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Michael Ndjinga, Marcial Nguemfouo

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Michael Ndjinga[1], Marcial Nguemfouo[2]

[1]University of Paris-Saclay, CEA Saclay, ISAS, DM2S, STMF,
91191 Gif-sur-Yvette, France,
michael.ndjinga@cea.fr

[2]University of Yaounde I, Department of Mathematics,
P.O. Box 812 Yaoundé, Cameroon,
marcial.nguemfouo@facsciences-uy1.cm

Abstract

We study the weak formulation of the Poisson problem on closed Lipschitz manifolds. Lipschitz manifolds do not admit tangent spaces everywhere and the definition of the Laplace-Beltrami operator is more technical than on classical differentiable manifolds [**?**, see, e.g.,]]Gesztesy. They however arise naturally after the triangulation of a smooth surface for computer vision or simulation purposes. We derive Stokes' and Green's theorems as well as a Poincaré's inequality on Lipschitz manifolds. The existence and uniqueness of weak solutions of the Poisson problem are given in this new framework for both the continuous and discrete problems. As an example of application, numerical results are given for the Poisson problem on the boundary of the unit cube.

Keywords– Lipschitz manifold, Laplace-Beltrami operator, Finite element method, Elliptic equation, closed surfaces

1 Introduction

Traditionally in the context of the finite element simulation of PDEs on surfaces [**?**, see, e.g.,]]Dziuk1988,Dziuk2013,ElliotStinner,NdjingaNguemfouo, a continuous surface Γ is first triangulated into a piecewise linear surface Γ_h that is no longer differentiable but merely a Lipschitz manifold. The solution u of a linear PDE on Γ can then be approximated by the solution u_h of the same PDE on Γ_h. Using a variational formulation, one then seek for $\tilde{u}_h$ the projection of u_h on the finite dimensional space generated by the nodal functions of Γ_h. The numerical resolution of the resulting linear system yields the nodal values of $\tilde{u}_h$.

In [11, 12], the well-posedness of the Poisson problem $-\triangle_\Gamma u = f$ on a smooth surface Γ immersed in $\mathbb{R}^d$ is studied, and a finite element approximation is proposed. In the context of immersed surfaces, [12] defines differential operators (tangential gradient and divergence) which are based on Fermi coordinates. Fermi coordinates [**?**, see Section 2.3 in]]Dziuk2013 are global coordinates on a hypersurface $\Gamma \in \mathbb{R}^d$ that avoid the use of local charts. Unfortunately, the existence of Fermi coordinates

requires Γ to be C^2, or more precisely to satisfy both an interior and an exterior sphere condition. The issue of polyhedral surfaces is raised in [11] (page 3), which quotes [3] for the definition of Sobolev spaces on $C^{0,1}$ manifolds. However, [3] deal with $C^{2,\kappa}$ manifolds (see Definition 2.10 and Section 4.2). The assumption of C^2 regularity made on the surface Γ in [11, 12] restricts the scope of surfaces Γ for which the methodology yields the well-posedness of the continuous problem. Furthermore, even if Γ is assumed C^2, after triangulation of Γ, the new surface Γ_h is piecewise linear, and no longer C^2. The study and characterisation of u_h the solution of the Poisson problem on Γ_h can not therefore be performed using the methodology in [11, 12] due to the lack of smoothness of Γ_h.

In order to perform calculus on more general surfaces that are not C^2, especially on triangulated surfaces, we can not rely on the approach of [11, 12] using Fermi coordinates on C^2 immersed surfaces. We chose instead to extend the definition of the classical differential operators to Lipschitz manifolds following [9] (Section 3), [8] (Appendix A) and [10].

Lipschitz manifolds are an intermediate structure between topological manifolds where the tangent space is defined nowhere and C^1 manifolds where the tangent space is defined everywhere. On Lipschitz manifolds the tangent space is defined almost everywhere, which is enough to define spaces of functions that are weakly differentiable. [8] defines Lebesgue L^p spaces, as well as Sobolev $W^{1,p}$ and $W^{-1,p}$ spaces. The main technical difficulty is that Lipschitz manifolds are not differentiable everywhere. For that reason, they make an intensive use of differentiability almost everywhere through the Rademacher's theorem. We use this new setting for the definition and study of the Poisson problem on a Lipschitz manifold, and propose a finite element method for its numerical simulation.

The article is organised as follows. In Section 2, following [8], we recall the definition of Lipschitz manifolds, and the associated differential and exterior calculi. We go on introducing the Sobolev spaces as well as the Laplace-Beltrami operator on a Lipschitz manifold. We extend [8] by proving Stokes' 1 and Green's 2 theorems as well as a Poincaré's inequality on Lipschitz manifolds. In Section 3, in the context of closed Lipschitz manifold we study the Poisson problem and give a theorem of existence and uniqueness of solutions. This theorem relies on the weak formulation of the problem that is available thanks to the Green's theorem. Section 4 is devoted to the introduction of the linear finite element discretization and Section 5 is concerned with the numerical simulation of the Poisson problem on a Lipschitz manifold : the boundary of the unit cube.

2 Calculus on Lipschitz manifolds

In standard calculus, the gradient, divergence and Laplace operators are classically defined on the Euclidean space $\mathbb{R}^d$. They can however also be defined on a C^1 manifold $\mathcal{M}$ in an intrinsic way (no immersion into $\mathbb{R}^d$), using a Riemannian metric [**?**, see for example Section 1.2 in]]Hebey2000 or the Hodge operator [**?**, see Section 11.2 in]]Aubin1998.

Following [8], we recall the definition of Lipschitz manifolds in section 2.1. Lipschitz differentiable forms are defined in section 2.3. Sobolev spaces on Lipschitz manifolds are introduced in section 2.4, and the Laplace-Beltrami operator in section 2.5.

2.1 Lipschitz manifolds

We recall that a **topological manifold** $\mathcal{M}$ of dimension $d \in \mathbb{N}^*$ is a topological space such that for every $x \in \mathcal{M}$ there exists an open set $U \subset \mathcal{M}$ and a map $\phi : U \to \mathbb{R}^d$ such that $x \in U$ and ϕ is an homeomorphism onto its image $\phi(U)$. The couple (U, ϕ), where $U \in \mathcal{M}$ is an open set around x and $\phi : U \to \phi(U) \subset \mathbb{R}^d$ is an homeomorphism, is called a **local coordinate chart**. An atlas on $\mathcal{M}$ is a countable familly $\{U_i, \phi_i\}_{i \in I}$ such that $\mathcal{M} = \cup_{i \in I} U_i$ and (U_i, ϕ_i) is a local coordinate chart for each $i \in I$.

We recall that a map $\phi : E \to F$ between two metric spaces E and F is a **bilipschitz map** if it is a Lipschitz homeomorphism onto its image $\phi(E)$, and its inverse is also Lipschitz.

Definition 1 (Lipschitz manifold)**.**
A **Lipschitz manifold** *is a topological manifold with an atlas (called a* **Lipschitz atlas***)* $\{U_i, \phi_i\}_{i \in I}$ *such that for any* $i, j \in I$ *the transition map* $\phi_i \circ \phi_j^{-1}$ *is bilipschitz from* $\phi_j(U_i \cap U_j)$ *to* $\phi_i(U_i \cap U_j)$*.*

Since differentiability will occur only almost everywhere, functional analysis on Lipschitz manifolds requires the definition of negligible sets.

Definition 2 (Negligible sets)**.**
Let Γ *be a Lipschitz manifold and* $\{U_i, \phi_i\}_{i \in I}$ *its Lipschitz atlas.*
A set $S \subset \Gamma$ *is negligible provided* $\phi_i(U_i \cap S)$ *has measure zero in* $\mathbb{R}^d$*.*

A property that is true on all but a negligible set of points of Γ is said to hold almost everywhere.

Given a Lipschitz manifold Γ with a Lipschitz atlas $\{U_i, \phi_i\}_{i \in I}$, a point $x \in \Gamma$ is said to be a **singular point** if there exists $i, j \in I$ such that $x \in U_i \cap U_j$ and $\phi_i \circ \phi_j^{-1}$ is not differentiable at $\phi_j(x)$. A point x that is not singular is called a **regular point**.

A consequence of the Rademacher's theorem [**?**, see Corollary 11.7 in]]Taylor2006 is that **the set of singular points is negligible**. Hence for any Lipschitz manifold, one can properly define a tangent space only almost everywhere (see Definition 5 below). Note that if the Lipschitz manifold is embedded in $\mathbb{R}^d$, it admits almost everywhere a normal vector.

Unlike smooth manifolds where differentiability can take place everywhere and be of any order, differentiability on Lipschitz manifolds can happen only at regular points and is necessarily of order one.

Definition 3 (Differentiable map)**.**
Let Γ *be a Lipschitz manifold. A map* $f : \Gamma \to \mathbb{R}$ *is said to be differentiable at* $x \in \Gamma$ *provided*

- x *is a regular point of* Γ
- *there exists a local chart* (U, ϕ) *such that* $x \in U$ *and* $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ *is differentiable at* $\phi(x)$

Definition 4 (Measurable and Lipschitz maps)**.**
Let Γ *be a Lipschitz manifold,* $m \in \mathbb{N}^*$*. A map* $f : \Gamma \to \mathbb{R}^m$ *is measurable (resp. Lipschitz) if for any local coordinate chart* (U, ϕ)*, the map* $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ *is measurable (resp. Lipschitz).*

Lipschitz maps are differentiable almost everywhere thanks to the Rademacher's theorem.

2.2 Tangent space

Let $x \in \Gamma$ be a regular point. A **path through** x is a continuous map $\gamma_x :]-\epsilon, \epsilon[\to \Gamma$, $\epsilon > 0$ with $\gamma_x(0) = x$ and such that there exists $i \in I$,

$x \in U_i$ and $\phi_i \circ \gamma_x$ is differentiable at 0.
We define the following equivalence relation between paths through x : γ_{1x} and γ_{2x} are equivalent if they share the same derivative at x :

$$\gamma_{1x} \equiv_x \gamma_{2x} \Leftrightarrow (\phi_i \circ \gamma_{1x})'(0) = (\phi_i \circ \gamma_{2x})'(0). \tag{1}$$

Following [8] appendix A and [9] Section 3.1, we define the tangent space almost everywhere as the space of equivalence classes $\dot{\gamma_x}$ for the relation (1).

Definition 5 (Tangent space)**.**
Let Γ be a Lipschitz oriented manifold. The tangent space of Γ at $x \in \Gamma$ is the vector space

$$\begin{array}{lcll} T_x\Gamma & = & \{\dot{\gamma_x}, \gamma_x \text{ a path through } x\} & \text{if } x \text{ is a regular point} \\ T_x\Gamma & = & \{0\} & \text{if } x \text{ is a singular point} \end{array} \tag{2}$$

If Γ is a Lipschitz oriented manifold of dimension $d \in \mathbb{N}^*$, its tangent space at every regular point is a vector space of dimension d. At every regular point $x \in \Gamma$ belonging to a local chart $(U, \phi = (\phi^1, \ldots, \phi^d))$, we can define a basis of the tangent space $T_x\Gamma$ by considering the d curves $\phi_x^j(t) = \phi_i^{-1}(t_1^x, \ldots, t_{j-1}^x, t_j^x + t, t_{j+1}^x, \ldots, t_d^x)$ where $(t_1^x, \ldots, t_{j-1}^x, t_j^x + t, t_{j+1}^x, \ldots, t_d^x) = \phi_i^{-1}(x)$. The curves $t \to \phi^j(t)$ form d independent classes for the relation (1). Hence $\dot{\phi_x^j}$ form a basis of $T_x\Gamma$.

Definition 6 (Strong gradient map)**.**
Let Γ be a Lipschitz manifold, $x \in \Gamma$ a regular point and $f : \Gamma \to \mathbb{R}$ be differentiable at x.
The linear tangent map of f at x denoted $Grad_x f$ is the map sending the equivalence class $\dot{\gamma}$ of the relation $\equiv_x$ to the equivalence class $f \dot{\circ} \gamma$ of the relation $\equiv_{f(x)}$:

$$Grad_x f : T_x(\Gamma) \to \mathbb{R} \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma_x} \to \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t)|_{t=0}. \tag{4}$$

On a Lipschitz manifold of dimension d, if $x \in \Gamma$ is a regular point, $T_x\Gamma$ is a real vector space of dimension d and basis $\dot{\phi_x^1}, \ldots \dot{\phi_x^d}$. Following a classical duality argument, the strong gradient of f at x, which is a linear form, can therefore be represented by a d-dimensional real vector denoted $\overrightarrow{\nabla}_x f$ such that

$$Grad_x f \left(\sum_{i=1}^{d} a_i \dot{\phi_x^i} \right) = (a_1, \ldots, a_d) \cdot \overrightarrow{\nabla}_x f.$$

2.3 Measurable and Lipschitz differential forms

On any vector space E of dimension d, we introduce for any integer $l \in \{0, 1, \ldots, d\}$ the l-th exterior power $\Lambda^l E$ as the vector space of l-linear alternate forms on E.
Because of the lack of differentiability of Lipschitz manifolds, we cannot work with smooth differential forms as done classically. Instead, following [9] (Section 3.2), we introduce measurable and Lipschitz differential forms as follows.

Definition 7 (Measurable and Lipschitz differential forms).
Let $p \in [1, \infty]$, $l \in \{0, 1, \dots, d\}$, Γ be a closed Lipschitz manifold of dimension d, and $\{U_i, \phi_i\}_{i \in I}$ its Lipschitz atlas.

A measurable (resp. Lipschitz) differential form ω of degree l is a mapping defined almost everywhere on Γ such that

- *$\omega \in \Lambda^l T_x \Gamma$ for almost every $x \in \Gamma$*
- *for any local chart $(U, \phi = (\phi^1, \dots, \phi^d))$ and any multi-index J of length l, there exists a measurable (resp. Lipschitz) function a_J such that almost everywhere*

$$\omega = \sum_{|J|=l} a_J \mathrm{d}\phi^J. \tag{5}$$

Due to the lack of regularity, we cannot define the exterior derivative as an endomorphism on differential forms as done classically [**?**, see, e.g., Theorem 5.42 in]]Lafontaine2015. Instead, the derivative of a Lipschitz differentiable form is a measurable differentiable form defined as follows.

Definition 8 (Exterior derivative of a Lipschitz differentiable form).
Let Γ be a Lipschitz manifold of dimension d, and $\{U_i, \phi_i\}_{i \in I}$ its Lipschitz atlas. Let $l \in \{0, 1, \dots, d\}$, and ω a Lipschitz differential form of degree l on Γ. The exterior derivative of ω is the measurable differential form $\mathrm{d}\omega$ of degree $l+1$ such that on any local chart $(U, \phi = (\phi^1, \dots, \phi^d))$

$$\mathrm{d}\omega = \sum_{|J|=l} \sum_{j \in J} \frac{\partial a_J}{\partial \phi^j} \mathrm{d}\phi^j \wedge \mathrm{d}\phi^J, \tag{6}$$

where a_J are the Lipschitz coefficients of ω : $\omega = \sum_{|J|=l} a_J \mathrm{d}\phi^J$.

In definition 8, the differentiability of a Lipschitz form follows from the Rademacher's theorem.

The orientability of a manifold is a necessary condition for the existence of a volume form [**?**, see, e.g., Theorem 6.5 in]]Lafontaine2015, which in turns yields a measure and integration theory on the manifold. The classical definition of orientability involves the positiveness everywhere of the Jacobian of the transition maps $\phi_i \circ \phi_j^{-1}$. However, in the case of Lipschitz manifolds, the transition maps are differentiable only almost everywhere.

Definition 9 (Orientation).
A Lipschitz manifold Γ with Lipschitz atlas $\{U_i, \phi_i\}_{i \in I}$ is oriented provided the change of coordinates $\phi_i \circ \phi_j^{-1}$ has positive jacobian on $U_i \cap U_j$ almost everywhere.

An orientation is therefore given by an atlas such that the coordinate charts $(U_i, \phi_i = (\phi_{i1}, \dots, \phi_{id}))$ define a positively oriented basis $\left(\frac{\mathrm{d}}{\mathrm{d}\phi_{i1}}, \dots, \frac{\mathrm{d}}{\mathrm{d}\phi_{id}}\right)$ of the tangent space almost everywhere. On any oriented Lipschitz manifold Γ of dimension d, there exists a Lipschitz differentiable form $\mathrm{d}V_\Gamma(x) \in \Lambda^d T_x \Gamma$ such that $\mathrm{d}V_\Gamma(x) \neq 0$ for almost every $x \in \Gamma$. The proof is a direct adaptation of Theorem 6.5 in [1]. Such a form is called a **volume form** [**?**, see, e.g., Definition 6.3 in]]Lafontaine2015 and takes the form $\mathrm{d}V_\Gamma = f \mathrm{d}\phi^1 \wedge \dots \wedge \mathrm{d}\phi^d$, where f is measurable.

Integrating volume forms requires the pullback operator. Following the definition 5.18 in [1] we define the pullback of a volume form by a Lipschitz map.

Definition 10 (Pullback of a volume form)**.**
Let Γ be a Lipschitz manifold of dimension $d \in \mathbb{N}^*$, U an open subset of $\mathbb{R}^d$ and let $g : U \to \Gamma$ be a Lipschitz map.
The pullback of a measurable differentiable form ω on Γ by g is the measurable differentiable form $g^*\omega$ on $\mathbb{R}^d$ defined by

$$(g^*\omega)_x(v_1, \dots, v_q) = \omega_{g(x)}(\nabla_{g(x)}g \cdot v_1, \dots, \nabla_{g(x)}g \cdot v_q) \quad a.e. x \in U, \forall v_1, \dots, v_q \in \mathbb{R}^d. \tag{7}$$

Note that the pullback of a Lipschitz differential form by a Lipschitz map is merely a measurable differential form (not Lipschitz). For instance, the pullback of a volume form $\mathrm{d}V_\Gamma = f\mathrm{d}\phi^1 \wedge \cdots \wedge \mathrm{d}\phi^d$ by a Lipschitz map g is the form $(g^*\omega)_x = f \circ g(x) \det(\nabla_{g(x)}g)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d$ [**?**, see, e.g., Proposition 6.12 in]]Lafontaine2015, which is not necessarily Lipschitz.

On $\mathbb{R}^d$, volume forms take the form $\mathrm{d}V_{\mathbb{R}^d}(x) = f(x)\mathrm{d}x^1 \cdots \mathrm{d}x^d$ and their integral is simply $\int_{\mathbb{R}^d} \mathrm{d}V_{\mathbb{R}^d} = \int_{\mathbb{R}^d} f(x)\mathrm{d}x^1 \dots \mathrm{d}x^d$. On a general Lipschitz manifold, the **integral of a volume form** [**?**, see, e.g., Definition 6.16 in]]Lafontaine2015, is defined by

$$\int_\Gamma f\mathrm{d}\lambda_\Gamma = \sum_{i \in I} \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\theta_i f \mathrm{d}V_\Gamma), \tag{8}$$

where $(\theta_i)_{i\in I}$ is a Lipschitz partition of unity on Γ, subordinate to the cover $(U_i)_{i\in I}$: $supp(\theta_i) \subset U_i$, $0 \le \theta_i \le 1$ and $\sum_{i\in I} \theta_i = 1$ [**?**, see Proposition 6.14 in]]Lafontaine2015.
The value of the integral (8) does not depend on the choice of coordinate charts nor the partition of unity [**?**, see the proof of Theorem 6.15 in]]Lafontaine2015.

We are now ready to state Stokes' theorem for Lipschitz forms, which is required for the integration by part (see the proof of Green's theorem 2).

Theorem 1 (Stokes' theorem)**.**
Let Γ be a compact oriented Lipschitz manifold of dimension d. Let ω be a Lipschitz form of degree $d-1$ on Γ. Then

$$\int_\Gamma \mathrm{d}\omega = 0. \tag{9}$$

Proof. Since Γ is compact, there is a finite set I' and a finite number of charts $(U_i, \phi_i)_{i\in I'}$ that covers Γ : $\Gamma = \cup_{i\in I'} U_i$. Let $(\alpha_i)_{i\in I'}$ be a smooth partition of unity subordinate to that cover. The existence of $(\alpha_i)_{i\in I'}$ is given by Proposition 6.14 in [1]. Since $\omega = \sum_{i\in I'} \alpha_i\omega$, it is sufficient to prove the theorem for $\alpha_i\omega$ that is, for a Lipschitz differentiable form supported within an open subset U_i.

We assume therefore in the following that $supp(\omega) \subset U_i$ where $(U_i, \phi_i = (\phi_i^1, \dots, \phi_i^d))$ is a local chart. Since ω is a Lipschitz differentiable form of degree $d-1$, for any $j \in \{1, 2, \dots, d\}$, there exists a Lipschitz function a_j supported in U_i such that almost everywhere

$$\omega(x) = \sum_{j=1}^{d} a_j(x)\mathrm{d}\phi^1 \wedge \mathrm{d}\phi^2 \wedge \cdots \wedge \widehat{\mathrm{d}\phi^j} \wedge \cdots \wedge \mathrm{d}\phi^d, \quad x \in \Gamma. \tag{10}$$

Since ω is Lipschitz, it admits an exterior derivative which is

$$\begin{aligned} \mathrm{d}\omega(x) &= \sum_{j=1}^{d} \frac{\partial a_j}{\partial \phi^j}(x)\, \mathrm{d}\phi^j \wedge \mathrm{d}\phi^1 \wedge \mathrm{d}\phi^2 \wedge \cdots \wedge \widehat{\mathrm{d}\phi^j} \wedge \cdots \wedge \mathrm{d}\phi^d \quad (11) \\ &= \left(\sum_{j=1}^{d} (-1)^{j-1} \frac{\partial a_j}{\partial \phi^j}(x) \right) \mathrm{d}\phi^1 \wedge \mathrm{d}\phi^2 \wedge \cdots \wedge \mathrm{d}\phi^d, \quad (12) \end{aligned}$$

From the definition of the tangent map (Definition 6) we have

$$\frac{\partial a_j}{\partial \phi^j}(\phi_i^{-1}(x)) = \frac{\partial a_j \circ \phi_i^{-1}}{\partial x^j}(x). \quad (13)$$

The pullback of the exterior derivative is

$$\begin{aligned} (\phi_i^{-1})^*(\mathrm{d}\omega) &= \left(\sum_{j=1}^{d} (-1)^{j-1} \frac{\partial a_j}{\partial \phi^j} \right) (\phi_i^{-1}(x))\, (\phi_i^{-1})^*(\mathrm{d}\phi^1) \wedge (\phi_i^{-1})^*(\mathrm{d}\phi^2) \wedge \cdots \wedge (\phi_i^{-1})^*(\mathrm{d}\phi^d) \quad (14) \\ &= \left(\sum_{j=1}^{d} (-1)^{j-1} \frac{\partial a_j}{\partial \phi^j} \right) (\phi_i^{-1}(x))\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^d, \quad (15) \end{aligned}$$

since $(\phi_i^{-1})^*(\mathrm{d}\phi_i^j) = \mathrm{d}\phi^j \circ \phi_i^{-1} = \mathrm{d}x^j$.
From the definition of form integrals (8) :

$$\begin{aligned} \int_{U_i} \mathrm{d}\omega &= \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\mathrm{d}\omega) \quad (16) \\ &= \int_{\phi_i(U_i)} \left(\sum_{j=1}^{d} (-1)^{j-1} \frac{\partial a_j}{\partial \phi^j} \right) (\phi_i^{-1}(x))\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^d \quad (17) \end{aligned}$$

Using the tangent map property (13) we obtain

$$\int_{U_i} \mathrm{d}\omega = \int_{\phi_i(U_i)} \left(\sum_{j=1}^{d} (-1)^{j-1} \frac{\partial a_j \circ \phi_i^{-1}}{\partial x^j} \right) (x)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^d \quad (18)$$

Since a_j is compactly supported within the open set U_i, a_j is zero in a neighborhood of ∂U_i and therefore on ∂U_i we have $a_j = 0$ and $\nabla a_j = 0$. Similarly on $\partial\phi_i(U_i)$, $a_j \circ \phi_i^{-1} = 0$ and $\nabla a_j \circ \phi_i^{-1} = 0$, where ∇a_j denotes the tangent map of a_j. ∇a_j goes from $T_x\Gamma$ to $\mathbb{R}$ and can therefore be identified with a vector denoted $\left(\frac{\partial a_j}{\partial \phi^1}, \ldots, \frac{\partial a_j}{\partial \phi^d} \right)$.
We can therefore extend $a_j \circ \phi_i^{-1}$ to $\mathbb{R}^d$ as a Lipschitz function f_j with compact support :

$$f_j(x) = \begin{cases} a_j \circ \phi_i^{-1}(x) & \text{if} \quad x \in U_i \\ 0 & \text{if} \quad x \notin U_i \end{cases}. \quad (19)$$

Since a_j is Lipschitz and ϕ_i^{-1} is Lipschitz by definition of a Lipschitz chart, f_j is Lipschitz as composition of Lipschitz functions and then is differentiable by Rademacher's Theorem.

Now (18) becomes

$$\int_{U_i} \mathrm{d}\omega = \int_{\mathbb{R}^d} \left(\sum_{j=1}^{d} (-1)^{j-1} \frac{\partial f_j}{\partial x^j} \right) (x)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^d \tag{20}$$

$$= \sum_{j=1}^{d} (-1)^{j-1} \int_{\mathbb{R}^d} \frac{\partial f_j}{\partial x^j}(x)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^d \tag{21}$$

$$= \sum_{j=1}^{d} (-1)^{j-1} \int_{\mathbb{R}^{d-1}} \left(\int_{-\infty}^{+\infty} \frac{\partial f_j}{\partial x^j}(x)\, \mathrm{d}x^j \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^d \tag{22}$$

where (22) is a consequence of Fubini's theorem.
The result follows from the fact that f_j has compact support.

□

2.4 Sobolev spaces on Lipschitz manifolds

The Lebesgue spaces are defined by a pullback of the Euclidean Lebesgue space.

Definition 11 (Lebesgue space $L^p(\Gamma)$)**.**
Let $p \in [1, \infty]$, Γ be a compact oriented Lipschitz manifold and $\{U_i, \phi_i\}_{i \in I}$ its Lipschitz atlas.
The space $L^p(\Gamma)$ is defined as the set of real valued functions f defined almost everywhere on Γ such that for any local chart $\{U_i, \phi_i\}$, $f \circ \phi_i^{-1} \in L^p(\phi_i(U_i))$.

The associated norm is

$$\|f\|_p = \left(\int_\Gamma |f|^p \, \mathrm{d}\lambda_\Gamma \right)^{\frac{1}{p}}. \tag{23}$$

The integral above is defined through a volume form (see formula 8).

The strong gradient $\overrightarrow{\nabla}_x \phi$ was defined in (6) for a differentiable function f at a regular point x. The usual definition of Sobolev Spaces [?, see]]Adams2003 is based on smooth test functions. Since differentiability is limited to order one on Lipschitz manifolds we choose to replace smooth test functions by Lipschitz test functions. A function $f \in L^p(\Gamma)$ is said to be **weakly differentiable** if there exists $\vec{g} \in L^p(\Gamma)^d$ such that for any Lipschitz function $\phi : \Gamma \to \mathbb{R}$ we have

$$\int_\Gamma f \overrightarrow{\nabla}_x \phi \, \mathrm{d}\lambda_\Gamma = - \int_\Gamma \phi \vec{g} \, \mathrm{d}\lambda_\Gamma. \tag{24}$$

$\vec{g}$ is called the **weak gradient** of f and denoted $\overrightarrow{\nabla}_\Gamma f$.

Because Lipschitz manifolds admit only one order of differentiability, we can study only the space of order one weakly differentiable functions. The Sobolev space $W^{1,p}(\Gamma)$ is defined in a similar way to the Euclidean case.

Definition 12 (Sobolev space $W^{1,p}(\Gamma)$)**.**
Let $p \in [1, \infty]$, Γ be a compact oriented Lipschitz manifold. The space $W^{1,p}(\Gamma)$ is defined as the set of weakly differentiable functions equipped with the norm

$$\|f\|^p_{W^{1,p}(\Gamma)} = \|f\|^p_p + \|\overrightarrow{\nabla}_\Gamma f\|^p_p. \tag{25}$$

We introduce the following classical notations in the Hilbertian case : $H^1(\Gamma) = W^{1,2}(\Gamma)$.

From the Definition 12 of Sobolev spaces, it is straightforward that

$$\overrightarrow{\nabla}_\Gamma : W^{1,p}(\Gamma) \to L^p(\Gamma)^d. \tag{26}$$

The definition of the weak divergence by a duality approach as well as the study of weak solution of the Poisson problem both require Sobolev spaces with negative index. Following [6] sections 3.7 to 3.13, we define $W^{-1,p}(\Gamma)$ using duality.

Definition 13 (Dual Sobolev space $W^{-1,p}(\Gamma)$)**.**
Let $p \in [1, \infty[$, Γ be a compact oriented Lipschitz manifold.
The space $W^{-1,p}(\Gamma)$ is defined as the dual of the space $W^{1,p}(\Gamma)$: $W^{-1,p}(\Gamma) = (W^{1,p'}(\Gamma))'$, where p' is the conjugate of p : $\frac{1}{p'} + \frac{1}{p} = 1$.

It is equipped with the norm

$$\|L\|_{W^{-1,p}(\Gamma)} = \sup_{f \in W^{1,p}(\Gamma), f \neq 0} \frac{|L(f)|}{\|f\|_{W^{1,p}(\Gamma)}}. \tag{27}$$

$W^{-1,p}(\Gamma)$ is the extension to $W^{1,p}(\Gamma)$ of distributions that act normally on infinitely smooth test functions [**?**, see Section 3.10 in]Adams2003. The **weak divergence** is defined for $p \in]1, \infty]$ as the adjoint of the weak gradient

$$\nabla_\Gamma \cdot : L^p(\Gamma)^d \to W^{-1,p}(\Gamma). \tag{28}$$

Indeed, if $p' < \infty$ is the conjugate of p, $\overrightarrow{\nabla}_\Gamma : W^{1,p'}(\Gamma) \to L^{p'}(\Gamma)^d$ yields an adjoint operator $\nabla_\Gamma \cdot$ from $(L^{p'}(\Gamma)^d)'$ (which is isometrically equivalent to $L^p(\Gamma)^d$) into $(W^{1,p'}(\Gamma))' = W^{-1,p}(\Gamma)$.

The surface divergence of a general function $\vec{f} \in L^p(\Gamma)^d$ is therefore a linear operator acting on $W^{1,p}(\Gamma)$. However, for $\vec{f} \in W^{1,p}(\Gamma)^d$, $\nabla \cdot \vec{f}$ can be identified by duality with a function in $L^p(\Gamma)$.

The following Green's theorem connects the divergence and the gradient on Lipschitz manifolds. From a functional analytic point of view, it is an extension of the definition of weak differentiability (24) to test functions in $W^{1,p}(\Gamma)$.

Theorem 2 (Green's theorem)**.** *Let Γ be a compact oriented Lipschitz manifold of dimension d, $u \in W^{1,p}(\Gamma)$ and $\vec{v} \in W^{1,p'}(\Gamma)^d$. Then*

$$\int_\Gamma \vec{v} \cdot \overrightarrow{\nabla}_\Gamma u \, \mathrm{d}\lambda_\Gamma = -\int_\Gamma u \nabla_\Gamma \cdot \vec{v} \, \mathrm{d}\lambda_\Gamma. \tag{29}$$

Proof. Define the following Lipschitz differential forms of order $(d-1)$:

$$\omega_i = u v_i \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^d, \quad i = 1, \ldots, d,$$

where $v_i, i = 1, \ldots, d$ are the components of v.
Since

$$\mathrm{d}\omega_i = \frac{\partial}{\partial x^i}(u v_i) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^d = \frac{\partial}{\partial x^i}(u v_i) \, \mathrm{d}\lambda_\Gamma,$$

we have

$$\nabla_\Gamma \cdot (u\vec{v}) \, \mathrm{d}\lambda_\Gamma = \sum_{i=1}^d \frac{\partial}{\partial x^i}(u v_i) \, \mathrm{d}\lambda_\Gamma = \sum_{i=1}^d \mathrm{d}\omega_i.$$

Given that in a similar way to the strong gradient, the weak gradient satisfies the product rule

$$\nabla_\Gamma \cdot (u\vec{v}) = u\nabla_\Gamma \cdot \vec{v} + \vec{v} \cdot \overrightarrow{\nabla}_\Gamma u,$$

the result follows from Stokes theorem 1 :

$$\int_\Gamma \vec{v} \cdot \overrightarrow{\nabla}_\Gamma u \, \mathrm{d}\lambda_\Gamma + \int_\Gamma u\nabla_\Gamma \cdot \vec{v} \, \mathrm{d}\lambda_\Gamma = \int_\Gamma \nabla_\Gamma \cdot (u\vec{v}) \, \mathrm{d}\lambda_\Gamma = \sum_{i=1}^{d} \int_\Gamma \mathrm{d}\omega_i = 0.$$

□

A consequence of the Green's theorem 2 is that the operator $-\Delta_\Gamma$ is symmetric and positive. Green's theorem 2 will be used to perform the integration by part needed to obtain the weak formulation (38) of the Laplace-Beltrami operator (Theorem 3).

2.5 The Laplace-Beltrami operator on Lipschitz manifolds

The Laplace-Beltrami operator is classically defined as a second order differential operator on C^2 manifolds [**?**, see for instance]]Aubin1998. The lack of smoothness on a Lipschitz manifold prevents us from making sense of the Laplace–Beltrami operator as a differential operator of order two. One way of defining the Laplace-Beltrami operator on a Lipschitz manifold Γ is

$$\triangle_\Gamma := \nabla_\Gamma \cdot \overrightarrow{\nabla}_\Gamma : W^{1,p}(\Gamma) \to W^{-1,p}(\Gamma). \tag{30}$$

The following theorem, taken from [8] (Theorem 1.3) gives some important properties of the Laplace-Beltrami operator on Lipschitz manifolds on the Sobolev space $H^1(\Gamma) = W^{1,2}(\Gamma)$.

Theorem 3 (Laplace-Beltrami operator on Lipschitz manifolds)**.**
Let Γ be a compact, connected, oriented Lipschitz manifold. The operator $\triangle_\Gamma := \nabla_\Gamma \cdot \overrightarrow{\nabla}_\Gamma$ is well defined, self adjoint and bounded from $W^{1,p}(\Gamma)$ to $W^{-1,p}(\Gamma)$.
Moreover the operator $-\triangle_\Gamma$ has a purely discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \ldots \tag{31}$$

with $\lambda_j \to \infty$ as $j \to \infty$.
Furthermore, there is an Hilbertian basis $(\psi_j)_{j\geq 0}$ of $L^2(\Gamma)$ composed of eigenvectors $\psi_j \in H^1(\Gamma)$ of $-\triangle_\Gamma$.

A corollary of theorem 3 is the existence of the Poincaré constant on $H^1(\Gamma)$ which is the first non zero eigenvalue λ_1 of $-\triangle_\Gamma$.

Corollary 1 (Poincaré's inequality)**.**
Let Γ be a compact, connected, oriented Lipschitz manifold. There exists a constant $C > 0$ such that

$$\forall u \in H^1(\Gamma), \quad \|u - |\Gamma|^{-1} \int_\Gamma u\|_{L^2(\Gamma)} \leq C\|\nabla_\Gamma u\|_{L^2(\Gamma)}. \tag{32}$$

A classical consequence of Poincaré's inequality is that provided $\int_\Gamma u = 0$, the L^2 norm of $\nabla_\Gamma u$ is equivalent to the H^1 norm of u :

$$\int_\Gamma u = 0 \Rightarrow \|\nabla_\Gamma u\|_{L^2(\Gamma)} \leq \|u\|_{H^1(\Gamma)} \leq (1 + C)\|\nabla_\Gamma u\|_{L^2(\Gamma)}. \tag{33}$$

3 The Poisson problem on a closed Lipschitz surface

Now that we have defined the Laplace-Beltrami operator in section 2.5, we can study the Poisson problem on closed Lipschitz manifolds. We start by setting the problem and the relevant functional spaces in section 3.1. We then give the weak formulation of the problem in section 3.2. We end up by giving an existence result in section 3.3.

3.1 Definition and functional spaces

Let Γ be a closed Lipschitz manifold. Since Γ is closed, it has no boundary ($\partial\Gamma = \emptyset$), and all the constant functions u are in the kernel of Δ_Γ. Δ_Γ is therefore not invertible on the space of functions $u \in H^1(\Gamma)$.
We choose to impose the global condition $\int_\Gamma u = 0$ to guarantee the uniqueness of solution. Hence we define

$$L^2_0(\Gamma) = \left\{ f \in L^2(\Gamma), \int_\Gamma f = 0 \right\}, \qquad H^1_0(\Gamma) = \left\{ u \in H^1(\Gamma), \int_\Gamma u = 0 \right\} \tag{34}$$

Consider the following Poisson problem for a given $f \in L^2_0(\Gamma)$:

$$\text{Find } u \in H^1_0(\Gamma) \text{ such that} \qquad -\Delta_\Gamma u = f. \tag{35}$$

In the next section we give the weak form of the Poisson problem (35).

3.2 Weak form of the Poisson problem

The classical Laplace-Beltrami operator acts on C^2 functions. A classical solution of (35) is therefore a function $u \in C^2(\Gamma)$. Unfortunately such a strong solution doesn't always exists even if f is assumed continuous (see Section 3.1.2 in [7] in the Euclidean case).

We now define a weak form of the Laplace Beltrami operator which acts on $H^1_0(\Gamma)$ instead of $C^2(\Gamma)$. The **weak Laplace-Beltrami operator on** Γ : $H^1_0(\Gamma) \to (H^1_0(\Gamma))'$ is the operator sending $u \in H^1_0(\Gamma)$ to the linear functional

$$\begin{aligned} H^1_0(\Gamma) &\to \mathbb{R} && (36)\\ v &\to \int_\Gamma \overrightarrow{\nabla}_\Gamma u \cdot \overrightarrow{\nabla}_\Gamma v \, \mathrm{d}\lambda_\Gamma, && (37) \end{aligned}$$

where $\overrightarrow{\nabla}_\Gamma u$ is the weak gradient of u on Γ given by (24).

The variational formulation for (35) is the following.

$$\text{Find } u \in H^1_0(\Gamma) \text{ such that} \quad \forall v \in H^1_0(\Gamma), \int_\Gamma \overrightarrow{\nabla}_\Gamma u \cdot \overrightarrow{\nabla}_\Gamma v \, \mathrm{d}\lambda_\Gamma = \int_\Gamma f v \, \mathrm{d}\lambda_\Gamma. \tag{38}$$

We obtain (38) from (35) using the Green's theorem 2.

A solution $u \in H^1_0(\Gamma)$ of (38) is called a **weak solution** of the Poisson problem (35). Indeed, it is only one time weakly differentiable whereas a strong (classical) solution is twice strongly differentiable.

In the next section we are going to prove following [12], that the Poisson problem with a zero mean right hand side admits, a unique solution with zero mean on a closed Lipschitz manifold.

3.3 Existence result

The existence and uniqueness of weak solutions for the variational formulation (38) of problem (35) on non smooth manifolds Γ is to our knowledge open. Classical existence results require the smoothness of Γ [**?**, see for instance Chapter 4, Section 1.2 in]]Aubin1998. The following statement is taken from [11] Theorem 1 b).

Theorem 4 (Existence and uniqueness of weak solutions on a C^3 hypersurface ([11])).
Let Γ be a closed embedded C^3 hypersurface in $\mathbb{R}^3$. For every $f \in L^2_0(\Gamma)$, there exists a unique weak solution $u \in H^1_0(\Gamma)$ of $-\triangle u = f$ on Γ.

Proof. : see Theorem 1 b) in [11]. □

Unfortunately on a Lipschitz manifold, similar existence results can not be found in the literature. Besides, as mentioned in [11] page 3, it is important to check that the space $H^1(\Gamma_h)$ is well defined, where Γ_h results from a triangulation of Γ.

The following statement is taken from [11] Theorem 2 b).

Theorem 5 (Existence and uniqueness of weak solutions on a triangulated hypersurface ([11])).
Let Γ_h be a closed embedded $C^{0,1}$ hypersurface in $\mathbb{R}^3$. For every $f_h \in L^2(\Gamma_h)$ with $\int_{\Gamma_h} f_h = 0$, there exists a weak solution $u_h \in H^1(\Gamma_h)$ of $-\triangle u_h = f_h$ on Γ_h. Furthermore u_h is unique up to a constant.

[11] claims that the proof is a "simple application of usual Hilbert space methods". However these Hilbert space methods require a geometrical and functional setting for calculus on Lipschitz manifolds that is not easily found in the literature. Moreover Stokes' theorem 1 and Poincaré's inequality (Corollary 1) on Lipschitz manifolds are never stated and do not seem trivial to us.

Our existence theorem on Lipschitz manifolds relies on the Stokes' theorem 1 and on the Lax-Milgram's theorem.

Theorem 6 (Existence theorem on Lipschitz manifolds).
Let Γ be a closed Lipschitz manifold. Let $f \in L^2_0(\Gamma)$. There exists a unique weak solution $u \in H^1_0(\Gamma)$ of $-\Delta_\Gamma u = f$ on Γ. Furthermore, u depends continuously on the right hand side:

$$\exists C' > 0, \|u\|_{H^1_0(\Gamma)} \leq C' \|f\|_{L^2_0(\Gamma)}. \tag{39}$$

Proof. In order to use the Lax-Milgram's theorem, using the bilinear form

$$\begin{array}{ccc} a(\cdot,\cdot) : H^1_0(\Gamma) \times H^1_0(\Gamma) & \to & \mathbb{R} \\ (u,v) & \to & \int_\Gamma \overrightarrow{\nabla}_\Gamma u \cdot \overrightarrow{\nabla}_\Gamma v \, \mathrm{d}\lambda_\Gamma \end{array}$$

and the linear form

$$\begin{array}{ccc} b(\cdot) : H^1_0(\Gamma) & \to & \mathbb{R} \\ v & \to & \int_\Gamma f v \, \mathrm{d}\lambda_\Gamma \end{array}$$

we rewrite the weak formulation (38) of (35) as

$$a(u,v) = b(v),\ \forall v \in H^1_0(\Gamma)$$

- The continuity of the bilinear form $a(\cdot,\cdot)$ is a consequence of the Cauchy-Schwarz's inequality :

$$\begin{aligned} |a(u,v)| &= |\int_\Gamma \overrightarrow{\nabla}_\Gamma u \cdot \overrightarrow{\nabla}_\Gamma v \, \mathrm{d}\lambda_\Gamma| \\ &\leq \|\overrightarrow{\nabla}_\Gamma u\|_{L^2_0(\Gamma)} . \|\overrightarrow{\nabla}_\Gamma v\|_{L^2_0(\Gamma)} \\ &\leq \|u\|_{H^1_0(\Gamma)} \|v\|_{H^1_0(\Gamma)} \end{aligned}$$

- We obtain the coercivity of $a(\cdot,\cdot)$ thanks to the Poincaré's inequality (Corollary 1) or more precisely inequality (33):

$$\begin{aligned} a(u,u) &= \left(\overrightarrow{\nabla}_\Gamma u, \overrightarrow{\nabla}_\Gamma u\right)_{L^2_0(\Gamma)} \\ &= \|\overrightarrow{\nabla}_\Gamma u\|^2_{L^2_0(\Gamma)} \\ &\geq \frac{1}{(1+C)^2} \|u\|^2_{H^1_0(\Gamma)} \end{aligned}$$

- The continuity of $b(\cdot)$ is the consequence of the Cauchy-Schwarz's inequality and the Poincaré's inequality (Corollary 1)

$$\begin{aligned} |b(v)| = |(f,v)_{L^2_0(\Gamma)}| &\leq \|f\|_{L^2_0(\Gamma)} \|v\|_{L^2_0(\Gamma)} \\ &\leq \|f\|_{L^2_0(\Gamma)} \|v\|_{H^1_0(\Gamma)} \end{aligned}$$

Since the requirement of the Lax-Milgram's theorem are satisfied, the existence of a unique solution to the weak formulation (38) satisfying the stability estimate (39) holds with $C' = \frac{1}{(1+C)^2}$. □

In the next section we introduce the linear finite element method.

4 The finite element method

The finite element method (FEM) is a numerical method for solving problems of engineering and mathematical physics that are formulated by PDEs [**?**, see, e.g.,]]Allaire2007,ElliotStinner).

When the domain is a hypersurface Γ, we assume as done in [11], the existence of a polyhedric surface Γ_h that approximates Γ and is made of triangles with nodes on Γ. We can approximate functions $u \in H^1_0(\Gamma)$ by functions $u_h \in H^1_0(\Gamma_h)$. We can then approximate functions $u_h \in H^1_0(\Gamma_h)$ by their projection on $PL_0(\Gamma_h)$ the space of continuous piecewise affine functions with zero mean.

Considering the weak Laplace-Beltrami operator on the piecewise linear manifold Γ_h, we seek for $\tilde{u}_h \in PL_0(\Gamma_h)$ the projection on $PL_0(\Gamma_h)$ of the solution $u_h \in H^1_0(\Gamma_h)$ of the Poisson problem $-\triangle_{\Gamma_h} u_h = f_h$. The finite element approximation therefore projects the original problem set on $H^1_0(\Gamma)$ onto the finite dimensional space $PL_0(\Gamma_h)$ using a discrete variational formulation.

The **discrete variational formulation** of the Poisson equation (35) is the following.

$$\text{Find } \tilde{u}_h \in PL_0(\Gamma_h) \text{ such that } \forall \tilde{v}_h \in PL_0(\Gamma_h), \quad \int_{\Gamma_h} \overrightarrow{\nabla}_{\Gamma_h} \tilde{u}_h \cdot \overrightarrow{\nabla}_{\Gamma_h} \tilde{v}_h = \int_{\Gamma_h} f_h \tilde{v}_h. \tag{40}$$

In a similar fashion as in equations (36-37), the **discrete Laplace-Beltrami operator** $-\tilde{\triangle}_h : PL_0(\Gamma_h) \to PL_0(\Gamma_h)'$ is the operator sending $\tilde{u}_h \in PL_0(\Gamma_h)$ to the linear functional

$$\begin{aligned} PL_0(\Gamma_h) &\to \mathbb{R} && (41)\\ \tilde{v}_h &\to \int_{\Gamma_h} \overrightarrow{\nabla}_{\Gamma_h}\tilde{u}_h \cdot \overrightarrow{\nabla}_{\Gamma_h}\tilde{v}_h, && (42)\end{aligned}$$

where $\overrightarrow{\nabla}_{\Gamma_h}\tilde{u}_h$ is the weak gradient of $\tilde{u}_h$ on Γ_h given by (24).

$-\tilde{\triangle}_h$ acts between finite dimensional spaces and approximates the continuous weak Laplace-Beltrami operator (36-37). It can be represented by a **stiffness matrix** $A_{\triangle_{\Gamma_h}}$ with size n the number of nodes of Γ_h.

The unknown function $\tilde{u}_h$ belongs to the finite dimensional space $PL_0(\Gamma_h)$. Define the **unknown vector** $(u_1, ..., u_n)$ as the vector of components of $\tilde{u}_h$ on the nodal basis:

$$\tilde{u}_h = \sum_{i=1}^{n} u_i \phi_i. \tag{43}$$

Since $PL_0(\Gamma_h)$ is generated by the nodal functions $\phi_i : \Gamma_h \to \mathbb{R}, \quad i = 1, ..., n$ such that $\phi_i(x_j) = \delta_{ij}$, the solution $\tilde{u}_h$ of the discrete Poisson problem (40) must therefore satisfy the following system of equations

$$\forall i \in \{1, ..., n\}, \qquad \int_{\Gamma_h} \overrightarrow{\nabla}_{\Gamma_h}\tilde{u}_h \cdot \overrightarrow{\nabla}_{\Gamma_h}\phi_i = \int_{\Gamma_h} f_h \phi_i, \tag{44}$$

which takes the algebraic form

$$A_{\triangle_{\Gamma_h}} X = b_h, \text{ where } X = {}^t(u_1, ..., u_n). \tag{45}$$

From (44) the coefficients of the stiffness matrix $A_{\triangle_{\Gamma_h}} = (a_{ij})_{i,j=1,...,n}$, and of the right hand side vector $b_h = {}^t(b_1, ..., b_n)$ are therefore given by

$$\begin{aligned} a_{ij} &= \int_{\Gamma_h} \overrightarrow{\nabla}_{\Gamma_h}\phi_i \cdot \overrightarrow{\nabla}_{\Gamma_h}\phi_j, && (46)\\ b_j &= \int_{\Gamma_h} f\phi_j. && (47)\end{aligned}$$

The stiffness matrix $A_{\triangle_{\Gamma_h}}$ is symmetric positive but not invertible [**?**, see Theorem 4.1 in]]NdjingaNguemfouo). However the finite element linear system admits a unique solution provided the right hand side has zero mean [**?**, see Theorem 4.2 in]]NdjingaNguemfouo, hence the existence of the discrete solution $\tilde{u}_h$ as stated in the following theorem.

Theorem 7 (Existence of the discrete solution)**.**
Let $f_h \in PL_0(\Gamma_h)$, there exists a unique discrete solution $\tilde{u}_h \in PL_0(\Gamma_h)$ to the discrete Poisson problem (40).

The computation of the coefficients a_{ij} is usually performed by expressing the integral in (46) as a sum over the triangles T_k composing Γ_h:

$$a_{ij} = \sum_{T_k} \int_{T_k} \overrightarrow{\nabla}_{\Gamma_h}\phi_i \cdot \overrightarrow{\nabla}_{\Gamma_h}\phi_j. \tag{48}$$

The gradients $\overrightarrow{\nabla}_{\Gamma_h}\phi_i$ in (48) are thus computed in each separate triangle T_k where their value is well defined and constant. There is no need to compute the gradient at the edge between two triangles T_k, where it is not properly defined. Therefore the lack of smoothness of Γ_h does not yield any singularity in the computation of the stiffness matrix coefficients.

The coefficients a_{ij} of $A_{\triangle_{\Gamma_h}}$ can be computed explicitely as a function of the mesh nodes coordinates and are given in [14].

5 Numerical experiments on the unit cube boundary

In order to illustrate the applicability of our theoretical study we present in this section the numerical simulation of a Poisson problem on the boundary of the unit cube.

5.1 Setting of the problem

We consider the unit cube $\Omega_{cube} = [0,1] \times [0,1] \times [0,1] \subset \mathbb{R}^3$ and its boundary

$$\Gamma = \partial\Omega_{cube}.$$

Γ is a topological manifold but not a differential manifold because of the presence of sharp edges where Γ admits no tangent space. Γ is however a Lipschitz manifold,which admits tangent spaces except at the edges. As a Lipschitz manifold, Γ can be endowed with a Laplace-Beltrami operator $\triangle_\Gamma = \nabla \cdot \overrightarrow{\nabla}$, defined as the combination of a surface divergence $\nabla_\Gamma \cdot$, and of a surface gradient $\overrightarrow{\nabla}_\Gamma$ [?, see]]Gesztesy.

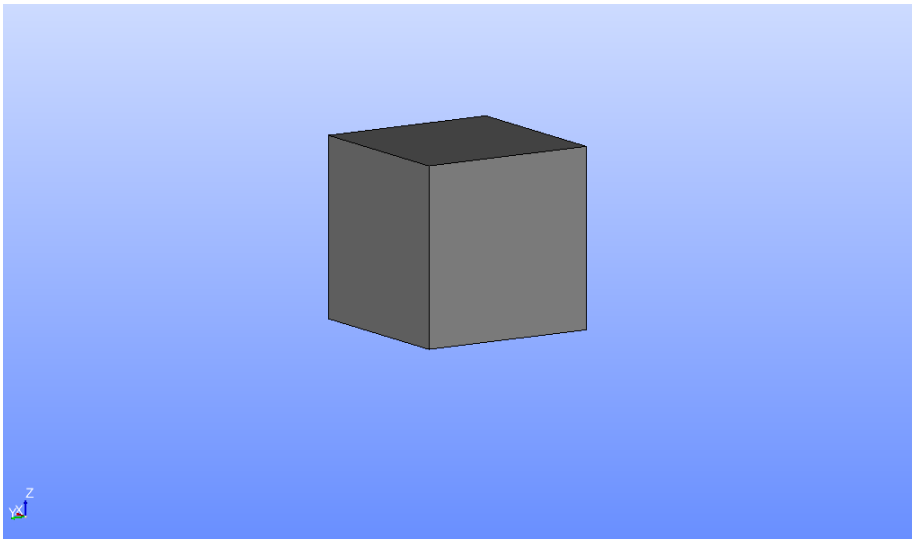


Figure 1: The unit cube in SALOME CAO module

We consider f the restriction to Γ of the smooth function $\cos(2\pi x)\cos(2\pi y)\cos(2\pi z)$. The explicit expressions of f on each face of Γ are

$$f(x,y,z) = \begin{cases} \cos(2\pi x)\cos(2\pi y) & \text{if} \quad z = 0 \text{ or } z = 1 \\ \cos(2\pi x)\cos(2\pi z) & \text{if} \quad y = 0 \text{ or } y = 1 \\ \cos(2\pi y)\cos(2\pi z) & \text{if} \quad x = 0 \text{ or } x = 1 \end{cases} . \tag{49}$$

f is an eigenfunction of the Laplace-beltrami operator on Γ since

$$\begin{aligned}
\triangle_\Gamma f(x,y,z) &= \begin{cases} \partial_{xx}\cos(2\pi x)\cos(2\pi y) + \partial_{yy}\cos(2\pi x)\cos(2\pi y) & \text{if} \quad z=0 \text{ or } z=1 \\ \partial_{xx}\cos(2\pi x)\cos(2\pi z) + \partial_{zz}\cos(2\pi x)\cos(2\pi z) & \text{if} \quad y=0 \text{ or } y=1 \\ \partial_{yy}\cos(2\pi y)\cos(2\pi z) + \partial_{zz}\cos(2\pi y)\cos(2\pi z) & \text{if} \quad x=0 \text{ or } x=1 \end{cases} \\
&= \begin{cases} -2(2\pi)^2\cos(2\pi x)\cos(2\pi y) & \text{if} \quad z=0 \text{ or } z=1 \\ -2(2\pi)^2\cos(2\pi x)\cos(2\pi z) & \text{if} \quad y=0 \text{ or } y=1 \\ -2(2\pi)^2\cos(2\pi y)\cos(2\pi z) & \text{if} \quad x=0 \text{ or } x=1 \end{cases} \\
&= -8\pi^2 f.
\end{aligned}$$

We are going to solve the following Poisson problem on Γ :

$$-\triangle_\Gamma u = f, \tag{50}$$

$$\int_\Gamma u = 0,$$

where the right hand side f defined in (49) and the unknown $u \in H^1(\Gamma)$ are **zero mean functions**.

Our objective is to solve numerically the Poissson problem (35) using the finite element method described in [11].

- For the numerical resolution of our discrete problem, we use an iterative solver (Conjugate Gradient) because the stiffness matrix $A_{\triangle_{\Gamma_h}}$ is large and sparse [**?**, see]]Saad1996 .
- For the design and meshing of the domain we use the GEOMETRY and MESH modules of the software SALOME 9.5 [**?**, see]]Ribes,Salome.
- For the visualisation of the result, we use the PARAVIS module included in SALOME [**?**, see]]Salome.
- For the coding of the script, we use Python language and the open-source Linux based library [18] which is very practical for the manipulation of large matrices, vectors, meshes and fields. It (SOLVERLAB) can handle finite element and finite volume discretizations, read general $1D, 2D$ and $3D$ geometries and meshes generated by SALOME.

5.2 Meshing of the domain

Below are the meshes used in our convergence analysis.

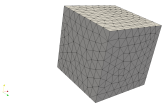
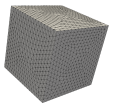
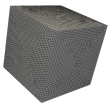
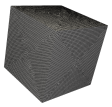
meshCubeSkin 1	meshCubeSkin 2	meshCubeSkin 3	meshCubeSkin 4
			
412 cells	1923 cells	7042 cells	13225 cells

Figure 2: Meshes of the unit cube boundary

5.3 Visualisation of the results

Below are visualisations of the numerical results obtained on the different meshes

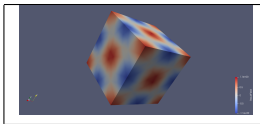
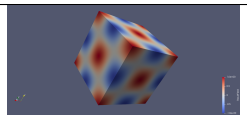
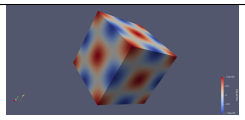
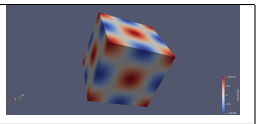
			
412 cells	1923 cells	7042 cells	13225 cells

Figure 3: Numerical results of the finite elements on the unit cube boundary

Below are clipings of the previous numerical results.

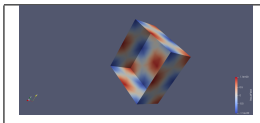
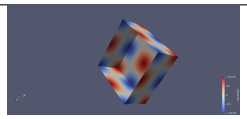
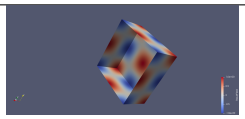
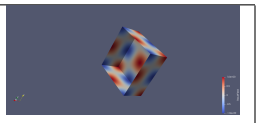
			
412 cells	1923 cells	7042 cells	13225 cells

Figure 4: Clipings of the numerical results on the unit cube boundary

5.4 Numerical convergence

When refining the mesh, we observe the convergence of the method with a numerical order of 1.91.

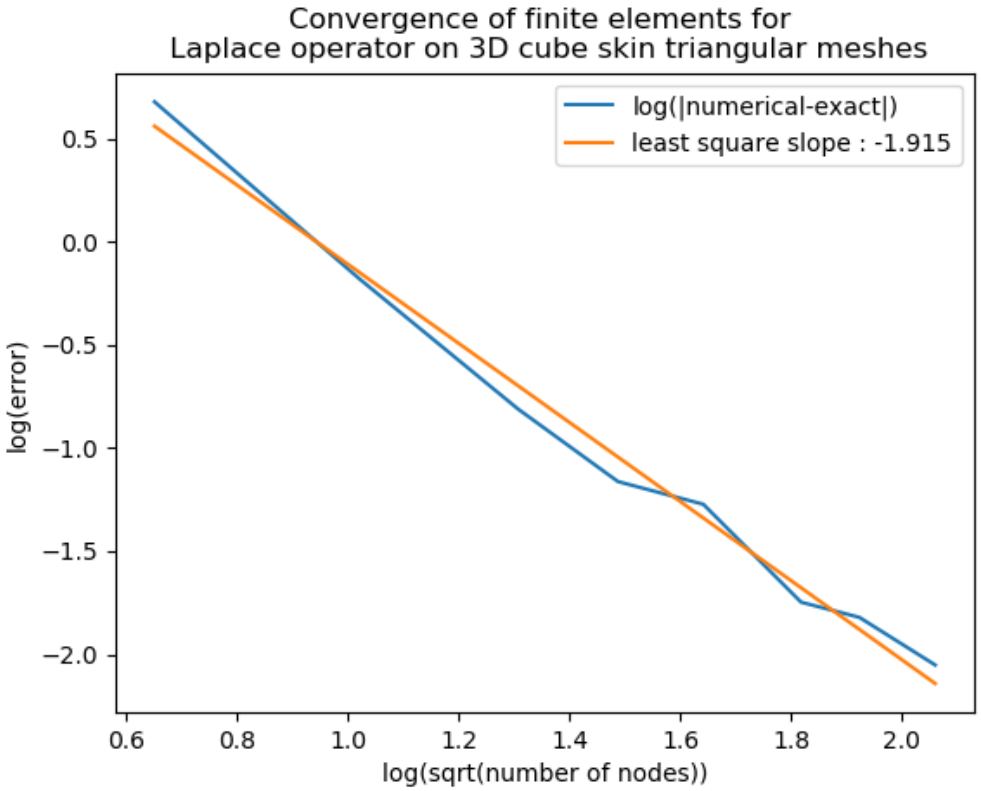


Figure 5: Convergence of the finite element method on the cube boundary

6 Conclusion and perspectives

The notion of Lipschitz manifold has been studied by some authors [8, 9] and is usefull in laying the foundation of the finite element method on general compact manifolds. We showed how it allows the proper definition of tangent spaces, gradient and divergence operators almost everywhere. Differentiability almost everywhere thanks to Rademacher's theorem is the key ingredient of the calculus on Lipschtz manifolds.

We have proposed proofs of the Stokes' (1) and Green's (2) theorems as well as a Poincaré's inequality. This enables the weak formulation of the Poisson problem. The well-posedness of the weak formulation of the Poisson problem is then obtained thanks to the Lax-Milgram's theorem.

We gave an example of numerical simulation on a Lipschitz (non smooth) manifold to illustrate the fact that the lack of smoothness does not yield any numerical singularity. The numerical results of the Poisson problem on the unit cube boundary showed that the finite element

approximation converges towards the exact solution.

This study has set the theoretical ground for further numerical analysis of more complex PDEs on Lipschitz manifolds. Note however that on Lipschitz manifolds the maximum order of differentiability is one which is enough for the classical weak formulation of second order elliptic PDEs. This is however a technical difficulty for general high order PDEs on Lipschitz manifolds. One workaround for PDEs of order three and more could however be to reset the PDE into a system of low order PDEs.

Yet, with this new approach we were not able to extend the regularity and convergence theorems of [11, 12] because the lack of smoothness of the Lipschitz manifold Γ prohibits the use of the lift operator. Further research should therefore be devoted to the theoretical analysis of the convergence of both the approximate manifold Γ_h and solution u_h towards the exact manifold Γ and solution u.

Declarations

Some journals require declarations to be submitted in a standardised format. Please check the Instructions for Authors of the journal to which you are submitting to see if you need to complete this section. If yes, your manuscript must contain the following sections under the heading 'Declarations':

- Funding **Not applicable**
- Conflict of interest/Competing interests The authors declare no conflict of interest regarding the publication of this paper
- Ethics approval
- Consent to participate
- Consent for publication
- Availability of data and materials The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.
- Code availability
- Authors' contributions

Not applicable

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