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Constante de stabilité pour les éléments finis de Fortin-Soulie

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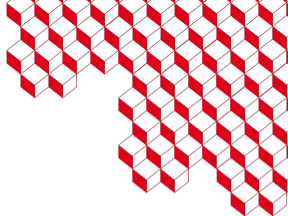
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Stability estimates for Fortin-Soulie mixed finite elements

Improved stability estimates

11^{ème} Biennale de la SMAI, 22-26 mai 2023

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CEA DES/ISAS/DM2S/STMF SACLAY

Outline

Context

Nonconforming discretization

Numerical results

Conclusion



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Context



- TrioCFD software:
simulation of unsteady turbulent flows at low Mach number in industrial configurations.
- Solve in (\mathbf{u}, p) (fluid velocity and pressure) such that:

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \mathbf{grad}) \mathbf{u} + \mathbf{grad} p = \mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \\ \mathbf{u}(t=0) = \mathbf{u}_0 \end{cases}, \text{ with appropriate BC.}$$

Space discretization on a grid with MAC scheme or on simplexes with $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ FE.
Resolution using a three-step scheme.

- R&D works on spatial numerical schemes.
 $\mathbf{P}_{nc}^1 - P_{MPFA}^0$ scheme, with Andrew Peitavy (PhD in progress).
DGFEM, with Mayssa Mroueh (PhD in progress).

Improved stability estimates for solving Stokes problem with Fortin-Soulie finite elements
Jamelot'23 (submitted).

Stokes Problem, variational formulation

- Solve in $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that:
$$\begin{cases} -\nu \Delta \mathbf{u} + \mathbf{grad} p &= \mathbf{f}, \\ \operatorname{div} \mathbf{u} &= 0. \end{cases}$$

$\mathbf{u} \in \mathbf{V}$ where: $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}$, $\mathbf{H}_0^1(\Omega) = \mathbf{V} \oplus \mathbf{V}^\perp$.

- Let $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$, which is an Hilbert space endowed with the norm

$$\|(\mathbf{u}, p)\|_{\mathcal{X}} = \left(\|\mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-2} \|p\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

(*FV - S*) Solve in $(\mathbf{u}, p) \in \mathcal{X}$ such that $\forall (\mathbf{v}, q) \in \mathcal{X}$: $a_S((\mathbf{u}, p), (\mathbf{v}, q)) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)}$.

$$a_S : \begin{cases} \mathcal{X} \times \mathcal{X} & \rightarrow \mathbb{R} \\ ((\mathbf{u}', p'), (\mathbf{v}, q)) & \mapsto \nu (\mathbf{u}', \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (\operatorname{div} \mathbf{v}, p')_{L^2(\Omega)} - (\operatorname{div} \mathbf{u}', q)_{L^2(\Omega)} \end{cases}$$

- Poincaré-Steklov inequality: $\exists C_{PS} > 0 \mid \|\mathbf{v}\|_{L^2(\Omega)} \leq C_{PS} \|\mathbf{Grad} \mathbf{v}\|_{L^2(\Omega)}$.
 $(\mathbf{v}, \mathbf{w})_{\mathbf{H}_0^1(\Omega)} = (\mathbf{Grad} \mathbf{v}, \mathbf{Grad} \mathbf{w})_{L^2(\Omega)}$ and $\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} = \|\mathbf{Grad} \mathbf{v}\|_{L^2(\Omega)}$.

- Girault-Raviart'86: the operator div is an isomorphism from \mathbf{V}^\perp on $L_0^2(\Omega)$.

$$\forall q \in L_0^2(\Omega), \exists! \mathbf{v}_q \in \mathbf{V}^\perp \mid \operatorname{div} \mathbf{v}_q = q, \quad \|\mathbf{v}_q\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|q\|_{L^2(\Omega)}.$$

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Notations

- Let $\Omega \subset \mathbb{R}^d$ be a connected, bounded, open polygon or polyhedron with a Lipschitz boundary. Family of meshes $(\mathcal{T}_h)_h$, with $h \rightarrow 0$.
- Simplicial regular triangulation: $\mathcal{T}_h = \bigcup_K K$.

Vertices set: $\mathcal{S}_h = \bigcup_S S$.

Facets set: $\mathcal{F}_h = \bigcup_F F = \mathcal{F}_h^i \cup \mathcal{F}_h^b$; \mathcal{F}_h^i (resp. \mathcal{F}_h^b): inner (resp. boundary) facets.

Orthonormal vectors to facets: $(\mathbf{n}_F)_{F \in \mathcal{F}_h}$ (\mathbf{n}_F outward oriented if $F \in \partial\Omega$).

- h_K : diameter of element K , ρ_K : diameter of the inscribed sphere of element K ,

$$\exists \sigma > 0 \mid \forall h, \forall K \in \mathcal{T}_h, \quad \sigma_K = \frac{h_K}{\rho_K} \leq \sigma.$$

- For $d = 2$, $\sigma \geq \left(\frac{\pi}{2\theta}\right)^{1/2}$ where θ is the smallest angle of the triangulation.
- For $d = 3$, $\sigma \geq \left(\frac{\pi}{2\omega}\right)^{1/3}$ where ω is the smallest solid angle of the triangulation.

Bilinear form $(\mathbf{u}_h, \mathbf{v}_h)_h$ and operator div_h

- Piecewise regular functions.
- $\mathcal{P}_h H^1 = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in H^1(K)\}$, $\mathcal{P}_h \mathbf{H}^1 = (\mathcal{P}_h H^1)^d$.

$$\forall v, w \in \mathcal{P}_h H^1, \quad (v, w)_h = \sum_{K \in \mathcal{T}_h} (\mathbf{grad} v, \mathbf{grad} w)_{L^2(K)}, \quad \|v\|_h^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{grad} v\|_{L^2(K)}^2,$$

$$\forall \mathbf{v}, \mathbf{w} \in \mathcal{P}_h \mathbf{H}^1, \quad (\mathbf{v}, \mathbf{w})_h = \sum_{K \in \mathcal{T}_h} (\mathbf{Grad} \mathbf{v}, \mathbf{Grad} \mathbf{w})_{L^2(K)}, \quad \|\mathbf{v}\|_h^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{Grad} \mathbf{v}\|_{L^2(K)}^2.$$

- Interface jumps for $v \in \mathcal{P}_h H^1$:

For $F \in \mathcal{F}_h^i$, $\bar{F} = \bar{K}_L \cap \bar{K}_R$, \mathbf{n}_F oriented from K_L to K_R $[v]_{|F} = (v|_{K_L} - v|_{K_R})_{|F}$.

For $F \in \mathcal{F}_h^b$, $[v]_{|F} = v|_F$.

- $\mathcal{P}_h \mathbf{H}(\text{div}) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid \forall K \in \mathcal{T}_h, \mathbf{v}|_K \in \mathbf{H}(\text{div}; K)\}$.

Operator div_h : $\forall \mathbf{v} \in \mathcal{P}_h \mathbf{H}(\text{div}), \forall q \in L^2(\Omega), \quad (\text{div}_h \mathbf{v}, q) = \sum_{K \in \mathcal{T}_h} (\text{div} \mathbf{v}, q)_{L^2(K)}$.

Nonconforming mixed finite elements (i) Crouzeix-Raviart'73

- Nonconforming approximation of $H^1(\Omega)$:

$$X_h = \left\{ v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P^k(K) \quad \text{and} \quad \forall F \in \mathcal{F}_h^i, \forall q \in P^{k-1}(F) \quad \int_F [v] q = 0 \right\}.$$

- Nonconforming approximation of $H_0^1(\Omega)$:

$$X_{0,h} = \left\{ v \in X_h \mid \forall F \in \mathcal{F}_h^b, \forall q \in P^{k-1}(F) \quad \int_F v q = 0 \right\}.$$

- Thanks to the patch test, one can prove that:

The mapping $v_h \rightarrow \|v_h\|_h$ is a norm on $X_{0,h}$.

We have a discrete Poincaré-Steklov inequality [Ern-Guermond'21, Lemma 36.6](#):

$$\forall v \in X_{0,h}, \quad \|v\|_{L^2(\Omega)} \leq C_{PS}^{nc} \|v\|_h.$$

- Approximation of $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$:

Nonconforming approximation of $\mathbf{H}^1(\Omega)$ and $\mathbf{H}_0^1(\Omega)$: $\mathbf{X}_h = (X_h)^d$ and $\mathbf{X}_{0,h} = (X_{0,h})^d$.

Approximation of $L_0^2(\Omega)$: $Q_h = \{q \in L_0^2(\Omega) \mid \forall K \in \mathcal{T}_h, \quad q|_K \in P^{k-1}(K)\}$.

$$\mathcal{X}_h := \mathbf{X}_{0,h} \times Q_h, \quad \|(\mathbf{v}, q)\|_{\mathcal{X}_h} = \left(\|\mathbf{v}\|_h^2 + \nu^{-2} \|q\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Nonconforming mixed finite elements (ii) Crouzeix-Raviart'73

- $(FV_h - S)$ Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that $\forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h: a_{S,h}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \ell_{\mathbf{f}}(\mathbf{v}_h)$.

$$a_{S,h} : \begin{cases} \mathcal{X}_h \times \mathcal{X}_h & \rightarrow \mathbb{R} \\ ((\mathbf{u}'_h, p'_h), (\mathbf{v}_h, q_h)) & \mapsto \nu (\mathbf{u}'_h, \mathbf{v}_h)_h - (\operatorname{div}_h \mathbf{v}_h, p'_h) - (\operatorname{div}_h \mathbf{u}'_h, q_h) \end{cases}$$

For $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\ell_{\mathbf{f}}(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}$.

For $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $\ell_{\mathbf{f}}(\mathbf{v}_h) = \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}_0^1(\Omega)}$ **Veeseer-Zanotti'18.**

- Suppose there exists a Fortin operator $\Pi_{nc} : \mathbf{H}^1(\Omega) \rightarrow \mathbf{X}_h$ such that

$$\begin{cases} \forall \mathbf{v} \in \mathbf{H}^1(\Omega) & (\operatorname{div}_h \Pi_{nc} \mathbf{v}, q) = (\operatorname{div} \mathbf{v}, q)_{L^2(\Omega)}, \quad \forall q \in Q_h, \\ \exists \tilde{C} > 0 \mid \forall \mathbf{v} \in \mathbf{H}^1(\Omega) & \|\Pi_{nc} \mathbf{v}\|_h \leq \tilde{C} \|\mathbf{Grad} \mathbf{v}\|_{\mathbb{L}^2(\Omega)}. \end{cases}$$

Let $p'_h \in Q_h \subset L_0^2(\Omega)$. Consider $\mathbf{v}_{p'_h} \in \mathbf{V}^\perp$ such that

$$\operatorname{div} \mathbf{v}_{p'_h} = p'_h \quad \text{and} \quad \|\mathbf{v}_{p'_h}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|p'_h\|_{L^2(\Omega)}.$$

Let $\mathbf{v}_{h,p'_h} = \Pi_{nc} \mathbf{v}_{p'_h}$. We have, setting $\tilde{C}_{\operatorname{div}} = \tilde{C} C_{\operatorname{div}}$:

$$\forall q \in Q_h \quad (\operatorname{div}_h \mathbf{v}_{h,p'_h}, q) = (p'_h, q)_{L^2(\Omega)} \quad \text{and} \quad \|\mathbf{v}_{h,p'_h}\|_h \leq \tilde{C}_{\operatorname{div}} \|p'_h\|_{L^2(\Omega)}.$$

Mixed finite elements $\mathbf{P}_{nc}^1 - P^0$ Crouzeix-Raviart'73, Example 4

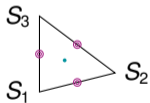
- Approximation of $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$: $\mathcal{X}_{cr} = \mathbf{X}_{cr} \times Q_{cr}$.

$$\mathbf{X}_{cr} = (\mathbf{X}_{cr})^d, \quad \mathbf{X}_{cr} = \left\{ v_h \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h \quad v_{h|K} \in P^1(K); \forall F \in \mathcal{F}_h \quad \int_F [v_h] = 0 \right\},$$

$$Q_{cr} = \left\{ q_h \in L_0^2(\Omega) \mid \forall K \in \mathcal{T}_h \quad q_{h|K} \in P^0(K) \right\}.$$

$$\mathbf{X}_{cr} = \text{vect} \left((\psi_F)_{F \in \mathcal{F}_h} \right), \text{ with: } \psi_{F|K} := \begin{cases} 0 & \text{si } F \notin \partial K, \\ 1 - d\lambda_{S_F,K} & \text{si } F \in \partial K. \end{cases}$$

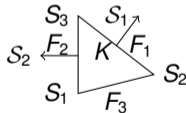
- Degrees of freedom:



Velocity, pressure.

We have: $\mathbf{grad} \psi_{F|K} = -d \mathbf{grad} \lambda_{S_F,K} = |K|^{-1} S_{F,K}$.

Barycentric coordinates:



$$\lambda_{S_i,K} = (d|K|)^{-1} (\vec{OS}_i - \mathbf{x}) \cdot S_{F_i,K}.$$

Fortin operator $\mathbf{P}_{nc}^1 - P^0$

- Interpolation operator for scalar functions: $\pi_{cr} : H^1(\Omega) \rightarrow X_{cr}$, $\pi_{cr} v = \sum_{F \in \mathcal{F}_h} \left(\frac{1}{|F|} \int_F v \right) \psi_F$.

Apel-Nicaise-Schöberl'01, Lemma 2:

$$\mathbf{grad} \pi_{cr} v|_K = |K|^{-1} \int_K \mathbf{grad} \pi_{cr} v = |K|^{-1} \int_{\partial K} \pi_{cr} v \mathbf{n} = |K|^{-1} \int_{\partial K} v \mathbf{n} = |K|^{-1} \int_K \mathbf{grad} v.$$

Hence: $|\mathbf{grad} \pi_{cr} v|_K| \leq |K|^{-1/2} \|\mathbf{grad} v\|_{L^2(K)} \Rightarrow \|\mathbf{grad} \pi_{cr} v\|_{L^2(K)} \leq \|\mathbf{grad} v\|_{L^2(K)}$.

For all $v \in H^1(\Omega)$, we have: $\|\pi_{cr} v\|_h \leq \|\mathbf{grad} v\|_{L^2(\Omega)}$.

- Fortin operator / Interpolation operator for vector functions: $\Pi_{cr} \mathbf{v} = (\pi_{cr} v_i)_{i=1}^d$.

$$\forall K \in \mathcal{T}_h \int_K \operatorname{div} \Pi_{cr} \mathbf{v} = \int_K \sum_{F \in \partial K} |K|^{-1} \frac{\int_F \mathbf{v} \cdot \mathcal{S}_{F,K}}{|F|} = \sum_{F \in \partial K} \int_F \mathbf{v} \cdot \mathbf{n}_{F,K} = \int_K \operatorname{div} \mathbf{v}.$$

For all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $q_h \in Q_{cr}$, we have:

$$\|\Pi_{cr} \mathbf{v}\|_h \leq \|\mathbf{Grad} \mathbf{v}\|_{\mathbb{L}^2(\Omega)} \quad \text{and} \quad (\operatorname{div}_h \Pi_{cr} \mathbf{v}, q_h) = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}$$

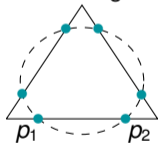
Mixed finite elements $\mathbf{P}_{nc}^2 - P_{disc}^1$ Fortin-Soulie'83

- Approximation of $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$: $\mathcal{X}_{fs} = \mathbf{X}_{fs} \times Q_{fs}$.

$$\mathbf{X}_{fs} = (X_{fs})^d, X_{fs} = X_{0,lg} \oplus \Phi_h, \quad Q_{fs} = \left\{ q_h \in L_{zvm}^2(\Omega) \mid \forall K \in \mathcal{T}_h, q_h \in P^1(K) \right\}.$$

$$\text{We set: } \begin{cases} X_{0,lg} = \left\{ v_h \in H^2(\Omega) \mid \forall K \in \mathcal{T}_h, v_h|_K \in P^2(K), v_h|_{\partial\Omega=0} \right\}, \\ \Phi_h = \left\{ \phi_h \in L^2(\Omega) \mid \phi_h|_K = \alpha_K \phi_K, \alpha_K \in \mathbb{R} \right\} \text{ where } \phi_K = 2 - 3 \sum_{S \in \partial K} \lambda_{S,K}^2. \end{cases}$$

- Gauss-Legendre points:



- Degrees of freedom:



Velocity, pressure.

- Integration: $\forall q \in P^3(F), \int_F q = \frac{|F|}{2} (q(p_1) + q(p_2))$.

$$\forall F \in \partial K, \forall q \in P^1(F), \int_F \phi_K q = 0.$$

Fortin operator $\mathbf{P}_{nc}^2 - P_{disc}^1$

- Interpolation operator for scalar functions:

$$\tilde{\pi}_{fs} : H^1(\Omega) \rightarrow X_{fs} \mid \begin{cases} \forall \mathbf{S} \in \mathcal{V}_h, & (\tilde{\pi}_{fs}(\mathbf{v}))(\mathbf{S}) = (\pi_{\text{Scott-Zhang}}(\mathbf{v}))(\mathbf{S}) \\ \forall F \in \mathcal{F}_h, & \int_F \tilde{\pi}_{fs}(\mathbf{v}) = \int_F \mathbf{v} \\ \forall K \in \mathcal{T}_h, & \int_K \tilde{\pi}_{fs}(\mathbf{v}) = \int_K \mathbf{v} \end{cases} .$$

- Interpolation operator for vector functions: $\tilde{\Pi}_{fs} : \mathbf{H}^1(\Omega) \rightarrow \mathbf{X}_{fs} \mid \tilde{\Pi}_{fs}(\mathbf{v}) = {}^t(\tilde{\pi}_{fs}(v_x), \tilde{\pi}_{fs}(v_y))$.
- Fortin operator:

$$\Pi_{fs} : \mathbf{H}^1(\Omega) \rightarrow \mathbf{X}_{fs} \mid \begin{cases} \forall \mathbf{S} \in \mathcal{V}_h, & (\Pi_{fs}(\mathbf{v}))(\mathbf{S}) = (\Pi_{\text{Scott-Zhang}}(\mathbf{v}))(\mathbf{S}) \\ \forall F \in \mathcal{F}_h, & \int_F \Pi_{fs}(\mathbf{v}) = \int_F \mathbf{v} \\ \forall K \in \mathcal{T}_h, & \int_K \mathbf{x} \operatorname{div} \Pi_{fs}(\mathbf{v}) = \int_K \mathbf{x} \operatorname{div} \mathbf{v} \end{cases} .$$

- For all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $q_h \in Q_{fs}$, we have: $(\operatorname{div}_h \Pi_{fs} \mathbf{v}, q_h) = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}$.
- Theorem:** Let $\sigma_D > 0$. For all $\mathbf{v} \in \bigcap_{0 < s < \sigma_D} \mathbf{H}^{1+s}(\Omega)$, we have:

$$\begin{aligned} \forall s \in]0, \sigma_D[, \quad \forall K \in \mathcal{T}_h, & \quad \|\mathbf{Grad}(\Pi_{fs} \mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K)} \leq (\sigma_K)^2 (h_K)^s |\mathbf{v}|_{1+s, K}, \\ \forall s \in]0, \sigma_D[, \quad \exists C_{FS} = \mathcal{O}(\sigma^2), & \quad \|\Pi_{fs} \mathbf{v} - \mathbf{v}\|_h \leq C_{FS} h^s |\mathbf{v}|_{1+s, \Omega}. \end{aligned}$$

Stability constant of Fortin operator $\mathbf{P}_{nc}^2 - P_{disc}^1$ (i)

- $\|\mathbf{Grad}(\Pi_{fs}(\mathbf{v}) - \mathbf{v})\|_{\mathbb{L}^2(K)} \lesssim \|\mathbf{Grad}(\Pi_{fs}(\mathbf{v}) - \tilde{\Pi}_{fs}(\mathbf{v}))\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{Grad}(\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v})\|_{\mathbb{L}^2(K)}$.
- For all $\hat{\mathbf{v}} \in \mathbf{P}^2(\hat{K})$, $\tilde{\Pi}_{fs}(\hat{\mathbf{v}}) = \hat{\mathbf{v}}$.

Thanks to Bramble-Hilbert/Deny-Lions Lemma, we have [Ern-Guermond'21, Lemma 11.9](#):

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \|\mathbf{Grad}(\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v})\|_{\mathbb{L}^2(K)} \lesssim \sigma_K |\mathbf{v}|_{1,K},$$

$$\forall \mathbf{v} \in \mathbf{H}^2(\Omega), \quad \|\mathbf{Grad}(\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v})\|_{\mathbb{L}^2(K)} \lesssim \sigma_K h_K |\mathbf{v}|_{2,K}.$$

- Using interpolation theory [Tartar'07, Lemma 22.2](#):

$$\forall \mathbf{v} \in \bigcap_{0 < s < \sigma_D} \mathbf{H}^{1+s}(\Omega), \quad \|\mathbf{Grad}(\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v})\|_{\mathbb{L}^2(K)} \lesssim \sigma_K (h_K)^s |\mathbf{v}|_{1+s,K}.$$

- $\tilde{\Pi}_{fs}(\mathbf{v}) = \sum_{S \in \mathcal{V}_h} \mathbf{v}_S \phi_S + \sum_{F \in \mathcal{F}_h} \mathbf{v}_F \phi_F + \sum_{K \in \mathcal{T}_h} \tilde{\mathbf{v}}_K \phi_K, \quad \Pi_{fs}(\mathbf{v}) = \sum_{S \in \mathcal{V}_h} \mathbf{v}_S \phi_S + \sum_{F \in \mathcal{F}_h} \mathbf{v}_F \phi_F + \sum_{K \in \mathcal{T}_h} \mathbf{v}_K \phi_K.$

Stability constant of Fortin operator $\mathbf{P}_{nc}^2 - P_{disc}^1$ (ii)

- We have: $\Pi_{fs}(\mathbf{v}) - \tilde{\Pi}_{fs}(\mathbf{v}) = (\mathbf{v}_K - \tilde{\mathbf{v}}_K) \phi_K$, hence:

$$\begin{aligned} \|\mathbf{Grad} (\Pi_{fs}(\mathbf{v}) - \tilde{\Pi}_{fs}(\mathbf{v}))\|_{\mathbb{L}^2(K)} &= \|\mathbf{Grad} ((\mathbf{v}_K - \tilde{\mathbf{v}}_K) \phi_K)\|_{\mathbb{L}^2(K)} \\ &\lesssim \|\mathbf{grad} \phi_K\|_{\mathbb{L}^2(K)} |\mathbf{v}_K - \tilde{\mathbf{v}}_K|, \\ &\lesssim \sigma_K |\mathbf{v}_K - \tilde{\mathbf{v}}_K|. \end{aligned}$$

- Let us estimate $|\mathbf{v}_K - \tilde{\mathbf{v}}_K|$.

$$\int_K (\Pi_{fs}(\mathbf{v}) - \tilde{\Pi}_{fs}(\mathbf{v})) = \int_K \phi_K (\mathbf{v}_K - \tilde{\mathbf{v}}_K) = \frac{|K|}{4} (\mathbf{v}_K - \tilde{\mathbf{v}}_K)$$

$$\int_K (\Pi_{fs}(\mathbf{v}) - \tilde{\Pi}_{fs}(\mathbf{v})) = \int_K (\Pi_{fs}(\mathbf{v}) - \mathbf{v}) = \int_{\partial K} \mathbf{x} (\Pi_{fs}(\mathbf{v}) - \mathbf{v}) \cdot \mathbf{n}_{|\partial K} = \int_{\partial K} \mathbf{x} (\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v}) \cdot \mathbf{n}_{|\partial K}$$

- **Lemma (*)**: Let $\mathbf{w} \in \mathbf{H}^1(\Omega)$ and $q \in P^1(K)$. Setting $\underline{\mathbf{w}} = \mathbf{w}|_K - \frac{\int_K \mathbf{w}}{|K|}$, we have:

$$\left| \int_{\partial K} q \underline{\mathbf{w}} \cdot \mathbf{n} \right| \leq |K| \|\mathbf{Grad} \underline{\mathbf{w}}\|_{\mathbb{L}^2(K)}.$$

Stability constant of Fortin operator $\mathbf{P}_{nc}^2 - P_{disc}^1$ (iii)

- Using Lemma (*), we get:

$$|\mathbf{v}_K - \tilde{\mathbf{v}}_K| = 4 |K|^{-1} \left| \int_{\partial K} \mathbf{x} \left(\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v} \right) \cdot \mathbf{n}_{|\partial K} \right| \leq 4 \|\mathbf{Grad} \left(\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v} \right)\|_{\mathbb{L}^2(K)}.$$

- We deduce that $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$:

$$\|\mathbf{Grad} \left(\Pi_{fs}(\mathbf{v}) - \tilde{\Pi}_{fs}(\mathbf{v}) \right)\|_{\mathbb{L}^2(K)} \lesssim \sigma_K |\mathbf{v}_K - \tilde{\mathbf{v}}_K| \lesssim \sigma_K \|\mathbf{Grad} \left(\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v} \right)\|_{\mathbb{L}^2(K)}.$$

- Stability constant for Π_{fs} : $\forall \mathbf{v} \in \bigcap_{0 < s < \sigma_D} \mathbf{H}^{1+s}(\Omega)$,

$$\|\mathbf{Grad} \left(\Pi_{fs}(\mathbf{v}) - \mathbf{v} \right)\|_{\mathbb{L}^2(K)} \lesssim (\sigma_K + 1) \|\mathbf{Grad} \left(\tilde{\Pi}_{fs}(\mathbf{v}) - \mathbf{v} \right)\|_{\mathbb{L}^2(K)},$$

$$\lesssim (\sigma_K)^2 (h_K)^s |\mathbf{v}|_{1+s,K}.$$

$$\|\mathbf{Grad} \left(\Pi_{fs}(\mathbf{v}) - \mathbf{v} \right)\|_{\mathbb{L}^2(\Omega)} \lesssim \sigma^2 h^s |\mathbf{v}|_{1+s,\Omega} \text{ by summation.}$$

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Numerical results

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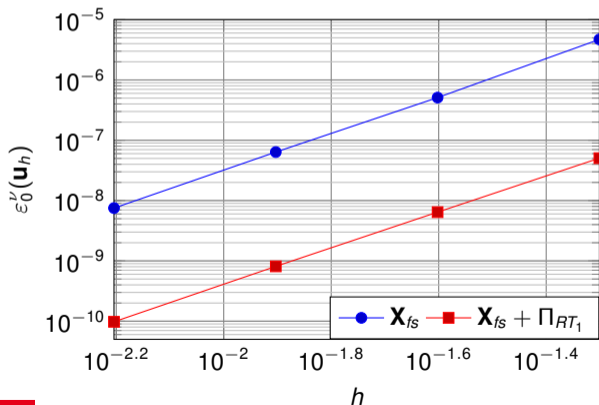


Numerical results with a regular solution

- Stokes Problem in $\Omega = (0, 1)^2$ with:

$$\mathbf{u} = \begin{pmatrix} (1 - \cos(2\pi x)) \sin(2\pi y) \\ (\cos(2\pi y) - 1) \sin(2\pi x) \end{pmatrix}, \quad p = \sin(2\pi x) \sin(2\pi y), \quad \mathbf{f} = -\nu \Delta \mathbf{u} + \mathbf{grad} p.$$

- Plots $\varepsilon_0^\nu(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} / \|(\mathbf{u}, p)\|_{\mathcal{X}}$ for $\nu = 1.00 e - 4$. Convergence rate: $\tau = 3$.



\mathbf{X}_{fs} : $\mathbf{u}_h \notin \mathbf{H}(\text{div}; \Omega)$.

$\mathbf{X}_{fs} + \Pi_{RT_1}$: $\mathbf{u}_h \in \mathbf{H}(\text{div}; \Omega)$.

Pressure robust disc.:

Linke et al

Π_{RT_1} computation:

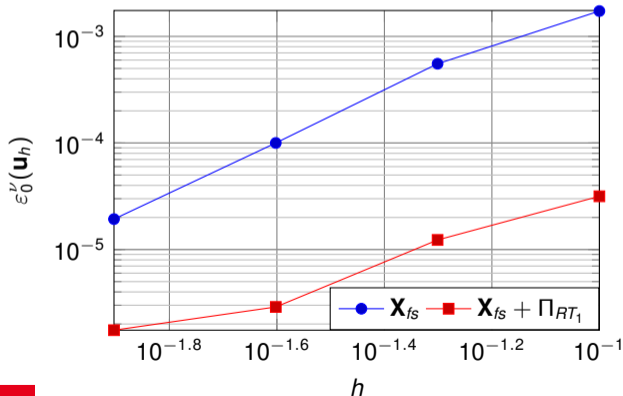
Gatica'14, Lemma 3.11

Numerical results with a low-regular solution

- Stokes Problem in $\Omega = (0, 1)^2$. We set: $\mathbf{x}_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$, $(r, \theta) = \left(|\mathbf{x} - \mathbf{x}_0|, \arctan\left(\frac{y-y_0}{x-x_0}\right) \right)$.

Consider the prescribed solution (with inhomogeneous Dirichlet BC):

$$\mathbf{u} = r^\alpha \mathbf{e}_\theta, \quad p = r^\beta - \int_{\Omega} r^\beta, \quad \mathbf{f} = -\nu(\alpha^2 - 1)\rho^{\alpha-2} \mathbf{e}_\theta + \beta \mathbf{e}_r.$$



$$\alpha = 0.45 \text{ and } \beta = 1,$$

$$\nu = 1.00 e - 4,$$

$$\epsilon_0^\nu(\mathbf{u}) = \frac{\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}}{\|(\mathbf{u}, p)\|_X}.$$

$$\tau_{fs} = 2.18,$$

$$\tau_{fs+rt} = 1.39.$$

Outline

Context

Nonconforming discretization

Numerical results

Conclusion



Conclusion

- Local stability estimates:
For order 1 in $2D$ and $3D$ without mesh regularity assumption.
For order 2 in $2D$ with mesh regularity assumption.
- Higher order and higher dimension:
Heavy mathematical material needed [Sauter et al.](#)
Splitting the normal and tangential components could help to get (local) stability estimates.
- Numerical results could be enhanced using BDM projection instead of RT.
[Brezzi-Douglas-Marini'85](#), [Boffi-Brezzi-Fortin'13](#).