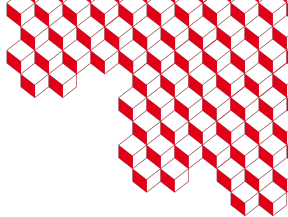




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Towards a monolithic multigrid method for Oseen problems

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Outline

Introduction

Proposed approach

Brief theoretical overview

Numerical Results

From Two- to Multigrid

Conclusions



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Brief theoretical overview

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Oseen problems

The Oseen problem appears in

- viscous incompressible flows with convection
- Picard linearisation of Navier-Stokes equations

and can be written as:

$$\begin{aligned} -\nu \Delta \vec{u} + \vec{v} \cdot \nabla \vec{u} + \nabla p &= \vec{f}, \\ \nabla \cdot \vec{u} &= 0, \end{aligned}$$

in Ω with the boundary conditions on $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$:

$$\begin{aligned} \vec{u} &= \vec{w} \text{ on } \partial\Omega_D, \\ \frac{\partial \vec{u}}{\partial n} - \vec{n}p &= \vec{s} \text{ on } \partial\Omega_N. \end{aligned}$$

Discrete Oseen problems

Discretizations lead to the **saddle-point** system:

$$\begin{pmatrix} F & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}$$

- F : convection-diffusion block (non symmetric)
- B and B^T : discrete divergence and gradient
- C : stabilization block (0 for naturally stable methods)
- Variable convective flow: no linear-time solver robust wrt both the mesh size and the viscosity ν (*i.e.* Reynolds number)
- AMG methods: struggle with the C pressure block: 0 or small elements.

AMG methods

Multigrid methods are efficient solvers for elliptic PDEs.

Two main components: the **smoother** and the **coarse-grid correction**

- **smoother**: simple iterative scheme (Jacobi, GS) corrects high frequency part of the error
⇒ the error well approximated on a coarser grid
- **coarse-grid construction**: prolongation operator P ($\mathbf{v} = P\mathbf{v}_c$)
⇒ coarse matrix $A_c = P^T A P$
- **coarse-grid correction**: amounts to solve system with A_c as system matrix.

Multigrid (MG): recursive application.

Algebraic multigrid (AMG): uses only the system matrix A

⇒ **black-box** approach.

Aggregation

- Classic AMG: P interpolation operator \Rightarrow often leads to denser A_c
- Aggregation: definition of small subsets of the unknowns (= aggregates)
- Prolongation defined as constant by aggregate $\Rightarrow P$ logical matrix and P_{ij} indicates whether the i^{th} (fine) unknown belongs to the j^{th} aggregate.
- A_c keeps the same sparsity pattern as A .

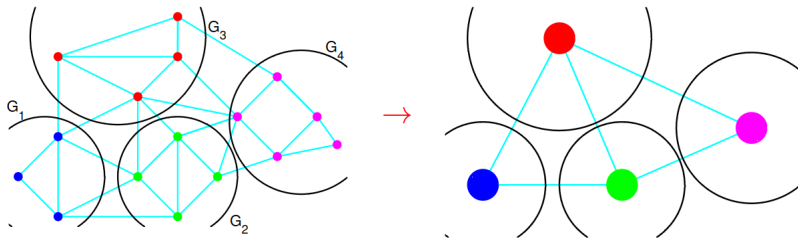


Figure 1: Representation of plain aggregation.

Outline

Introduction

Proposed approach

Brief theoretical overview

Numerical Results

From Two- to Multigrid

Conclusions



Transform-then-solve approach

Algebraic transformation of the system to give it a more "suitable" structure:

- Sign of the pressure equation is changed
- Change of variables:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} I & -D_{\mathbf{u}}^{-1}B^T \\ & I \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{pmatrix},$$

where $D_{\mathbf{u}}$ is typically the diagonal of the F block. The matrix system becomes:

$$A = \begin{pmatrix} F & B^T \\ -B & C \end{pmatrix} \begin{pmatrix} I & -D_{\mathbf{u}}^{-1}B^T \\ & I \end{pmatrix} = \begin{pmatrix} F & (I - FD_{\mathbf{u}}^{-1})B^T \\ -B & C + BD_{\mathbf{u}}^{-1}B^T \end{pmatrix}$$

Transform-then-solve approach

$$A = \begin{pmatrix} F & (I - FD_u^{-1})B^T \\ -B & C + BD_u^{-1}B^T \end{pmatrix}$$

Diagonal blocks: structure compatible with AMG methods:

- F : convection-diffusion matrix
- $\hat{C} = C + BD_u^{-1}B^T$: similar to discrete Laplacian.

Naive approach: **independent** definition of the MG components:

- Velocity: based on $F \Rightarrow P_u$ (and D_u)
- Pressure: based on $\hat{C} \Rightarrow P_p$ (and D_p)

$$P = \begin{pmatrix} P_u & \\ & P_p \end{pmatrix} \Rightarrow \text{does not work for NS}$$

Outline

Introduction

Proposed approach

Brief theoretical overview

Numerical Results

From Two- to Multigrid

Conclusions



Convergence theory: context

Theoretical framework:

- Two-grid method
- Jacobi-type smoother $\omega^{-1}D = \omega^{-1} \begin{pmatrix} D_u & \\ & D_p \end{pmatrix}$
- Prolongation of the form $P = \begin{pmatrix} P_u & \\ & P_p \end{pmatrix}$
- Galerkin coarse grid correction $A_c = P^T A P$
- One (pre)-smoothing step only.

Convergence theorem [PLB,Notay - 2022]

The norm of the iteration matrix is bounded

$$\min_{\tau} \|I - \tau K\| \leq \sqrt{1 - \alpha(\delta_{\mathbf{u}}, \delta_p, \gamma_{\mathbf{u}}, \gamma_p)}$$

if all 4 constants, which depend on the multigrid components, are bounded

- $\delta_{\mathbf{u}}(F_S, P_{\mathbf{u}}, D_{\mathbf{u}})$: if $P_{\mathbf{u}}$ and $D_{\mathbf{u}}$ form an efficient TG for F_S
- $\delta_p(\hat{C}, P_p, D_p)$: if P_p and D_p form an efficient TG for \hat{C}
- $\gamma_{\mathbf{u}}(D_{\mathbf{u}}, F)$: if $D_{\mathbf{u}}$ is a good smoother for F , e.g. $\text{diag}(F)$
- $\gamma_p(D_p, S, P_p)$: ?

Evaluation of the constants

- Numerical evaluation for model problem: finite differences on staggered grid, constant convection flow (θ orientation), Jacobi smoother
- Only free parameter to control $\gamma_p(D_p, S, P_p)$: **aggregation** of the pressure P_p

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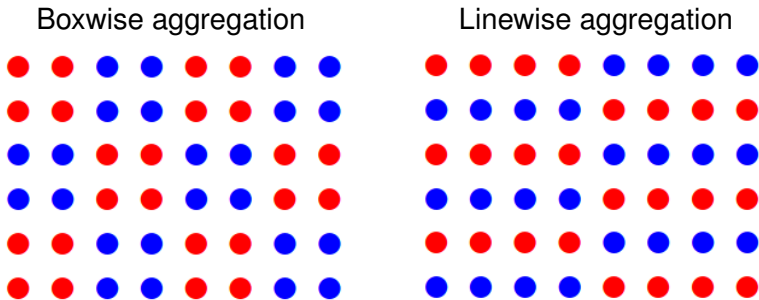


Figure 2: Schematic representation of the two choices of aggregation.

Evaluation of the constants

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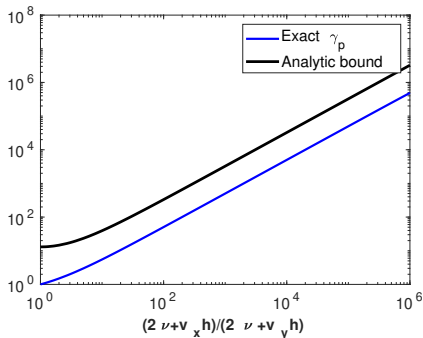
		Boxwise	Linewise	Boxwise	Linewise
		δ_p	δ_p	γ_p	γ_p
$\theta = 0$	$\nu = 10^{-2}$	5.7	9.1	17.0	3.5
	$\nu = 10^{-4}$	5.7	9.1	192.8	6.0
	$\nu = 10^{-6}$	5.7	9.1	219.6	6.1
$\theta = \pi/4$	$\nu = 10^{-2}$	5.7	9.1	1.3	1.3
	$\nu = 10^{-4}$	5.7	9.1	1.3	1.3
	$\nu = 10^{-6}$	5.7	9.1	1.3	1.3

Table 1: Evaluation of the constants related to the convergence analysis ($h = 1/16$).

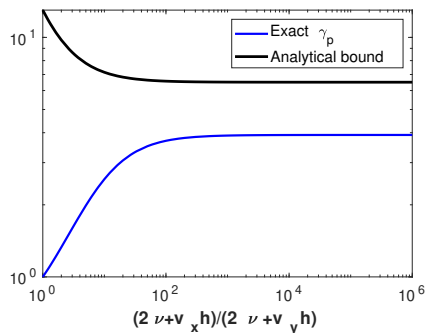
Analysis of the constants

Proof of uniform convergence for constant convection flow if aggregation in the direction of the flow: $\gamma_p \leq \frac{13}{2} \left(1 + \frac{2\nu + v_y h}{2\nu + v_x h} \right) < 13$.

Boxwise aggregation



Linewise aggregation



Definition of the aggregation

Aggregation of velocity: driven by convection (bound $\delta_{\mathbf{u}}$)

\Rightarrow algebraic aggregation based on F

Aggregation of pressure: driven by convection (bound γ_p)

\Rightarrow algebraic aggregation **NOT** based on $\hat{C} = C + BD_{\mathbf{u}}^{-1}B^T$

Definition of the aggregation

Aggregation of velocity: driven by convection (bound $\delta_{\mathbf{u}}$)

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How to define the aggregation of the pressure unknowns to be driven by convection?

Use of a geometric artifice:

- Build auxiliary convection-diffusion matrix X on pressure points
- Algebraically aggregate the points based on auxiliary matrix X .

\Rightarrow Similar aggregation pattern to velocity

Outline

Introduction

Proposed approach

Brief theoretical overview

Numerical Results

From Two- to Multigrid

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Numerical experiments: settings



- Convection-diffusion problems on staggered grid (FD): recirculating flows
- Smoother: 1 SOR iteration for pre- and post-smoothing with $\omega = 2/3$
- Aggregation defined by applying AGMG [Notay, Napov] software to
 - velocity block $F: P_u$
 - auxiliary matrix $X: P_p$

⇒ aggregates aligned with the flow (quality aware aggregation)
- Galerkin coarse grid matrix
- Two-grid method used as a preconditioner for GCR (mathematically equivalent to GMRES).

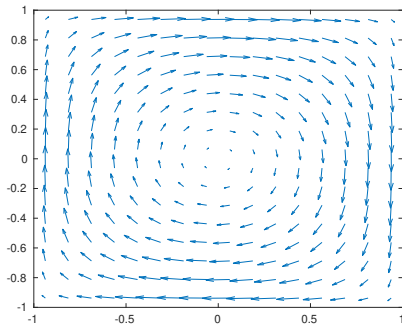
Numerical experiments: recirculating flows

$$2D1: \vec{v} = \begin{pmatrix} x(1-x)(2y-1) \\ -(2x-1)y(1-y) \end{pmatrix}$$

$$2D2: \vec{v} = \begin{pmatrix} \cos(\pi x) \sin(\pi y) \\ -\sin(\pi x) \cos(\pi y) \end{pmatrix}$$

$$ESW^1: \vec{v} = \begin{pmatrix} 2y(1-x^2) \\ -2x(1-y^2) \end{pmatrix}$$

Recirculating Flow ESW



¹ESW: Elman, Silvester, Wathen, 2005.

Numerical experiments: recirculating flow (cont'd)

Flow \ ν	$h = 1/128$				$h = 1/256$			
	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-2}	10^{-4}	10^{-6}	10^{-8}
2D1	15	36	48	49	14	30	49	51
2D2	14	39	47	49	15	37	46	53
ESW	14	40	42	42	13	38	44	44

Table 2: Recirculating flows: number of iterations until $\|\mathbf{r}_{\text{rel}}\| \leq 10^{-6}$.

Outline

Introduction

Proposed approach

Brief theoretical overview

Numerical Results

From Two- to Multigrid

Conclusions



Transition from two- to multigrid

$$A_c = \begin{pmatrix} P_u^T & \\ & P_p^T \end{pmatrix} \begin{pmatrix} F & (I - FD_u^{-1})B^T \\ -B & C + BD_u^{-1}B^T \end{pmatrix} \begin{pmatrix} P_u & \\ & P_p \end{pmatrix} = \begin{pmatrix} F_c & BT_c \\ -B_c & \hat{C}_c \end{pmatrix}$$

For significant convection, linewise aggregates are required, e.g. along x -axis:

$$\hat{C} : \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \Rightarrow \hat{C}_c : \begin{bmatrix} & 4 & \\ 1 & -10 & 1 \\ & 4 & \end{bmatrix}$$

Anisotropy in the **orthogonal** direction

Limits of the transform-then-solve approach

- Anisotropy introduced in the (isotropic) pressure block
- Aggregation driven by the velocity not suited to coarser levels
- Degradation of the δ_u but γ_p stays well bounded

	Level 1		Level 2		Level 3		Level 4	
	γ_p	δ_p	γ_p	δ_p	γ_p	δ_p	γ_p	δ_p
$\nu = 10^{-2}$	1.8	6.8	1.7	31.2	1.7	181.3	1.7	982.5
$\nu = 10^{-4}$	3.8	6.8	4.6	36.9	2.8	1354.9	1.7	2962.0
$\nu = 10^{-6}$	3.9	6.8	5.4	37.1	5.4	381.9	2.6	4591.7
$\nu = 10^{-8}$	3.9	6.8	5.4	37.1	5.6	382.2	2.7	4627.0

Table 3: Evaluation of the constants related to the convergence analysis ($h = 1/32$).

Naive attempts

Introduction of anisotropy in \hat{C} causes the method to break down,

- 1 Adaptation of the aggregation at coarse levels:
 - boxwise aggregation (to mitigate the anisotropy)
 - orthogonal linewise aggregation (to counter anisotropy)

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A mediocre coarse grid correction can still lead to a viable multigrid method with a more involved smoother

- 2 Vanka or Braess-Sarrasin smoothers
- 3 Go back to theory

More involved smoothers

Jacobi-type smoother:

$$\begin{pmatrix} D_u & \\ & M_p \end{pmatrix}$$

- $D_u = \text{diag}(F)$
- M_p : approximation of the pressure block by the AGMG preconditioner
 - δ_p should be close to 1 for all ν
 - γ_p stays bounded for all ν

$\theta \backslash \nu$		$h = 1/64$				$h = 1/128$			
		10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-2}	10^{-4}	10^{-6}	10^{-8}
0		80	71	72	71	89	113	113	113
$\pi/16$		69	63	63	63	105	86	85	85
$\pi/4$		74	69	69	69	104	97	97	97

Table 4: Constant convection flow: number of **multigrid** iterations until $\|\mathbf{r}_{\text{rel}}\| \leq 10^{-6}$.

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Introduction

Proposed approach

Brief theoretical overview

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Conclusions



Conclusions and perspectives

Conclusions:

- Theoretical analysis of the convergence (**in norm**) for Oseen problems
- **Robustness** of the resulting two-grid method on recirculating flows
- Transition to **multigrid** NOT straightforward

Perspectives:

- Two-grid as a **tool** as a building block
- Full **algebraization** of the method
- Application to Navier-Stokes equations (Picard linearization).



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Questions?