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Record statistics of integrated random walks and the random acceleration process

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Abstract.

We address the theory of records for integrated random walks with finite variance. The long-time continuum limit of these walks is a non-Markov process known as the random acceleration process or the integral of Brownian motion. In this limit, the renewal structure of the record process is the cornerstone for the analysis of its statistics. We thus obtain the analytical expressions of several characteristics of the process, notably the distribution of the total duration of record runs (sequences of consecutive records), which is the continuum analogue of the number of records of the integrated random walks. This result is universal, i.e., independent of the details of the parent distribution of the step lengths.

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1. Introduction

An observation in a time series is called an (upper) record if it is greater than all previous observations in the series. Two simple situations arise either when the observations are independent identically distributed (iid) random variables $\eta_1, \eta_2, \dots, \eta_n$, drawn from a given continuous distribution Φ , or when they are the successive positions of a random walk V_1, V_2, \dots, V_n built from the previous variables, with

$$V_n - V_{n-1} = \eta_n,$$

hence

$$V_n = V_0 + \eta_1 + \dots + \eta_n, \quad (1.1)$$

where the number n of steps is a discrete time. The number M_n of records scales as $\ln n$ in the first case, independently of the choice of the step length distribution Φ , and as \sqrt{n} in the second case, again independently of this distribution, if, e.g., the latter is symmetric.‡

Consider now the *integrated random walk* X_1, X_2, \dots, X_n defined as

$$X_n - X_{n-1} = V_n,$$

or

$$X_n - 2X_{n-1} + X_{n-2} = \eta_n,$$

hence

$$X_n = X_0 + V_1 + \dots + V_n. \quad (1.2)$$

While the V_n process is Markovian, the X_n process does not possess this property. Nonetheless the couple (V_n, X_n) is Markovian, since at each step its evolution is entirely determined by the noise η_n , as the following recursion shows

$$(V_0, X_0) \xrightarrow{\eta_1} (V_1, X_1) \xrightarrow{\eta_2} (V_2, X_2) \xrightarrow{\eta_3} \dots$$

The aim of the present work is to investigate the statistics of records for the integrated random walk X_n . We assume henceforward that the step length distribution Φ is symmetric with finite variance $\langle \eta^2 \rangle = 2D$, i.e., such that the random walk V_n is diffusive, with diffusion coefficient D . For this class of distributions, asymptotic properties of the random walk V_n and of the integrated random walk X_n are described by their continuum analogues, Brownian motion, and the integral of Brownian motion—or random acceleration process—respectively. As will be seen below, the number M_n of records for the integrated process scales as n . We shall focus our attention on the exact asymptotic distribution of this number of records, and of related observables, in the limit of long times.

A sample path of such an integrated random walk X_n is depicted in figure 1 and the corresponding path of the random walk V_n is depicted in figure 2. These figures reveal some salient features. In figure 1 full symbols are the successive records of X_n . Record runs (sequences of consecutive records) begin at a green dot and end at a red one. In figure 2 red dots are marked at the same epochs as the red dots of figure 1. They correspond to the beginnings of downcrossing steps. Thus, at red dots the process (V_n, X_n) *almost* restarts afresh. This renewal structure is indeed only

‡ Further details and a comprehensive bibliography are presented a little further down in this introduction.

confirming the $\ln n$ scaling mentioned earlier.

Records for one-dimensional random walks constitute a second facet of the theory of records, whose developments ran parallel to and independently from the studies of records for iid random variables. A remarkable historical coincidence, which seemingly has gone unnoticed so far, is that the paper of Chandler [1] on records for iid random variables and the paper by Blackwell [9], which laid the foundations of the study of records for random walks, were both simultaneously presented for publication in June 1952. Blackwell introduced the times of occurrence and values of records, ‘*aptly christened the ladder random variables of the random walk*’ by Feller in [10], to quote Spitzer [11].

As a matter of fact, looking at the occurrences of ‘*record values*’ in the monographs of Feller [10, 12], it is found that this terminology refers, on the one hand, as expected, to the simple situation of iid random variables described above, but also, on the other hand, to the *ladder points* of one-dimensional random walks. To quote Feller [12, Ch. XII], ‘*Looking at the graph of a random walk one notices as a striking feature the points where S_n [the position of the walk after n steps, denoted as V_n above] reaches a record value, that is, where S_n exceeds all previously attained values S_0, \dots, S_{n-1} . These are the ladder points (...). The theoretical importance of ladder points derives from the fact that the sections between them are probabilistic replicas of each other, and therefore important conclusions concerning the random walk can be derived from a study of the first ladder point*’. As emphasised by Spitzer [11], it was in connection with renewal theory that ladder random variables were first studied by Blackwell. Renewal theory [10, 12, 13, 14] is precisely the cornerstone for the investigation of the statistics of records—or ladder points—for random walks, as the second sentence of Feller quoted above suggests, and as is amply detailed in chapters XII and XVIII of [12].

All the tools necessary to investigate the statistics of records for random walks are contained in [12]. In a nutshell, the distribution $f(n)$ of waiting times between two records is given by Sparre Andersen theory [15, 16, 12]. This distribution is universal, i.e., independent of the parent distribution Φ , provided the latter is continuous and symmetric. Hence the distribution of the number of records M_n after n steps of the walk (or in other words the number of renewals up to time n), is universal, too. The expression of its generating function

$$\sum_{n \geq 0} \mathbb{P}(M_n = m) z^n = \tilde{f}(z)^{m-1} \frac{1 - \tilde{f}(z)}{1 - z}, \quad (1.5)$$

(where $\tilde{f}(z) = 1 - \sqrt{1 - z}$ is the generating function of the $f(n)$) is an immediate consequence of the renewal structure of the sequence of records (or ladder points) (see, e.g., [17, § 3], [18], [19, § 3]). From (1.5) the mean number of records ensues easily,

$$\langle M_n \rangle \underset{n \rightarrow \infty}{\approx} \frac{2\sqrt{n}}{\sqrt{\pi}},$$

as well as the expression of $\mathbb{P}(M_n = m)$, whose asymptotic scaling form reads

$$\mathbb{P}(M_n = m) \underset{n \rightarrow \infty}{\approx} \frac{1}{\sqrt{n}} g\left(\frac{m}{\sqrt{n}}\right), \quad g(x) = \frac{e^{-x^2/4}}{\sqrt{\pi}} \quad (x \geq 0),$$

(see, e.g., [12, page 373], [17, § 3], [18], [19, § 3]), confirming the \sqrt{n} scaling mentioned earlier. The number of records can still be expressed as the sum (1.3), however now

the indicator variables I_1, I_2, \dots are no longer independent. The probability r_n of occurrence of a record at time n is still equal to $\langle I_n \rangle$, with the explicit expression [19]

$$r_n = q(n) = \frac{1}{2^{2n}} \binom{2n}{n} \underset{n \rightarrow \infty}{\approx} \frac{1}{\sqrt{\pi n}},$$

where $q(n)$, the probability that the random walk, starting at the initial position V_0 , stays below V_0 up to step n , is related to the first passage probability $f(n)$ by $f(n) = q(n-1) - q(n)$.

More recently, investigations on the theory of records have permeated the physics community as being part of the broader field of extremal events, of natural interest in physics. Records can indeed be seen as extremal events monitored in time. We refer the reader to [19, 20] for an overview of the recent applications of the theory of records in statistical physics. In particular the theory of records for one-dimensional random walks has been revisited and enriched in a series of papers in the past two decades (see [19] and the references therein).

In contrast, to our knowledge, essentially nothing is known so far on the statistics of records for the integrated random walk or for the random acceleration process. These processes are known to be appreciably more difficult to study than random walks. A number of advances have nevertheless been made in the past or more recently on various aspects of these processes, both in mathematics [21, 22, 23, 24, 25, 26] and in physics, in particular on first-passage properties, inelastic collapse of particles, time at which the maximum is reached, statistics of the occupation time, dynamics with resetting [27, 28, 29, 30, 31, 32, 33] (see the recent review [34] for additional references).

The question of analysing the statistics of records for the integrated random walk was raised in [19] but left unanswered. It is the purpose of the present work to fill this gap, at least for diffusive random walks, where the variance of the step lengths is finite. We start with a summary of our findings.

Summary of the results of the present work

We shall show that for the random acceleration process, which is the continuum analogue of the integrated random walks defined above, the process of records has a renewal structure involving an infinite sequence of iid couples of intervals of time (τ_n, δ_n) , whose joint law is given in (2.13). As depicted in figure 3, the endpoints of the time intervals τ_n (respectively, δ_n) are marked by green dots (respectively, red dots). A record run (a sequence of consecutive records) starts at every green dot and stops at the next red one. This renewal structure—foreshadowed by the discrete process—is the cornerstone for the analysis that follows. Henceforth we shall call the red dots renewal events (or renewals for short).

The simplest observable to consider is the number N_t of renewals, or equivalently of records runs, occurring between 0 and t . The determination of its distribution is a classic in renewal theory (see, e.g., [12, 17]). In particular its average grows as

$$\langle N_t \rangle \approx A t^{1/4},$$

where the exponent 1/4 is the well-known persistence exponent of the random acceleration process [22, 25, 27], whereas the non-universal prefactor A depends on the parent distribution Φ , as demonstrated in Table 2.

As previously stated, M_n , the number of records up to time n in the discrete theory, grows linearly with n . More precisely, its distribution scales as

$$\mathbb{P}(M_n = m) \underset{n \rightarrow \infty}{\approx} \frac{1}{n} f_R \left(\frac{m}{n} \right), \quad (1.6)$$

where the universal scaling function f_R is the probability density of the ratio

$$R = \lim_{n \rightarrow \infty} \frac{M_n}{n} = \lim_{t \rightarrow \infty} \frac{S_t}{t}, \quad (1.7)$$

in which S_t , the sum of all (finished or not finished) record runs, is the continuum analogue of M_n . Thus R is the fraction of time during which the process sets a record. The density $f_R(x)$ is an asymmetric U-shaped curve depicted in figure 6, whose analytic expression is given in (5.13). Its first moment

$$\langle R \rangle = r_\infty = 1 - \frac{\sqrt{6}}{4} = 0.387627\dots \quad (1.8)$$

is a universal number, whose natural interpretation is the limit, when the discrete time $n \rightarrow \infty$, of the probability r_n of occurrence of a record at n , or probability of record breaking, defined as for the cases of iid random variables or random walks,

$$r_n = \mathbb{P}(X_n > \max(X_1, \dots, X_{n-1})). \quad (1.9)$$

In other words,

$$\langle R \rangle = \lim_{n \rightarrow \infty} \frac{\langle M_n \rangle}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n r_i = r_\infty. \quad (1.10)$$

As a consequence, the mean number of records grows linearly as

$$\langle M_n \rangle \underset{n \rightarrow \infty}{\approx} n r_\infty.$$

We complement this study by the determination of the distributions of the epochs T_{N_t} , of the last renewal before t , and \tilde{T}_{N_t} , of the last dot before t , regardless of its colour. Both observables again grow linearly with time. The densities $f_U(x)$ and $f_V(x)$ of the corresponding limiting ratios

$$U = \lim_{t \rightarrow \infty} \frac{T_{N_t}}{t}, \quad V = \lim_{t \rightarrow \infty} \frac{\tilde{T}_{N_t}}{t},$$

given in (6.4) and (6.12) and depicted in figure 7, are universal. Armed with this knowledge we recover, by a different method, a result given in [31] on the time t_m for the random acceleration process to reach its maximum. The density of the rescaled time

$$W = \lim_{t \rightarrow \infty} \frac{t_m}{t}$$

has a universal expression given in (6.14), involving r_∞ and the density f_U , which sheds new light on the result given in [31].

A last remark is in order. In the present study, universality for integrated random walks only manifests itself asymptotically. This is particularly evident when considering the probability of record breaking r_n , defined in (1.9). Except for $r_1 = \mathbb{P}(X_1 > 0) = 1/2$, the probability r_n of having a record at any finite n is non universal. For instance, for $n = 2$, $X_2 = V_1 + V_2 = 2\eta_1 + \eta_2$, thus, using symmetries, we have

$$r_2 = \frac{1}{2} - \mathbb{P}(\eta_1 > 0, \eta_1 < \eta_2 < 2\eta_1). \quad (1.11)$$

This probability is non-universal, as demonstrated in Table 1. Universality is reached asymptotically, i.e., $r_n \rightarrow r_\infty$.

The paper is structured as follows. Section 2 gives preliminaries on the random acceleration process, with emphasis on the distributions of three important random

distribution	r_2
uniform	$7/16 = 0.4375$
triangular	$43/96 = 0.447916 \dots$
exponential	$11/24 = 0.458333 \dots$
Gaussian	$5/8 - (\arctan 2)/(2\pi) = 0.448791 \dots$
binary	$1/4 = 0.25$

Table 1. Exact value of the probability r_2 of having a record at time $n = 2$ for various symmetric step length distributions Φ : uniform, triangular (the law of the sum of two uniform variables), exponential, Gaussian and binary (± 1). The latter distribution, besides the fact that it is not continuous, hence that (1.11) does not hold, appears as an outlier, inasmuch as the value $r_2 = \mathbb{P}(\eta_1 = \eta_2 = +1) = 1/4$ is quite different from those for the continuous distributions, which vary over a rather narrow range.

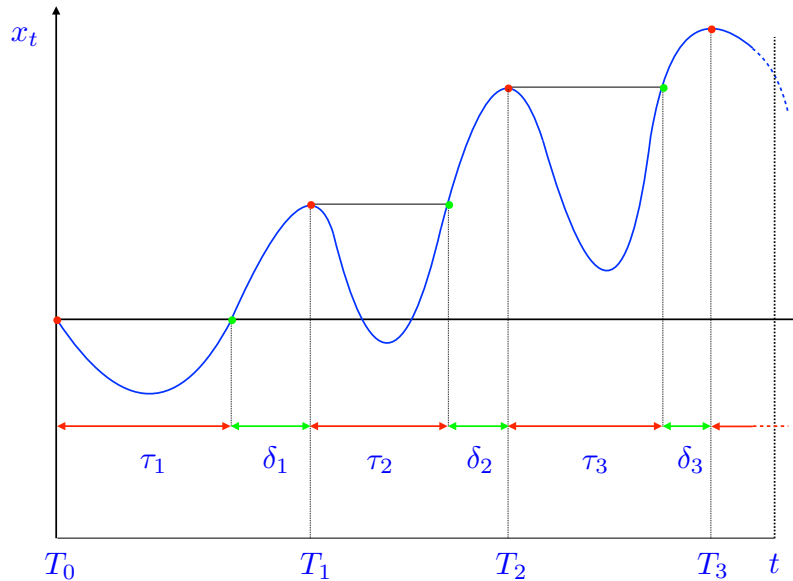


Figure 3. Schematic drawing of a path of the random acceleration process. Sections between two red dots are probabilistic replicas of each other. In each section, records for x_t take place between a green and a red dot—this defines the lengths δ_n of the record runs. In the present example, discarding the red dot at the origin, the number N_t of renewals (red dots) between 0 and t is equal to 3.

variables, depicted in figure 4. Section 3 highlights the renewal structure underlying the process of records in the random acceleration process. Section 4 provides an analytical treatment of the distribution of the number of renewals. Section 5, which is the main section, gives the exact distribution of the number of records in the

asymptotic regime. Section 6 gives the distributions of the epochs T_{N_t} and \tilde{T}_{N_t} and the distribution of the time at which the random acceleration process reaches its maximum. We discuss some possible extensions in section 7. Three appendices contain more technical matters.

2. Elements on the random acceleration process

Throughout this work, asymptotic analytical results on the statistics of records of integrated random walks with finite variance will be obtained using their continuum limit, the random acceleration process.

2.1. Definition of the process

Consider a particle submitted to a random force, whose position x_t obeys the stochastic equation of motion

$$\frac{d^2 x_t}{dt^2} = \eta_t, \quad (2.1)$$

where η_t is a normalised Gaussian white noise, i.e.,

$$\langle \eta_t \rangle = 0, \quad \langle \eta_t \eta_{t'} \rangle = \delta(t - t').$$

This is the original Langevin equation without damping force. Equivalently, the position x_t of the particle and its velocity v_t jointly obey the first-order (i.e., Markovian) dynamics

$$\frac{dv_t}{dt} = \eta_t, \quad \frac{dx_t}{dt} = v_t, \quad (2.2)$$

with initial condition (v_0, x_0) . Hence

$$v_t = v_0 + W_t, \quad x_t = x_0 + v_0 t + \int_0^t du W_u,$$

where the integral of the noise,

$$W_t = \int_0^t du \eta_u$$

is normalised Brownian motion, such that $D = 1/2$. The process x_t is usually referred to as the *integral of Brownian motion* or the *random acceleration process*. We have

$$\langle v_t \rangle = v_0, \quad \langle x_t \rangle = x_0 + v_0 t, \quad \text{Var } v_t = \langle W_t^2 \rangle = t, \quad \text{Var } x_t = \frac{t^3}{3}.$$

The fluctuating parts are the leading ones at long times, so that v_t and x_t respectively grow as $t^{1/2}$ and as $t^{3/2}$.

2.2. Some preliminary results

We gather here some results that we shall need in the sequel. Consider the randomly accelerated particle starting from the origin with initial velocity v_0 . From now on, v_0 will represent an initial microscopic velocity acting as a cutoff, the role of which is to regularise the theory, as is necessary when dealing, for example, with first-passage observables in Brownian motion (see section 3.3 for a detailed discussion). We shall henceforth use the notation

$$\varepsilon = \sqrt{|v_0|}. \quad (2.3)$$

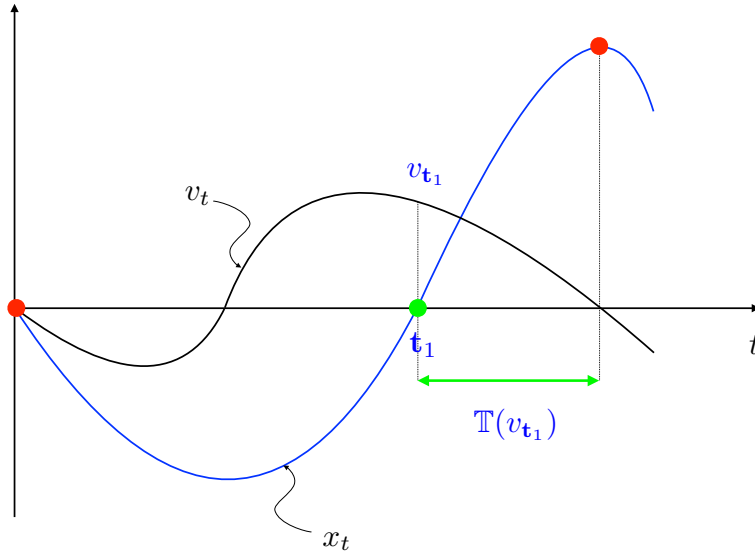


Figure 4. Three fundamental random variables. Epoch of the first zero crossing t_1 of x_t ; corresponding velocity v_{t_1} of the randomly accelerated particle (or position of Brownian motion); first hitting time $\mathbb{T}(v_{t_1})$ of the origin for Brownian motion starting from v_{t_1} at time $t = 0$. (Schematic drawing.)

We shall also restrict ourselves to paths of x_t with v_0 negative, as in figure 4, which, as explained later, is a simplification for the analysis which follows.

As will be made clear in the next section, there are three fundamental random variables to consider for the sequel (see figure 4).

- The first one is the time of occurrence of the first zero crossing t_1 of x_t .
- The second one is the corresponding velocity v_{t_1} of the randomly accelerated particle (or position of Brownian motion).
- The third one is the first hitting time $\mathbb{T}(v_{t_1})$ of the origin for Brownian motion starting from v_{t_1} at time $t = 0$. Note that this random variable depends itself on another random variable (namely v_{t_1}). This is also the time taken by x_t to go from the green dot to the red one, starting from $x_{t_1} = 0$ (see figure 4).

The expression of the joint distribution of the epoch t_1 and of the corresponding velocity v_{t_1} is a classical result due to McKean [21, 24] which states that, taking $v_0 < 0$ and $v > 0$ as in figure 4,

$$f_{t_1, v_{t_1}}(\tau, v) = \frac{\sqrt{3}v}{\pi\tau^2} e^{-2(v^2 - |v_0|v + v_0^2)/\tau} \operatorname{erf} \sqrt{\frac{6|v_0|v}{\tau}}. \quad (2.4)$$

The marginal distribution of v_{t_1} ensues by integration of (2.4) on τ :

$$f_{v_{t_1}}(v) = \frac{3\varepsilon}{2\pi} \frac{v^{3/2}}{v^3 + |v_0|^3}. \quad (2.5)$$

There is no explicit expression of the marginal distribution of \mathbf{t}_1 , henceforth denoted for short as

$$\rho(\tau) \equiv f_{\mathbf{t}_1}(\tau),$$

however, in the regime where both τ and v are large, such that $|v_0| \ll \tau \sim v^2$, (2.4) simplifies to

$$f_{\mathbf{t}_1, v_{\mathbf{t}_1}}(\tau, v) \approx \frac{6\sqrt{2}\varepsilon}{\pi^{3/2}} \frac{v^{3/2}}{\tau^{5/2}} e^{-2v^2/\tau}. \quad (2.6)$$

Thus, by integration on v , the asymptotic expression of the marginal $\rho(\tau)$ ensues

$$\rho(\tau) \underset{\tau \rightarrow \infty}{\approx} \frac{c}{\tau^{5/4}}, \quad c = \frac{3\Gamma(1/4)\varepsilon}{2^{11/4}\pi^{3/2}}. \quad (2.7)$$

As for the third variable, we have the well-known result for the probability density of the first hitting time $\mathbb{T}(v)$ of the origin for Brownian motion starting from $v > 0$ [12],

$$f_{\mathbb{T}(v)}(\delta) = \frac{v e^{-v^2/(2\delta)}}{\sqrt{2\pi\delta^3}}, \quad (2.8)$$

which thereby provides the expression of the conditional probability density of the first hitting time $\mathbb{T}(v_{\mathbf{t}_1})$ of the origin for Brownian motion starting from $v_{\mathbf{t}_1} = v$,

$$f_{\mathbb{T}(v_{\mathbf{t}_1})|v_{\mathbf{t}_1}}(\delta|v) = \frac{v e^{-v^2/(2\delta)}}{\sqrt{2\pi\delta^3}}. \quad (2.9)$$

Using (2.5) and (2.9), the density $f_{\mathbb{T}(v_{\mathbf{t}_1})}(\delta)$ is obtained in the form of the integral of the product

$$f_{\mathbb{T}(v_{\mathbf{t}_1})}(\delta) = \int dv f_{\mathbb{T}(v_{\mathbf{t}_1})|v_{\mathbf{t}_1}}(\delta|v) f_{v_{\mathbf{t}_1}}(v) = \frac{1}{\delta} g\left(\frac{\varepsilon}{\delta^{1/4}}\right), \quad (2.10)$$

which leads to an explicit albeit complicated expression of $g(x)$. For $x \rightarrow 0$ this function behaves as

$$g(x) \underset{x \rightarrow 0}{\approx} \frac{3\Gamma(1/4)}{2^{9/4}\pi^{3/2}} x,$$

which implies that

$$f_{\mathbb{T}(v_{\mathbf{t}_1})}(\delta) \underset{\delta \rightarrow \infty}{\approx} \frac{c\sqrt{2}}{\delta^{5/4}}. \quad (2.11)$$

Note that while the tail index of the conditional density (2.9) (where $v_{\mathbf{t}_1} = v$ has a fixed value) is equal to $1/2$, the tail index of the density (2.10) of the random variable $\mathbb{T}(v_{\mathbf{t}_1})$ is equal to $1/4$.

Anticipating on what follows, we shall be interested in the joint density of \mathbf{t}_1 and $\mathbb{T}(v_{\mathbf{t}_1})$. This density, denoted for short by $\rho(\tau, \delta)$, is obtained by integration of the product of (2.4) and (2.9):

$$\rho(\tau, \delta) \equiv f_{\mathbf{t}_1, \mathbb{T}(v_{\mathbf{t}_1})}(\tau, \delta) = \int dv f_{\mathbb{T}(v_{\mathbf{t}_1})|v_{\mathbf{t}_1}}(\delta|v) f_{\mathbf{t}_1, v_{\mathbf{t}_1}}(\tau, v). \quad (2.12)$$

Its asymptotic expression can be obtained by multiplying (2.6) by (2.9) and integrating upon v , which leads to the scaling form

$$\rho(\tau, \delta) \approx \frac{9\Gamma(3/4)\varepsilon}{2^{1/4}\pi^2} \frac{\delta^{1/4}}{\tau^{3/4}(\tau + 4\delta)^{7/4}}. \quad (2.13)$$

This expression is a key ingredient in all subsequent developments.

The marginals ensuing from (2.13) by integration upon each variable yield (2.7) and (2.11) back. The intervals of time \mathbf{t}_1 and $\mathbb{T}(v_{\mathbf{t}_1})$ separately have heavy-tailed distributions with tail coefficients proportional to ε and the same tail index $1/4$. The latter is the well-known survival (or persistence) exponent of the random acceleration process [21, 22, 25, 27], i.e., the decay exponent of the survival probability, or probability that the process x_t has not returned to its starting point $x = 0$ until time t ,

$$p(t) = \mathbb{P}(\mathbf{t}_1 > t) = \int_t^\infty d\tau \rho(\tau), \quad (2.14)$$

falling off as

$$p(t) \underset{t \rightarrow \infty}{\approx} \frac{4c}{t^{1/4}}.$$

Thus, in Laplace space,

$$\mathcal{L}_t p(t) = \hat{p}(s) \underset{s \rightarrow 0}{\approx} \frac{a}{s^{3/4}}, \quad (2.15)$$

with

$$a = 4\Gamma(3/4)c = \frac{3\varepsilon}{2^{1/4}\sqrt{\pi}}, \quad (2.16)$$

from which the scaling form of $\hat{p}(s)$ follows

$$1 - \hat{p}(s) = s\hat{p}(s) \approx as^{1/4}. \quad (2.17)$$

As a matter of interest, let us remark that the expression of the survival probability (2.14) has an explicit integral representation in Laplace space. It reads (see, e.g., [24, 30])

$$\hat{p}(s) = \frac{1}{s} \left(1 - \frac{3}{\pi} \int_0^\infty du \cosh u \frac{\sinh(3u/2)}{\sinh 3u} \cos \left(v_0 \sqrt{8s} \sinh u \right) \right).$$

For $s \rightarrow 0$ this expression yields back (2.15), (2.16) [30].

3. Renewal structure and observables of interest

3.1. Renewal structure for records

A schematic drawing of a path of the process is depicted in figure 3. This path is the continuum analogue of the path depicted in figure 1. Two series of conspicuous points are represented by red and green dots. By convention, the origin is marked as a red dot. The first green dot corresponds to the first crossing of the origin. The following red dot is the first maximum of the path beyond this first green dot. Then, considering the latter as the new origin, the process starts afresh. In other words, sections between two red dots are probabilistic replicas of each other. *Record runs*, that is sequences of consecutive records of the discrete process, translate into parts of these sections comprised between green and red dots. This renewal structure will be the basis of all further considerations. The red dots are renewal events (or renewals for short).

This can be formalised as follows (see figure 4). The randomly accelerated particle is launched from the origin with a negative initial velocity $v_0 = -\varepsilon^2$, where ε is a

microscopic cutoff (see (2.3) and section 3.3). The first section is made of the following sequence of points (v_t, x_t) :

$$(v_0, x_0 = 0) \rightarrow \underbrace{(v_{\mathbf{t}_1}, x_{\mathbf{t}_1} = 0)}_{\text{green}} \rightarrow \underbrace{(v_{\mathbf{t}_1 + \mathbb{T}(v_{\mathbf{t}_1)}}, x_{\mathbf{t}_1 + \mathbb{T}(v_{\mathbf{t}_1)}) = 0}_{\text{red}}, \quad (3.1)$$

where, as defined earlier, \mathbf{t}_1 is the epoch of first passage by the origin of x_t and $\mathbb{T}(v_{\mathbf{t}_1})$ is the hitting time of the origin for Brownian motion starting at $v_{\mathbf{t}_1}$. This velocity is positive, and typically much larger than the microscopic initial velocity v_0 . The last two points of the above sequence correspond respectively to the first green dot and to the following red dot. This section is then repeated, i.e., the process starts afresh, the particle is launched with initial velocity $v_0 = -\varepsilon^2$ from the last red dot considered as the new origin.

This scheme justifies in retrospect our choice of a negative initial velocity $v_0 = -\varepsilon^2$ at $t = 0$, as mentioned above. Had we taken another prescription at $t = 0$, this would have only changed the distribution of the first time interval (between $t = 0$ and the first red dot), keeping the distribution of all subsequent couples (τ_n, δ_n) unchanged. Changing the (somewhat arbitrary) initial conditions of the continuum process would therefore only induce additional corrections to scaling falling off as $1/N_t \sim t^{-1/4}$ in relative value.

We shall denote by τ_1, τ_2, \dots the successive copies of \mathbf{t}_1 and by $\delta_1, \delta_2, \dots$ the successive copies of $\mathbb{T}(v_{\mathbf{t}_1})$. We can view the process as being in one of two states: *on* or *off*. Initially it is *off*, and it remains so for a time τ_1 ; it then goes *on* and remains so for a time δ_1 ; and so forth. The sequence of τ_n gives the lengths of the no-record runs, or *off* states, while the sequence of δ_n gives the lengths of the record runs, or *on* states. Green and red dots signal the switchover points of the process from one state to the other.

To summarise, the cornerstone of the analysis that follows is the renewal structure of the sequence of iid couples of intervals of time (τ_n, δ_n) whose common density $\rho(\tau, \delta)$ is given in (2.13). Finally, we shall denote the waiting times between two renewals (that is, the duration of the sections between two red dots) as $\sigma_n = \tau_n + \delta_n$.

3.2. Factorisation of the joint distribution $\rho(\tau, \delta)$ in the asymptotic regime

The expression (2.13) of the density $\rho(\tau, \delta)$ actually exhibits a stronger form of scaling, besides the power laws derived above. This expression is indeed a homogeneous function of its arguments τ and δ . In other words, \mathbf{t}_1 and the dimensionless ratio

$$Z = \frac{\mathbb{T}(v_{\mathbf{t}_1})}{\mathbf{t}_1}$$

become asymptotically independent as \mathbf{t}_1 gets larger and larger. Their joint law reads

$$f_{\mathbf{t}_1, Z}(\tau, z) \approx \rho(\tau) f_Z\left(z = \frac{\delta}{\tau}\right), \quad (3.2)$$

with

$$f_Z(z) = \frac{12 \Gamma(3/4)^2}{\pi^{3/2}} \frac{z^{1/4}}{(1 + 4z)^{7/4}}. \quad (3.3)$$

The latter distribution is normalised, as should be.

In Laplace space, the transform $\hat{\rho}(s, u)$ of the joint density $\rho(\tau, \delta)$ has a scaling form which can be derived as follows. We have

$$\begin{aligned}
 1 - \hat{\rho}(s, u) &= \int_0^\infty d\tau \int_0^\infty d\delta \rho(\tau, \delta) (1 - e^{-(s\tau + u\delta)}) \\
 &\approx \int_0^\infty dz f_Z(z) \int_0^\infty d\tau \rho(\tau) (1 - e^{-(s+uz)\tau}) \\
 &\approx \int_0^\infty dz f_Z(z) (s + uz) \int_0^\infty d\tau p(\tau) e^{-(s+uz)\tau} \\
 &\approx \int_0^\infty dz f_Z(z) (s + uz) \hat{p}(s + uz) \\
 &\approx a \int_0^\infty dz f_Z(z) (s + uz)^{1/4}.
 \end{aligned}$$

The second line is derived from (3.2), the third one by means of an integration by parts, and the fifth one by substituting $s + uz$ for s in (2.15).

Introducing the dimensionless ratio

$$\xi = \frac{u}{s}, \quad (3.4)$$

we finally obtain

$$1 - \hat{\rho}(s, u) \approx a s^{1/4} h(\xi), \quad (3.5)$$

with

$$h(\xi) = \int_0^\infty dz f_Z(z) (1 + \xi z)^{1/4} = \langle (1 + \xi Z)^{1/4} \rangle. \quad (3.6)$$

This function admits the closed-form expression (see (C.2))

$$h(\xi) = \frac{1 + \sqrt{\xi}}{\sqrt{1 + \frac{1}{2}\sqrt{\xi}}}. \quad (3.7)$$

It is an algebraic function of degree four, obeying the biquadratic equation (see (C.3))

$$(\xi - 4)h^4 + 8h^2 - 4(\xi - 1)^2 = 0. \quad (3.8)$$

A first consequence of the above is that the common distribution $f(\sigma)$ of the total waiting times $\sigma_n = \tau_n + \delta_n$ is also heavy-tailed with tail index $1/4$, and amplitude proportional to ε . Its Laplace transform indeed reads

$$\hat{f}(s) = \hat{\rho}(s, s), \quad (3.9)$$

thus from (3.5) we get

$$1 - \hat{\rho}(s, s) \approx a h(1) s^{1/4}, \quad (3.10)$$

and therefore

$$f(\sigma) \approx \frac{c h(1)}{\sigma^{5/4}}, \quad (3.11)$$

with (see (1.8), (5.10)),

$$h(1) = \langle (1 + Z)^{1/4} \rangle = \frac{2\sqrt{6}}{3} = \frac{1}{1 - r_\infty}. \quad (3.12)$$

The tail index $1/4$ of this distribution is the same as that of the distributions of the intervals τ_n and δ_n , i.e., it is the persistence exponent of the random acceleration

process. Since this index is less than unity, the first moment of $f(\sigma)$ is divergent, hence the renewal process built upon the waiting times σ_n does not equilibrate, but rather keeps a sensitive memory of its initial state. The same holds for the complete renewal process built upon the couples (τ_n, δ_n) . As a consequence, a large class of observables (see, e.g., section 3.4) are scale invariant.

3.3. On the role of the cutoff

Let us come back to the prescription which consists in launching the random acceleration process from $x_0 = 0$ with an initial microscopic velocity $v_0 = -\varepsilon^2$ both initially and at every red dot, where ε is the cutoff defined in (2.3).

The integrated random walk process breaks the continuum scale invariance of the random acceleration process, if only because it is defined at discrete integer times n . In order to get meaningful predictions from the continuum theory, one must therefore break scale invariance by introducing a microscopic scale, be it either spatial, temporal, or both. A minimal prescription consists in imposing an initial velocity v_0 . This is manifest in the expression (2.4) of the joint law of τ_1 and v_1 . This law degenerates to $\delta(\tau)\delta(v)$ in the $v_0 \rightarrow 0$ limit.

This is also manifest in the expression (2.8) of the density of the first hitting time $\mathbb{T}(v)$ of the origin for the Brownian velocity. Intuitively, if the particle was launched at the origin with zero velocity, it would cross the origin almost immediately after, hence the hitting time of the origin could not be finite.

In some sense, the cutoff ε makes the connection between the discrete and continuum formalisms. It enters the tail parameters of all power-law distributions: (2.6), (2.7), (2.11), (2.13), (3.11), as well as all non-universal results, such as the expression (4.4) of the mean number of records. The value of the cutoff ε appearing in these non-universal observables turns out to depend on the distribution of step lengths (see Table 2). In this respect, the situation is quite similar to that met in a recent study of the statistics of records for planar random walks [35]. There, too, it turns out to be necessary to introduce a cutoff into the continuum theory, whose numerical value is different, e.g., for lattice Polya walks and for off-lattice Pearson walks with steps of unit length.

3.4. Observables of interest

The observables studied in the sequel are defined as follows.

Number of renewals and epochs of last events. The first and simplest observable of interest, denoted by N_t , is the number of renewals (red dots in figure 3, discarding the red dot at the origin), or record runs, which occurred between 0 and t , that is the random variable for the largest n for which $T_n \leq t$, where the epoch T_n of the n -th renewal is

$$T_n = \underbrace{(\tau_1 + \delta_1)}_{\sigma_1} + \underbrace{(\tau_2 + \delta_2)}_{\sigma_2} + \cdots + \underbrace{(\tau_n + \delta_n)}_{\sigma_n}, \quad (3.13)$$

and $T_0 = 0$. For instance, $N_t = 3$ in figure 3.

The epoch of the last renewal before t , that is of the N_t -th renewal, therefore reads

$$T_{N_t} = \underbrace{(\tau_1 + \delta_1)}_{\sigma_1} + \underbrace{(\tau_2 + \delta_2)}_{\sigma_2} + \cdots + \underbrace{(\tau_{N_t} + \delta_{N_t})}_{\sigma_{N_t}}. \quad (3.14)$$

While T_n is the sum of a fixed number n of random variables σ_n , T_{N_t} is the sum of a random number N_t of such random variables.

A related observable, denoted by \tilde{T}_{N_t} , is the epoch of the last dot before t , regardless of its colour (green or red), that is, the epoch of the last change of state of the process from *off* to *on* or from *on* to *off*, or else the last endpoint of an interval τ_n or δ_n .

Number of records. Within the continuum formalism, the number M_n of records up to time n of the integrated random walk is represented by the sum S_t of all the intervals of time δ_n spent between green and red dots for $n = 1, \dots, N_t$, possibly up to a correction for the last interval. In other words, S_t is the total duration of all (complete or incomplete) record runs, or equivalently the total duration spent by the process in the *on* state.

Two cases are to be considered when dealing with the quantities S_t and \tilde{T}_{N_t} . Either t falls in the interval τ_{N_t+1} , i.e., outside a record run as in figure 3 (the process is *off*), or it falls in the interval δ_{N_t+1} , i.e., inside a record run (the process is *on*). The first case occurs with asymptotic probability $1 - r_\infty$, the second case with asymptotic probability r_∞ .

(i) In the first case, time t is between a red and a green dot, hence $T_{N_t} < t < T_{N_t} + \tau_{N_t+1}$. We have

$$\begin{aligned} S_t &= \delta_1 + \cdots + \delta_{N_t}, \\ \tilde{T}_{N_t} &= T_{N_t}. \end{aligned} \quad (3.15)$$

(ii) In the second case, time t is between a green and a red dot, hence $T_{N_t} + \tau_{N_t+1} < t < T_{N_t} + \tau_{N_t+1} + \delta_{N_t+1}$. We have

$$\begin{aligned} S_t &= \delta_1 + \cdots + \delta_{N_t} + t - T_{N_t} - \tau_{N_t+1} \\ &= t - (\tau_1 + \cdots + \tau_{N_t+1}), \\ \tilde{T}_{N_t} &= T_{N_t} + \tau_{N_t+1}. \end{aligned} \quad (3.16)$$

The distributions of these observables will be determined in the following sections 4, 5, 6.

4. Number of renewals

Following the definition given in the previous section, for a given time t , N_t is the unique integer such that $T_{N_t} \leq t < T_{N_t+1}$, with the definition (3.14). Let

$$p_n(t) = \mathbb{P}(N_t = n) = \mathbb{P}(T_n \leq t < T_{n+1}) \quad (4.1)$$

denote the probability that N_t equals some integer n . In Laplace space, we have

$$\hat{p}_n(s) = \mathcal{L}_t p_n(t) = \left\langle \int_{T_n}^{T_{n+1}} dt e^{-st} \right\rangle$$

$$\begin{aligned}
&= \left\langle \frac{1 - e^{-s(\tau_{n+1} + \delta_{n+1})}}{s} e^{-sT_n} \right\rangle \\
&= \frac{1 - \hat{\rho}(s, s)}{s} \hat{\rho}(s, s)^n,
\end{aligned} \tag{4.2}$$

which is well normalised. This expression involves the joint law $\rho(\tau, \delta)$ only through the combination $\hat{\rho}(s, s) = \hat{f}(s)$ (see (3.9)) in accordance with the fact that the time intervals between successive renewals are the total waiting times $\sigma_n = \tau_n + \delta_n$.

Let us focus our attention on the mean number $\langle N_t \rangle$ of renewals between 0 and t . Its Laplace transform reads

$$\mathcal{L}_t \langle N_t \rangle = \sum_{n \geq 0} \hat{p}_n(s) = \frac{\hat{\rho}(s, s)}{s(1 - \hat{\rho}(s, s))}. \tag{4.3}$$

Using (3.10), this reads

$$\mathcal{L}_t \langle N_t \rangle \approx \frac{1}{ah(1)s^{5/4}},$$

therefore

$$\langle N_t \rangle \approx A t^{1/4}, \tag{4.4}$$

with (see (2.16), (3.12))

$$A = \frac{1}{ah(1)\Gamma(5/4)} = \frac{2^{3/4}\sqrt{\pi}}{\underbrace{\Gamma(1/4)\sqrt{3}}_{0.474685\dots}} \frac{1}{\varepsilon}. \tag{4.5}$$

The mean number of renewals grows as a power law whose exponent $1/4$ is the tail index of the law of the waiting times σ_n . The predicted amplitude A is the ratio of a universal number by the cutoff ε . Measuring the mean number of renewals therefore gives access to the value of ε , which is expected to depend on microscopic details of the discrete process, i.e., on the parent distribution Φ of step lengths.

Figure 5 shows numerical data for the mean number $\langle N(n) \rangle$ of renewals of integrated random walks in discrete time against $n^{1/4}$, for the step length distributions already considered in Table 1: uniform, triangular, exponential, Gaussian and binary. All datasets exhibit a very accurate asymptotic linear growth as a function of $n^{1/4}$. The slopes A of least-square fits over the range $10^2 < n < 10^4$ (regression lines are not shown) and the corresponding values of the cutoff ε , according to (4.5), are given in Table 2. The binary distribution again appears as an outlier.

distribution	A	ε
uniform	0.623	0.762
triangular	0.602	0.789
exponential	0.551	0.861
Gaussian	0.592	0.801
binary	0.891	0.533

Table 2. Numerical values of the amplitude A of the power-law growth (4.4) of the mean number of renewals, as extracted from the data shown in figure 5, and of the corresponding value of the cutoff ε , according to (4.5), for various symmetric step length distributions.

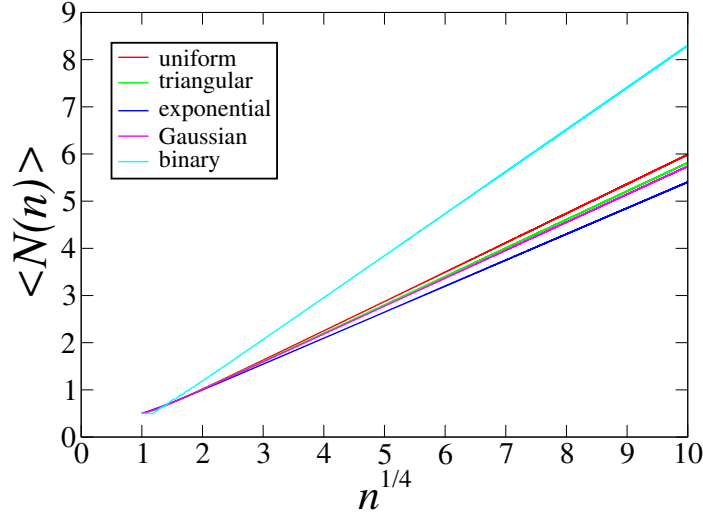


Figure 5. Mean number $\langle N(n) \rangle$ of renewals of integrated random walks in discrete time against $n^{1/4}$ up to $n = 10^4$, for various symmetric distributions of the elementary steps (see legend).

The full statistics of the number of renewals at large times can be derived from an appropriate scaling analysis of the exact expression (4.2). Omitting every detail, we obtain the following scaling formula [17]

$$N_t \approx \frac{1}{ah(1)} t^{1/4} X \approx A \Gamma(5/4) t^{1/4} X,$$

where the dimensionless reduced variable X is distributed according to the universal law

$$f_X(x) = \int \frac{dz}{2\pi i z^{3/4}} e^{z - xz^{1/4}}.$$

This probability density can be expressed as a linear combination of three hypergeometric functions of type ${}_0F_2$ [36, 37, 38]. We have the identity

$$X \equiv (L_{1/4})^{-1/4},$$

where $L_{1/4}$ is distributed according to the one-sided Lévy stable law of index 1/4 and an appropriate scale factor.

5. Number of records and total duration of record runs

The sum S_t of the durations of all (complete or incomplete) record runs, which is the continuum analogue of the number of records for integrated random walks, is the central observable of interest. The purpose of this section is the determination of the density $f_{S_t}(t, y)$ of S_t , defined as

$$\mathbb{P}(S_t \in (y, y + dy)) = f_{S_t}(t, y) dy.$$

This quantity is entirely determined by the knowledge of the density $\rho(\tau, \delta)$. In Laplace space we find that

$$\begin{aligned}\hat{f}_{S_t}(s, u) &= \mathcal{L}_t \mathcal{L}_y f_{S_t}(t, y) = \mathcal{L}_t \langle e^{-uS_t} \rangle \\ &= \frac{u(1 - \hat{\rho}(s)) + s(1 - \hat{\rho}(s, s + u))}{s(s + u)(1 - \hat{\rho}(s, s + u))},\end{aligned}\quad (5.1)$$

as we now show by considering the two cases discussed in section 3.4.

(i) In the first case, using (3.15),

$$\begin{aligned}\hat{f}_{S_t}(s, u)_{(i)} &= \sum_{n \geq 0} \mathcal{L}_t \langle e^{-uS_t} I(T_n < t < T_n + \tau_{n+1}) \rangle \\ &= \sum_{n \geq 0} \left\langle e^{-u(\delta_1 + \dots + \delta_n)} \int_{T_n}^{T_n + \tau_{n+1}} dt e^{-st} \right\rangle \\ &= \sum_{n \geq 0} \left\langle e^{-u(\delta_1 + \dots + \delta_n)} e^{-sT_n} \frac{1 - e^{-s\tau_{n+1}}}{s} \right\rangle \\ &= \sum_{n \geq 0} \left\langle e^{-s(\tau_1 + \dots + \tau_n)} e^{-(s+u)(\delta_1 + \dots + \delta_n)} \frac{1 - e^{-s\tau_{n+1}}}{s} \right\rangle \\ &= \sum_{n \geq 0} \hat{\rho}(s, s + u)^n \frac{1 - \hat{\rho}(s)}{s} \\ &= \frac{1}{1 - \hat{\rho}(s, s + u)} \frac{1 - \hat{\rho}(s)}{s}.\end{aligned}\quad (5.2)$$

In the first line $I(\cdot)$ is the indicator function of the event in the parentheses.

(ii) In the second case, using (3.16),

$$\begin{aligned}\hat{f}_{S_t}(s, u)_{(ii)} &= \sum_{n \geq 0} \mathcal{L}_t \langle e^{-uS_t} I(T_n + \tau_{n+1} < t < T_n + \tau_{n+1} + \delta_{n+1}) \rangle \\ &= \sum_{n \geq 0} \left\langle e^{u(\tau_1 + \dots + \tau_{n+1})} \int_{T_n + \tau_{n+1}}^{T_n + \tau_{n+1} + \delta_{n+1}} dt e^{-(s+u)t} \right\rangle \\ &= \sum_{n \geq 0} \hat{\rho}(s, s + u)^n \frac{\hat{\rho}(s) - \hat{\rho}(s, s + u)}{s + u} \\ &= \frac{1}{1 - \hat{\rho}(s, s + u)} \frac{\hat{\rho}(s) - \hat{\rho}(s, s + u)}{s + u}.\end{aligned}\quad (5.3)$$

The two expressions (5.2) and (5.3) add up to (5.1).

In order to analyse the sum S_t at long times, we consider the scaling regime where both Laplace variables s and u are small. In this regime, the expression (5.1) can be simplified by means of the estimates (2.17) and (3.5), i.e.,

$$1 - \hat{\rho}(s) \approx as^{1/4}, \quad 1 - \hat{\rho}(s, s + u) \approx as^{1/4}h(\xi + 1),$$

where $\xi = u/s$ and $h(\xi)$ is given in (3.7). We thus obtain

$$\hat{f}_{S_t}(s, u) \approx \frac{1}{s} g_R(\xi), \quad (5.4)$$

with

$$\begin{aligned} g_R(\xi) &= \frac{1}{\xi+1} \left(1 + \frac{\xi}{h(\xi+1)} \right) \\ &= \frac{1}{\xi+1} \left(1 + \left(1 - \left(1 - \frac{1}{2}\xi \right) \sqrt{\xi+1} \right)^{1/2} \right). \end{aligned} \quad (5.5)$$

It can be shown by eliminating radicals that g_R is an algebraic function of degree four, obeying

$$4(\xi+1)^3 g_R^4 - 16(\xi+1)^2 g_R^3 + 16(\xi+1) g_R^2 - (\xi-2)^2 = 0. \quad (5.6)$$

The scaling expression (5.4) is entirely parameter-free. It implies that, in the continuum theory, the asymptotic fraction of time spent during record runs (i.e., between green and red dots),

$$R = \lim_{t \rightarrow \infty} \frac{S_t}{t}, \quad (5.7)$$

has a non-trivial universal distribution, with density $f_R(x)$ defined by

$$\mathbb{P}(R \in (x, x+dx)) = f_R(x)dx \quad (0 < x < 1).$$

This random variable R is also the limiting ratio

$$R = \lim_{n \rightarrow \infty} \frac{M_n}{n}, \quad (5.8)$$

where M_n is the total number of records of the integrated random walk up to discrete time n . In particular, $\langle R \rangle = r_\infty$ (see (1.10)).

The existence of the limiting distribution $f_R(x)$ entails that

$$\begin{aligned} \hat{f}_{S_t}(s, u) &= \mathcal{L}_t \langle e^{-uS_t} \rangle \underset{s, u \rightarrow 0}{\approx} \mathcal{L}_t \langle e^{-utR} \rangle \\ &\approx \mathcal{L}_t \int_0^\infty dx e^{-utx} f_R(x) \approx \int_0^\infty dx f_R(x) \int_0^\infty dt e^{-t(s+ux)} \\ &\approx \left\langle \frac{1}{s+uR} \right\rangle. \end{aligned}$$

An identification with (5.4) yields

$$g_R(\xi) = \left\langle \frac{1}{1+\xi R} \right\rangle. \quad (5.9)$$

The explicit expression of $g_R(\xi)$ given in (5.5) allows the determination of the moments of R as well as of its full distribution (see Appendix B for details).

The moments of R are readily derived by expanding $g_R(\xi)$ as a power series. We thus obtain

$$\langle R \rangle = 1 - \frac{1}{h(1)} = 1 - \frac{\sqrt{6}}{4} = 0.387627\dots, \quad (5.10)$$

and more generally

$$\langle R^n \rangle = 1 - a_n \sqrt{6}, \quad (5.11)$$

where

$$a_0 = 0, \quad a_1 = \frac{1}{4}, \quad a_2 = \frac{7}{24}, \quad a_3 = \frac{359}{1152}, \quad a_4 = \frac{2239}{6912}, \dots$$

These positive rational numbers obey the four-term linear recursion (see (C.9))

$$(16n^2 - 9)a_{n-1} + (16n^2 + 48n + 25)a_n - 16(n+1)(5n+7)a_{n+1} + 48(n+1)(n+2)a_{n+2} = 0. \quad (5.12)$$

Using (B.3), one finds the following universal result

$$f_R(x) = \frac{\gamma_R(x)}{x^{3/4}(1-x)^{1/2}} \quad (0 < x < 1) \quad (5.13)$$

for the probability density of the asymptotic ratio R , with

$$\gamma_R(x) = \frac{1+2x}{2\pi\sqrt{2x^{3/2} + \sqrt{1+3x}}}. \quad (5.14)$$

The limiting behaviours of the density $f_R(x)$ read

$$f_R(x) \underset{x \rightarrow 0}{\approx} \frac{1}{2\pi x^{3/4}}, \quad f_R(x) \underset{x \rightarrow 1}{\approx} \frac{3}{4\pi(1-x)^{1/2}}.$$

The exponents appearing in the denominators of these expressions can be interpreted as being equal to $1 - \theta$, where θ is a persistence exponent, by analogy with the singular behaviour of the distribution of the occupation time of aging processes at its two ends [39, 40]. For $x \rightarrow 0$, $\theta = 1/4$, while for $x \rightarrow 1$, $\theta = 1/2$. The first case corresponds to the persistence exponent of the random acceleration process, the second one to that of Brownian motion.

Figure 6 shows histogram plots of the distribution of the total number of records for integrated random walks made of $n = 10^4$ steps, with symmetric uniform and exponential step distributions (see legend). Numerical data are rescaled according to (5.8). Each dataset contains 50 bins. Every second bin of each dataset is plotted alternatively. Both rescaled histograms are in excellent agreement with the theoretical prediction (5.13) (full curve).

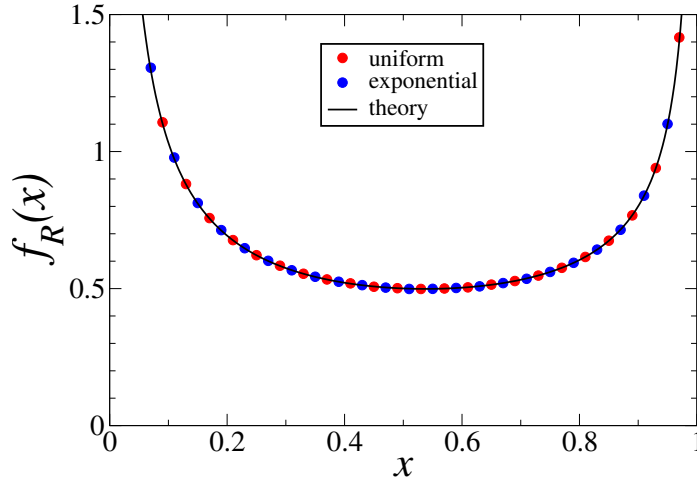


Figure 6. Symbols: histogram plots of the distribution of the total number of records for integrated random walks made of $n = 10^4$ steps, with symmetric uniform and exponential step distributions (see legend), rescaled according to (5.8). Full curve: theoretical prediction (5.13).

6. Epochs of last events

The last observables of interest are the two natural epochs introduced in section 3.4. The derivations of their distributions closely follow the approach described in section 5.

6.1. Epoch T_{N_t} of the last renewal

The definition of the epoch T_{N_t} of the last renewal (red dot) before time t does not need to distinguish the two cases discussed in section 3.4. The expression of the density in Laplace space is obtained in the manner set out above,

$$\begin{aligned}
 \hat{f}_{T_{N_t}}(s, u) &= \sum_{n \geq 0} \mathcal{L}_t \left\langle e^{-uT_n} I(T_n < t < T_{n+1}) \right\rangle \\
 &= \sum_{n \geq 0} \left\langle e^{-uT_n} \int_{T_n}^{T_{n+1}} dt e^{-st} \right\rangle \\
 &= \sum_{n \geq 0} \left\langle e^{-(s+u)T_n} \frac{1 - e^{-s(\tau_{n+1} + \delta_{n+1})}}{s} \right\rangle \\
 &= \sum_{n \geq 0} \left\langle e^{-(s+u)(\tau_1 + \delta_1 + \dots + \tau_n + \delta_n)} \frac{1 - e^{-s(\tau_{n+1} + \delta_{n+1})}}{s} \right\rangle \\
 &= \sum_{n \geq 0} \hat{\rho}(s + u, s + u)^n \frac{1 - \hat{\rho}(s, s)}{s} \\
 &= \frac{1}{1 - \hat{\rho}(s + u, s + u)} \frac{1 - \hat{\rho}(s, s)}{s}.
 \end{aligned}$$

This last expression involves the joint law $\rho(\tau, \delta)$ only through the combination $\hat{\rho}(s, s) = \hat{f}(s)$ (see (3.9)). This is again to be expected, as the time intervals between successive red dots are the total waiting times $\sigma_n = \tau_n + \delta_n$. Its asymptotic analysis proceeds as previously. In the scaling regime where s and u are small, we get

$$\hat{f}_{T_{N_t}}(s, u) \approx \frac{1}{s} g_U(\xi),$$

where $\xi = u/s$ and

$$g_U(\xi) = (\xi + 1)^{-1/4}. \quad (6.1)$$

The dimensionless ratio

$$U = \lim_{t \rightarrow \infty} \frac{T_{N_t}}{t} \quad (6.2)$$

therefore has a universal distribution, such that

$$g_U(\xi) = \left\langle \frac{1}{1 + \xi U} \right\rangle.$$

The moments of U

$$\langle U^n \rangle = \frac{\Gamma(n + 1/4)}{\Gamma(1/4)n!}, \quad (6.3)$$

are rational numbers,

$$\langle U \rangle = \frac{1}{4}, \quad \langle U^2 \rangle = \frac{5}{32}, \quad \langle U^3 \rangle = \frac{15}{128}, \quad \langle U^4 \rangle = \frac{195}{2048}, \quad \dots$$

The corresponding density reads

$$f_U(x) = \frac{1}{\pi\sqrt{2}} x^{-3/4} (1-x)^{-1/4} = \beta_{\frac{1}{4}, \frac{3}{4}}(x) \quad (0 < x < 1), \quad (6.4)$$

where

$$\beta_{a,b}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}$$

is the beta distribution on $[0, 1]$. We have thus recovered—in the present case for $\theta = 1/4$ —the beta distribution $f_U(x) = \beta_{1-\theta, \theta}(x)$ of the reduced epoch U of the last renewal for an arbitrary tail index $\theta < 1$ (see, e.g., [17]).

6.2. Epoch \tilde{T}_{N_t} of the last change of state

In Laplace space the expression of the density of \tilde{T}_{N_t} reads

$$\hat{f}_{\tilde{T}_{N_t}}(s, u) = \frac{1 - \hat{\rho}(s) + \hat{\rho}(s+u) - \hat{\rho}(s+u, s)}{s(1 - \hat{\rho}(s+u, s+u))}, \quad (6.5)$$

as we now show by considering again the two cases discussed in section 3.4.

(i) In the first case, using (3.15),

$$\begin{aligned} \hat{f}_{\tilde{T}_{N_t}}(s, u)_{(i)} &= \sum_{n \geq 0} \mathcal{L}_t \left\langle e^{-uT_n} I(T_n < t < T_n + \tau_{n+1}) \right\rangle \\ &= \sum_{n \geq 0} \left\langle e^{-uT_n} \int_{T_n}^{T_n + \tau_{n+1}} dt e^{-st} \right\rangle \\ &= \sum_{n \geq 0} \left\langle e^{-(s+u)T_n} \frac{1 - e^{-s\tau_{n+1}}}{s} \right\rangle \\ &= \sum_{n \geq 0} \left\langle e^{-(s+u)(\tau_1 + \delta_1 + \dots + \tau_n + \delta_n)} \frac{1 - e^{-s\tau_{n+1}}}{s} \right\rangle \\ &= \sum_{n \geq 0} \hat{\rho}(s+u, s+u)^n \frac{1 - \hat{\rho}(s)}{s} \\ &= \frac{1}{1 - \hat{\rho}(s+u, s+u)} \frac{1 - \hat{\rho}(s)}{s}. \end{aligned} \quad (6.6)$$

(ii) In the second case, using (3.16),

$$\begin{aligned} \hat{f}_{\tilde{T}_{N_t}}(s, u)_{(ii)} &= \sum_{n \geq 0} \mathcal{L}_t \left\langle e^{-u(T_n + \tau_{n+1})} I(T_n < t - \tau_{n+1} < T_n + \delta_{n+1}) \right\rangle \\ &= \sum_{n \geq 0} \left\langle e^{-u(T_n + \tau_{n+1})} \int_{T_n + \tau_{n+1}}^{T_n + \tau_{n+1} + \delta_{n+1}} dt e^{-st} \right\rangle \\ &= \sum_{n \geq 0} \left\langle e^{-(s+u)(T_n + \tau_{n+1})} \frac{1 - e^{-s\delta_{n+1}}}{s} \right\rangle \\ &= \sum_{n \geq 0} \left\langle e^{-(s+u)(\tau_1 + \delta_1 + \dots + \tau_n + \delta_n + \tau_{n+1})} \frac{1 - e^{-s\delta_{n+1}}}{s} \right\rangle \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n \geq 0} \hat{\rho}(s+u, s+u)^n \frac{\hat{\rho}(s+u) - \hat{\rho}(s+u, s)}{s} \\
 &= \frac{1}{1 - \hat{\rho}(s+u, s+u)} \frac{\hat{\rho}(s+u) - \hat{\rho}(s+u, s)}{s}. \tag{6.7}
 \end{aligned}$$

The two expressions (6.6) and (6.7) add up to (6.5).

The asymptotic analysis proceeds as previously. In the scaling regime where s and u are small, we get

$$\hat{f}_{\tilde{T}_{N_t}}(s, u) \approx \frac{1}{s} g_V(\xi),$$

where $\xi = u/s$ and

$$\begin{aligned}
 g_V(\xi) &= \frac{1}{h(1)} \left((\xi+1)^{-1/4} + h \left(\frac{1}{\xi+1} \right) - 1 \right) \\
 &= \frac{\sqrt{6}}{4} \left((\xi+1)^{-1/4} + \frac{1 + \sqrt{\xi+1}}{\sqrt{\xi+1 + \frac{1}{2}\sqrt{\xi+1}}} - 1 \right). \tag{6.8}
 \end{aligned}$$

Note that the first term in the right side is equal to $(1 - r_\infty)g_U(\xi)$, as it should. The dimensionless ratio

$$V = \lim_{t \rightarrow \infty} \frac{\tilde{T}_{N_t}}{t} \tag{6.9}$$

therefore has a universal distribution, such that

$$g_V(\xi) = \left\langle \frac{1}{1 + \xi V} \right\rangle.$$

The moments of V are readily derived by expanding $g_V(\xi)$ as a power series. We thus obtain

$$\langle V \rangle = \frac{\sqrt{6}}{16} + \frac{1}{6} = 0.319759\dots,$$

and more generally

$$\langle V^n \rangle = (1 - r_\infty) \langle U^n \rangle + b_n \tag{6.10}$$

for $n \geq 1$ (see (6.3)), where

$$b_1 = \frac{1}{6}, \quad b_2 = \frac{11}{96}, \quad b_3 = \frac{155}{1728}, \quad b_4 = \frac{12395}{165888}, \quad \dots$$

These positive rational numbers obey the four-term linear recursion (see (C.13))

$$\begin{aligned}
 &32(n-1)(2n-1)b_{n-1} - (176n^2 + 16n + 5)b_n \\
 &+ 32(n+1)(5n+3)b_{n+1} - 48(n+1)(n+2)b_{n+2} = 0. \tag{6.11}
 \end{aligned}$$

Using again (B.3), some algebra yields the density

$$f_V(x) = \frac{\gamma_V(x)}{x^{3/4}(1-x)^{1/4}} \quad (0 < x < 1), \tag{6.12}$$

with

$$\gamma_V(x) = \frac{\sqrt{3}}{4\pi} + \frac{\sqrt{6}}{4\pi x^{1/4} \sqrt{4-3x}} \left(\sqrt{4-3x} - 2(1-x)^{3/2} \right)^{1/2}. \tag{6.13}$$

Again one notes that the first contribution to f_V is equal to $(1 - r_\infty)f_U$.

Figure 7 shows histogram plots of the distribution of the epochs T_{N_t} and \tilde{T}_{N_t} for integrated random walks made of $n = 10^4$ steps, with symmetric uniform and exponential step distributions (see legend). Numerical data are rescaled according to (6.2), (6.9). Both rescaled histograms are again in excellent agreement with the predictions (6.4), (6.12) (full curves).

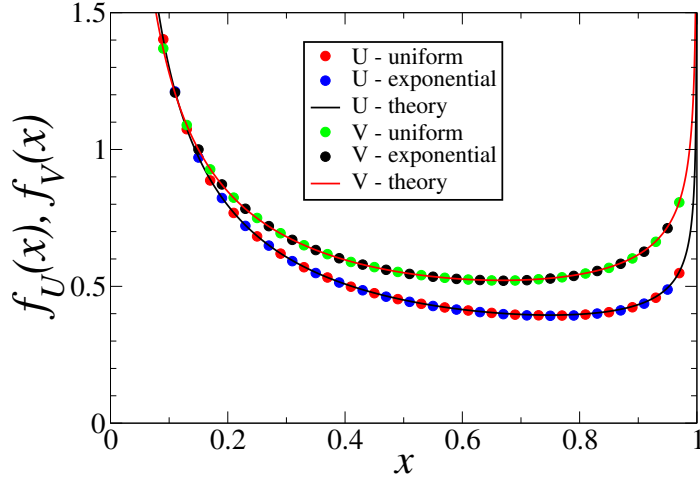


Figure 7. Symbols: histogram plots of the distribution of the epochs T_{N_t} of the last renewal and \tilde{T}_{N_t} of the last marked point for integrated random walks made of $n = 10^4$ steps, with symmetric uniform and exponential step distributions (see legend), rescaled according to (6.2), (6.9). Full curves: predictions (6.4), (6.12).

6.3. Time t_m to reach the maximum

Let us denote by t_m the time at which the random acceleration process reaches its maximum during the time interval $(0, t)$, and by

$$W = \lim_{t \rightarrow \infty} \frac{t_m}{t}$$

the rescaled time. According to the dichotomy defined in section 3.4, either t falls in the interval τ_{N_t+1} , or it falls in the interval δ_{N_t+1} , with respective probabilities $1 - r_\infty$ and r_∞ . In the first case $t_m = T_{N_t}$, while in the second case $t_m = t$. It follows that

$$f_W(x) = r_\infty \delta(x - 1) + (1 - r_\infty) f_U(x). \quad (6.14)$$

We thus swiftly recover one of the main results of [31], by a method which is completely different from that presented in this reference and which has the advantage of highlighting the probabilistic content of (6.14).

7. Discussion

In this paper we have investigated the statistics of upper records for integrated random walks with finite variance. Our main focus was on the asymptotic regime of long times, where the discrete integrated random walk process is described by its continuum

analogue, the random acceleration process. Within this setting, the asymptotic statistics of records relies on the renewal structure of the process, involving a sequence of iid couples of intervals of time (τ_n, δ_n) , defining an infinite alternating sequence of green and red dots. Runs of consecutive records of the discrete process take place in all time intervals δ_n between green and red dots. For reference, let us mention that this alternating scheme is precisely what is referred to in the mathematical literature as an *alternating renewal process* [13]. The alternating renewal process considered in the present work is generic in the sense that the *off* and *on* random variables, namely the time intervals τ_n and δ_n , respectively, are not independent.

Let us come back on the three cases defined in the Introduction, namely iid random variables $\eta_1, \eta_2, \dots, \eta_n$, the successive positions of a random walk V_1, V_2, \dots, V_n , built from the previous variables, and the integrated random walk X_1, X_2, \dots, X_n , built from the latter. The probability of record breaking at time n and the mean number of records up to n for these three cases are respectively

$$\begin{aligned} r_n &= \frac{1}{n}, & \langle M_n \rangle_{n \rightarrow \infty} &\approx \ln n, \\ r_n &\underset{n \rightarrow \infty}{\approx} \frac{1}{\sqrt{\pi n}}, & \langle M_n \rangle_{n \rightarrow \infty} &\approx \frac{2\sqrt{n}}{\sqrt{\pi}}, \\ r_n &\underset{n \rightarrow \infty}{\approx} r_\infty, & \langle M_n \rangle_{n \rightarrow \infty} &\approx n r_\infty. \end{aligned} \quad (7.1)$$

The classes of universality of these three cases are different and follow a hierarchy of decreasing generality. For iid random variables, the parent distribution of step lengths Φ is assumed to be continuous. For random walks, it is continuous and symmetric, thus Lévy flights belong to this class. For integrated random walks, it is symmetric with finite variance. For all three cases, discrete distributions have to be considered separately. For integrated random walks universality only holds asymptotically, while for the other two cases it holds at any finite time.

In view of these considerations two natural extensions arise. The first one consists in considering integrated random walks with heavy-tailed parent distribution Φ . The second one consists in considering $(k-1)$ -fold integrated random walks. For both cases, the question is to know how the last line of (7.1) is modified. We discuss both situations in turn.

We first address the case of integrated Lévy flights where the parent step length distribution is still symmetric, but is now heavy-tailed, with a tail index $\alpha < 2$, so that $\langle \eta^2 \rangle$ is divergent. In this situation, the velocity and position of the walker scale respectively as $V_n \sim n^{1/\alpha}$ and $X_n \sim n^{(\alpha+1)/\alpha}$. The survival probability (2.14), which falls off as $t^{-1/4}$ in the diffusive case, is known to decay as $t^{-\theta}$, where the persistence exponent $\theta = 1/(2 + \alpha)$ varies continuously with α [26]. This situation is qualitatively different from the situation studied in this work. Indeed, trajectories of integrated Lévy flights are discontinuous in the continuum limit, whereas those of the random acceleration process are continuous. In spite of this, we expect that the number of records of integrated Lévy flights with $\alpha < 2$ still grows linearly, and that the corresponding ratio R has a universal, α -dependent, distribution. In particular, the asymptotic probability of record breaking $r_\infty = \langle R \rangle$ is expected to be universal, and to exhibit a smooth dependence on the tail index α over a rather small range, interpolating between $r_\infty = 0.387627\dots$ (see (1.8)) for $\alpha = 2$ and $r_\infty \rightarrow 1/2$ as $\alpha \rightarrow 0$. In this limit, the process indeed grows so fast that the record-breaking condition at time n becomes local and amounts to $V_n > 0$.

It turns out that the case of $(k - 1)$ -fold integrated random walks is somewhat similar. Its continuum limit obeys $d^k x_t / dt^k = \eta_t$. Usual random walks and integrated random walks considered in this work respectively correspond to $k = 1$ and $k = 2$. For general k , the position of the walker scales as $X_n \sim n^{k-1/2}$. The survival probability decays as $t^{-\theta_k}$, where the persistence exponent θ_k is a decreasing function of the integer k . Besides the exact values $\theta_1 = 1/2$ (Brownian motion) and $\theta_2 = 1/4$ (random acceleration process), we have $\theta_3 \approx 0.220$, $\theta_4 \approx 0.210$, $\theta_5 \approx 0.204$, and so on [41]. The limit of these exponents, $\lim_{k \rightarrow \infty} \theta_k = 3/16 = 0.1875$, is related to the diffusion equation in the plane. Its exact value has been derived only recently [42, 43]. Here, too, the number of records is expected to grow linearly for all $k \geq 2$, with a universal, k -dependent, asymptotic probability of record breaking growing from $r_\infty = 0.387627\dots$ for $k = 2$ to $r_\infty \rightarrow 1/2$ as $k \rightarrow \infty$, for the same reason as above.

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Appendix A. A word on notations

Asymptotic equivalence

The symbol \approx stands for asymptotic equivalence; the symbol \sim is weaker and means ‘of the order of’.

Probability densities, Laplace transforms, limiting distributions

The probability density function of the continuous random variable X is denoted by $f_X(x)$, with

$$f_X(x) = \frac{d}{dx} \mathbb{P}(X < x).$$

In the course of this work, we encounter several positive time-dependent continuous random variables, denoted generically by Y_t . The probability density function of such a random variable is denoted by $f_{Y_t}(t, y)$ where time t appears as a parameter. The Laplace transform with respect to y of this density is

$$\hat{f}_{Y_t}(t, u) = \mathcal{L}_y f_{Y_t}(t, y) = \langle e^{-uY_t} \rangle = \int_0^\infty dy e^{-uy} f_{Y_t}(t, y),$$

and its double Laplace transform with respect to t and y is denoted by

$$\hat{f}_{Y_t}(s, u) = \mathcal{L}_{t,y} f_{Y_t}(t, y) = \mathcal{L}_t \langle e^{-uY_t} \rangle = \int_0^\infty dt e^{-st} \int_0^\infty dy e^{-uy} f_{Y_t}(t, y). \quad (\text{A.1})$$

Assume that Y_t scales asymptotically as t . As $t \rightarrow \infty$ the density $f_{t^{-1}Y_t}(t, x = y/t)$ of the rescaled variable Y_t/t converges to a limit, denoted by

$$f_X(x) = \lim_{t \rightarrow \infty} f_{t^{-1}Y_t}(t, x = y/t). \quad (\text{A.2})$$

Appendix B. Inversion of the scaling form of a double Laplace transform

For completeness, we reproduce hereafter Appendix B of [17].

Consider the probability density function $f_{Y_t}(t, y)$ of the positive random variable Y_t , and assume that its double Laplace transform (A.1) with respect to t and y has the scaling behaviour

$$\hat{f}_{Y_t}(s, u) = \frac{1}{s} g\left(\frac{u}{s}\right) \quad (\text{B.1})$$

in the regime $s, u \rightarrow 0$, with u/s arbitrary. Then the following properties hold.

(i) When $t \rightarrow \infty$ the random variable Y_t/t possesses a limiting distribution given by (A.2).

(ii) The scaling function g is related to f_X by

$$g(\xi) = \left\langle \frac{1}{1 + \xi X} \right\rangle = \int_0^\infty dx \frac{f_X(x)}{1 + \xi x}. \quad (\text{B.2})$$

(iii) This can be inverted as

$$f_X(x) = -\frac{1}{\pi x} \lim_{\epsilon \rightarrow 0} \text{Im} g\left(-\frac{1}{x + i\epsilon}\right). \quad (\text{B.3})$$

(iv) Finally the moments of X can be obtained, when they exist, by expanding $g(\xi)$ as a Taylor series, since (B.2) implies that

$$g(\xi) = \sum_{k \geq 0} (-\xi)^k \langle X^k \rangle. \quad (\text{B.4})$$

These properties can be easily understood as follows.

(i) First, a direct consequence of the scaling form (B.1) is that Y_t scales as t , as can be seen by Taylor expanding the right side of this equation, which generates the moments of Y_t in the Laplace space conjugate to t . Therefore (A.2) holds.

(ii) Then, (B.2) is a simple consequence of (A.2), since

$$\hat{f}_{Y_t}(s, u) = \int_0^\infty dt e^{-st} \langle e^{-uY_t} \rangle = \int_0^\infty dt e^{-st} \langle e^{-utX} \rangle = \left\langle \frac{1}{s + uX} \right\rangle.$$

(iii) Now,

$$f_X(x) = \langle \delta(X - x) \rangle = -\frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \text{Im} \left\langle \frac{1}{x + i\epsilon - X} \right\rangle.$$

The right side can be rewritten using (B.2), yielding (B.3).

Appendix C. Some detailed derivations

This appendix is devoted to the detailed derivations of a few results used in the body of the paper.

Appendix C.1. Derivation of the algebraic expression (3.7) of the function $h(\xi)$

The function $h(\xi)$ is defined by the integral expression (3.6), where the distribution $f_Z(z)$ is given by (3.3). This reads

$$h(\xi) = \frac{12\Gamma(3/4)^2}{\pi^{3/2}} \int_0^\infty dz \frac{z^{1/4}(1 + \xi z)^{1/4}}{(1 + 4z)^{7/4}}.$$

Setting $z = u/(4(1-u))$ and $\xi = 4(1-\zeta)$, we obtain

$$\begin{aligned} h &= \frac{3\Gamma(3/4)^2}{\sqrt{2}\pi^{3/2}} \int_0^1 du u^{1/4}(1-u)^{-3/4}(1-\zeta u)^{1/4} \\ &= \frac{3}{\sqrt{2}} F\left(-\frac{1}{4}, \frac{5}{4}; \frac{3}{2}; \zeta\right). \end{aligned} \quad (\text{C.1})$$

The hypergeometric function boils down to something more elementary. More precisely, we are facing the first of the 15 entries of the so-called Schwarz Table of all cases where the hypergeometric series reduces to an algebraic function (see, e.g., [44, Vol. I, Sec. 2.7.2]).

This reduction can be shown by elementary means as follows. Starting from the hypergeometric differential equation obeyed by (C.1), i.e.,

$$\zeta(1-\zeta) \frac{d^2 h}{d\zeta^2} + \left(\frac{3}{2} - 2\zeta\right) \frac{dh}{d\zeta} + \frac{5}{16} h = 0,$$

and setting $\xi = 4\cos^2\alpha$, i.e., $\zeta = \sin^2\alpha$, with $0 \leq \alpha \leq \pi/2$ for definiteness, we obtain

$$\frac{d^2 h}{d\alpha^2} + 2\cot\alpha \frac{dh}{d\alpha} + \frac{5}{16} h = 0.$$

Setting $h = v/(\sin\alpha)$, we obtain the simple differential equation

$$\frac{d^2 v}{d\alpha^2} + \frac{9}{4} v = 0,$$

whose solutions are $\sin(3\alpha/2)$ and $\cos(3\alpha/2)$. The regularity of h and its value $h = 1$ for $\xi = 0$, i.e., $\alpha = \pi/2$, yield

$$h = \sqrt{2} \frac{\sin(3\alpha/2)}{\sin\alpha}.$$

Some trigonometric identities finally yield

$$h = \frac{1 + 2\cos\alpha}{\sqrt{1 + \cos\alpha}} = \frac{1 + \sqrt{\xi}}{\sqrt{1 + \frac{1}{2}\sqrt{\xi}}}. \quad (\text{C.2})$$

It can be shown by eliminating radicals that $h(\xi)$ is an algebraic function of degree four, obeying the biquadratic equation

$$(\xi - 4)h^4 + 8h^2 - 4(\xi - 1)^2 = 0. \quad (\text{C.3})$$

Appendix C.2. Derivation of the recursion (5.12) for the coefficients a_n

The gist of Appendix C.2 and Appendix C.3 resides in the fact that algebraic functions obey linear differential equations with polynomial coefficients. As a consequence, the coefficients of their power-series expansions obey linear recursions. These properties were known to Abel as early as 1827 (see [45] for an account of historical and algorithmic aspects). In modern times they are only seldom mentioned or used. The present case provides an example of a situation where they are useful.

The function $g_R(\xi)$ obeys the fourth-order algebraic equation (see (5.6))

$$P(\xi, g_R) = 4(\xi+1)^3 g_R^4 - 16(\xi+1)^2 g_R^3 + 16(\xi+1) g_R^2 - (\xi-2)^2 = 0. \quad (\text{C.4})$$

The linear differential equation obeyed by $g_R(\xi)$ can be derived in three steps.

First, its first derivative reads

$$\begin{aligned}\frac{dg_R}{d\xi} &= -\frac{\partial P/\partial \xi}{\partial P/\partial g_R} \\ &= -\frac{6(\xi+1)^2 g_R^4 - 16(\xi+1)g_R^3 + 8g_R^2 + 2 - \xi}{8(\xi+1)g_R((\xi+1)g_R - 1)((\xi+1)g_R - 2)}.\end{aligned}\quad (\text{C.5})$$

This expression is a rational function of g_R . It can therefore be reduced to the form

$$\frac{dg_R}{d\xi} = A_3(\xi)g_R^3 + A_2(\xi)g_R^2 + A_1(\xi)g_R + A_0(\xi), \quad (\text{C.6})$$

where the A_i are rational functions of ξ . This can be done by expressing that the difference between (C.5) and (C.6) is a multiple of $P(\xi, g_R)$. This condition yields coupled linear equations for the $A_i(\xi)$, whose solution yields

$$\frac{dg_R}{d\xi} = -\frac{N(\xi, g_R)}{4\xi(\xi+1)(\xi-2)(\xi-3)}, \quad (\text{C.7})$$

with

$$\begin{aligned}N(\xi, g_R) &= 12(\xi+1)^2 g_R^3 - 36(\xi+1)g_R^2 \\ &\quad + (\xi^3 - 11\xi^2 + 243\xi + 12)g_R + 3(\xi-2)^2.\end{aligned}$$

Second, higher-order derivatives of the function $g_R(\xi)$ can be readily evaluated by applying iteratively the total derivative operator

$$\frac{d}{d\xi} = \frac{\partial}{\partial \xi} + \frac{dg_R}{d\xi} \frac{\partial}{\partial g_R}$$

to the expression (C.7). In the present situation, it is sufficient to go up to the second derivative.

Third, eliminating nonlinear terms (those proportional to g_R^2 and g_R^3) between the resulting expressions of the first and second derivatives, we obtain the desired linear differential equation in the form

$$\begin{aligned}16(\xi+1)^2(\xi-3)\frac{d^2 g_R}{d\xi^2} \\ + 16(\xi+1)(3\xi-7)\frac{dg_R}{d\xi} + (7\xi-25)g_R + 9 = 0.\end{aligned}\quad (\text{C.8})$$

Finally, inserting the power-series expansion

$$g_R(\xi) = \sum_{n \geq 0} (1 - a_n \sqrt{6})(-\xi)^n$$

(see (5.9), (5.11)) into (C.8), we obtain the following four-term linear recursion for the coefficients a_n :

$$\begin{aligned}(16n^2 - 9)a_{n-1} + (16n^2 + 48n + 25)a_n \\ - 16(n+1)(5n+7)a_{n+1} + 48(n+1)(n+2)a_{n+2} = 0.\end{aligned}\quad (\text{C.9})$$

Appendix C.3. Derivation of the recursion (6.11) for the coefficients b_n

The following analysis is in the same vein as the previous section. We start by splitting $g_V(\xi)$ given in (6.8) according to

$$g_V(\xi) = \frac{\sqrt{6}}{4} (g_U(\xi) - 1) + g_B(\xi), \quad (\text{C.10})$$

with

$$g_B(\xi) = \frac{\sqrt{6}}{4} \frac{1 + \sqrt{\xi + 1}}{\sqrt{\xi + 1 + \frac{1}{2}\sqrt{\xi + 1}}}.$$

It can be shown by eliminating radicals that g_B is an algebraic function of degree four, obeying the biquadratic equation

$$16(\xi + 1)(4\xi + 3)g_B^4 - 48(\xi + 1)^2 g_B^2 + 9\xi^2 = 0. \quad (\text{C.11})$$

The linear differential equation obeyed by $g_B(\xi)$ can be derived by means of the three-step procedure presented in Appendix C.2. We thus obtain

$$16(\xi + 1)^2(4\xi + 3) \frac{d^2 g_B}{d\xi^2} + 96(\xi + 1)^2 \frac{d g_B}{d\xi} + 5g_B = 0. \quad (\text{C.12})$$

Inserting the power-series expansion

$$g_B(\xi) = \sum_{n \geq 0} b_n (-\xi)^n$$

(see (6.10), (C.10)) into (C.12), we obtain the following four-term linear recursion for the coefficients b_n :

$$\begin{aligned} &32(n-1)(2n-1)b_{n-1} - (176n^2 + 16n + 5)b_n \\ &+ 32(n+1)(5n+3)b_{n+1} - 48(n+1)(n+2)b_{n+2} = 0. \end{aligned} \quad (\text{C.13})$$

We have $b_0 = 1$, whereas the b_n enter the expression (6.10) of the moments of V for $n \geq 1$ only.

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