HAL
open science

Survival probability of random walks and Lévy flights with stochastic resetting

Claude Godrèche, Jean-Marc Luck

To cite this version:

Claude Godrèche, Jean-Marc Luck. Survival probability of random walks and Lévy flights with stochastic resetting. Journal of Statistical Mechanics: Theory and Experiment, 2022, 2022, pp.073201. 10.1088/1742-5468/ac7a2a . cea-03648077

HAL Id: cea-03648077

https://cea.hal.science/cea-03648077v1

Submitted on 21 Apr 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:2204.07392v1 [cond-mat.stat-mech] 15 Apr 2022

Survival probability of random walks and Lévy flights with stochastic resetting

Claude Godrèche and Jean-Marc Luck

Université Paris-Saclay, CNRS, CEA, Institut de Physique Théorique,
91191 Gif-sur-Yvette, France

Abstract. We perform a thorough analysis of the survival probability of symmetric random walks with stochastic resetting, defined as the probability for the walker not to cross the origin up to time n. For continuous symmetric distributions of step lengths with either finite (random walks) or infinite variance (Lévy flights), this probability can be expressed in terms of the survival probability of the walk without resetting, given by Sparre Andersen theory. It is therefore universal, i.e., independent of the step length distribution. We analyze this survival probability at depth, deriving both exact results at finite times and asymptotic late-time results. We also investigate the case where the step length distribution is symmetric but not continuous, focussing our attention onto arithmetic distributions generating random walks on the lattice of integers. We investigate in detail the example of the simple Polya walk and propose an algebraic approach for lattice walks with a larger range.

18 April 2022

E-mail: `claude.godreche@ipht.fr`, `jean-marc.luck@ipht.fr`

1. Introduction

This work is a sequel to the recent study of the statistics of the maximum and of the number of records for random walks with stochastic resetting [1], a topic of much current interest (see [2] for a review). Consider a random walk starting from the origin, defined by the recursion

$$x_{n+1} = \begin{cases} 0 & \text{with prob. } r, \\ x_n + \eta_{n+1} & \text{with prob. } 1-r. \end{cases} \tag{1.1}$$

At each time step, the walker is reset to the origin with probability r. The step lengths η_n are iid (independent and identically distributed) random variables with an arbitrary symmetric distribution with density $\rho(\eta)$.

The central object of interest of the present work is the survival probability (or persistence probability) of the random walker, that is, the probability for the walker not to cross the origin, up to time n:

$$Q_n = \mathbb{P}(x_1 \geq 0, \ldots, x_n \geq 0) = \mathbb{P}(x_1 \leq 0, \ldots, x_n \leq 0). \tag{1.2}$$

The generating series of the sequence Q_n has the expression

$$\tilde{Q}(z) = \sum_{n \geq 0} Q_n z^n = \frac{\tilde{q}((1-r)z)}{1 - rz\,\tilde{q}((1-r)z)}, \tag{1.3}$$

where $\tilde{q}(z)$ is the generating series of the sequence of survival probabilities q_n of the same walker in the absence of resetting. The identity (1.3) can be deduced either from a renewal approach (see section 3), or from the formalism used in [1] for investigating the statistics of the maximum. The expression of $\tilde{q}(z)$ stems from Sparre Andersen theory (see section 2). If the step length distribution is continuous and symmetric, with either finite (random walks) or infinite variance (Lévy flights), $\tilde{q}(z)$ is given by (2.6), and so q_n and Q_n are universal: q_n only depends on n, whereas Q_n also depends on the resetting probability r. If the step length distribution possesses a discrete component, $\tilde{q}(z)$ is given by the general expression (2.12) and is no longer universal.

The relation (1.3) is the starting point of a thorough analysis of several novel aspects of the problem, as we now summarize. Section 2 is a brief reminder of the predictions of Sparre Andersen theory on the survival probability q_n of symmetric random walks in the absence of resetting, the step length distribution being either continuous or not. In section 3, using a direct renewal approach, we establish in full generality the identity (1.3) between the survival probabilities of the random walk in the presence and in the absence of resetting. We also recall how the same result can be deduced from the study made in [1]. The case of a symmetric continuous step length distribution is considered in section 4. We analyze the universal expression of the survival probability Q_n at depth, deriving both exact results at finite times and asymptotic late-time results. Section 5 is devoted to the simple Polya walk on the lattice of integers, for which the survival probability is studied following the same setting. More general arithmetic step length distributions are dealt with in section 6 by means of a novel algebraic approach.

2. Elements of Sparre Andersen theory

This section is a brief reminder of some aspects of Sparre Andersen theory on sums of random variables [3, 4, 5]. Consider a random walk without resetting, whose position

at time n,

$$x_n = \eta_1 + \cdots + \eta_n, \tag{2.1}$$

is the sum of n iid random variables whose common distribution is symmetric, with density $\rho(\eta)$. Introduce the probabilities

$$\begin{aligned} \pi_n &= \mathbb{P}(x_n \geq 0) = \mathbb{P}(x_n \leq 0), \\ q_n &= \mathbb{P}(x_1 \geq 0, \ldots, x_n \geq 0) = \mathbb{P}(x_1 \leq 0, \ldots, x_n \leq 0). \end{aligned} \tag{2.2}$$

A combinatorial theorem due to Sparre Andersen states that these probabilities are related by the identity [5, ch. XII]

$$\tilde{q}(z) = \sum_{n \geq 0} q_n z^n = \exp\left(\sum_{n \geq 1} \frac{\pi_n}{n} z^n \right). \tag{2.3}$$

The first few of these relations read

$$q_0 = 1, \quad q_1 = \pi_1, \quad q_2 = \frac{\pi_2 + \pi_1^2}{2}, \quad q_3 = \frac{2\pi_3 + 3\pi_1\pi_2 + \pi_1^3}{6}. \tag{2.4}$$

Let us now discuss the consequences of the identity (2.3), depending on the nature of the distribution $\rho(\eta)$.

Continuous symmetric distributions. If the step length distribution is symmetric and continuous, the position x_n is non-zero with certainty at all times $n \geq 1$, so that we have

$$\pi_n = \frac{1}{2} \qquad (n \geq 1). \tag{2.5}$$

The series entering the right-hand side of (2.3) therefore reads

$$\tilde{q}(z) = \sum_{n \geq 0} q_n z^n = \frac{1}{\sqrt{1-z}}, \tag{2.6}$$

hence

$$q_n = b_n, \tag{2.7}$$

where b_n denotes the binomial probability

$$b_n = \frac{(2n)!}{(2^n n!)^2} = \frac{\binom{2n}{n}}{2^{2n}}. \tag{2.8}$$

This is the well-known universal expression of the survival probability for a symmetric continuous distribution, with either finite or infinite variance [3, 4, 5]. The following power-law decay and asymptotic series of corrections are therefore universal:

$$q_n = \frac{1}{\sqrt{\pi n}} \left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} + \cdots \right). \tag{2.9}$$

Arbitrary symmetric distributions. In the general case where the step length distribution is symmetric, but not necessarily continuous, its density $\rho(\eta)$ may contain Dirac delta functions, either at the origin ($\eta = 0$) or at pairs of symmetric positions ($\eta = \pm a$). In such a circumstance, the probability

$$P_n = \mathbb{P}(x_n = 0) \tag{2.10}$$

is non-zero for some n. Equations (2.5) and (2.6) become

$$2\pi_n = \mathbb{P}(x_n \geq 0) + \mathbb{P}(x_n \leq 0) = 1 + P_n \tag{2.11}$$

and

$$\tilde{q}(z) = \sum_{n\geq 0} q_n z^n = \frac{1}{\sqrt{1-z}} \exp\left(\frac{1}{2}\sum_{n\geq 1}\frac{P_n}{n}\, z^n\right). \tag{2.12}$$

We have in particular

$$\tilde{q}(z) \approx \frac{E}{\sqrt{1-z}} \qquad (z \to 1), \tag{2.13}$$

hence

$$q_n \approx \frac{E}{\sqrt{\pi n}}, \tag{2.14}$$

where

$$E = \exp\left(\frac{1}{2}\sum_{n\geq 1}\frac{P_n}{n}\right) \geq 1 \tag{2.15}$$

is the asymptotic enhancement factor of the survival probability q_n with respect to the formula (2.9), which holds for symmetric continuous distributions (see [5, ch. XVIII], and [1, 6]).

The special class of arithmetic step length distributions, that is, discrete probability distributions yielding random walks on the lattice of integers, will be considered in section 6.

3. A direct renewal approach

The sequence of resetting events is a usual renewal process on the integers. The description of such a process is a simple transcription of that given for renewal processes with continuous random variables (see, e.g., [7]). The durations $T_1, T_2, \ldots$ between successive resetting events are iid with common geometric distribution

$$f_k = \mathbb{P}(T = k) = r(1-r)^{k-1} \qquad (k \geq 1), \tag{3.1}$$

whose complementary distribution function reads

$$g_k = \mathbb{P}(T > k) = \sum_{j>k} f_j = (1-r)^k \qquad (k \geq 0). \tag{3.2}$$

For any given time $n \geq 0$, the number of resetting events up to n is the unique integer $N_n \geq 0$ such that

$$T_1 + \cdots + T_{N_n} \leq n < T_1 + \cdots + T_{N_n+1}, \tag{3.3}$$

and the age B_n of the process at time n, also dubbed the backward recurrence time, is such that

$$n = T_1 + \cdots + T_{N_n} + B_n, \qquad B_n = 0, 1, \ldots, T_{N_n+1} - 1. \tag{3.4}$$

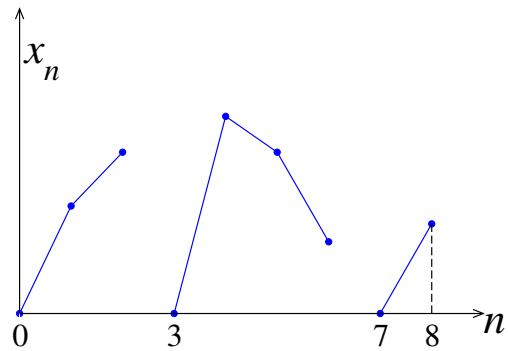

Figure 1. Example of a random walk of $n = 8$ steps which does not cross the origin, with two resetting events at times 3 and 7. In terms of a renewal process (see text), we have $N_8 = 2$, $T_1 = 3$, $T_2 = 4$ and $B_8 = 1$.

These definitions are illustrated in figure 1.

Let us now relate the survival probability Q_n of the walker in the presence of resetting to the survival probability q_n of the same walker in the absence of resetting. For a given number of steps n, let us denote a realization of the process by

$$\mathcal{C} = \{N_n = m, T_1 = k_1, \ldots, T_m = k_m, B_n = b\}. \tag{3.5}$$

The joint probability distribution associated to (3.5) reads

$$P(\mathcal{C}) = f_{k_1} \ldots f_{k_m}\, g_b\, \delta\Big(\sum_{i=1}^{m} k_i + b, n\Big), \tag{3.6}$$

where δ is the Kronecker delta symbol.

In the example shown in figure 1, $\mathcal{C} = \{N_8 = 2, T_1 = 3, T_2 = 4, B_8 = 1\}$, which gives a contribution $q_2 q_3 q_1$ to Q_8. For a generic realization $\mathcal{C}$, this contribution reads

$$Q(\mathcal{C}) = q_{k_1-1} \ldots q_{k_m-1}\, q_b, \tag{3.7}$$

because every duration $T_i = k_i$ between two consecutive resetting events brings a factor q_{k_i-1}, since the walker survives during $k_i - 1$ steps before a resetting occurs. The last incomplete lapse of time $B_n = b$ brings a factor q_b.

The survival probability Q_n is obtained by averaging (3.7) over all realizations $\mathcal{C}$:

$$\begin{aligned} Q_n &= \sum_{\mathcal{C}} P(\mathcal{C}) Q(\mathcal{C}) \qquad (3.8)\\ &= \sum_{m\geq 0} \sum_{k_1,\ldots,k_m} \sum_{b} f_{k_1} \ldots f_{k_m} g_b\; q_{k_1-1} \ldots q_{k_m-1} q_b\, \delta\Big(\sum_{i=1}^{m} k_i + b, n\Big). \end{aligned}$$

The associated generating series follows easily:

$$\begin{aligned} \tilde{Q}(z) &= \sum_{m\geq 0} \sum_{k_1,\ldots,k_m} \sum_{b} f_{k_1} z^{k_1} \ldots f(k_m) z^{k_m} g_b z^b\; q_{k_1-1} \ldots q_{k_m-1} q_b \\ &= \beta(z) \sum_{m\geq 0} \alpha(z)^m = \frac{\beta(z)}{1-\alpha(z)}, \end{aligned} \tag{3.9}$$

with

$$\alpha(z) = \sum_{k\geq 1} f_k\, q_{k-1}\, z^k = rz\, \tilde{q}((1-r)z),$$
$$\beta(z) = \sum_{b\geq 0} g_b\, q_b\, z^b = \tilde{q}((1-r)z). \tag{3.10}$$

We thus obtain the key result

$$\tilde{Q}(z) = \frac{\tilde{q}((1-r)z)}{1 - rz\, \tilde{q}((1-r)z)}, \tag{3.11}$$

relating the generating series of $\tilde{Q}(z)$ and $\tilde{q}(z)$, already announced in (1.3).

As the resetting probability r varies, the expression (3.11) interpolates between the following two limiting situations. In the absence of resetting ($r = 0$), we have $\tilde{Q}(z) = \tilde{q}(z)$, and so $Q_n = q_n$ for all n. In the other limit ($r = 1$), where the walker stays put at the origin, (3.11) yields $\tilde{Q}(z) = 1/(1-z)$, and so $Q_n = 1$ for all n, as expected.

The first few relations between the Q_n and the q_n read

$$Q_1 = (1-r)q_1 + r,$$
$$Q_2 = (1-r)^2 q_2 + 2r(1-r)q_1 + r^2,$$
$$Q_3 = (1-r)^3 q_3 + r(1-r)^2(q_1^2 + 2q_2) + 3r^2(1-r)q_1 + r^3. \tag{3.12}$$

There is an alternative route to obtaining (3.11), based upon the relationship between the survival probability Q_n and the distribution of the maximum of the walk,

$$M_n = \max(0, \ldots, x_n). \tag{3.13}$$

All surviving walks on the negative side up to time n, obeying $x_1 \leq 0, \ldots, x_n \leq 0$, have $M_n = 0$, whereas all non-surviving walks have $M_n > 0$. We have therefore

$$Q_n = \mathbb{P}(M_n = 0). \tag{3.14}$$

The generating series of this quantity, obtained by means of a renewal integral equation, identifies to (3.11) [1].

To close, let us mention that rational expressions similar to (3.11) relating generating series for quantities with and without stochastic resetting are in fact ubiquitous [1, 2, 8, 9, 10, 11]. The above direct renewal approach applies to all these situations.

4. Symmetric continuous step length distributions

4.1. Generating series

In this section we consider the case of symmetric continuous step length distributions, with either finite or infinite variance. The survival probability q_n in the absence of resetting is given by the universal formula (2.7), the generating series $\tilde{q}(z)$ of which is (2.6). Inserting this latter expression into (3.11) yields the following generating series of the survival probabilities Q_n for symmetric continuous step length distributions:

$$\tilde{Q}(z) = \frac{1}{\sqrt{1-(1-r)z} - rz}. \tag{4.1}$$

This universal expression is one of the cornerstones of the present work. We now investigate its consequences in detail.

4.2. Exact finite-time results

The survival probability Q_n only depends on time n and on the resetting probability r. By expanding the formula (4.1) as a power series in z, it is readily seen that Q_n is a polynomial of degree n in r, which has the same universality as the expression (2.7) to which it reduces for $r = 0$. It is plotted against r in figure 2 up to $n = 10$. Its first values read

$$\begin{aligned}
Q_0 &= 1, \\
Q_1 &= \frac{1}{2}(1+r), \\
Q_2 &= \frac{1}{8}(3+2r+3r^2), \\
Q_3 &= \frac{1}{16}(5+r+7r^2+3r^3), \\
Q_4 &= \frac{1}{128}(35-12r+66r^2+20r^3+19r^4), \\
Q_5 &= \frac{1}{256}(63-59r+166r^2-6r^3+75r^4+17r^5).
\end{aligned} \tag{4.2}$$

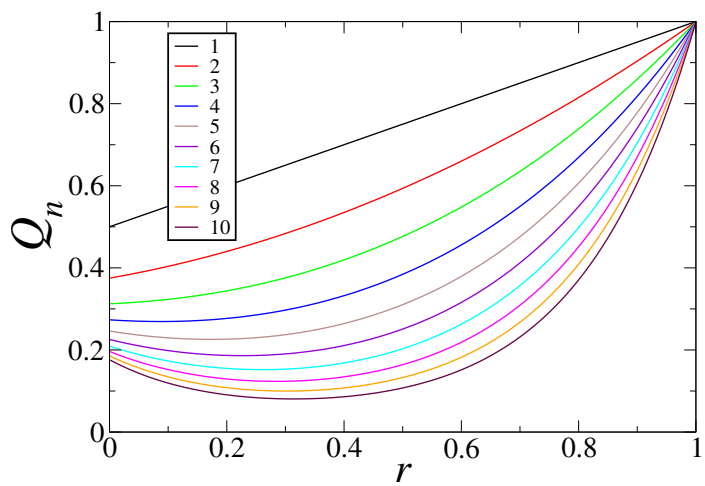

Figure 2. Survival probability Q_n for symmetric continuous step length distributions against resetting probability r for times n up to 10 (see legend).

It turns out that Q_n obeys a four-term linear recursion, whose origin can be traced back to the works by Abel on algebraic functions (see [12] for an overview including historical and algorithmic aspects). In the present situation, the derivation goes as follows. First, by eliminating the square root in (4.1), we obtain a quadratic equation for $\tilde{Q}(z)$:

$$(1-(1-r)z-r^2z^2)\tilde{Q}(z)^2 - 2rz\tilde{Q}(z) - 1 = 0. \tag{4.3}$$

Second, by differentiating the above equation w.r.t. z and judiciously eliminating non-linear terms, we obtain a linear first-order differential equation for $\tilde{Q}(z)$:

$$2(1-(1-r)z)(1-(1-r)z-r^2z^2)\tilde{Q}'(z)$$

$$+ (r - 1 + (1 + r)(1 - 3r)z + 3r^2(1 - r)z^2)\tilde{Q}(z)$$
$$- r(2 - (1 - r)z) = 0. \tag{4.4}$$

Third, by expanding the above differential equation as a power series in z, we obtain the four-term linear recursion

$$2nQ_n - (4n - 3)(1 - r)Q_{n-1} + ((2n - 3)(1 - 2r) - 3r^2)Q_{n-2}$$
$$+ (2n - 3)r^2(1 - r)Q_{n-3} = 2r\delta_{n1} - r(1 - r)\delta_{n2}. \tag{4.5}$$

The behavior of the survival probability Q_n as $r \to 0$ or $r \to 1$ can be studied by appropriately expanding the generating series (4.1). In the weak-resetting regime ($r \to 0$), we obtain

$$\tilde{Q}(z) = \frac{1}{\sqrt{1 - z}} + \left(\frac{z}{1 - z} - \frac{z}{2(1 - z)^{3/2}}\right) r + \cdots, \tag{4.6}$$

hence

$$Q_n = b_n + c_n r + \cdots, \qquad c_n = 1 - nb_n. \tag{4.7}$$

The first three correction terms $c_1 = 1/2$, $c_2 = 1/4$, $c_3 = 1/16$ are positive, whereas $c_4 = -3/32$ and all subsequent ones are negative. This corroborates the observation (see figure 2) that Q_n is monotonically increasing with r for $n = 1$, 2 and 3, whereas it exhibits a minimum for a non-trivial r_n for all $n \geq 4$. The value r_n of the resetting probability at which Q_n is minimal is plotted against $1/n$ in figure 3 (red dataset). For large times, r_n approaches the limit (4.17) (red arrow), at which the asymptotic decay rate K (4.13) is maximal.

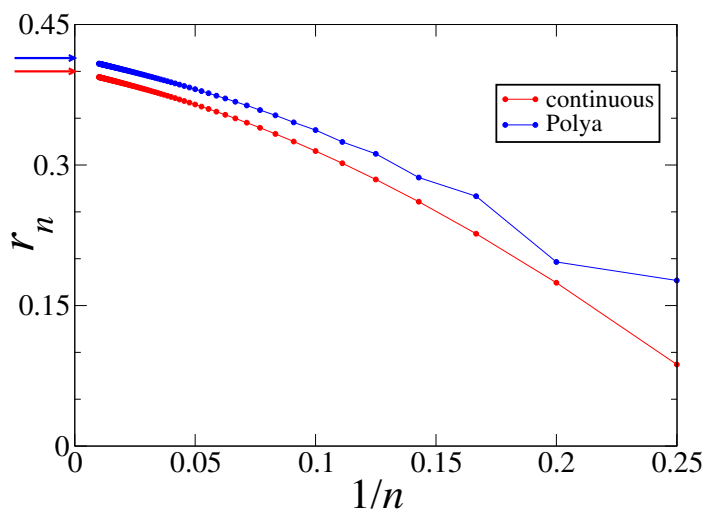

Figure 3. Values r_n of the resetting probability at which Q_n is minimal against $1/n$ for $n \geq 4$. Red: data for for symmetric continuous step length distributions. Blue: data for the simple Polya walk (see section 5.2). Arrows: limits (4.17) and (5.30).

In the strong-resetting regime ($r \to 1$), we obtain

$$\tilde{Q}(z) = \frac{1}{1 - z} - \frac{z}{2(1 - z)^2}(1 - r) + \cdots, \tag{4.8}$$

hence

$$Q_n = 1 - \frac{n}{2}\,(1-r) + \cdots \tag{4.9}$$

When $r = 1$, as said above, the walker stays put at the origin, so that $Q_n = 1$ at all times. The correction term expresses that there are on average $n(1-r)$ time steps where the walker is not being reset, and the walker's displacement at each of those times is positive (or negative) with probability $\pi_1 = 1/2$.

4.3. Asymptotic late-time results

Let us now turn to the asymptotic behavior of the survival probabilities Q_n at late times. For any fixed value of the resetting probability r, the closest singularity of the generating series (4.1) is a simple pole located at

$$z_0 = \frac{\sqrt{5r^2 - 2r + 1} + r - 1}{2r^2}, \tag{4.10}$$

whereas the branch-point singularity of the square root lies further away, at

$$z_c = \frac{1}{1-r}. \tag{4.11}$$

We have indeed $1 < z_0 < z_c$. Therefore Q_n decays exponentially as

$$Q_n \approx A\,\mathrm{e}^{-Kn}, \tag{4.12}$$

with

$$K = \ln z_0 = \ln \frac{\sqrt{5r^2 - 2r + 1} + r - 1}{2r^2}. \tag{4.13}$$

The amplitude A can also be worked out and reads

$$A = \frac{2r}{\sqrt{5r^2 - 2r + 1}}. \tag{4.14}$$

An exponential decay law of the survival probability was to be expected. In the presence of resetting, the walker indeed reaches a nonequilibrium steady state characterized by a non-trivial stationary distribution of its position [2]. As is well known, for stationarity processes, the survival probability (or persistence probability) generically falls off exponentially in time. This picture is corroborated by the fact that the number of factors in (3.7) scales linearly with time.

The decay rate K (4.13) entering the exponential law (4.12) is plotted against r in figure 4 (red curve). It vanishes as $r \to 0$ and $r \to 1$, according to

$$\begin{aligned} K &= r - \frac{r^2}{2} + \cdots && (r \to 0), \\ K &= \frac{1-r}{2} - \frac{7(1-r)^3}{48} + \cdots && (r \to 1), \end{aligned} \tag{4.15}$$

and reaches its maximum,

$$K_{\max} = \ln \frac{5}{4} \approx 0.223143, \tag{4.16}$$

for the value

$$r = \frac{2}{5} \tag{4.17}$$

of the resetting probability.

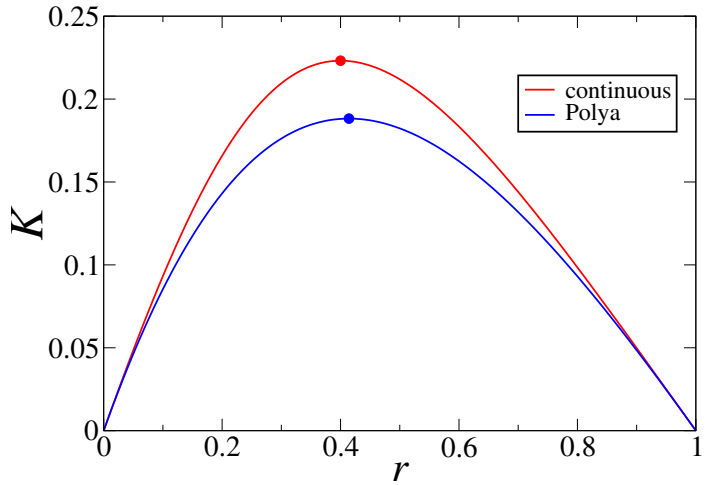

Figure 4. Asymptotic decay rate K of the survival probabilities against resetting probability r. Red: universal expression (4.13) for symmetric continuous step length distributions. Blue: expression (5.26) for the simple Polya walk. Symbols: maximal values $K_{\max}$ (see (4.16), (4.17) and (5.29), (5.30)).

Finally, a crossover takes place in the scaling regime where n is large, whereas r is small. Both singularities of the generating series (4.1) merge in this regime, as we have $K \approx z_0 - 1 \approx r$, whereas $z_c - z_0 \approx r^2$ is much smaller. Setting $z = \mathrm{e}^{-s}$, the leading-order singular behavior of (4.1) reads

$$\tilde{Q}(z) \approx \frac{1}{\sqrt{s+r}}, \tag{4.18}$$

hence

$$Q_n \approx \frac{\mathrm{e}^{-rn}}{\sqrt{\pi n}}. \tag{4.19}$$

The power-law decay (2.9) of the survival probability q_n in the absence of resetting is thus reduced by an exponential factor involving the mean number of resettings nr. The full crossover between the decay laws (4.12) and (4.19) is captured by the more complete scaling formula

$$\tilde{Q}(z) \approx \frac{1}{\sqrt{s+r}-r} \approx \frac{\sqrt{s+r}+r}{s+r(1-r)}, \tag{4.20}$$

featuring a pole at $s = -r(1-r)$ and a square-root branch point at $s = -r$, and yielding

$$Q_n \approx \mathrm{e}^{-rn}\left(\frac{1}{\sqrt{\pi n}} + r\,\mathrm{e}^{r^2 n}\left(1+\mathrm{erf}(r\sqrt{n})\right)\right). \tag{4.21}$$

The first and second term of this crossover formula respectively match the asymptotic decay of the expressions (4.19) and (4.12), to leading order at small r. The crossover time between both regimes diverges as $n \sim 1/r^2$.

5. The simple Polya walk

5.1. Generating series

The simple Polya walk on the lattice of integers is generated by the binary step length distribution

$$\rho(\eta) = \frac{1}{2}\left(\delta(\eta-1)+\delta(\eta+1)\right). \tag{5.1}$$

In this case, the probability P_n introduced in (2.10) can be determined as follows. In the absence of resetting, the walker's position x_n is zero if the number of steps $n = 2k$ is even, and the walk consists of k steps to the right and k steps to the left. This reads (see (2.8) for the definition of b_k)

$$P_{2k} = b_k, \qquad P_{2k+1} = 0. \tag{5.2}$$

We have thus, using (2.6),

$$\sum_{k\geq 1} \frac{b_k}{k}\, y^k = \int_0^y \left(\frac{1}{\sqrt{1-x}} - 1\right)\frac{\mathrm{d}x}{x} = -2\ln\frac{1+\sqrt{1-y}}{2}. \tag{5.3}$$

The sum involved in (2.12) therefore reads

$$\begin{aligned}\frac{1}{2}\sum_{n\geq 1}\frac{P_n}{n}\, z^n &= \frac{1}{4}\sum_{k\geq 1}\frac{b_k}{k}\, z^{2k} \\ &= -\frac{1}{2}\ln\frac{1+\sqrt{1-z^2}}{2} \\ &= \ln\frac{\sqrt{1+z}-\sqrt{1-z}}{z}.\end{aligned} \tag{5.4}$$

We are thus left with the expressions

$$\tilde{q}(z) = \frac{2}{\sqrt{1-z^2}+1-z} = \frac{1}{z}\left(\frac{1+z}{\sqrt{1-z^2}} - 1\right). \tag{5.5}$$

The second formula is more amenable to power-series expansion, as it shows explicitly the even and odd components of $\tilde{q}(z)$.

The survival probability q_n without resetting inherits the dependence on the parity of n of the probability P_n given in (5.2). It reads

$$q_{2k-1} = q_{2k} = b_k, \tag{5.6}$$

and obeys the power-law fall-off

$$q_n \approx \sqrt{\frac{2}{\pi n}}. \tag{5.7}$$

The enhancement factor with respect to the universal result (2.9) therefore reads

$$E = \sqrt{2}. \tag{5.8}$$

The corrections to the leading behavior (5.7) depend on the parity of n, according to

$$q_n = \sqrt{\frac{2}{\pi n}}\begin{cases}\left(1 - \dfrac{1}{4n} + \dfrac{1}{32n^2} + \cdots\right) & (n \text{ even}),\\[2ex] \left(1 - \dfrac{3}{4n} + \dfrac{25}{32n^2} + \cdots\right) & (n \text{ odd}).\end{cases} \tag{5.9}$$

Finally, inserting (5.5) into (3.11), we obtain

$$\begin{aligned}\tilde{Q}(z) &= \frac{2}{\sqrt{1-(1-r)^2z^2}+1-(1+r)z} \\ &= 1+\frac{(1+r)z+(1+r^2)z^2}{\sqrt{1-(1-r)^2z^2}+1-(1+r^2)z^2}.\end{aligned} \tag{5.10}$$

These expressions are another key outcome of this work. We now investigate their consequences in detail.

5.2. Exact finite-time results

The survival probability Q_n of the Polya walk with resetting only depends on time n and on the resetting probability r. By expanding the second line of (5.10) as a power series in z, we obtain $Q_0 = 1$, as should be, and

$$Q_{2k+1} = (1+r)R_k, \qquad Q_{2k+2} = (1+r^2)R_k \qquad (k \geq 0), \tag{5.11}$$

where the generating series of the R_k reads, upon setting $y = z^2$,

$$\tilde{R}(y) = \sum_{k\geq 0} R_k y^k = \frac{1}{\sqrt{1-(1-r)^2y}+1-(1+r^2)y}. \tag{5.12}$$

As it turns out, R_k is a reciprocal polynomial of degree $2k$ in r, and so Q_n is a polynomial of degree n in r, whose structure depends on the parity of n, according to (5.11). For $r = 0$, we have $R_k = b_{k+1}$, in agreement with (5.6). For $r = 1$, we have $R_k = 1/2$, and so $Q_n = 1$, as expected.

The survival probability Q_n is plotted against r in figure 5 up to $n = 10$. Its first values read

$$\begin{aligned}&Q_0 = 1, \\ &Q_1 = (1+r)R_0, \qquad Q_2 = (1+r^2)R_0, \\ &Q_3 = (1+r)R_1, \qquad Q_4 = (1+r^2)R_1, \\ &Q_5 = (1+r)R_2, \qquad Q_6 = (1+r^2)R_2, \\ &Q_7 = (1+r)R_3, \qquad Q_8 = (1+r^2)R_3,\end{aligned} \tag{5.13}$$

with

$$\begin{aligned}R_0 &= \frac{1}{2}, \\ R_1 &= \frac{1}{8}(3-2r+3r^2), \\ R_2 &= \frac{1}{16}(5-8r+14r^2-8r^3+5r^4). \\ R_3 &= \frac{1}{128}(35-94r+205r^2-228r^3+205r^4-94r^5+35r^6).\end{aligned} \tag{5.14}$$

The formula (5.11) implies

$$(1+r)Q_{2k+2} - (1+r^2)Q_{2k+1} = 0. \tag{5.15}$$

This suggests to consider the same differences in the other parity sector, namely

$$\begin{aligned}\Delta_k &= (1+r)Q_{2k+3} - (1+r^2)Q_{2k+2} \\ &= (1+r)^2R_{k+1} - (1+r^2)^2R_k.\end{aligned} \tag{5.16}$$

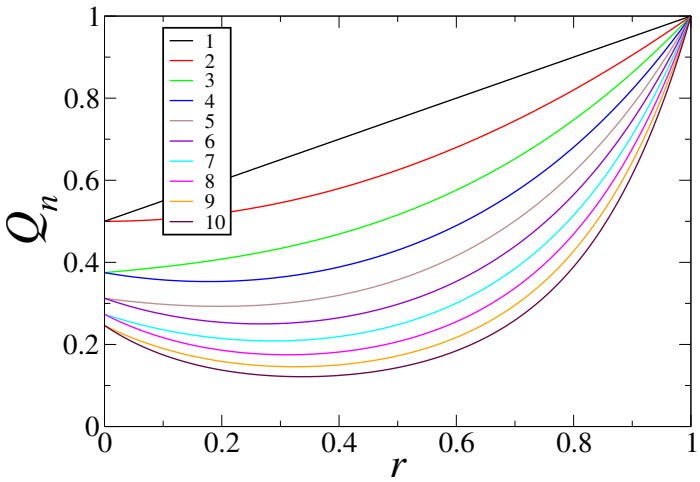

Figure 5. Survival probability Q_n of the simple Polya walk against resetting probability r for times n up to 10 (see legend).

The corresponding generating series reads

$$\tilde{\Delta}(y) = (1+r)^2 \frac{\tilde{R}(y) - R_0}{y} - (1+r^2)^2 \tilde{R}(y). \tag{5.17}$$

Using (5.12), we obtain

$$\tilde{\Delta}(y) = \frac{1}{y^2}\left(\sqrt{1-(1-r)^2 y} - 1 + \frac{1}{2}(1-r)^2 y\right), \tag{5.18}$$

hence

$$\Delta_k = -\frac{b_{k+2}}{2k+3}\,(1-r)^{2k+4}, \tag{5.19}$$

(see (2.8)). The resulting two-term linear recursion for the polynomials R_k,

$$(1+r)^2 R_{k+1} - (1+r^2)^2 R_k = -\frac{b_{k+2}}{2k+3}\,(1-r)^{2k+4} \quad (k \geq 0), \tag{5.20}$$

supersedes the three-term linear recursion that could be derived by means of the algebraic approach of section 4.2.

The behavior of Q_n as $r \to 0$ can be readily investigated by expanding the generating series (5.10) as

$$\begin{aligned}\tilde{Q}(z) = {} & \frac{1}{z}\left(\frac{1+z}{\sqrt{1-z^2}} - 1\right) \\ & + \frac{1}{z}\left(-\frac{1+z-z^2}{(1-z)\sqrt{1-z^2}} + \frac{1+z}{1-z}\right) r + \cdots,\end{aligned} \tag{5.21}$$

hence

$$Q_n = q_n + c_n r + \cdots, \qquad c_n = 2 - (n+2) q_n. \tag{5.22}$$

The survival probability q_n in the absence of resetting depends on the parity of n according to (5.6), whereas the relation between c_n and q_n holds irrespective of the parity of n. The correction terms $c_1 = 1/2$ and $c_3 = 1/8$ are positive, whereas c_2

vanishes, and $c_4 = -1/4$ and all subsequent ones are negative. This corroborates the observation (see figure 5) that Q_n is monotonically increasing with r for $n = 1$, 2 and 3, whereas it exhibits a minimum for a non-trivial r_n for all $n \geq 4$. The value r_n of the resetting probability at which Q_n is minimal has been plotted against $1/n$ in figure 3 (blue dataset). Oscillations according to the parity of n are clearly visible for the smaller values of n. For large times, r_n approaches the limit (5.30) (blue arrow), at which the asymptotic decay rate K (5.26) is maximal.

In the strong-resetting regime ($r \to 1$), the result (4.9) holds unchanged, as well as its interpretation.

5.3. Asymptotic late-time results

The analysis of the asymptotic behavior of the survival probability Q_n at late times closely follows the analysis made in section 4.3. For any fixed resetting probability r, the closest singularity of the generating series (5.10) is a simple pole located at

$$z_0 = \frac{1+r}{1+r^2}, \tag{5.23}$$

whereas the branch-point singularity of the square root lies further away, at

$$z_c = \frac{1}{1-r}. \tag{5.24}$$

The survival probability therefore falls off exponentially as

$$Q_n \approx A\,\mathrm{e}^{-Kn}, \tag{5.25}$$

with

$$K = \ln z_0 = \ln \frac{1+r}{1+r^2} \tag{5.26}$$

and

$$A = \frac{4r}{(1+r)^2}. \tag{5.27}$$

The expression (5.26) of the decay rate K can be alternatively derived from the recursion (5.20). This is however not the case for the amplitude A.

The decay rate K (5.26) entering the exponential law (5.25) has been plotted against r in figure 4 (blue curve). It vanishes as $r \to 0$ and $r \to 1$, according to

$$\begin{aligned} K &= r - \frac{3r^2}{2} + \cdots && (r \to 0), \\ K &= \frac{1-r}{2} - \frac{(1-r)^2}{8} + \cdots && (r \to 1), \end{aligned} \tag{5.28}$$

and reaches its maximum,

$$K_{\max} = \ln \frac{1+\sqrt{2}}{2} \approx 0.188226, \tag{5.29}$$

for the value

$$r = \sqrt{2} - 1 \approx 0.414213 \tag{5.30}$$

of the resetting probability.

Finally, a crossover similar to that described at the end of section 4.3 takes place in the scaling regime where n is large, whereas r is small.

6. Arithmetic distributions

This section is devoted to the special class of arithmetic step length distributions corresponding to random walks on the lattice of integers. A study of the statistics of records for these distributions in the absence of resetting is given in [6] (see also [5]), while the case with resetting is addressed in [1].

6.1. General formalism and results

We focus our attention onto symmetric arithmetic step length distributions with finite range J, of the form

$$\rho(\eta) = \sum_{j=1}^{J} \frac{a_j}{2} \left(\delta(\eta - j) + \delta(\eta + j)\right), \qquad \sum_{j=1}^{J} a_j = 1. \tag{6.1}$$

For the sake of simplicity, steps of length zero are not allowed. We consider the generic case where all the a_j are non-zero.

The Fourier transform of (6.1) reads

$$\widehat{\rho}(k) = \sum_{j=1}^{J} a_j \cos(jk) = \sum_{j=1}^{J} a_j T_j(\cos k). \tag{6.2}$$

We have indeed $\cos(jk) = T_j(\cos k)$, where the T_j are the Tchebyshev polynomials of the first kind. As a consequence, the Fourier transform $\widehat{\rho}(k)$ is a polynomial of degree J in $\cos k$. It is therefore an even 2π-periodic function of k. In the $k \to 0$ limit, we have

$$\widehat{\rho}(k) = 1 - Dk^2 + \cdots, \tag{6.3}$$

where

$$D = \frac{1}{2} \sum_{j=1}^{J} j^2 a_j \tag{6.4}$$

is the diffusion coefficient, such that $\langle \eta_n^2 \rangle = 2D$.

The connection with the formalism of section 2 goes as follows. The probability introduced in (2.10) reads

$$P_n = \int_0^{\pi} \widehat{\rho}(k)^n \, \frac{\mathrm{d}k}{\pi}, \tag{6.5}$$

and therefore

$$\sum_{n \geq 1} \frac{P_n}{n} \, z^n = -\int_0^{\pi} \ln(1 - z\widehat{\rho}(k)) \frac{\mathrm{d}k}{\pi}, \tag{6.6}$$

so that (2.12) and (2.15) respectively become

$$\tilde{q}(z) = \frac{1}{\sqrt{1-z}} \exp\left(-\frac{1}{2} \int_0^{\pi} \ln(1 - z\widehat{\rho}(k)) \frac{\mathrm{d}k}{\pi}\right) \tag{6.7}$$

and

$$E = \exp\left(-\frac{1}{2} \int_0^{\pi} \ln(1 - \widehat{\rho}(k)) \frac{\mathrm{d}k}{\pi}\right). \tag{6.8}$$

The last two formulas are given in [6].

Here, we propose to shed some new light on the problem by means of the following algebraic approach. Let us consider first the generating series $\tilde{q}(z)$, given by (6.7). The expression (6.2) of the Fourier transform $\widehat{\rho}(k)$ implies that $1 - z\widehat{\rho}(k)$ is a polynomial of degree J in $\cos k$, which can be factored as

$$1 - z\widehat{\rho}(k) = C \prod_{j=1}^{J} (1 - \Lambda_j \cos k), \tag{6.9}$$

where the prefactor C and the roots Λ_j depend on z and on the weights a_j. Setting $k = 0$, we get

$$1 - z = C \prod_{j=1}^{J} (1 - \Lambda_j). \tag{6.10}$$

Inserting (6.9) into (6.7), and using the integral

$$\int_0^\pi \ln(1 - \Lambda \cos k) \frac{\mathrm{d}k}{\pi} = -2 \ln \frac{\sqrt{1+\Lambda} - \sqrt{1-\Lambda}}{\Lambda}, \tag{6.11}$$

as well as the identity (6.10), we obtain the expression

$$\tilde{q}(z) = \frac{1}{C} \prod_{j=1}^{J} \frac{1}{\Lambda_j} \left(\sqrt{\frac{1+\Lambda_j}{1-\Lambda_j}} - 1 \right). \tag{6.12}$$

The generating series $\tilde{q}(z)$ is an algebraic function of z and the a_j with degree 2^J. The right-hand side of (6.12) is indeed a symmetric function of the roots Λ_j, where each square root has two branches, and therefore brings a factor two to the degree of $\tilde{q}(z)$.

The above property still holds in the presence of resetting. The key relation (3.11) between $\tilde{q}(z)$ and $\tilde{Q}(z)$ is rational, and so the generating series $\tilde{Q}(z)$ is an algebraic function of z, r and the a_j, with the same degree 2^J. As a consequence, Q_n obeys linear recursion relations, generalizing (4.5), whose complexity grows exponentially with the range J of the walk. Furthermore, the location z_0 of the closest singularity of the generating series $\tilde{Q}(z)$, governing the exponential decay of Q_n, is an algebraic function of the resetting probability r and the a_j, whose degree is however not determined by the present reasoning.

Consider now the asymptotic enhancement factor E, given by (6.8). The difference $1 - \widehat{\rho}(k)$ vanishes for $k = 0$, and can therefore be factored as

$$1 - \widehat{\rho}(k) = c(1 - \cos k) \prod_{j=1}^{J-1} (1 - \lambda_j \cos k). \tag{6.13}$$

where the prefactor c and the roots λ_j depend on the weights a_j. In the $k \to 0$ limit, using (6.3), (6.4), we obtain

$$2D = c \prod_{j=1}^{J-1} (1 - \lambda_j). \tag{6.14}$$

Inserting (6.13) into (6.8), and using (6.11) and (6.14), we obtain

$$E = \frac{1}{\sqrt{D}} \prod_{j=1}^{J-1} \frac{1 - \lambda_j}{\lambda_j} \left(\sqrt{\frac{1+\lambda_j}{1-\lambda_j}} - 1 \right). \tag{6.15}$$

The product

$$\langle h_1 \rangle = E\sqrt{D} \tag{6.16}$$

is therefore an algebraic function of the a_j with degree 2^{J-1}. This quantity is the mean value of the first positive abscissa h_1 of a random walk issued from the origin, irrespective of the time at which this position is reached [13, 14, 1] (see also [5, ch. XVIII]).

6.2. The simple Polya walk

A first illustration of the above approach is provided by the simple Polya walk, studied at length in section 5. In this case, we have $J = 1$, $a_1 = 1$, $D = 1/2$ and $\tilde{\rho}(k) = \cos k$. For generic values of z, we have $C = 1$ and $\Lambda_1 = z$, and so (6.12) reads

$$\tilde{q}(z) = \frac{1}{z}\left(\sqrt{\frac{1+z}{1-z}} - 1\right), \tag{6.17}$$

which is equivalent to (5.5). Moreover, (6.15) yields $\langle h_1 \rangle = E\sqrt{D} = 1$, as should be.

6.3. Lattice walks with range two

The next case, in order of increasing complexity, consists in lattice walks with range $J = 2$. Using the parametrization

$$a_1 = 1 - a, \qquad a_2 = a, \tag{6.18}$$

we have

$$D = \frac{1+3a}{2} \tag{6.19}$$

and

$$\tilde{\rho}(k) = (1-a)\cos k + a\cos(2k) = (1-a)\cos k + a(2\cos^2 k - 1). \tag{6.20}$$

For generic values of z, we have therefore

$$1 - z\tilde{\rho}(k) = 1 + az - (1-a)z\cos k - 2az\cos^2 k, \tag{6.21}$$

so that $C = 1 + az$, whereas Λ_1 and Λ_2 are the roots of the quadratic equation

$$(1+az)\Lambda^2 - (1-a)z\Lambda - 2az = 0. \tag{6.22}$$

By eliminating the roots Λ_1 and Λ_2 between (6.12) and (6.22), we obtain

$$\begin{aligned} &a^2z^2(1-z)^2\tilde{q}(z)^4 + 2az(1-z)^2\tilde{q}(z)^3 \\ &-2(1+a)z(1-z)\tilde{q}(z)^2 - 4(1-z)\tilde{q}(z) + 4 = 0. \end{aligned} \tag{6.23}$$

As anticipated, this is a polynomial equation with degree 4 for $\tilde{q}(z)$, the generating series for the survival probability q_n in the absence of resetting.

As far as the enhancement factor E is concerned, the constants entering the product (6.13) read $c = 1 + a$ and $\lambda_1 = -2a/(1+a)$, and so (6.15) yields

$$E = \frac{\sqrt{1+3a} - \sqrt{1-a}}{a\sqrt{2}}. \tag{6.24}$$

The enhancement factor reaches its maximum $E_{\max} = \sqrt{2}$, corresponding to the simple Polya walk (see (5.8)), for $a = 0$ and $a = 1$, and its minimum $E_{\min} = \sqrt{3/2}$ for $a = 2/3$. The product

$$\langle h_1 \rangle = E\sqrt{D} = \frac{2}{1 + \sqrt{\dfrac{1-a}{1+3a}}} \tag{6.25}$$

increases monotonically from 1 to 2 as a increases from 0 to 1.

In the presence of resetting, using the key relation (3.11), we can derive a polynomial equation with degree 4 for $\tilde{Q}(z)$, that is similar to (6.23), albeit too long to be reported here. The location $z_0 = \mathrm{e}^K$ of the closest pole of $\tilde{Q}(z)$, which governs the exponential decay $Q_n \sim \mathrm{e}^{-Kn}$, is found to obey the quadratic equation

$$\begin{aligned} &\left((1-r)^4 a^2 + 2r(1-r)^2 a + 2r^2(1+r^2)\right) z_0^2 \\ &-2\left((1-r)^3 a^2 + r(1-r)(2-r)a + r^2(1+r)\right) z_0 \\ &+(1-r)^2 a^2 + 2r(1-r)a = 0. \end{aligned} \tag{6.26}$$

The corresponding decay rate K vanishes as $r \to 0$ and as $r \to 1$, as

$$\begin{aligned} K &= r + \left(\frac{1}{2} - E^2\right) r^2 + \cdots && (r \to 0), \\ K &= \frac{1-r}{2} - \frac{1-2a(1-a)}{8}(1-r)^2 + \cdots && (r \to 1). \end{aligned} \tag{6.27}$$

The first expression involves the enhancement factor E given in (6.24). Both expansions agree with (5.28) for $a = 0$ and $a = 1$, as should be.

7. Discussion

In a recent work [1], we have revisited various features of the statistics of extremes and records of symmetric random walks with stochastic resetting. In the present paper, which is a sequel to the latter, we entirely focus our attention onto the survival probability (or persistence probability) Q_n, defined as the probability for the walker not to cross the origin up to time n. Throughout this work we consider for simplicity symmetric step length distributions, and the origin is both the starting point of the walker and its resetting point.

Stochastic resetting with probability r at each step has the peculiar feature that it does not alter the Markovian nature of free random walk. This entails, among other remarkable properties, the existence of the identity (1.3) (or (3.11)) between the generating series of the survival probabilities Q_n and q_n, respectively with and without resetting. In the present situation, this identity allowed us to investigate in detail many facets of the problem at hand.

For random walks and Lévy flights with symmetric and continuous step length distributions, the survival probabilities are universal: q_n only depends on time n according to (2.7), whereas Q_n only depends on n and r. The properties of Q_n are explored at depth in section 4, both at finite times and in the asymptotic regime of late times. Among their most noticeable properties, the Q_n are polynomials of degree n in r which obey a four-term linear recursion, as a consequence of the algebraic nature of the generating series $\tilde{Q}(z)$. They fall off exponentially in n, with the corresponding decay rate K vanishing both as $r \to 0$ and as $r \to 1$, and being maximal for $r = 2/5$.

Whenever the step length distribution is not continuous, the survival probabilities are no longer universal, but rather depend on details of the underlying distribution.

The class of arithmetic step length distributions, that is, discrete probability distributions yielding random walks on the lattice of integers, deserves special interest. The simplest of all lattice walks is the simple Polya walk, for which a thorough investigation of the survival probability Q_n is given in section 5, both at finite times and in the asymptotic regime of late times. The Q_n are again polynomials of degree n in r, whose structure depends on the parity of n. They fall off exponentially in n, with the corresponding decay rate K being maximal for $r = \sqrt{2} - 1$. In the generic situation of lattice walks with a larger range, we have shown that quantities such as the generating series $\tilde{Q}(z)$ and the asymptotic enhancement factor E are algebraic functions of the model parameters, and we determined their respective degrees. Algebraic functions were shown to play a key role in a germane problem, namely the encounters of the simple Polya walk with a ballistic obstacle [15]. The algebraic approach can be expected to facilitate further investigations of lattice walks.

References

[1] Godrèche C and Luck J M 2022 Maximum and records for random walks with stochastic resetting arXiv:2203.01102

[2] Evans M R, Majumdar S N and Schehr G 2020 *J. Phys. A: Math. Theor.* **53** 193001

[3] Sparre Andersen E 1953 *Math. Scand.* **1** 263–285

[4] Sparre Andersen E 1954 *Math. Scand.* **2** 194–222

[5] Feller W 1971 *An Introduction to Probability Theory and its Applications* 2nd ed vol 2 (New York: Wiley)

[6] Mounaix P, Majumdar S N and Schehr G 2020 *J. Phys. A: Math. Theor.* **53** 415003

[7] Godrèche C and Luck J M 2001 *J. Stat. Phys.* **104** 489–524

[8] Evans M R and Majumdar S N 2011 *Phys. Rev. Lett.* **106** 160601

[9] den Hollander F, Majumdar S N, Meylahn J M and Touchette H 2019 *J. Phys. A: Math. Theor.* **52** 175001

[10] Majumdar S N, Mori F, Schawe H and Schehr G 2021 *Phys. Rev. E* **103** 022135

[11] Majumdar S N, Mounaix P, Sabhapandit S and Schehr G 2022 *J. Phys. A: Math. Theor.* **55** 034002

[12] Bostan A, Chyzak F, Salvy B, Lecerf G and Schost E 2007 Differential Equations for Algebraic Functions *Proceedings of the 2007 International Symposium on Symbolic and Algebraic Computation* (New York: Association for Computing Machinery) (*Preprint* `arXiv:cs/0703121`)

[13] Spitzer F 1960 *Trans. Amer. Math. Soc.* **94** 150–169

[14] Spitzer F 2001 *Principles of Random Walk* (New York: Springer)

[15] Bauer M, Godrèche C and Luck J M 1999 *J. Stat. Phys.* **96** 963–1019