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# Numerical study of the dynamics of some long range spin glass models

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**Abstract.** We present results of a Monte Carlo study of the equilibrium dynamics of the one dimensional long-range Ising spin glass model. By tuning a parameter  $\sigma$ , this model interpolates between the mean field Sherrington-Kirkpatrick model and a proxy of the finite dimensional Edward-Anderson model. Activated scaling fits for the behavior of the relaxation time  $\tau$  as a function of the number of spins  $N$  (Namely  $\ln(\tau) \propto N^\psi$ ) give values of  $\psi$  that are not stable against inclusion of subleading corrections. Critical scaling ( $\tau \propto N^\rho$ ) gives more stable fits, at least in the non mean field region. We also present results on the scaling of the time decay of the critical remanent magnetization of the Sherrington-Kirkpatrick model, a case where the simulation can be done with quite large systems and that shows the difficulties in obtaining precise values for dynamical exponents in spin glass models.

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## 1. Introduction

The low temperature phase of spin glasses is characterized by an extremely slow dynamics. A popular method to study this dynamics consists of Monte Carlo simulations of small systems, followed by a finite size scaling analysis, leading to the determination of scaling exponents. In this paper we will show the difficulties in obtaining precise values for these dynamical exponents with two examples.

The first problem is the equilibrium dynamics of the one-dimensional long-range Ising spin glass (1dLR) model. The one-dimensional long-range Ising spin glass model [1] with  $N$  sites is a generalization of the Sherrington-Kirkpatrick spin-glass (SK) model [2]. The spins ( $\sigma_i = \pm 1$  with equal probability) are placed equidistantly on a circle of circumference  $N$ , and the Hamiltonian is given by the expression

$$\mathcal{H} = - \sum_{1 \leq i < j \leq N} \sigma_i \sigma_j J_{i,j} , \quad (1)$$

where the  $J_{i,j}$  are independent quenched random couplings distributed according to

$$J_{i,j} = c_N(\sigma) \epsilon_{i,j} / r_{i,j} , \quad r_{i,j} = N / \pi \sin((i - j)\pi / N) , \quad (2)$$

where the  $\epsilon_{i,j}$  are iid random variables with zero mean and unit variance (namely  $E(\epsilon_{i,j}) = 0$  and  $E(\epsilon_{i,j}^2) = 1$ ). The constant  $c_N$  is determined by the normalization condition  $\sum_{j \neq 1} E(J_{1,j}^2) = 1$ . Depending on the value of the real parameter  $\sigma \geq 0$ , the model behaves as follows:

- $0 \leq \sigma < 1/2$ . The static properties of the model do not depend on  $\sigma$  [3, 4] and are thus the same as for the SK model (obtained for  $\sigma = 0$ ). In particular the critical temperature is  $T_c = 1$  independent of  $\sigma$ . This is the SK like region;
- $1/2 \leq \sigma \leq 2/3$ . The model is in a mean field phase below some  $T_c > 0$ ;
- $2/3 \leq \sigma \leq 1$ . The model is in a non mean field phase below some  $T_c > 0$ ;
- $1 < \sigma$ . The model has no phase transition, namely  $T_c = 0$ .

This model has been extensively studied numerically (both in this version [5]-[8] and in dilute versions [9, 10] that are more suitable for efficient numerical simulations) since its behavior as a function of  $\sigma$  is analogous to the behavior of the Edwards Anderson Ising (EAI) spin glass model (the nature of the low temperature of this model has been the subject of a controversy for many years) as a function of the dimension  $d$ : in the 1dLR model the value  $\sigma = 2/3$  plays the role of the upper critical dimension of the EAI model ( $d_{ucd} = 6$  for this model) and the value  $\sigma = 1$  plays the role of the lower critical dimension ( $d_{lcd} = 2$  for this model). The hope has been expressed that there is an exact correspondence between  $\sigma$  and  $d$  and that there is in particular two values of  $\sigma$  defining 1dLR models with the same universal properties as the EAI model in dimensions 4 and 3 respectively. This hope is not substantiated by recent numerical results (using the diluted version of the model) however [10].

In a recent paper, Monthus and Garel [11] give numerical evidence that the equilibrium relaxation time  $\tau$  of the 1dLR model with Gaussian distributed quenched couplings scales like  $E(\ln(\tau)) \propto N^\psi$  with  $\psi \approx 1/3$ , like in the SK model (see [12, 13, 14, 15] and references therein), for all values of  $\sigma \in [0, 1]$ . This is a surprising result since the static properties of the model do depend on  $\sigma$  for  $\sigma > 2/3$ , and it contradicts the later analytical prediction that [16]  $\psi = 1 - \sigma$  in this region. The results of [11] have been obtained for the temperature  $T = T_c/2$ , but it is natural to expect that  $\psi$  is temperature independent below  $T_c$ . In [11] the relaxation time is defined from the long time exponential decay of the equilibrium spin spin autocorrelation function  $q_J(t)$  and is obtained numerically, for a given disorder sample  $J$ , using an eigenvalue technique introduced in [15]. The systems studied in [11] are however very small with  $6 \leq N \leq 20$ . Our purpose is to investigate the question further by direct Monte Carlo simulation of the dynamics of the model, a method that allows using quite larger system sizes. It is however subject to thermal errors and implies the use of the median to analyze the data, instead of the average, due to the presence of a tail of very slow disorder samples, whose relaxation time cannot be measured practically with our method.

The second problem is the critical relaxation of the SK model. This is a particularly interesting phenomenon from a numerical point of view since the scaling exponents are exactly known and, as there is neither the need to equilibrate the system nor to sample thoroughly the phase space, we can simulate quite large systems, by spin glass standards at least.

If a very strong constant magnetic field is applied to a spin glass, the individual spins align along this magnetic field. If afterward this magnetic field is switched off, the spin glass relaxes very slowly towards a state of non-zero remanent magnetization, with some excess internal energy relative to the internal energy at equilibrium. This phenomenon has been the object of detailed experimental studies [17].

At the critical point this phenomenon is simpler and well understood analytically [18, 19] for the SK model: Starting at time  $t = 0$  from a configuration where all spins are aligned, both the magnetization  $m(t)$ , the overlap between two clones  $q(t)$  and the internal energy  $e(t)$  relax algebraically towards their equilibrium values with simple exponents, namely  $m(t) \propto t^{-\delta_m}$ ,  $q(t) \propto t^{-\delta_q}$ ,  $e(t) - e(\infty) \propto t^{-\delta_e}$ , with  $\delta_e = \delta_q = 1$  and  $\delta_m = 5/4$ . It has been argued that below  $T_c$ ,  $m(t)$ ,  $q(t)$  and  $e(t)$  relax in two steps, first algebraically towards a non equilibrium  $N$  dependent value and then, on some exponentially large time scale (out of reach of numerical simulations), towards their equilibrium value [20, 18], however the numerical situation is not clear [21].

The outlook of this paper is as follows: in a first section we study the equilibrium dynamics of the 1dLR model. The precise determination of the scaling law governing the growth with  $N$  of the relaxation time of this model turns out to be quite difficult. In a second section we turn to another problem, namely the decay of the critical remanent magnetization of the SK model. This is an enlightening case where the exact values of the scaling exponents are known and we can test the precision of the Monte Carlo method. In a final section we present our conclusions.

## 2. Equilibrium dynamics of the 1dLR model

We measure the value of the spin spin autocorrelation function  $q_J(t)$  at equilibrium, and define the relaxation time  $\tau_J$  as the (unique) solution of the equation  $q_J(\tau_J) \equiv 1/2\sqrt{\langle q^2 \rangle_J}$ , where the average overlap squared is measured in the same disorder configuration. We have shown [14] that the alternative definition involving the disorder averaged  $\langle q^2 \rangle$ , namely  $q_J(\tau_J) \equiv 1/2\sqrt{E(\langle q^2 \rangle_J)}$  gives very similar results for the system sizes considered. As discussed in [13] the method we are using assumes that  $q(t)/\sqrt{\langle q^2 \rangle} \approx G(t/\tau)$  with some function  $G(\cdot)$ .

We now proceed to give some technical details of our simulation: The autocorrelation function  $q_J(t)$  is measured for integer values of the argument inside a time window of size  $W_{T,N} = 6000 R_{T,N}$ , where  $R_{T,N}$  is an integer scale factor adjusted in such a way that the window width scales roughly like  $\tau$  when the temperature  $T$  and the number of spins  $N$  vary, specifically we enforce the relation  $W_{T,N} \gtrsim 10 \text{ median}(\tau)$ . For small values of  $\tau$  however,  $R_{T,N}$  sticks to its lowest possible value  $R_{T,N} = 1$ , and the bound that we impose is definitively not saturated. Inside this time window  $q_J(t)$  is measured for 60 values of the argument (that are multiple of  $R_{T,N}$ ) whose logarithms are roughly uniformly distributed. In our computer program the relaxation time  $\tau_J$  is defined as the smallest integer  $t_{up}$  multiple of  $R_{T,N}$  such that  $q_J(t_{up}) < 1/2\sqrt{\langle q^2 \rangle_J}$ . As a check of our procedure we have also measured  $t_{int}$ , the result of a linear interpolation between  $t_{down} \equiv t_{up} - R_{T,N}$  and  $t_{up}$ , truncating the result to the lowest integer, namely in C language style notations:

$$t_{int} \equiv (int) \left[ t_{down} + \frac{q_J(t_{down})(t_{up} - t_{down})}{q_J(t_{down}) - q_J(t_{up})} \right]. \quad (3)$$

The difference between  $t_{up}$  and  $t_{int}$  gives an idea of the systematic errors induced by our selection of values of  $t$  for the measurement of  $q_J(t)$ . Needless to say we must make such a selection and cannot compute and store  $q_J(t)$  for all values of  $t$  inside the chosen window. We note that in contrast to our approach the method of [11] gives real valued relaxation times.

We take the quenched couplings to be Gaussian distributed, like in [11] (but in the  $\sigma = 0$  case, where we consider binary distributed couplings, in order to compare with [13, 14]), and consider  $N_{dis}$  independent disorder samples, with  $N_{dis} = 1024$  in most cases. We first bring the system to equilibrium, using the parallel tempering algorithm, that is currently the best existing algorithm for this purpose. The lowest temperature is  $T_c/2$  and we have  $N_T = 16$  values of  $T$  separated by a fixed interval  $\Delta T = T_c/10$ , namely the highest temperature is  $T = 2T_c$ , well inside the paramagnetic phase where the dynamics is fast. We perform  $2 \cdot 10^5$  parallel tempering sweeps (a parallel tempering sweep consists of  $N_T - 1$  conditional swaps of configurations plus one Metropolis system sweep), the second half being used to measure static quantities like  $\langle q^2 \rangle$  that we will use later. For the largest systems, we have used 31 values of the temperature separated by the fixed interval  $\Delta T = T_c/20$ , and perform  $4 \cdot 10^5$  parallel tempering sweeps. We have monitored the time  $n_T^c$  spent by a given Markov chain  $c$

at each temperature. With the parallel tempering algorithm as  $N$  increases, at fixed  $\Delta T$ , one notices that these  $(N_T)^2$  numbers, as measured inside the simulation, spread more and more around their common mean. Very soon some  $n_T^c$  becomes zero and the algorithm clearly breaks down. In our simulation, out of many thousand chains, no chain has a ratio  $r_c \equiv \max_T(\{n_T^c\})/\min_T(\{n_T^c\}) > 64$ , and only 3 a ration above 32, a reasonable result according to our experience with this algorithm, if not ideal. This equilibrium procedure is done for two independent copies of the system with the same disorder samples, two clones.

Next we measure (for each clone) the autocorrelation function  $q_J(t)$  along a long chain of length  $N_{sweeps}$  of at least  $10^7$  Metropolis sweeps, and always larger than ten window length. Measurements are made every  $5R_{T,N}$  sweeps. We have checked that a ten fold increase in the chain length changes very little to the estimated median and to the estimated statistical error on this median, indeed in this simulation the main source of statistical error is the fluctuations of the disorder and not the thermal noise [13, 14]. The data for  $q_J(t)$  are averaged over the two clones.

We base our statistical analysis of the dynamics of the model on the median of the relaxation time distribution. Considering the median rather than the average leads to an immense saving in computer time: we only need the value of  $\tau$  for at least the 50% faster disorder samples (samples with the smaller values of  $\tau$ ), and can forget about the slower samples (that we take to have  $\tau = +\infty$ ). Our empirical choice of window length turns out to be large enough for this aim.

To compute the median we first sort the  $N_{dis}$  data for  $\tau$ , and define the median as the average of the values for data number  $N_{dis}/2 - 1$  and  $N_{dis}/2$  (In our simulation the number of disorder samples  $N_{dis}$  is always even). For small system sizes usually the values of  $\tau$  for the data number  $N_{dis}/2 - 1$  and  $N_{dis}/2$  are the same (remember that our  $\tau$ 's are integer valued), and the median is accordingly not a very good measure. With the parameters we have chosen for the time window however this never occurs for larger systems (that are what matters for our analysis of the scaling of the relaxation time). The statistical errors on  $\text{median}(\ln(\tau))$  are obtained from a bootstrap analysis. We have done the statistical analysis for both  $t_{int}$  and  $t_{up}$  prescriptions and found results that differ by quite less than the estimated statistical error. This check is important as it shows that our selection of values of  $t$  for the measurements does not produce significant systematic errors. The plots presented here have been made using  $t_{int}$ .

In order to compare directly with the results of [11] we have performed runs at four values of  $\sigma$ , namely  $\sigma = 0$  (SK model),  $\sigma = 0.25$  (inside the SK like region),  $\sigma = 0.75$  and  $\sigma = 0.85$  both in the non mean field region.

Let us start with the 1dLR  $\sigma = 0.75$  and  $T = 0.7$  case. We have data for systems with  $N = 8, 16, 24, \dots, 1024$ . We fit these data, using gnuplot 4.6, to the usual two-parameter form  $\text{median}(\ln(\tau)) = CN^\psi$ . We have monitored the evolution of the fitted value of  $\psi$  and of the reduced chi squared  $\chi^2/ndf$ , as  $N_{min}$  the lowest value of  $N$  included in the fit is increased. A stable plateau is reached for  $\psi$  when  $N_{min} = 128$ , with a reasonable reduced chi squared  $\approx 0.9$ . The value of  $\psi$  for such a fit ( $128 \leq N \leq 1024$ )

$\sigma$	$\mathbb{P}(J)$	Upgrading	$N_{max}$	$\ln(\tau)$
0.75	Gauss	Random	1024	12.0
0.75	Gauss	Systematic	512	11.1
0.25	Gauss	Systematic	1024	8.5
0	$\pm 1$	Random	768	8.4
0	$\pm 1$	Systematic	768	7.4

**Table 1.** Details of the simulation of the 1dLR model (with  $T = 0.7T_c$ ):  $\sigma$ , coupling probability distribution, type of upgrading, largest system size simulated, and the median of the relaxation time distribution for  $N = 512$ . The systematic upgrading leads to a faster dynamics.

is  $\psi = 0.20$ , with an estimated statistical error ( $\pm 0.003$ ) on the third digit (But the systematic errors are larger, as we will see later). As an example of the stability of the fit we note that a quite lower cutoff  $N_{\min} = 32$  would give  $\psi = 0.22$  (but an unacceptable  $\chi^2/ndf \approx 6$ ). Figure 1 shows a plot of  $\ln(\text{median}(\ln(\tau)))$  as a function of  $\ln(N)$  together with the best two-parameter fit. There are sizable deviations of the data from the fitted curve for low values of  $N$  (and a fit to the low portion of the curve would give a much larger value of  $\psi$  than the value we obtain). It is natural to try a fit including some subleading corrections. The data are however not good enough for a fit to the sum of two power laws, and we have tried a fit to a power law plus a constant, namely  $\text{median}(\ln(\tau)) = A + CN^\psi$ . We find a plateau for  $\psi$ , with a reasonable reduced chi squared  $\approx 0.3$ , for  $N_{\min} = 32$ . We will use in the following the same values of  $N_{\min}$ , namely 128 and 32 for fits with two parameters and with three parameters respectively. The value of  $\psi$  we obtain with this three-parameter fit is  $\psi = 0.10 \pm 0.01$ . This is many standard deviations away from the previous result.

We have redone the simulation using systematic upgrading for the dynamics rather than random upgrading, still using the same thermalized initial configurations (but with a maximal value of 512 for  $N$ ). In the former case the spin configuration at time  $t + 1$  is obtained from the spin configuration at time  $t$  by applying the (single spin) algorithm to every sites successively and in a fixed order. In the later case the (single spin) algorithm is applied to a spin chosen at random, this procedure being repeated exactly  $N$  times. It is well known that the former does not satisfy detailed balance (but do preserve the equilibrium state) whereas the later does preserve detailed balance. It is usually assumed that both dynamics lead to the same scaling exponents, due to universality, and we assume that this is true and compare the results from the two dynamics, in order to have an idea of systematic errors, obviously not included in the gnuplot statistical error estimates. We first remark that the difference between the random and systematic updating estimates decreases as  $N_{\min}$  is increased, as it should. A two-parameter fit of the systematic dynamics data ( $128 \leq N \leq 512$ ) gives  $\psi = 0.22 \pm 0.1$  and a three-parameter fits ( $32 \leq N \leq 512$ ) gives  $\psi = 0.08 \pm 0.01$ . Both

results are in reasonable agreement with their random upgrading counterparts, but leave us with some unexplained discrepancy between the two-parameter and three-parameter fits.

The fact that we obtain very small values of  $\psi$  with the three-parameter fit indicates that may be  $\psi$  is indeed exactly zero, and that we have critical scaling, namely  $\tau \propto N^\rho$ . It turns out that critical scaling fits work nicely. A two-parameter critical fit for random update gives  $\rho = 2.29 \pm 0.05$  with a fair reduced chi squared ( $\approx 1.4$ ) for  $N_{\min} = 128$ , and a three-parameter fit gives  $\rho = 2.24 \pm 0.03$  for  $N_{\min} = 32$  (see Fig. 2) with a fair reduced chi squared ( $\approx 1.5$ ). Systematic upgrade gives  $\rho = 2.15 \pm 0.04$  and  $\rho = 2.13 \pm 0.02$  respectively. There is a nice overall agreement between the above four estimates of  $\rho$ , with systematic errors of few percents.

The values obtained for the exponent  $\rho$  are not absurd, it has been argued that the non equilibrium correlation length of the 3d [22] and 4d [23] EAI model grows like  $\xi(t) \propto t^{1/z(T)}$  where  $t$  is the time since the quench, and  $z(T) \approx z(T_c)T_c/T$  with  $z(T_c) = 6.86 \pm 0.16$  in 3d and  $z(T_c) \approx 5.4$  in 4d. That  $\xi(t) \propto t^{1/z(T)}$  in an infinite volume non equilibrium situation means hand-wavily that at time  $t$  the length scales below  $\xi(t)$  are equilibrated. Crudely speaking it indicates that at equilibrium on a system of size  $L$  the relaxation time  $\tau$  fulfills the relation  $L \propto \tau^{1/z(T)}$ , namely  $\rho = z(T)/d$ , not far from the values of the critical scaling fits of our 1dLR data (according to [10] the 1dLR model with  $\sigma = 0.79$  is a proxy to the 4d EAI model, and  $z(T_c) \approx 5.4$  translates into  $\rho(T/T_c = 0.7) = 5.4/(0.7 d) = 1.9$ ).

We have made simulations for  $T = 0.5$  with the same value of  $\sigma$  but we can only simulate systems up to  $N = 192$ , where already  $\text{median}(\ln(\tau)) = 15.2$ , namely  $\text{median}(\tau) \approx 4 \cdot 10^6$ , compared to  $\text{median}(\ln(\tau)) = 13.8$  for  $N = 1024$  in the  $T = 0.7$  case. Such a value for  $N$  is (juggling from the  $T = 0.7$  case) quite too small to obtain sensible results for  $\psi$ . We also made preliminary runs for  $\sigma = 0.85$  with  $T = 0.7$  and  $0.5$  with the same no-go conclusion.

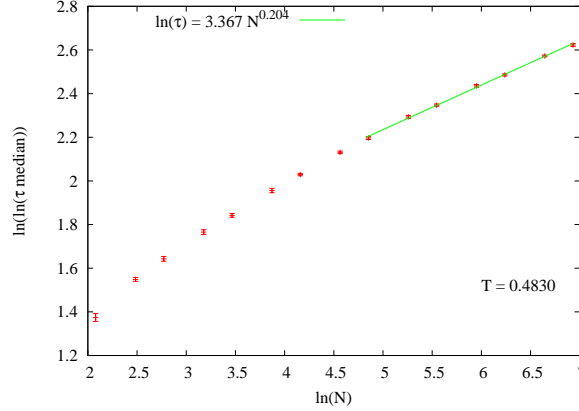
Simulations for lower values of  $\sigma$  are easier. We have data for  $\sigma = 0.25$  and  $T = 0.7$ , namely in a region where the statics is the same as the one of the SK model. We use random upgrading with systems up to  $N = 1024$ . Two-parameter fits give  $\psi = 0.25 \pm 0.01$ . Three-parameter fits give  $\psi = 0.19 \pm 0.02$  (see Fig. 3). In this case there is no satisfactory critical fit: a two-parameter fit (still with  $N \geq 128$ ) has a reduced chi squared of 5, and a three-parameter fit ( $N \geq 32$ ) a reduced chi squared of 5 again.

In order to compare with existing results for the SK model we have done a simulation of this model with binary distributed couplings, using systematic upgrading and  $N$  up to 768. A two-parameter fit gives  $\psi = 0.30 \pm 0.01$  (see Fig. 4), in agreement with the existing literature [12, 13, 14]. A three parameter fit gives however a smaller value  $\psi = 0.17 \pm 0.03$ . Repeating the simulation with random upgrading we obtain  $\psi = 0.26$  and  $\psi = 0.17 \pm 0.02$  with two-parameter and three-parameter fits respectively. In this case again, no satisfactory critical fit is obtained.

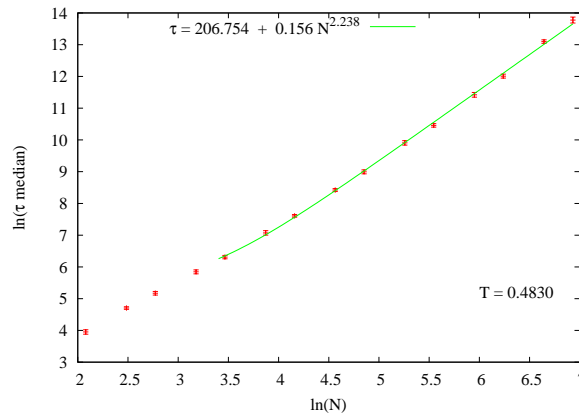
The conclusion of this section is that our data disagree with the claim that  $\psi = 1/3$  in the spin glass phase for all  $0 \leq \sigma < 1$ . We are left with the following possibilities



i) Either the activated three-parameter fits are not trustworthy, and  $\psi$  do decrease from a value close to  $1/3$  for the SK case to lower values for larger values of  $\sigma$ ; ii) Or  $\psi$  is quite smaller than previously thought, possibly exactly zero in the non mean field region; iii) Another possibility is that the coefficient in front of  $N^\psi$  is extremely small as found in [24] for the SK model at  $T = T_c/20$ . In order to distinguish between these scenarios one would need some new ideas, as brute force is not a possibility here.



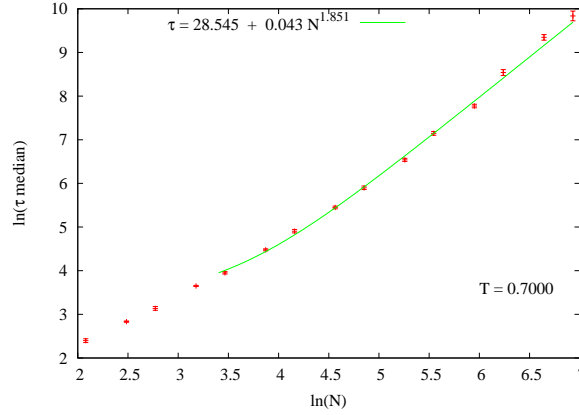
**Figure 1.** Data for  $\ln(\text{median}(\ln(\tau)))$  as a function of  $\ln(N)$  for the 1dLR model with  $\sigma = 0.75$ ,  $T = 0.7 T_c$  and values of  $N$  between 8 and 1024. The quenched random couplings are Gaussian distributed and the random updating scheme is used. The green line (color on-line) is a two-parameter activated fit of the data for  $N \geq 128$ .



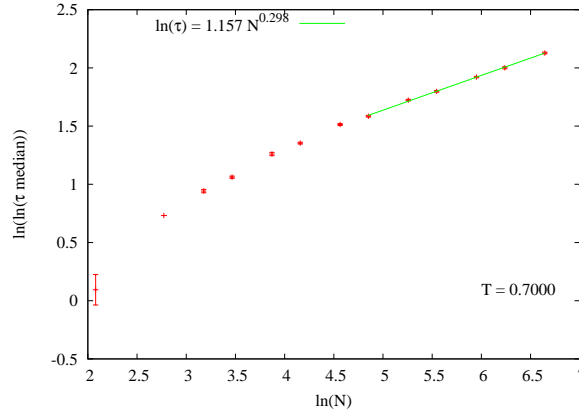
**Figure 2.** Data for  $\text{median}(\ln(\tau))$  as a function of  $\ln(N)$  for the 1dLR model with  $\sigma = 0.75$  and  $T = 0.7 T_c$  as in Fig. 1. The green line (color on-line) is a three-parameter critical fit of the data for  $N \geq 32$ .

### 3. The relaxation of the critical SK model

We simulate the SK model with binary distributed couplings, using the Heat Bath algorithm with random site updating, starting from a configuration where all spins are set to one. The number of sites  $N$  ranges from  $N = 1024$  to  $2^{17} = 131072$ . We perform



**Figure 3.** Same plot as Fig. 2 but in the SK phase. The critical ansatz fails to reproduce the data. Here  $\sigma = 0.25$ ,  $T = 0.7 T_c$  and values of  $N$  are between 8 and 1024. The quenched random couplings are Gaussian distributed and the random updating scheme is used. The green line (color on-line) is a three-parameter scaling fit of the data for  $N \geq 32$ .

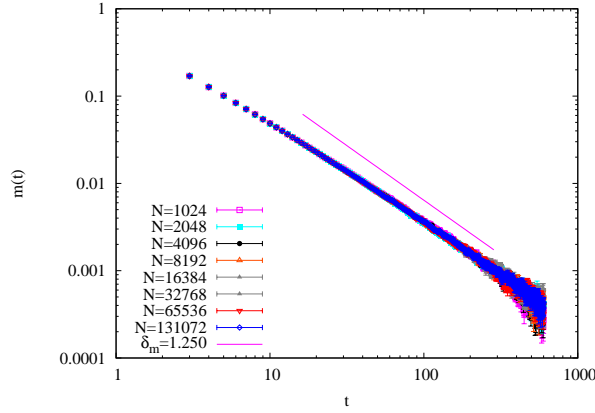


**Figure 4.** Data for  $\ln(\text{median}(\ln(\tau)))$  as a function of  $\ln(N)$  for the 1dLR model with  $\sigma = 0$  (SK model),  $T = 0.7 T_c$  and values of  $N$  between 8 and 768. The quenched random couplings are binary distributed and the systematic updating scheme is used. The green line (color on-line) is a two-parameter activated fit of the data for  $N \geq 128$ .

600 Monte Carlo time steps, and average the results over  $N_{dis} = 1310720/N$  disorder samples. This scaling of  $N_{dis}$  is such that the estimated statistical errors are roughly  $N$  independent.

In figure 5 we show  $m(t)$ , the magnetization as a function of  $t$ . The data show no meaningful finite size dependence, and are in rough agreement with a  $m(t) \propto t^{-5/4}$  behavior. They show however some bending that makes the precise determination of the exponent  $\delta_m$  ambiguous. Data for  $e(t)$  and  $q(t)$  are in similar agreement with the expected exponents  $\delta_e = \delta_q = 1$ .

In figure 6 we emphasize the deviations of the behavior of  $m(t)$  from the expected law by plotting  $m(t)t^{5/4}$  as a function of  $t$  for our largest systems. Note that the ordinate range was between  $10^{-4}$  and 1 in Fig. 5 and is now between 0.6 and 1.5 in Fig. 6.

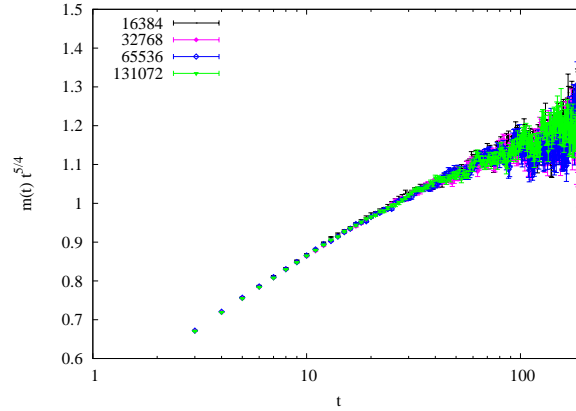


**Figure 5.** Decay of the magnetization  $m(t)$  as a function of  $t$  at the critical point for the SK model. The predicted scaling  $m(t) \propto t^{-5/4}$  is shown. This is in rough but not perfect agreement with the data.

The ratio clearly depends on  $t$  up to  $t \approx 100$ , where the noise become overwhelming. According to [19], the magnetization should scale with finite  $N$  like

$$m(t) = N^{-5/6} F_m(t/N^{2/3}) . \quad (4)$$

The ratio  $m(t)t^{5/4}$  in figure 6 shows no finite size effect but is clearly not independent of  $t$ . This means that we have sizable scaling violation, and we cannot determine the exponent from the small  $t/N^{2/3}$  behavior of  $m(t)$  in a scaling plot, as is usually done (see e.g [25]).



**Figure 6.** Ratio of  $m(t)$  as in Fig. 5 divided by the expected  $t^{-5/4}$  behavior, as a function of  $t$ .

To go further on, one should study the behavior of  $m(t)$  for larger values of  $t$ . This led us to perform additional simulations of the largest systems  $N = 16384, 32768, 65536$ , and  $131072$  with two improvements: the first is to replace [18] the magnetization, namely the overlap between spins at time  $t$  and spins at time 0, by the overlap between spins at time  $t$  and spins at time  $t_w = 3$  (the precise value of  $t_w$  being

irrelevant),

$$m(t) = \frac{1}{N} \sum_i \sigma_i(t) \sigma_i(t_w) , \quad (5)$$

this should not change the exponents governing the large  $t$  behavior, but reduce considerably the SN ratio by increasing the signal. Next we average the data for  $m(t)$  in bins of given integer part of  $\ln(t/4)/(\ln(2)/2)$ , this reduce the fluctuations and does not affect the large  $t$  behavior. We have also multiplied the number of disorder samples by 16 and the number of time steps by 4 for  $N = 16384$ , and 32768. The data can be found in Fig. 7, where both  $m(t)$  and  $t$  are averages inside bins. The points have been slightly shifted (horizontally) in a  $N$  dependent way, in order to increase the clarity of the figure. Up to  $t \approx 100$  there is no visible finite size effect (even with the 16 fold increase of the number of disorder samples, and the corresponding decrease of the estimated errors) and the data are steadily increasing with  $t$ . We interpret the data for larger values of  $t$  as the onset of the asymptotic  $m(t) \propto t^{-5/4}$  behavior, and not as a finite size effect, the argument being that if it was a finite size effect the data for  $N = 16384$  should be below the one at  $N = 32768$ . The question should hopefully be settled by extending the time range to say  $t = 10000$  with excellent precision up to  $N = 131072$ . This would be however a formidable task. Using current data it is interesting to show a scaling plot of  $m(t)t^{5/4}$  as a function of  $t/N^{2/3}$  (see Fig. 8). Strong violations of scaling are apparent for small values of the argument. The data can be interpreted as a very slow approach to scaling as  $N$  grows, towards a limit where  $m(t)t^{5/4}$  is independent of  $t$ . This shows how difficult a precise determination of  $\delta_m$  is. Even the modest task of determining the exponent governing the first non leading term, assuming that the leading term is exactly  $\delta_m = 5/4$  is difficult, as the statistical errors on  $m(t)$  increase strongly with  $t$ . Fig. 8 shows, as an example, the result of a fit of the  $N = 32768$  data to the form

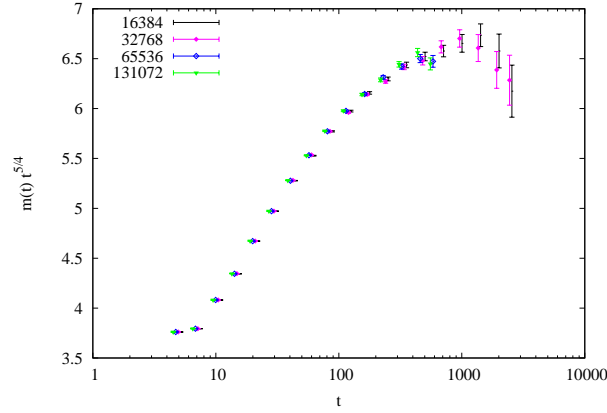
$$m(t)t^{5/4} = a - b(N^{2/3}/t)^\mu, \quad (6)$$

in the range  $t/N^{2/3} > 0.4$ . Here  $\mu = 0.51 \pm 0.01$ . The estimates of the value of this exponent and of the plateau height depend strongly on the range of data included in the fit however.

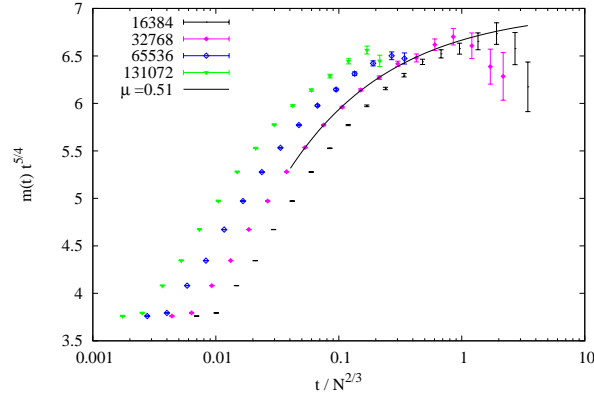
As a side remark we note that we made some preliminary runs using Metropolis instead of Heat Bath and/or systematic updating instead of random updating. It turns out that scaling violations (measured for example as the ratio of  $m(t)t^{5/4}$  between  $t/N^{2/3} = 0.01$  and 0.1) are smaller with systematic updating. It is expected that scaling violations are not universal and can accordingly depend significantly on the choice of a dynamics, we have no explanation however why systematic updating gives less scaling violations than random updating.

#### 4. Conclusions

We study by Monte Carlo method the equilibrium dynamics of the one dimensional long-range Ising spin glass model. Varying the parameter  $\sigma$  governing the decay with



**Figure 7.** Magnetization  $m(t)$  divided by the expected  $t^{-5/4}$  behavior, as a function of  $t$  for our largest systems. Here  $m(t)$  is defined as the overlap between configurations at time  $t$  and  $t_w = 3$ , and the data for both  $m(t)$  and  $t$  are binned (see text). The points have been slightly horizontally shifted in a  $N$  dependent way, in order to increase the clarity of the figure.



**Figure 8.** Scaling plot of  $m(t)t^{5/4}$  (as in Fig. 7) as a function of  $t/N^{2/3}$ . The data can be interpreted as showing a very slow  $N \rightarrow \infty$  limiting constant behavior. The result of a fit of the  $N = 32768$  data to the form  $m(t)t^{5/4} = a - b(N^{2/3}/t)^\mu$  is also shown. The data for the largest values of  $t/N^{2/3}$  have a negligible weight in the fit.

distance of the spin spin coupling of this model is similar to varying the dimensionality of the canonical Edwards-Anderson Ising spin glass model. We extend up to systems with  $N = 1024$  spins the results obtained by Monthus and Garel in [11], who argued that the dynamics is activated with  $\ln(\tau) \propto N^\psi$  with  $\psi = 1/3$  for all values of  $\sigma$ , from the Sherrington-Kirkpatrick limit ( $\sigma = 0$ ) to the physical dimension. We find some unexpected instability in the fits, possibly indicating critical scaling for  $\tau$  (namely  $\tau \propto N^\rho$ ) at least in the non mean-field region. It would mean that the 1dLR model dynamics is similar to the EAI dynamics in the RSB picture.

We then turn to an illustrative example of the general difficulties of obtaining precise estimates of scaling exponents from numerical simulations of disordered systems, by studying the decay of the critical remanent magnetization of the Sherrington-Kirkpatrick

model. Here the exact value of the exponent is known and one can simulate systems of very large sizes (up to  $2^{17}$  here). In this example the asymptotic regime is only approached for systems with hundred of thousands spins, that are definitively out of reach in usual situations, including the equilibrium dynamics of the one dimensional long-range Ising spin glass model.

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- [1] G. Kotliar, P.W. Anderson and D.L. Stein: Phys. Rev. **B 27**, 602 (1983).
- [2] D. Sherrington and S. Kirkpatrick: Phys. Rev. Lett. **35** 1792 (1975); Phys. Rev. B **17**, 4384 (1978).
- [3] T. Mori: Phys. Rev. **E 84**, 031128 (2011).
- [4] M. Wittmann and A. P. Young: Phys. Rev. **E 85**, 041104 (2012).
- [5] H. G. Katzgraber and A. P. Young: Phys. Rev. **B 67**, 134410 (2003).
- [6] H. G. Katzgraber, M. Korner, F. Liers and A.K. Hartmann: Prog. Theor. Phys. Sup. 157, **59** (2005).
- [7] H. G. Katzgraber and A. P. Young: Phys. Rev. **B 72**, 184416 (2005).
- [8] M. A. Moore: Phys. Rev. **B 82**, 014417 (2010).
- [9] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi and J. J. Ruiz-Lorenzo: Phys. Rev. Lett. **101**, 107203 (2008).
- [10] R. A. Baños, L. A. Fernandez, V. Martin-Mayor and A. P. Young: Phys. Rev. **B 86**, 134416 (2012).
- [11] C. Monthus and T. Garel: Phys. Rev. **B 89**, 014408 (2014).
- [12] A. Billoire and E. Marinari: J. Phys. A Math. Gen. **34**, L727 (2001).
- [13] A. Billoire: J. Stat. Mech. P11034 (2010).
- [14] A. Billoire: J. Phys. A Math. Gen. **44**, 075001 (2011).
- [15] C. Monthus and T. Garel: J. Stat. Mech. P12017 (2009).
- [16] C. Monthus: J. Stat. Mech. P08009 (2014).
- [17] J.-J. Préjean and J. Souletie: J. Phys. France **41**, 1335 (1980); A. Berton, J. Chaussy, J. Odin., R. Rammal, and R. Tournier: J. Phys. France Lett. **40**, 391 (1982); R. Omari, J.-J. Préjean, and J. Souletie: J. Phys. France **44**, 1069 (1984).
- [18] G. Parisi, P. Ranieri, F. Ricci-Tersenghi and J.J. Ruiz-Lorenzo: Journal of Physics A: Math. Gen. **30**, 7115 (1997).
- [19] F. Caltagirone, G. Parisi and T. Rizzo: Phys. Rev. **E 87**, 032134 (2013).
- [20] E. Marinari, G. Parisi and D. Rossetti: The European Physical Journal B, Volume 2, **495** (1998).
- [21] A. Billoire, G. Parisi, and F. Ricci-Tersenghi, unpublished.
- [22] F. Belletti et al: Phys. Rev. Lett. **101**, 157201 (2008).
- [23] G. Parisi, F. Ricci Tersenghi and J.J. Ruiz-Lorenzo: Journal of Physics A: Math and Gen. **29**, 7943 (1996).
- [24] T. Aspelmeier, R. A. Blythe, A. J. Bray and M. A. Moore: Phys. Rev. **B 74**, 184411 (2006).
- [25] A. Billoire and I.A. Campbell: Phys. Rev. **B 84** 054442 (2011).