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Moving grids and cell volume calculations

Stéphane Del Pino*, Bruno Després† and Alexiane Plessier‡

March 4, 2024

1 Introduction

Numerical approximation of continuum mechanics on moving grid is a classic topic that has been investigated for decades. There are two different formulations that naturally deal with mesh motion

1. ALE (Arbitrary Lagrangian Eulerian), that formulates continuum mechanics in a moving frame, and
2. Lagrangian (more precisely semi-Lagrangian or updated Lagrangian), where the frame is moving at the flow velocity.

Obviously Lagrangian is a special case of ALE, but it is singular in the fact that the frame motion is an unknown of the calculation.

At time t , in dimension $d \in \{1, 2, 3\}$, let us consider an arbitrary domain $\omega(t) \in \mathbb{R}^d$ that moves at a given velocity $\mathbf{u} : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. Let us denote by $V_{\omega(t)} := \int_{\omega(t)} 1 \, d\mathbf{x}$, the volume (or the measure) of ω at time t . Under classical regularity assumptions, one has

$$\forall t \geq 0, \quad V'_{\omega(t)} = \frac{d}{dt} \int_{\omega(t)} 1 \, d\mathbf{x} = \int_{\partial\omega(t)} \langle \mathbf{u}(t, \mathbf{x}), \mathbf{n}(t, \mathbf{x}) \rangle \, ds,$$

where \mathbf{n} denotes the outgoing unit normal to $\partial\omega$.

Let us choose two arbitrary times t^1 and t^2 , one obviously has

$$V_{\omega(t^2)} = V_{\omega(t^1)} + \int_{t^1}^{t^2} V'_{\omega(t)} \, dt. \tag{1}$$

The aim of this paper is to retrieve this kind of identity in the discrete case. This study focuses on the displacement of polygonal¹ cells defined by the motion of their vertices.

We first recall the two formulae that are classically used to compute cell volumes in the context of explicit semi-Lagrangian schemes. The first formula consists in computing the cell volume according to the vertex positions at some time t^n . The second one, accounts for volume fluxes to update the volume at time t^{n+1} , knowing the volume at time t^n . These formula correspond respectively to the left-hand side and right-hand side of the continuous formula (1). Then, for $d > 1$, we express the $\mathcal{O}(\Delta t^2)$ difference between these two volumes, when the natural explicit quadrature formula is used. Finally, we propose a modification of the volume fluxes formula that depends only on the vertex positions at time t^n and t^{n+1} . It restores the identity (1) at the discrete level.

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¹Polytopal in 3D.

2 Notations

Before discussing the discrete case, let us introduce some notations. Let \mathcal{M} be a mesh. We denote by

- \mathcal{J} the set of all cells of \mathcal{M} ,
- \mathcal{F} the set of all faces of \mathcal{M} (for $2 \leq d \leq 3$), and
- \mathcal{R} the set of all nodes of \mathcal{M} .

One also defines

- $j \in \mathcal{J}$ a cell,
- $f \in \mathcal{F}$ a face, and
- $r \in \mathcal{R}$ a node.

This allows to introduce local sets of elements, for instance

- $\mathcal{R}_j \subset \mathcal{R}$ is the set of nodes that define the cell $j \in \mathcal{J}$,
- $\mathcal{F}_r \in \mathcal{F}$ is the set of faces that are connected to the node $r \in \mathcal{R}$,
- ...

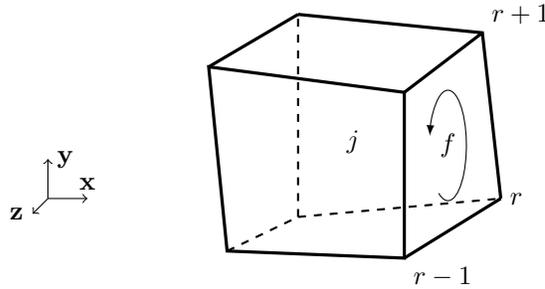


Figure 1: Illustration of the local node ordering with regard to a face of a cell. The ordering is locally defined by browsing the nodes r of the face f in the order that defines the normal outgoing from the cell j . This ordered set is denoted \mathcal{R}_{jf} .

Finally for 3D calculations we introduce the handy notation $[[\cdot]]^{jf}$.

Notation 1 (For 3D formulas). Let $j \in \mathcal{J}$ be a 3D cell of a mesh and $f \in \mathcal{F}_j$ be a face of j . We define the operator $[[\cdot]]^{jf}$ by

$$\forall s \in \mathcal{R}_{jf}, \quad [[\phi]]_s^{jf} = \phi_{s+1} - \phi_{s-1},$$

where \mathcal{R}_{jf} is the ordered set of nodes of the face f . The order is set to define the normal to f outgoing from the cell j . The notation $s+1$ (resp. $s-1$) designates the node after (resp. before) the node s in \mathcal{R}_{jf} (see Figure 1), and periodicity is implied: $s = s + \#\mathcal{R}_{jf}$.

3 Volume calculations of a moving polygonal cell

From the discrete point of view, one generally considers polygonal cells that remain polygonal during the calculation. Such cells can be defined according to their vertices². A direct consequence is the definition of the volume V_j of a cell j according to their vertices positions $(\mathbf{x}_r)_{r \in \mathcal{R}_j}$

$$V_j = V\left((\mathbf{x}_r)_{r \in \mathcal{R}_j}\right).$$

For a set of nodal velocities $(\mathbf{u}_r)_{r \in \mathcal{R}_j}$, chain rule states that the volume variation writes

$$V_j' = \sum_{r \in \mathcal{R}_j} \langle \nabla_{\mathbf{x}_r} V_j, \mathbf{u}_r \rangle,$$

with $\mathbf{u}_r = \frac{d}{dt} \mathbf{x}_r$. Thus it is common to introduce the notation $\mathbf{C}_{jr} := \nabla_{\mathbf{x}_r} V_j$, so that one writes

$$V_j' = \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}, \mathbf{u}_r \rangle,$$

and one shows, see for instance [1], that in dimension d , the volume expresses as

$$V_j = \frac{1}{d} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}, \mathbf{x}_r \rangle.$$

From the continuous in time point of view, one retrieves formula (8) in the case of polygonal cell motion

$$V_j^{n+1} = V_j^n + \int_{t^n}^{t^{n+1}} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}(t), \mathbf{u}_r(t) \rangle dt, \quad (2)$$

where $V_j^n = \frac{1}{d} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{x}_r^n \rangle$, and superscripts account classically for $\phi^n = \phi(t^n)$.

3.1 \mathbf{C}_{jr} vectors

In order to deal with discrete in time versions of formula (2), we first give expressions of the \mathbf{C}_{jr} vectors in dimensions $d = \{1, 2, 3\}$.

As mentioned previously, the 3D case is not straightforward since one has to choose the geometry of non planar faces. In practice, 3D cells are defined by their vertex position and connectivity choice. Even if 3D cells are usually designed as polyedra in the literature, this is inappropriate since polyedra are *sensu stricto* defined as half-plane intersections, *i.e.* polyedra have flat faces. Unstructured meshes are generally made of faces which are not coplanar, and actually there is no unique way to define the geometry of a cell by just knowing its vertices. Let us recall three possible definitions, see Figure 2. In finite element analysis [2], one defines the \mathbb{Q}_1 -element as the image of the trilinear transformation of the unit cube $\hat{Q} :=]0, 1[^3$ which satisfies $\mathbf{x}_r = T_j(\hat{\mathbf{x}}_r)$, where $\hat{\mathbf{x}}_r$ and \mathbf{x}_r are the vertices of the reference cell and their corresponding image. Due to its definition, it requires T_j to be bijective, that is $|T_j'| > 0$ in \hat{Q} . Yield faces are hyperbolic paraboloids, which are double ruled but non-planar. Observe that if $|T_j'| \not\geq 0$, the yield face can be strictly included in $T_j(\hat{Q})$. Another possibility is to consider the ‘‘T2’’ definition of faces which consists in meshing conformly each face with triangles lying on face vertices. This is the simplest decomposition but one has to choose arbitrarily each face decomposition. This approach is used in [7]. Due to the arbitrariness of the decomposition, it can break numerically the symmetry of the flow, this is why in practice one prefers ‘‘2T2’’ decomposition which

²If in 2D, this definition is natural (considering straight edges), in 3D one has to precise the geometry of faces which may be non-planar.

consists in defining the volume as the mean of all possible choices. The disadvantage of such an approach is that one no longer knows clearly the boundary of the defined cell.

Last example we consider is the so called “T4” approach, see for instance [8], which consists in the decomposition of each polygonal face in triangles as depicted on Figure 2. The geometry is defined by constructing triangles based on each edge of the face and connected to the isobarycenter of the face’s vertices. As it is remarked in [5] one can show that “T4” gives the same volume as \mathbb{Q}_1 -element and it is also the case for the \mathbb{P}_1 -element. Thus with regard to the volume measure, this cell definition can be viewed as a generalization of linear isoparametric elements to arbitrary polytopal.

This is why we consider the “T4” case in this paper. The calculation of \mathbf{C}_{jr} is established for general polytopal cells: faces that can have an arbitrary number of edges. To the best of our knowledge, the “T4” is explained only for warped hexahedra in the literature.

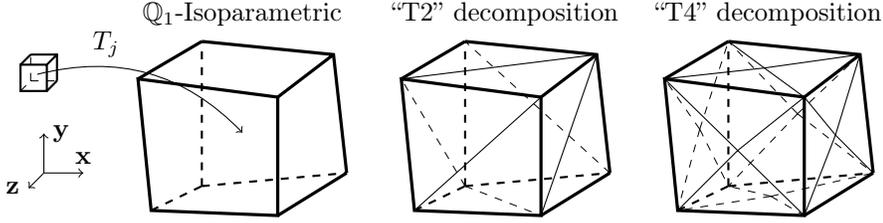


Figure 2: Examples of hexahedral cell definition. **Left:** Isoparametric trilinear transformation. **Middle:** Simplest faces simplicial decomposition. **Right:** Isobarycentric faces simplicial decomposition.

Property 1. Let $d \in \{1, 2, 3\}$ be the space dimension, $j \in \mathcal{J}$, $r \in \mathcal{R}_j$ and $\mathbf{C}_{jr} = \nabla_{\mathbf{x}_r} V_j$.

- In dimension $d = 1$, since $V_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ then

$$C_{j,j\pm\frac{1}{2}} = \pm 1. \quad (3)$$

- In dimension $d = 2$ if j is the polygonal cell defined by the nodes $r \in \mathcal{R}_j$ then

$$\mathbf{C}_{jr} = -\frac{1}{2}(\mathbf{x}_{r+1} - \mathbf{x}_{r-1})^\perp, \quad (4)$$

where $r + 1$ (resp. $r - 1$) denote the node before (resp. after) r in \mathcal{R}_j .

- In dimension $d = 3$, if j is a “T4” polytopal cell, then

$$\mathbf{C}_{jr} = \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left(\sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jf} - [\mathbf{x}]_s^{jf} \right) \wedge \mathbf{x}_s \right), \quad (5)$$

where \mathcal{F}_{jr} denotes the set of faces of cell j connected to the node r .

Proof. The cases $d = 1$ and $d = 2$ are straightforward and classical. The case $d = 3$ is established in Appendix A. \square

Remark 1. One should note that while deriving the \mathbf{C}_{jr} vectors in 3D, we obtain the expression of the so-call \mathbf{N}_{jfr} vectors:

$$\mathbf{N}_{jfr} := \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jf} - [\mathbf{x}]_s^{jf} \right) \wedge \mathbf{x}_s - \frac{1}{6}[\mathbf{x}]_r^{jf} \wedge \mathbf{x}_r.$$

These vectors can be used to define the Eucclhyd [6, 7, 4] scheme on arbitrary “T4” cells.

3.2 Time discretization

Let us now examine a first order in time explicit approximation of the right hand side of (2). This is of special interest since it is standard when building first order finite volume semi-Lagrangian schemes such as [3, 1] or [6, 7, 4] for instance. For these schemes the continuity equation is approximated as

$$M_j \tau_j^{n+1} = M_j \tau_j^n + \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r^n \rangle,$$

where M_j is the Lagrangian mass of the cell j and $\tau = \rho^{-1}$ is the specific volume. Actually this can be rewritten as

$$\bar{V}_j = V_j^n + \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r^n \rangle, \quad (6)$$

which is a first formula for the cell volume at time t^{n+1} . However for these schemes, the cells are displaced by setting $\mathbf{x}_r^{n+1} = \mathbf{x}_r^n + \Delta t \mathbf{u}_r^n$ and thus the new cell volume can also be defined by

$$V_j^{n+1} = \frac{1}{d} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^{n+1}, \mathbf{x}_r^{n+1} \rangle. \quad (7)$$

Property 2. Let \bar{V}_j and V_j^{n+1} be defined respectively by (6) and (7), then one has the exact formula

$$V_j^{n+1} - \bar{V}_j = \Delta t^2 \mathcal{G}_j, \quad (8)$$

where \mathcal{G}_j depends on the dimension d

$$\begin{cases} d = 1, & \mathcal{G}_j = 0, \\ d = 2, & \mathcal{G}_j = \frac{1}{2} \sum_{r \in \mathcal{R}_j} \langle \mathbf{u}_r^{n+1}, (\mathbf{u}_r^n)^\perp \rangle, \\ d = 3, & \mathcal{G}_j = \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_j} \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{u}^n]_r^{jf} - [\mathbf{u}^n]_s^{jf} \right) \wedge \mathbf{u}_s^n, \mathbf{x}_r^n + \frac{\Delta t}{3} \mathbf{u}_r^n \right\rangle, \end{cases}$$

where the object $[\cdot]^{jf}$ in dimension $d = 3$ has been introduced in Notation 1.

Proof. The proof is technical and is given in Appendix B. \square

Property 2 establishes that in dimension $d \geq 2$, first order explicit schemes defined using (6) experience a second order in time discrepancy for the volume calculation. Indeed, after the mesh motion, the new volume is usually updated by (7), using new node positions.

For (first order) explicit schemes, this is not an issue. Due to the CFL condition this difference converges rapidly. However, this can be more problematic in the case of implicit schemes. In that case, the time step may be large and cause instabilities (density can become negative or physical entropy can decrease).

To address this problem, the idea is to substitute the explicit \mathbf{C}_{jr}^n vectors by new ones that we denote \mathbf{D}_{jr} in order to establish

$$V_j^{n+1} = V_j^n + \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{D}_{jr}, \mathbf{u}_r \rangle,$$

where $\forall r \in \mathcal{R}$, $\mathbf{u}_r = \frac{1}{\Delta t} (\mathbf{x}_r^{n+1} - \mathbf{x}_r^n)$.

This is formalized as the following Problem.

Problem 1. Find specific vectors $\mathbf{D}_{jr} \in \mathbb{R}^d$ (for all $j \in \mathcal{J}$ and for all $r \in \mathcal{R}_j$) such that

$$\forall j \in \mathcal{J}, \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{D}_{jr}, \mathbf{u}_r \rangle = \frac{1}{d} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^{n+1}, \mathbf{x}_r^{n+1} \rangle - \frac{1}{d} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{x}_r^n \rangle. \quad (9)$$

Actually these vectors are defined uniquely for $1 \leq d \leq 3$ and are only defined according to the positions of the vertices of the cell at time t^n and t^{n+1} .

Proposition 1. *Problem (1) admits a unique solution.*

- In dimension $d = 1$,

$$\forall j \in \mathcal{J}, \quad \mathbf{D}_{j,j \pm \frac{1}{2}} = \pm 1. \quad (10)$$

- In dimension $d = 2$,

$$\forall j \in \mathcal{J}, \forall r \in \mathcal{R}_j, \quad \mathbf{D}_{jr} = \frac{1}{2} (\mathbf{C}_{jr}^n + \mathbf{C}_{jr}^{n+1}), \quad (11)$$

where $\mathbf{C}_{jr}^n = -\frac{1}{2}(\mathbf{x}_{r+1}^n - \mathbf{x}_{r-1}^n)^\perp$.

- In dimension $d = 3$,

$$\forall j \in \mathcal{J}, \forall r \in \mathcal{R}_j, \quad \mathbf{D}_{jr} = \frac{1}{2} (\mathbf{C}_{jr}^n + \mathbf{C}_{jr}^{n+1}) + \mathbf{C}_{jr}^\delta, \quad (12)$$

where

$$\begin{aligned} \mathbf{C}_{jr}^n &= \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left(\sum_{s \in \mathcal{R}_{jf}} \left(2\llbracket \mathbf{x} \rrbracket_r^{jfn} - \llbracket \mathbf{x} \rrbracket_s^{jfn} \right) \wedge \mathbf{x}_s^n \right), \quad \text{and} \\ \mathbf{C}_{jr}^\delta &= - \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \frac{1}{6} \left[2 \left(\llbracket \mathbf{x} \rrbracket_r^{jfn+1} - \llbracket \mathbf{x} \rrbracket_r^{jfn} \right) - \left(\llbracket \mathbf{x} \rrbracket_s^{jfn+1} - \llbracket \mathbf{x} \rrbracket_s^{jfn} \right) \right] \wedge (\mathbf{x}_s^{n+1} - \mathbf{x}_s^n). \end{aligned}$$

Proof. The proof is calculative especially in 3D. Thus, for the sake of clarity, it is given in Appendix C. \square

The \mathbf{C}_{jr} vectors satisfy the two following properties

$$\forall j \in \mathcal{J}, \quad \sum_{r \in \mathcal{R}_j} \mathbf{C}_{jr} = \mathbf{0}, \quad \text{and} \quad \forall r \in \mathcal{R}, \quad \sum_{j \in \mathcal{J}_r} \mathbf{C}_{jr} = \mathbf{0}. \quad (13)$$

These properties are important ingredients to establish conservation and stability of explicit finite volume semi-Lagrangian schemes such as [3, 1] or [6, 7, 4].

One easily shows that \mathbf{D}_{jr} vectors satisfy the same kind of relations,

$$\forall j \in \mathcal{J}, \quad \sum_{r \in \mathcal{R}_j} \mathbf{D}_{jr} = \mathbf{0}, \quad \text{and} \quad \forall r \in \mathcal{R}, \quad \sum_{j \in \mathcal{J}_r} \mathbf{D}_{jr} = \mathbf{0}. \quad (14)$$

Proof. See Appendix C.2. \square

A Construction of \mathbf{C}_{jr} in 3D

Here we prove the case $d = 3$ of Property 1. The cases $d \in \{1, 2\}$ are straightforward and have been published many times.

Let f be a face of the 3D cell j defined by N_f vertices $(\mathbf{x}_s)_{1 \leq s \leq N_f}$ which are ordered in the direct way according to the outgoing normal to j , see Figure 3. Thus, let \mathcal{R}_{jf} denote this ordered set of nodes. Let us define its center by $\mathbf{x}_G := \frac{1}{N_f} \sum_s \mathbf{x}_s$. For convenience, one also sets $\mathbf{x}_0 = \mathbf{x}_{N_f}$. Let us assume that vertex r is displaced to the new position $\tilde{\mathbf{x}}_r := \mathbf{x}_r + \delta \mathbf{x}_r$. The new face \tilde{f} lies on points $(\mathbf{x}_s)_{s \neq r}$, $\tilde{\mathbf{x}}_r$ and its center is $\tilde{\mathbf{x}}_G = \frac{1}{N_f} (\sum_{s \neq r} \mathbf{x}_s + \tilde{\mathbf{x}}_r)$. Obviously, the displacement of vertex G is nothing else but $\delta \mathbf{x}_G := \tilde{\mathbf{x}}_G - \mathbf{x}_G = \frac{1}{N_f} (\tilde{\mathbf{x}}_r - \mathbf{x}_r)$, so that $\delta \mathbf{x}_G = \frac{1}{N_f} \delta \mathbf{x}_r$. So, the volume spanned by the displacement of vertex r is the sum of the volume of N_f tetrahedra $(\mathbf{x}_G, \mathbf{x}_{s-1}, \mathbf{x}_s, \tilde{\mathbf{x}}_G)$ to which one has to add the volume of both tetrahedra $(\mathbf{x}_r, \tilde{\mathbf{x}}_G, \mathbf{x}_{r-1}, \tilde{\mathbf{x}}_r)$ and $(\mathbf{x}_r, \mathbf{x}_{r+1}, \tilde{\mathbf{x}}_G, \tilde{\mathbf{x}}_r)$.

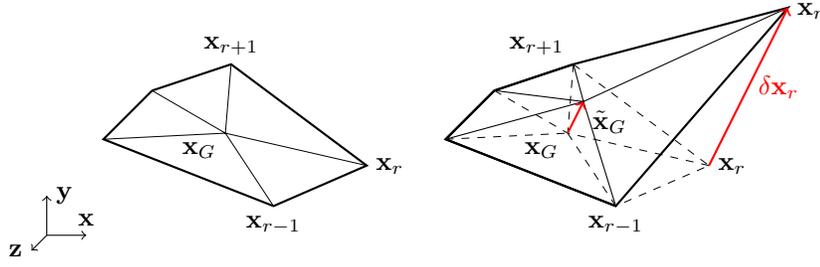


Figure 3: Volume change due to a vertex displacement.

Let $V_{s-1,s}$ denote the volume of the tetrahedron $(\mathbf{x}_G, \tilde{\mathbf{x}}_G, \mathbf{x}_{s-1}, \mathbf{x}_s)$. One has

$$\begin{aligned} 6V_{s-1,s} &= \det(\mathbf{x}_{s-1} - \mathbf{x}_G, \mathbf{x}_s - \mathbf{x}_G, \delta\mathbf{x}_G), \\ &= \det(\mathbf{x}_{s-1}, \mathbf{x}_s - \mathbf{x}_G, \delta\mathbf{x}_G) - \det(\mathbf{x}_G, \mathbf{x}_s - \mathbf{x}_G, \delta\mathbf{x}_G), \\ &= \det(\mathbf{x}_{s-1}, \mathbf{x}_s, \delta\mathbf{x}_G) - \det(\mathbf{x}_{s-1}, \mathbf{x}_G, \delta\mathbf{x}_G) + \det(\mathbf{x}_s, \mathbf{x}_G, \delta\mathbf{x}_G). \end{aligned}$$

Finally the sum of these tetrahedron volumes simplifies as

$$\sum_{s \in \mathcal{R}_{jf}} V_{s-1,s} = \frac{1}{6N_f} \sum_{s \in \mathcal{R}_{jf}} \det(\mathbf{x}_{s-1}, \mathbf{x}_s, \delta\mathbf{x}_r) = \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \det(\mathbf{x}_s, [\mathbf{x}]_s^{jf}, \delta\mathbf{x}_r).$$

The two remaining tetrahedron volumes sum write

$$\begin{aligned} 6(V_{r-1} + V_{r+1}) &= \det(\tilde{\mathbf{x}}_G - \mathbf{x}_r, \mathbf{x}_{r-1} - \mathbf{x}_r, \delta\mathbf{x}_r) + \det(\mathbf{x}_{r+1} - \mathbf{x}_r, \tilde{\mathbf{x}}_G - \mathbf{x}_r, \delta\mathbf{x}_r), \\ &= -\det(\mathbf{x}_{r-1} - \mathbf{x}_r, \tilde{\mathbf{x}}_G - \mathbf{x}_r, \delta\mathbf{x}_r) + \det(\mathbf{x}_{r+1} - \mathbf{x}_r, \tilde{\mathbf{x}}_G - \mathbf{x}_r, \delta\mathbf{x}_r), \\ &= \det([\mathbf{x}]_r^{jf}, \mathbf{x}_G + \delta\mathbf{x}_G - \mathbf{x}_r, \delta\mathbf{x}_r), \\ &= \det([\mathbf{x}]_r^{jf}, \mathbf{x}_G + \frac{1}{N_f} \delta\mathbf{x}_r - \mathbf{x}_r, \delta\mathbf{x}_r), \\ &= \det([\mathbf{x}]_r^{jf}, \mathbf{x}_G - \mathbf{x}_r, \delta\mathbf{x}_r), \end{aligned}$$

so, finally

$$(V_{r-1} + V_{r+1}) = \frac{1}{6N_f} \sum_{s \in \mathcal{R}_{jf}} \det([\mathbf{x}]_r^{jf}, \mathbf{x}_s, \delta\mathbf{x}_r) - \frac{1}{6} \det([\mathbf{x}]_r^{jf}, \mathbf{x}_r, \delta\mathbf{x}_r).$$

It follows that the volume spanned by the displacement of node r for the face f reads

$$V_r^f = \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \det(2[\mathbf{x}]_r^{jf} - [\mathbf{x}]_s^{jf}, \mathbf{x}_s, \delta\mathbf{x}_r) - \frac{1}{6} \det([\mathbf{x}]_r^{jf}, \mathbf{x}_r, \delta\mathbf{x}_r),$$

which rewrites

$$V_r^f = \left\langle \left(\frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} (2[\mathbf{x}]_r^{jf} - [\mathbf{x}]_s^{jf}) \wedge \mathbf{x}_s - \frac{1}{6} [\mathbf{x}]_r^{jf} \wedge \mathbf{x}_r \right), \delta\mathbf{x}_r \right\rangle.$$

This allows to define vectors

$$\mathbf{N}_{jfr} := \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} (2[\mathbf{x}]_r^{jf} - [\mathbf{x}]_s^{jf}) \wedge \mathbf{x}_s - \frac{1}{6} [\mathbf{x}]_r^{jf} \wedge \mathbf{x}_r,$$

that are used to define the Eucclhyd scheme for this kind of cells.

Also, the cell's volume variation due to the displacement of node r is written

$$\delta V_r = \left\langle \sum_{f \in \mathcal{F}_{jr}} \left(\frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jf} - [\mathbf{x}]_s^{jf} \right) \wedge \mathbf{x}_s - \frac{1}{6} [\mathbf{x}]_r^{jf} \wedge \mathbf{x}_r \right), \delta \mathbf{x}_r \right\rangle,$$

where we denoted $\mathcal{F}_{jr} = \mathcal{F}_j \cap \mathcal{F}_r$. Now, since the faces connected to node r are oriented directly according to the outgoing normal, one has $\sum_{f \in \mathcal{F}_{jr}} [\mathbf{x}]_r^{jf} = \mathbf{0}$ and then

$$\delta V_r = \left\langle \sum_{f \in \mathcal{F}_{jr}} \left(\frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jf} - [\mathbf{x}]_s^{jf} \right) \wedge \mathbf{x}_s \right), \delta \mathbf{x}_r \right\rangle.$$

Recalling that by definition $\mathbf{C}_{jr} := \nabla_{\mathbf{x}_r} V_j$, we have shown that for the chosen geometry in 3D, one has

$$\mathbf{C}_{jr} = \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left(\sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jf} - [\mathbf{x}]_s^{jf} \right) \wedge \mathbf{x}_s \right),$$

which ends the proof.

B Proof of Property 2

We compute the volume calculation difference by comparing the two formulas, that is according to vertex positions

$$\forall j \in \mathcal{J}, \quad V_j^{n+1} = \frac{1}{d} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^{n+1}, \mathbf{x}_r^{n+1} \rangle, \quad (15)$$

or updated by the formula

$$\forall j \in \mathcal{J}, \quad \bar{V}_j = V_j^n + \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r \rangle, \quad (16)$$

where the new position is defined by

$$\forall r \in \mathcal{R}, \quad \mathbf{x}_r^{n+1} = \mathbf{x}_r^n + \Delta t \mathbf{u}_r.$$

B.1 Case $d = 1$

In 1D, denoting vertices using the classical $j + \frac{1}{2}$ notation, one has $C_{j, j \pm \frac{1}{2}} = \pm 1$. Thus formulas (15) and (16) rewrite

$$V_j^{n+1} = x_{j+\frac{1}{2}}^{n+1} - x_{j-\frac{1}{2}}^{n+1} \quad \text{and} \quad \bar{V}_j = V_j^n + \Delta t \left(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}} \right).$$

Since $V_j^n = x_{j+\frac{1}{2}}^n - x_{j-\frac{1}{2}}^n$, one obviously has $\mathcal{G}_j = 0$.

B.2 Case $d = 2$

For 2D polygonal cells, $\mathbf{C}_{jr}^n = -\frac{1}{2}(\mathbf{x}_{r+1}^n - \mathbf{x}_{r-1}^n)^\perp$. So, one has

$$V_j^{n+1} = -\frac{1}{2} \sum_{r \in \mathcal{R}_j} \frac{1}{2} \left\langle (\mathbf{x}_{r+1}^{n+1} - \mathbf{x}_{r-1}^{n+1})^\perp, \mathbf{x}_r^{n+1} \right\rangle,$$

$$V_j^{n+1} = -\frac{1}{2} \sum_{r \in \mathcal{R}_j} \frac{1}{2} \left\langle (\mathbf{x}_{r+1}^n + \Delta t \mathbf{u}_{r+1} - \mathbf{x}_{r-1}^n - \Delta t \mathbf{u}_{r-1})^\perp, (\mathbf{x}_r^n + \Delta t \mathbf{u}_r) \right\rangle,$$

$$\begin{aligned}
V_j^{n+1} &= \frac{1}{2} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{x}_r^n \rangle + \frac{\Delta t}{2} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r \rangle \\
&\quad - \frac{\Delta t}{2} \sum_{r \in \mathcal{R}_j} \frac{1}{2} \langle (\mathbf{u}_{r+1} - \mathbf{u}_{r-1})^\perp, \mathbf{x}_r^n \rangle - \frac{\Delta t^2}{4} \sum_{r \in \mathcal{R}_j} \langle (\mathbf{u}_{r+1} - \mathbf{u}_{r-1})^\perp, \mathbf{u}_r \rangle.
\end{aligned}$$

After development and reindexation of the telescopic sums, one finds

$$- \sum_{r \in \mathcal{R}_j} \frac{1}{2} \langle (\mathbf{u}_{r+1} - \mathbf{u}_{r-1})^\perp, \mathbf{x}_r^n \rangle = \sum_{r \in \mathcal{R}_j} \left\langle \mathbf{u}_r, \frac{1}{2} (\mathbf{x}_{r-1}^n - \mathbf{x}_{r+1}^n)^\perp \right\rangle = \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r \rangle.$$

Also, noticing that $\sum_{r \in \mathcal{R}_j} \langle (\mathbf{u}_{r+1} - \mathbf{u}_{r-1})^\perp, \mathbf{u}_r \rangle = 2 \sum_{r \in \mathcal{R}_j} \langle \mathbf{u}_{r+1}^\perp, \mathbf{u}_r \rangle$, one gets

$$V_j^{n+1} = \underbrace{V_j^n + \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r \rangle}_{\mathcal{V}_j} + \Delta t^2 \underbrace{\frac{1}{2} \sum_{r \in \mathcal{R}_j} \langle \mathbf{u}_{r+1}, \mathbf{u}_r^\perp \rangle}_{\mathcal{G}_j}.$$

B.3 Case $d = 3$

Let us examine $V_j^{n+1} - V_j^n$ in dimension 3. One has $V_j^{n+1} = \frac{1}{3} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^{n+1}, \mathbf{x}_r^{n+1} \rangle$, so

$$V_j^{n+1} = \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left(\sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn+1} - [\mathbf{x}]_s^{jfn+1} \right) \wedge \mathbf{x}_s^{n+1} \right), \mathbf{x}_r^{n+1} \right\rangle.$$

Since $\mathbf{x}_s^{n+1} = \mathbf{x}_s^n + \Delta t \mathbf{u}_s$ and writing $[\mathbf{x}]_s^{jfn+1} = [\mathbf{x}]_s^{jfn} + \Delta t [\mathbf{u}]_s^{jf}$, one obtains after elementary manipulations

$$\begin{aligned}
V_j^{n+1} - V_j^n &= \frac{\Delta t}{3} \left\{ \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \right. \\
&\quad + \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \mathbf{u}_s, \mathbf{x}_r^n \right\rangle \\
&\quad + \left. \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{u}]_r^{jf} - [\mathbf{u}]_s^{jf} \right) \wedge \mathbf{x}_s^n, \mathbf{x}_r^n \right\rangle \right\} \\
&\quad + \frac{\Delta t^2}{3} \left\{ \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{u}]_r^{jf} - [\mathbf{u}]_s^{jf} \right) \wedge \mathbf{u}_s, \mathbf{x}_r^n \right\rangle \right. \\
&\quad + \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{u}]_r^{jf} - [\mathbf{u}]_s^{jf} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad + \left. \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \mathbf{u}_s, \mathbf{u}_r \right\rangle \right\} \\
&\quad + \frac{\Delta t^3}{3} \left\{ \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{u}]_r^{jf} - [\mathbf{u}]_s^{jf} \right) \wedge \mathbf{u}_s, \mathbf{u}_r \right\rangle \right\}
\end{aligned}$$

Thus, this defines, I_1 , I_2 and I_3 such that $V_j^{n+1} - V_j^n = \frac{\Delta t}{3}I_1 + \frac{\Delta t^2}{3}I_2 + \frac{\Delta t^3}{3}I_3$. Let us study each term of the sum. To analyze these three terms, let us establish few simple technical results.

First we will need the following Lemma.

Lemma 1. *Let $\forall s \in \mathcal{R}_{jf}$, $\mathbf{a}_s, \mathbf{b}_s \in \mathbb{R}^3$, then one has*

$$\sum_{s \in \mathcal{R}_{jf}} \llbracket \mathbf{a} \rrbracket_s^{jf} \wedge \mathbf{b}_s = \sum_{s \in \mathcal{R}_{jf}} \llbracket \mathbf{b} \rrbracket_s^{jf} \wedge \mathbf{a}_s. \quad (17)$$

Proof. Proving this result only requires a few algebra. □

I_1 and I_2 terms have the following structure

$$\begin{aligned} I = \sum_{r \in \mathcal{R}_j} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} & \left(\left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2\llbracket \mathbf{a} \rrbracket_r^{jf} - \llbracket \mathbf{a} \rrbracket_s^{jf} \right) \wedge \mathbf{a}_s, \mathbf{b}_r \right\rangle \right. \\ & + \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2\llbracket \mathbf{a} \rrbracket_r^{jf} - \llbracket \mathbf{a} \rrbracket_s^{jf} \right) \wedge \mathbf{b}_s, \mathbf{a}_r \right\rangle \\ & \left. + \left\langle \sum_{s \in \mathcal{R}_{jf}} \left(2\llbracket \mathbf{b} \rrbracket_r^{jf} - \llbracket \mathbf{b} \rrbracket_s^{jf} \right) \wedge \mathbf{a}_s, \mathbf{a}_r \right\rangle \right) \end{aligned}$$

that one can rewrite, changing summation order

$$\begin{aligned} I = \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} & \left(\left\langle \left(2\llbracket \mathbf{a} \rrbracket_r^{jf} - \llbracket \mathbf{a} \rrbracket_s^{jf} \right) \wedge \mathbf{a}_s, \mathbf{b}_r \right\rangle \right. \\ & + \left\langle \left(2\llbracket \mathbf{a} \rrbracket_r^{jf} - \llbracket \mathbf{a} \rrbracket_s^{jf} \right) \wedge \mathbf{b}_s, \mathbf{a}_r \right\rangle \\ & \left. + \left\langle \left(2\llbracket \mathbf{b} \rrbracket_r^{jf} - \llbracket \mathbf{b} \rrbracket_s^{jf} \right) \wedge \mathbf{a}_s, \mathbf{a}_r \right\rangle \right). \end{aligned}$$

First, looking at the second term of the sum, which is a determinant, one has

$$\forall r, s \in \mathcal{R}_{jf}, \quad \left\langle \left(2\llbracket \mathbf{a} \rrbracket_r^{jf} - \llbracket \mathbf{a} \rrbracket_s^{jf} \right) \wedge \mathbf{b}_s, \mathbf{a}_r \right\rangle = \left\langle \mathbf{a}_r \wedge \left(2\llbracket \mathbf{a} \rrbracket_r^{jf} - \llbracket \mathbf{a} \rrbracket_s^{jf} \right), \mathbf{b}_s \right\rangle,$$

and exchanging r and s indices, one gets

$$\begin{aligned} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \left(2\llbracket \mathbf{a} \rrbracket_r^{jf} - \llbracket \mathbf{a} \rrbracket_s^{jf} \right) \wedge \mathbf{b}_s, \mathbf{a}_r \right\rangle &= \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \mathbf{a}_s \wedge \left(2\llbracket \mathbf{a} \rrbracket_s^{jf} - \llbracket \mathbf{a} \rrbracket_r^{jf} \right), \mathbf{b}_r \right\rangle, \\ &= \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \left(\llbracket \mathbf{a} \rrbracket_r^{jf} - 2\llbracket \mathbf{a} \rrbracket_s^{jf} \right) \wedge \mathbf{a}_s, \mathbf{b}_r \right\rangle. \end{aligned}$$

The last term in the sum I can also be rearranged as

$$\sum_{r, s \in \mathcal{R}_{jf}} \left\langle \left(2\llbracket \mathbf{b} \rrbracket_r^{jf} - \llbracket \mathbf{b} \rrbracket_s^{jf} \right) \wedge \mathbf{a}_s, \mathbf{a}_r \right\rangle = \sum_{r, s \in \mathcal{R}_{jf}} \left\langle 2\llbracket \mathbf{b} \rrbracket_r^{jf} \wedge \mathbf{a}_s, \mathbf{a}_r \right\rangle - \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \llbracket \mathbf{b} \rrbracket_s^{jf} \wedge \mathbf{a}_s, \mathbf{a}_r \right\rangle.$$

Now one has

$$\sum_{r, s \in \mathcal{R}_{jf}} \left\langle 2\llbracket \mathbf{b} \rrbracket_r^{jf} \wedge \mathbf{a}_s, \mathbf{a}_r \right\rangle = \sum_{r, s \in \mathcal{R}_{jf}} \left\langle 2\mathbf{a}_r \wedge \llbracket \mathbf{b} \rrbracket_r^{jf}, \mathbf{a}_s \right\rangle,$$

and according to Lemma 1

$$\begin{aligned} \sum_{r,s \in \mathcal{R}_{jf}} \langle 2[\mathbf{b}]_r^{jf} \wedge \mathbf{a}_s, \mathbf{a}_r \rangle &= \sum_{r,s \in \mathcal{R}_{jf}} \langle \mathbf{b}_r \wedge 2[\mathbf{a}]_r^{jf}, \mathbf{a}_s \rangle, \\ &= \sum_{r,s \in \mathcal{R}_{jf}} \langle 2[\mathbf{a}]_r^{jf} \wedge \mathbf{a}_s, \mathbf{b}_r \rangle. \end{aligned}$$

Finally, using Lemma 1 and similar algebra, the remaining term writes

$$\begin{aligned} - \sum_{r,s \in \mathcal{R}_{jf}} \langle [\mathbf{b}]_s^{jf} \wedge \mathbf{a}_s, \mathbf{a}_r \rangle &= - \sum_{r,s \in \mathcal{R}_{jf}} \langle [\mathbf{a}]_s^{jf} \wedge \mathbf{b}_s, \mathbf{a}_r \rangle \\ &= - \sum_{r,s \in \mathcal{R}_{jf}} \langle \mathbf{a}_r \wedge [\mathbf{a}]_s^{jf}, \wedge \mathbf{b}_s \rangle \\ &= \sum_{r,s \in \mathcal{R}_{jf}} \langle [\mathbf{a}]_r^{jf} \wedge \mathbf{a}_s, \mathbf{b}_r \rangle. \end{aligned}$$

Thus, one has

$$\sum_{r,s \in \mathcal{R}_{jf}} \langle (2[\mathbf{b}]_r^{jf} - [\mathbf{b}]_s^{jf}) \wedge \mathbf{a}_s, \mathbf{a}_r \rangle = \sum_{r,s \in \mathcal{R}_{jf}} \langle 3[\mathbf{a}]_r^{jf} \wedge \mathbf{a}_s, \mathbf{b}_r \rangle.$$

Putting all the pieces together and factorizing, one gets

$$I = 3 \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \langle (2[\mathbf{a}]_r^{jf} - [\mathbf{a}]_s^{jf}) \wedge \mathbf{a}_s, \mathbf{b}_r \rangle.$$

Thus, it comes that the volume variation reads

$$\begin{aligned} V_j^{n+1} - V_j^n &= \Delta t \sum_{r \in \mathcal{R}_j} \left\langle \overbrace{\sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} (2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn}) \wedge \mathbf{x}_s^n, \mathbf{u}_r}^{\mathbf{C}_{jr}^n} \right\rangle \\ &\quad + \Delta t^2 \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} (2[\mathbf{u}]_r^{jfn} - [\mathbf{u}]_s^{jfn}) \wedge \mathbf{u}_s, \mathbf{x}_r^n \right\rangle \\ &\quad + \frac{\Delta t^3}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} (2[\mathbf{u}]_r^{jfn} - [\mathbf{u}]_s^{jfn}) \wedge \mathbf{u}_s, \mathbf{u}_r \right\rangle. \quad (18) \end{aligned}$$

So, one gets

$$V_j^{n+1} - \bar{V}_j = \Delta t^2 \mathcal{G}_j.$$

with

$$\mathcal{G}_j = \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} (2[\mathbf{u}]_r^{jfn} - [\mathbf{u}]_s^{jfn}) \wedge \mathbf{u}_s, \mathbf{x}_r^n + \frac{\Delta t}{3} \mathbf{u}_r \right\rangle,$$

which ends the proof.

C Proof of Proposition 1

We first compute the \mathbf{D}_{jr} vectors and then prove that these vectors satisfy (14).

C.1 Calculation of \mathbf{D}_{jr} vectors

Proving that in dimension 1, $D_{j,j\pm\frac{1}{2}} = \pm 1$ is straightforward.

C.1.1 Dimension 2

In order to simplify the notations, in this paragraph, one will consider a local and direct numbering of the nodes of the cells. Thus $r + 1$ (resp. $r - 1$) will denote the next (resp. previous) node in of the cell j .

In the 2D polygonal case, one has

$$\mathbf{C}_{jr}^n = -\frac{1}{2}(\mathbf{x}_{r+1}^n - \mathbf{x}_{r-1}^n)^\perp,$$

and since $\mathbf{x}_r^{n+1} = \mathbf{x}_r^n + \Delta t \mathbf{u}_r$,

$$\mathbf{C}_{jr}^{n+1} = \underbrace{-\frac{1}{2}(\mathbf{x}_{r+1}^n - \mathbf{x}_{r-1}^n)^\perp}_{\mathbf{C}_{jr}^n} - \frac{\Delta t}{2}(\mathbf{u}_{r+1} - \mathbf{u}_{r-1})^\perp,$$

So, (9) recasts in this case as

$$\forall (\mathbf{u}_r)_{r \in \mathcal{R}_j}, \quad \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{D}_{jr}, \mathbf{u}_r \rangle = \frac{1}{2} \sum_{r \in \mathcal{R}_j} \left\langle \mathbf{C}_{jr}^n - \frac{\Delta t}{2}(\mathbf{u}_{r+1} - \mathbf{u}_{r-1})^\perp, \mathbf{x}_r^n + \Delta t \mathbf{u}_r \right\rangle - \frac{1}{2} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{x}_r^n \rangle,$$

which expands to

$$\forall (\mathbf{u}_r)_{r \in \mathcal{R}_j}, \quad \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{D}_{jr}, \mathbf{u}_r \rangle = \frac{\Delta t}{2} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r \rangle - \frac{\Delta t}{4} \sum_{r \in \mathcal{R}_j} \left\langle (\mathbf{u}_{r+1} - \mathbf{u}_{r-1})^\perp, \mathbf{x}_r^n + \Delta t \mathbf{u}_r \right\rangle,$$

It is easy to check that $\sum_r \langle \mathbf{a}_r^\perp, \mathbf{b}_r \rangle = -\sum_r \langle \mathbf{b}_r^\perp, \mathbf{a}_r \rangle$, so since $\Delta t > 0$, one has

$$\forall (\mathbf{u}_r)_{r \in \mathcal{R}_j}, \quad \sum_{r \in \mathcal{R}_j} \langle \mathbf{D}_{jr}, \mathbf{u}_r \rangle = \frac{1}{2} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r \rangle + \frac{1}{4} \sum_{r \in \mathcal{R}_j} \left\langle (\mathbf{x}_r^n + \Delta t \mathbf{u}_r)^\perp, (\mathbf{u}_{r+1} - \mathbf{u}_{r-1}) \right\rangle.$$

Little algebra allows to show that

$$\sum_{r \in \mathcal{R}_j} \left\langle (\mathbf{x}_r^n + \Delta t \mathbf{u}_r)^\perp, (\mathbf{u}_{r+1} - \mathbf{u}_{r-1}) \right\rangle = 2 \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^{n+1}, \mathbf{u}_r \rangle.$$

Thus finally, one has

$$\forall (\mathbf{u}_r)_{r \in \mathcal{R}_j}, \quad \sum_{r \in \mathcal{R}_j} \langle \mathbf{D}_{jr}, \mathbf{u}_r \rangle = \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{2} (\mathbf{C}_{jr}^n + \mathbf{C}_{jr}^{n+1}), \mathbf{u}_r \right\rangle.$$

The unique solution for this system is $\forall j \in \mathcal{J}, \forall r \in \mathcal{R}_j, \mathbf{D}_{jr} = \frac{1}{2} (\mathbf{C}_{jr}^n + \mathbf{C}_{jr}^{n+1})$.

C.1.2 Dimension 3

In order to equalize the volumes, we want to find if there exists \mathbf{D}_{jr} vectors such that

$$V_j^{n+1} - \bar{V}_j = 0 \quad \iff \quad \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{D}_{jr}, \mathbf{u}_r \rangle = V_j^{n+1} - V_j^n.$$

A good starting point is to consider (18), where $V_j^{n+1} - V_j^n = I_1 + I_2 + I_3$, and first to reformulate the second term I_2 of the right hand side into the form $\Delta t \sum_r \langle \mathbf{a}_r, \mathbf{u}_r \rangle$. Once done, one will identify the \mathbf{D}_{j_r} vectors by considering the whole sum. So, let us denote

$$I_2 := \Delta t^2 \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{u}]_r^{jf} - [\mathbf{u}]_s^{jf} \right) \wedge \mathbf{u}_s, \mathbf{x}_r^n \right\rangle,$$

which rewrites

$$I_2 = \Delta t^2 \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \left(2[\mathbf{u}]_r^{jf} - [\mathbf{u}]_s^{jf} \right) \wedge \mathbf{u}_s, \mathbf{x}_r^n \right\rangle.$$

One now separates the sum as follows

$$I_2 = \Delta t^2 \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \left[\underbrace{2 \sum_{r, s \in \mathcal{R}_{jf}} \left\langle [\mathbf{u}]_r^{jf} \wedge \mathbf{u}_s, \mathbf{x}_r^n \right\rangle}_{I_{2,1} :=} - \underbrace{\sum_{r, s \in \mathcal{R}_{jf}} \left\langle [\mathbf{u}]_s^{jf} \wedge \mathbf{u}_s, \mathbf{x}_r^n \right\rangle}_{I_{2,2} :=} \right]. \quad (19)$$

Let us first consider $I_{2,1}$. Exchanging indices r and s , one has

$$I_{2,1} = \sum_{r, s \in \mathcal{R}_{jf}} \left\langle [\mathbf{u}]_s^{jf} \wedge \mathbf{u}_r, \mathbf{x}_s^n \right\rangle,$$

and since the term in the sum is a determinant

$$I_{2,1} = - \sum_{r, s \in \mathcal{R}_{jf}} \left\langle [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle,$$

which finally rewrites (using the Lemma 1)

$$I_{2,1} = - \sum_{r, s \in \mathcal{R}_{jf}} \left\langle [\mathbf{x}]_s^{jfn} \wedge \mathbf{u}_s, \mathbf{u}_r \right\rangle. \quad (20)$$

The calculations for $I_{2,2}$ proceed almost the same. Let us first use the determinant properties, exchanging second and third columns

$$I_{2,2} = - \sum_{r, s \in \mathcal{R}_{jf}} \left\langle [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_r^n, \mathbf{u}_s \right\rangle,$$

or, exchanging r and s

$$I_{2,2} = - \sum_{r, s \in \mathcal{R}_{jf}} \left\langle [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle. \quad (21)$$

Now injecting (20) and (21) in (19), one gets

$$I_2 = -\Delta t^2 \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle 2[\mathbf{x}]_s^{jfn} \wedge \mathbf{u}_s, \mathbf{u}_r \right\rangle + \Delta t^2 \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle,$$

and reordering the sums as initially, it gives

$$I_2 = -\Delta t^2 \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} 2[\mathbf{x}]_s^{jfn} \wedge \mathbf{u}_s, \mathbf{u}_r \right\rangle + \Delta t^2 \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle,$$

or using again Lemma 1

$$I_2 = -\Delta t \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} 2\Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle + \Delta t \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \Delta t [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle.$$

We now consider the term

$$I_3 = \frac{\Delta t^3}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{u}]_r^{jf} - [\mathbf{u}]_s^{jf} \right) \wedge \mathbf{u}_s, \mathbf{u}_r \right\rangle,$$

or

$$I_3 = \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\Delta t \mathbf{u}]_r^{jf} - [\Delta t \mathbf{u}]_s^{jf} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle.$$

Again using $\forall r \in \mathcal{R}, \mathbf{x}_r^{n+1} = \mathbf{x}_r^n + \Delta t \mathbf{u}_r$ it reads

$$I_3 = \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2 \left([\mathbf{x}]_r^{jfn+1} - [\mathbf{x}]_r^{jfn} \right) - \left([\mathbf{x}]_s^{jfn+1} - [\mathbf{x}]_s^{jfn} \right) \right) \wedge \left(\mathbf{x}_s^{n+1} - \mathbf{x}_s^n \right), \mathbf{u}_r \right\rangle.$$

Let us emphasize that at this point we have already found the form $\Delta t \sum_r \langle \mathbf{a}_r, \mathbf{u}_r \rangle$ that we were looking for and thus, we have an explicit form for a set of \mathbf{D}_{jr} vectors that ensures $V_j^{n+1} = \overline{V}_j$. We shall now give a simplified expression to ensure that these \mathbf{D}_{jr} vectors satisfy (13) and provide a consistent approximation of the divergence operator.

To do so, let us recall that $I_1 = \Delta t \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r \rangle$. So, we need to focus on the simplification of the two other terms. Let us first start with I_3 . A partial development of the differences leads to

$$I_3 = \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn+1} - [\mathbf{x}]_s^{jfn+1} \right) \wedge \left(\mathbf{x}_s^{n+1} - \mathbf{x}_s^n \right), \mathbf{u}_r \right\rangle \\ - \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \left(\mathbf{x}_s^{n+1} - \mathbf{x}_s^n \right), \mathbf{u}_r \right\rangle,$$

where one recognizes the expressions of \mathbf{C}_{jr}^n and \mathbf{C}_{jr}^{n+1} , so

$$I_3 = \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^{n+1}, \mathbf{u}_r \rangle + \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \langle \mathbf{C}_{jr}^n, \mathbf{u}_r \rangle \\ - \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn+1} - [\mathbf{x}]_s^{jfn+1} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\ - \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \mathbf{x}_s^{n+1}, \mathbf{u}_r \right\rangle.$$

Thus putting everything together one gets

$$\begin{aligned}
V_j^{n+1} - V_j^n &= \Delta t \sum_{r \in \mathcal{R}_j} \left\langle \frac{4}{3} \mathbf{C}_{jr}^n + \frac{1}{3} \mathbf{C}_{jr}^{n+1}, \mathbf{u}_r \right\rangle \\
&\quad - \Delta t \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} 2\Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad + \Delta t \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \Delta t [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn+1} - [\mathbf{x}]_s^{jfn+1} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \frac{\Delta t}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \mathbf{x}_s^{n+1}, \mathbf{u}_r \right\rangle.
\end{aligned}$$

In order to retrieve the 2D behavior we now write

$$V_j^{n+1} - V_j^n = \Delta t \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{2} \mathbf{C}_{jr}^n + \frac{1}{2} \mathbf{C}_{jr}^{n+1}, \mathbf{u}_r \right\rangle + \Delta t R_j.$$

The idea is to study R_j .

One has

$$\begin{aligned}
R_j &= \sum_{r \in \mathcal{R}_j} \left\langle \frac{5}{6} \mathbf{C}_{jr}^n, \mathbf{u}_r \right\rangle - \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{6} \overbrace{\sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn+1} - [\mathbf{x}]_s^{jfn+1} \right) \wedge \mathbf{x}_s^{n+1}}^{\mathbf{C}_{jr}^{n+1}}, \mathbf{u}_r \right\rangle \\
&\quad - \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} 2\Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad + \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \Delta t [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn+1} - [\mathbf{x}]_s^{jfn+1} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \mathbf{x}_s^{n+1}, \mathbf{u}_r \right\rangle.
\end{aligned}$$

Using $\forall r \in \mathcal{R}, \mathbf{x}_r^{n+1} = \mathbf{x}_r^n + \Delta t \mathbf{u}_r$, one gets

$$\begin{aligned}
R_j &= \sum_{r \in \mathcal{R}_j} \left\langle \frac{5}{6} \mathbf{C}_{jr}^n, \mathbf{u}_r \right\rangle - \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{6} \overbrace{\sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right)}^{\mathbf{C}_{jr}^n} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{6} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
&\quad - \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{6} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{6} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
&\quad - \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} 2\Delta t[\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad + \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \Delta t[\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \overbrace{\sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right)}^{\mathbf{C}_{jr}^n} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \overbrace{\sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right)}^{\mathbf{C}_{jr}^n} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
&\quad - \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle.
\end{aligned}$$

One observes that all the factors containing \mathbf{C}_{jr}^n cancel so one has

$$\begin{aligned}
R_j = & - \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{6} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
& - \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{6} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \sum_{r \in \mathcal{R}_j} \left\langle \frac{1}{6} \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
& - \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} 2\Delta t[\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& + \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \Delta t[\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \frac{1}{3} \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle.
\end{aligned}$$

One now inverts the sums, which gives

$$\begin{aligned}
R_j = & - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \frac{1}{6} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \frac{1}{6} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle 2\Delta t[\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& + \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \Delta t[\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \frac{1}{3} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \frac{1}{3} \left(2[\mathbf{x}]_r^{jfn} - [\mathbf{x}]_s^{jfn} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r, s \in \mathcal{R}_{jf}} \left\langle \frac{1}{6} \left(2\Delta t[\mathbf{u}]_r^{jf} - \Delta t[\mathbf{u}]_s^{jf} \right) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle.
\end{aligned}$$

A partial development of the sums gives

$$\begin{aligned}
R_j = & - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{3} [\mathbf{x}]_r^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle + \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{6} [\mathbf{x}]_s^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{3} \Delta t [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle + \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{6} \Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle 2 \Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& + \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \Delta t [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{2}{3} \Delta t [\mathbf{u}]_r^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle + \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{3} \Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{2}{3} [\mathbf{x}]_r^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle + \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{3} [\mathbf{x}]_s^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{6} (2 \Delta t [\mathbf{u}]_r^{jf} - \Delta t [\mathbf{u}]_s^{jf}) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle,
\end{aligned}$$

which simplifies into

$$\begin{aligned}
R_j = & - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle [\mathbf{x}]_r^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle + \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{2} [\mathbf{x}]_s^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{3}{2} \Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle \\
& - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{6} (2 \Delta t [\mathbf{u}]_r^{jf} - \Delta t [\mathbf{u}]_s^{jf}) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle.
\end{aligned}$$

Let us first remark that according to Lemma 1

$$\sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{2} [\mathbf{x}]_s^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle = \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{2} \Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle.$$

Also, one has

$$- \sum_{r,s \in \mathcal{R}_{jf}} \left\langle [\mathbf{x}]_r^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle = \sum_{r,s \in \mathcal{R}_{jf}} \left\langle [\mathbf{x}]_r^{jfn} \wedge \Delta t \mathbf{u}_r, \mathbf{u}_s \right\rangle,$$

so using again the Lemma 1 and exchanging indices r and s , one gets

$$- \sum_{r,s \in \mathcal{R}_{jf}} \left\langle [\mathbf{x}]_r^{jfn} \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle = \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \Delta t [\mathbf{u}]_s^{jf} \wedge \mathbf{x}_s^n, \mathbf{u}_r \right\rangle,$$

so that one finds

$$R_j = - \sum_{f \in \mathcal{F}_j} \frac{1}{12N_f} \sum_{r,s \in \mathcal{R}_{jf}} \left\langle \frac{1}{6} (2 \Delta t [\mathbf{u}]_r^{jf} - \Delta t [\mathbf{u}]_s^{jf}) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle.$$

Finally we reorder the sums to allow the identification of the \mathbf{D}_{jr} vectors

$$R_j = - \sum_{r \in \mathcal{R}_j} \left\langle \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \frac{1}{6} (2 \Delta t [\mathbf{u}]_r^{jf} - \Delta t [\mathbf{u}]_s^{jf}) \wedge \Delta t \mathbf{u}_s, \mathbf{u}_r \right\rangle.$$

Thus set

$$\mathbf{C}_{jr}^\delta := - \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \frac{1}{6} \left(2\Delta t [\mathbf{u}]_r^{jf} - \Delta t [\mathbf{u}]_s^{jf} \right) \wedge \Delta t \mathbf{u}_s. \quad (22)$$

It can trivially be rewritten according to the node positions only

$$\mathbf{C}_{jr}^\delta = - \sum_{f \in \mathcal{F}_{jr}} \frac{1}{12N_f} \sum_{s \in \mathcal{R}_{jf}} \frac{1}{6} \left[2 \left([\mathbf{x}]_r^{jf^{n+1}} - [\mathbf{x}]_r^{jf^n} \right) - \left([\mathbf{x}]_s^{jf^{n+1}} - [\mathbf{x}]_s^{jf^n} \right) \right] \wedge (\mathbf{x}_s^{n+1} - \mathbf{x}_s^n).$$

This gives the expression of the \mathbf{D}_{jr} vectors in 3D,

$$\forall j \in \mathcal{J}, \forall r \in \mathcal{R}_j, \quad \mathbf{D}_{jr} = \frac{1}{2} (\mathbf{C}_{jr}^n + \mathbf{C}_{jr}^{n+1}) + \mathbf{C}_{jr}^\delta.$$

C.2 Properties (13) of \mathbf{D}_{jr}

The result is obvious in the 1D case. Also, in the 2D case, since $\forall n > 0$, $\sum_{r \in \mathcal{J}_r} \mathbf{C}_{jr}^n = \mathbf{0}$ and $\sum_{j \in \mathcal{R}_j} \mathbf{C}_{jr}^n = \mathbf{0}$, one obviously has $\sum_{r \in \mathcal{J}_r} \mathbf{D}_{jr} = \mathbf{0}$ and $\sum_{j \in \mathcal{R}_j} \mathbf{D}_{jr} = \mathbf{0}$.

The 3D case requires a few more work.

$$\mathbf{D}_{jr} = \frac{1}{2} (\mathbf{C}_{jr}^n + \mathbf{C}_{jr}^{n+1}) + \mathbf{C}_{jr}^\delta.$$

Again, since $\forall n > 0$, $\sum_{r \in \mathcal{J}_r} \mathbf{C}_{jr}^n = \mathbf{0}$ and $\sum_{j \in \mathcal{R}_j} \mathbf{C}_{jr}^n = \mathbf{0}$, one only has to check that

$$\sum_{r \in \mathcal{J}_r} \mathbf{C}_{jr}^\delta = \mathbf{0} \quad \text{and} \quad \sum_{j \in \mathcal{R}_j} \mathbf{C}_{jr}^\delta = \mathbf{0}.$$

According to (5) and (22) one easily checks that \mathbf{C}_{jr}^n and \mathbf{C}_{jr}^δ share the same algebraic structure, which ends the proof.

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