of energy conservation within the framework of the quasilinear theory. Once again, to keep the problem as simple as possible, we

⁶By randomizing the particle phase over relatively short times, collisions often guarantee that the quasilinear theory is actually applicable.

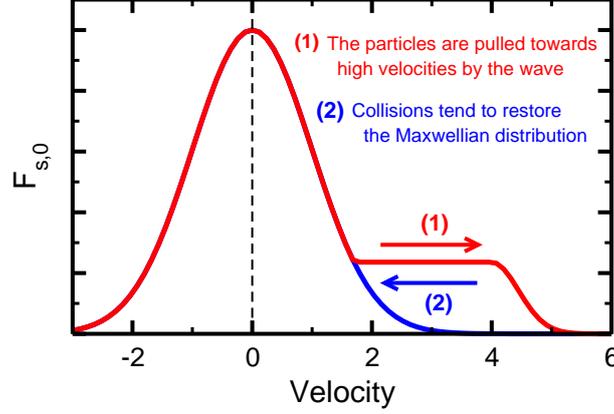


Figure 3.7: Respective effects of the wave and of collisions.

will restrict ourselves to electrostatic waves. The total kinetic energy of the particles comprising the plasma is given by

$$\varepsilon(t) \equiv \sum_s \frac{m_s}{2} \int d^3\mathbf{v} v^2 f_s(\mathbf{r}, \mathbf{v}, t). \quad (3.224)$$

As usual, the instantaneous energy is of little interest. The secular energy variation is obtained by applying the quasilinear averaging operation on $\tilde{\varepsilon}$ to obtain

$$\varepsilon(\bar{t}) \equiv \langle \tilde{\varepsilon}(t) \rangle_{\tau, \mathbf{r}} = \sum_s \frac{m_s}{2} \int d^3\mathbf{v} v^2 F_{s,0}(\mathbf{v}, \bar{t}), \quad (3.225)$$

where \bar{t} corresponds to the slow time, i.e. a time comparable to the collision period.

The time derivative of this quantity is given by

$$\frac{d\varepsilon}{d\bar{t}} = \frac{\partial \varepsilon}{\partial \bar{t}} = \sum_s \frac{m_s}{2} \int d^3\mathbf{v} v^2 \frac{\partial F_{s,0}}{\partial \bar{t}}. \quad (3.226)$$

When collisions are neglected, Eq. 3.204 leads to

$$\frac{d\varepsilon}{d\bar{t}} = \sum_s \frac{m_s}{2} \int d^3\mathbf{v} v^2 \frac{\partial}{\partial \mathbf{v}} \cdot \bar{\bar{\mathbf{D}}}_{ql} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}}, \quad (3.227)$$

giving, after integration by part

$$\frac{d\varepsilon}{d\bar{t}} = - \sum_s \frac{m_s}{2} \int d^3\mathbf{v} 2\mathbf{v} \cdot \bar{\bar{\mathbf{D}}}_{ql} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}}. \quad (3.228)$$

Using expression (3.202) for the quasilinear diffusion tensor in the previous expression yields

$$\frac{d\varepsilon}{d\bar{t}} = - \sum_s \frac{q_s^2}{2m_s} \int d^3\mathbf{v} \sum_{\mathbf{k}} |\varphi_{\omega, \mathbf{k}}|^2 \Re \left(\frac{i\mathbf{v} \cdot \mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}} \right) \mathbf{k} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}} e^{2\omega_i \bar{t}}. \quad (3.229)$$

Replacing $\mathbf{k} \cdot \mathbf{v}$ by $\omega + (\mathbf{k} \cdot \mathbf{v} - \omega)$ in the numerator leads to

$$\frac{d\varepsilon}{d\bar{t}} = - \sum_s \frac{q_s^2}{2m_s} \int d^3\mathbf{v} \sum_{\mathbf{k}} |\varphi_{\omega,\mathbf{k}}|^2 \Re \left(\frac{i\omega}{\omega - \mathbf{k} \cdot \mathbf{v}} \right) \mathbf{k} \cdot \frac{\partial F_{s,0}}{\partial \mathbf{v}} e^{2\omega_i \bar{t}}, \quad (3.230)$$

or equivalently

$$\frac{d\varepsilon}{d\bar{t}} = - \frac{\epsilon_0}{2} \sum_{\mathbf{k}} k^2 |\varphi_{\omega,\mathbf{k}}|^2 \Re \left(i\omega \sum_s \frac{q_s^2}{\epsilon_0 m_s k^2} \int d^3\mathbf{v} \frac{\mathbf{k} \cdot \partial F_{s,0} / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \right) e^{2\omega_i \bar{t}}. \quad (3.231)$$

Recalling that the dispersion relation of electrostatic waves can be written as (Eq. 3.36)

$$1 + \sum_s \frac{q_s^2}{\epsilon_0 m_s k^2} \int d^3\mathbf{v} \frac{\mathbf{k} \cdot \partial F_{s,0} / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} = 0, \quad (3.232)$$

we obtain

$$\frac{d\varepsilon}{d\bar{t}} = - \frac{\epsilon_0}{2} \sum_{\mathbf{k}} \omega_i k^2 |\varphi_{\omega,\mathbf{k}}|^2 e^{2\omega_i \bar{t}} = - \frac{\epsilon_0}{2} \sum_{\mathbf{k}} \omega_i |\mathbf{E}_{\omega,\mathbf{k}}|^2 e^{2\omega_i \bar{t}}, \quad (3.233)$$

The last part of this expression was obtained by reintroducing the Fourier components of the electric field: $\mathbf{E}_{\omega,\mathbf{k}} = -i\mathbf{k}\varphi_{\omega,\mathbf{k}}$.

The last expression can be cast in the form

$$\frac{d\varepsilon}{d\bar{t}} = - \frac{\epsilon_0}{2} \frac{\partial}{\partial \bar{t}} \frac{1}{2} \sum_{\mathbf{k}} |\mathbf{E}_{\omega,\mathbf{k}}|^2 e^{2\omega_i \bar{t}}. \quad (3.234)$$

Furthermore, from Eq. 3.188, we have

$$\langle |\mathbf{E}|^2 \rangle_{\tau,\mathbf{r}}(\bar{t}) = \frac{1}{2} \sum_{\mathbf{k}} |\mathbf{E}_{\omega,\mathbf{k}}|^2 e^{2\omega_i \bar{t}}, \quad (3.235)$$

so that

$$\frac{d\varepsilon}{d\bar{t}} = - \frac{\partial}{\partial \bar{t}} \left(\frac{\epsilon_0 \langle |\mathbf{E}|^2 \rangle_{\tau,\mathbf{r}}}{2} \right), \quad (3.236)$$

Between parentheses appears the energy density associated to the wave electric field

$$W = \frac{\epsilon_0 \langle |\mathbf{E}|^2 \rangle_{\tau,\mathbf{r}}}{2} \quad (3.237)$$

showing the energy lost by the wave is effectively transferred to the particles, or vice versa in the case of instabilities. In the case of electromagnetic waves, supplemental terms appear, corresponding to the energy transported by the wave (described by the Poynting flux), discussed in section 1.4. The calculation is more complicated, but still demonstrates that the energy is conserved, as it must.

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