Non stationarity of renewal processes with power-law tails

Claude Godrèche

Non stationarity of renewal processes with power-law tails

Claude Godrèche

Institut de Physique Théorique, Université Paris-Saclay, CEA and CNRS, 91191 Gif-sur-Yvette, France

Abstract. Renewal processes generated by a power-law distribution of intervals with tail index less than unity are genuinely non stationary, even when the system, subjected to periodic boundary conditions, is translation invariant. This issue is illustrated by a critical review of the recent paper by Barma, Majumdar and Mukamel 2019 *J. Phys. A* **52** 254001, devoted to the investigation of the properties of a specific one-dimensional equilibrium spin system with long-range interactions. We explain why discarding the non-stationarity of the process underlying the model leads to an incorrect expression of the critical spin-spin correlation function.

1. Introduction

Consider the class of models defined as the succession of a fluctuating number of time, or space, independent and identically distributed (iid) intervals conditioned by the value of their sum. These intervals describe for example the lapses of time between some recurrent events, or spin domains for particular one-dimensional magnetic systems. The correlation function for such *tied-down renewal processes* has been recently investigated in two references [1, 2], with different methods and different results. The aim of this note is to clarify this issue.

Reference [1] revisits the TIDSI model, a truncated version of a microscopic one-dimensional spin model with long-range interactions, dubbed the inverse distance squared Ising (IDSI) model [3, 4, 5]. This TIDSI model, which has been investigated in a series of papers in recent years [6, 7, 8], is made of a fluctuating number of domains, filling up the total size L of the system. A configuration is thus entirely specified by the number and sizes of these domains. The Boltzmann weight associated to the realisation $\{\ell_1, \ell_2, \dots, \ell_n, n\}$ of such a configuration reads [1]

$$p(\ell_1, \ell_2, \dots, \ell_n, n; L) = \frac{y^n g(\ell_1) \dots g(\ell_n) \delta\Big(\sum_{i=1}^n \ell_i, L\Big)}{Z(L)}, \tag{1.1}$$

where the denominator

$$Z(L) = \sum_{n \ge 1} \sum_{\ell_1 \dots \ell_n} y^n g(\ell_1) \dots g(\ell_n) \delta\Big(\sum_{i=1}^n \ell_i, L\Big) \tag{1.2}$$

ensures the normalisation and $\delta(\cdot, \cdot)$ is the Kronecker delta. In (1.1), y denotes the fugacity and $g(\ell)$ is given by

$$g(\ell) = \frac{1}{\ell^{1+\theta}}, \tag{1.3}$$

where $\ell = 1, 2, \ldots$, and the tail index θ is positive. In [1, 6, 7] the phase diagram is analysed according to the values of the fugacity y and the index θ, which are the two parameters of the model. For $y = y_c$, such that

$$y_c = \frac{1}{\sum_\ell g(\ell)} = \frac{1}{\zeta(1+\theta)},$$

where $\zeta(\cdot)$ is the Riemann zeta function, the system is critical, separating a paramagnetic phase from a ferromagnetic one.

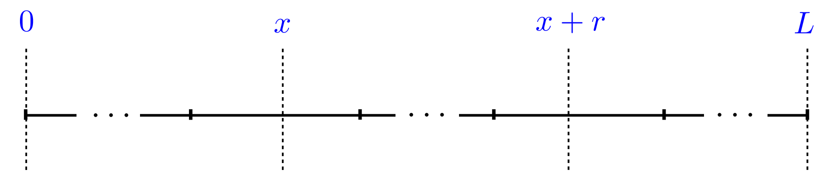

Figure 1. A tied-down renewal process is defined as the succession of a fluctuating number of time (or space) iid intervals between points (which can be events, domain walls, ...) conditioned to filling up the total size L of the system. The quantity of interest $p_m(x, r, L)$ is the probability that m points fall between the two arbitrary positions x and $x + r$. The correlation function $C(x, r, L)$ is defined in (3.1).

The main result of [1] is the expression of the *critical* correlation function between two spins at x and $x + r$, for $\theta < 1$, in the regime of short separations ($1 \ll r \ll L$), for a system with *periodic boundary conditions*. As stated in [1], since the system is translation invariant, this expression is independent of the position x of the first spin and only depends on the separation r. The expression derived in [1] is

$$C(r, L) \approx 1 - \frac{1}{\theta}\left(\frac{r}{L}\right)^{1-\theta}. \tag{1.4}$$

The same question, in more generality, was posed in [2] and solved by exact methods, for *open boundary conditions* (see figure 1). Restricting to the regime of short separations between x and $x + r$ ($1 \ll r \ll x \sim L$), the following result holds, when $\theta < 1$,

$$C(x, r, L) \approx 1 - \frac{\sin \pi\theta}{\pi(1-\theta)} \left[\frac{x}{L}\left(1 - \frac{x}{L}\right)\right]^{\theta-1} \left(\frac{r}{L}\right)^{1-\theta}, \tag{1.5}$$

giving clear evidence for the role of the position x, that is, for the non stationarity of the process.

The question therefore naturally arises of understanding the relationship, if any, between the two expressions (1.4) and (1.5). Averaging $C(x, r, L)$ in (1.5) on x yields

$$C(r, L) \approx 1 - \frac{\sin \pi\theta}{\pi(1-\theta)} \frac{\Gamma(\theta)^2}{\Gamma(2\theta)} \left(\frac{r}{L}\right)^{1-\theta}, \tag{1.6}$$

to be compared to (1.4). As will be explained later, averaging (1.5) on x is the correct way to take into account periodic boundary conditions, in case these are imposed on the system.

As we shall see, the central point, beyond the existence of a discrepancy between the two expressions (1.4) and (1.6), is the difference in nature of the methods used to derive them. While in [1] invariance by translation is invoked at the very start in order to justify the use of a stationary formalism where the dependence in x is discarded, in contrast, in [2] the dependence in x is essential when deriving (1.5), before any possible tracing of this expression upon x is performed, resulting in (1.6).

We shall explain below why the method used in [1] in order to derive (1.4)—the so-called *independent interval approximation*—is not exact, as claimed in this reference, and can only predict the scaling behaviour $C(r, L) - 1 \sim (r/L)^{1-\theta}$.

2. The class of tied-down renewal processes

Before discussing this issue, we first show that the TIDSI model defined by (1.1), (1.2), (1.3), *when taken at criticality*, belongs to the class of tied-down renewal processes considered in [2, 9] (independently of any considerations on the boundary conditions).

For this class of processes, a configuration is again specified by a fluctuating number of intervals (e.g., domains) $\tau_1, \tau_2, \ldots$ which are iid random variables with common probability distribution

$$f(\ell) = \mathrm{Prob}(\tau = \ell). \tag{2.1}$$

A realisation $\{\ell_1, \ell_2, \ldots, \ell_n, n\}$ of such a configuration, where the number of intervals takes the value n and the random variables τ_i take the values ℓ_i, has weight

$$p(\ell_1, \ell_2, \ldots, \ell_n, n; L) = \frac{f(\ell_1) \ldots f(\ell_n) \delta\Big(\sum_{i=1}^{n} \ell_i, L\Big)}{Z(L)}, \tag{2.2}$$

where

$$Z(L) = \sum_{n \geq 1} \sum_{\ell_1 \ldots \ell_n} f(\ell_1) \ldots f(\ell_n) \delta\Big(\sum_{i=1}^{n} \ell_i, L\Big). \tag{2.3}$$

These processes are renewal processes because the intervals are iid random variables, and they are tied down because these intervals are conditioned to sum up to a given value L, generalizing the tied-down random walk, starting from the origin and conditioned to end at the origin at a given time [10, 9] (see section 4). (For the tied-down random walk, intervals are temporal, while in the rest of this note all intervals are spatial.) The definitions (2.1), (2.2) and (2.3) easily generalize to the case where intervals are continuous random variables.

It is now evident that, at criticality, i.e., if $y = y_c$, (1.1) reduces to (2.2). Indeed, the product $y_c g(\ell)$ is normalised and can be identified with the probability distribution $f(\ell)$, where in this specific case we have

$$f(\ell) = \frac{1}{\zeta(1+\theta)} \frac{1}{\ell^{1+\theta}}. \tag{2.4}$$

Therefore, as already stated in [2, 9], the class of tied-down renewal processes defined by (2.1), (2.2) and (2.3) above, encompasses as a particular case the TIDSI model considered in [1, 6, 7, 8], at criticality, with the consequence that all results found in

[2, 9] can be immediately transposed to the latter. This applies, in the present case, to (1.5) and (1.6).

The fact, demonstrated above, that, at criticality, the TIDSI model belongs to the class of tied-down renewal processes was contradicted in [1]. As emphasised in this reference: *'A joint distribution similar to equation (10) was studied in the context of the tied-down renewal process [10], with the important difference that in the latter case the fugacity y was taken to be exactly 1. As we will see later, in our* TIDSI *model where y can vary, there is a mixed-order phase transition at a critical value $y = y_c$ (which need not be 1)'*‡.
This assertion does not hold true. The fact that $f(\ell)$ in (2.2) is a normalised probability distribution was missed in [1]. This corresponds to taking $y = y_c$ in (1.1), and not $y = 1$ as incorrectly stated in [1].

3. Translation invariance and non stationarity

We now analyse the nature of the methods leading respectively to (1.4) and (1.6). In section 3.1, we first define the correlation function $C(x, r, L)$ for an open system, we then explain the consequences of imposing periodic boundary conditions. The intermediate section 3.2 gives some preparatory material on renewal theory needed for section 3.3, where we give a concise overview of the method used in [1] in order to obtain (1.4), then conclude on its validity.

3.1. Correlation function for a finite system

Let us start with the open system $(0, L)$, depicted in figure 1. The correlation function is defined as

$$C(x, r, L) = \left\langle (-1)^{N(x,x+r)} \right\rangle = \sum_{m \geq 0} (-1)^m p_m(x, r, L), \tag{3.1}$$

where $N(x, x+r)$, the number of points (events, domain walls, ...) comprised between two arbitrary positions x and $x + r$, has probability distribution denoted as

$$p_m(x, r, L) = \mathrm{Prob}(N(x, x + r) = m).$$

As shown in [2], for a broad distribution $f(\ell)$ with tail index $\theta < 1$, if x, r, L are all simultaneously large, the correlation function $C(x, r, L)$ is dominated by $p_0(x, r, L)$, the probability that the interval $(x, x + r)$ does not contain any point. From now on we shall focus on this quantity. We assume that the first interval begins at site 0.

If periodic boundary conditions are imposed on the system, any configuration with weight (2.2) can be represented by (a fluctuating number of) intervals on a circle, where the two points 0 and L are identified. Since the position of this point on the circle is arbitrary, $p_0(x, r, L)$ should be averaged on x. Likewise (1.6) ensues from (1.5).

Another point should be checked beforehand. In the case of the TIDSI model, intervals represent spin domains. Therefore if the system is periodic, the number of domains between 0 and L should be even. It turns out that resuming the derivations made in [2] with n even does not change the expression (1.5), hence, after averaging on x, does not change (1.6) either.

‡ In the first sentence quoted above, equation (10) refers to (2.2), and reference [10] to [2].

Note finally that if L is sent to infinity, the conditioning disappears. So $p_0(x, r, L \to \infty) = p_0(x, r)$, which is the probability that the interval $(x, x+r)$ does not contain any point for an ordinary renewal process.

3.2. Elements of renewal theory

It is useful to recall some results of renewal theory before coming to the next section. Consider first an ordinary renewal process, i.e., without tied-down conditioning. The following dichotomy holds.

(i) If the distribution of intervals $f(\ell)$ has a finite first moment

$$\langle\tau\rangle = \int_0^\infty \mathrm{d}\ell\, \ell f(\ell), \tag{3.2}$$

e.g., when $f(\ell)$ is narrow, with finite moments, or broad with a power-law tail of index $\theta > 1$, then the process reaches a stationary regime, where $p_0(x, r) \to p_{0,\mathrm{stat}}(r)$, in the limit $x \to \infty$ (see e.g., [11]). In Laplace space

$$\mathop{\mathcal{L}}_{r} p_{0,\mathrm{stat}}(r) = \hat{p}_{0,\mathrm{stat}}(s) = \frac{1}{s} - \frac{1-\hat{f}(s)}{\langle\tau\rangle s^2}. \tag{3.3}$$

(ii) The situation is different when $f(\ell)$ is a broad distribution with tail index $\theta < 1$. Then $\langle\tau\rangle$ is no longer finite and $p_0(x, r)$ keeps a dependence in x even at large values of this variable. The process is genuinely non stationary [11, 12].

Now, let us come back to tied-down renewal processes (conditioned by the sum of intervals equal to L). The system is now finite. Let us denote by $\langle\tau\rangle^\star$ the first moment of the marginal distribution $f(\ell|L)$ of a single interval, obtained by tracing the full distribution $p(\ell_1, \ell_2 \cdots, \ell_n, n; L)$ on all ℓ_i but one. Alternatively, $\langle\tau\rangle^\star$ is equal to the product of L by the mean inverse number of intervals in $(0, L)$.

(i) If $\langle\tau\rangle$ is finite, then $\langle\tau\rangle^\star$ is asymptotically equal to $\langle\tau\rangle$ at large L.

(ii) If $\langle\tau\rangle$ is infinite, then [9]

$$\langle\tau\rangle^\star \approx \frac{\pi c}{\sin \pi\theta}\, L^{1-\theta}, \tag{3.4}$$

where c is the tail coefficient of the distribution $f(\ell) \approx c/\ell^{1+\theta}$. For the TIDSI model c is equal to y_c (see (2.4)). The prediction (3.4) is checked below on the example of the tied-down random walk, which corresponds to taking $\theta = 1/2$ (see figure 2).

3.3. Derivation of (1.4) in the independent interval approximation

We now come to the derivation of (1.4) given in [1] when $\theta < 1$. It proceeds as follows.

(i) The system is taken infinite, stationary and the distribution of sizes of domains $f(\ell)$ is assumed to have a finite first moment $\langle\tau\rangle$. Therefore the stationary probability that an interval of size r does not contain any point is, in Laplace space, given by (3.3), which expresses $\hat{p}_{0,\mathrm{stat}}(s)$ in terms of $\hat{f}(s)$ and $\langle\tau\rangle$.

(ii) In this formalism, $f(\ell)$ is thought as being $f(\ell|L)$, but nevertheless approximated by $f(\ell)$, if $\ell \ll L$, and $\langle\tau\rangle$ is thought as being $\langle\tau\rangle^\star$, and given by [1]

$$\langle\tau\rangle^\star \approx \frac{y_c}{1-\theta} L^{1-\theta}. \tag{3.5}$$

(iii) The expansion of the Laplace transform of $f(\ell)$ with respect to ℓ,

$$\hat{f}(s) \approx 1 + y_c\Gamma(-\theta)s^\theta, \tag{3.6}$$

is carried into the expression (3.3) of $\hat{p}_{0,\mathrm{stat}}(s)$. This gives

$$\hat{p}_{0,\mathrm{stat}}(s) \approx \frac{1}{s} + \frac{y_c\Gamma(-\theta)}{\langle\tau\rangle}s^{\theta-2},$$

where again $\langle\tau\rangle$ is replaced by $\langle\tau\rangle^\star$ given by (3.5), yielding finally [1]

$$p_{0,\mathrm{stat}}(r) \approx 1 - \frac{1}{\theta}\left(\frac{r}{L}\right)^{1-\theta}, \tag{3.7}$$

which is (1.4), since the correlation function is dominated by $p_{0,\mathrm{stat}}(r)$.

This heuristic line of reasoning—the so-called *independent interval approximation* (IIA)—would predict the correct asymptotic stationary regime if $f(\ell)$ had a finite first moment. But, in the present situation of a broad distribution with index $\theta < 1$, it does not lead to exact results because it does not treat the genuine non stationarity of the process adequately. In particular the various assumptions on which it is based do not form a coherent whole, as we now comment.

On the one hand, using (3.3) requires $\langle\tau\rangle$ to be finite. On the other hand, using the expansion (3.6) only makes sense if $\langle\tau\rangle$ is infinite. Taking $\langle\tau\rangle^\star$ in place of $\langle\tau\rangle$ does not circumvent this contradiction because the finiteness of $\langle\tau\rangle$ is an intrinsic property of the distribution $f(\ell)$, independent of the size of the system—i.e., holding even for an infinite system. In contrast, $\langle\tau\rangle^\star$ is a property of the finite system and depends on L.

In conclusion, the IIA used in [1] only predicts the power $1-\theta$ of r/L in (1.4) or (3.7). The correctness of the prefactor $1/\theta$ in front of $(r/L)^{1-\theta}$ is illusory. Actually, this prefactor should be modified because the expression (3.5) for $\langle\tau\rangle^\star$ is inaccurate and should be replaced by (3.4). By so doing, the prefactor $1/\theta$ in (1.4) or (3.7) is changed to $\sin\pi\theta/[\pi\theta(1-\theta)]$, which does not give the correct expression for the correlation function either. Another shortfall of the method is that it is unable to predict the non-stationary correlation function $C(x, r, L)$ for an open system§.

4. The tied-down random walk as a benchmark

We can illustrate and validate the results seen above, such as (1.5), (1.6) for the correlation function and (3.4) for $\langle\tau\rangle^\star$, on the case of tied-down processes with $\theta = 1/2$.

The tied-down random walk is the simplest representative of this class, with the advantage of being intuitively understandable [10, 9]. The walk (with steps ±1) is said to be *tied down* because it is conditioned to return to the origin at a given time, L. The weight of a configuration of the walk is given by (2.2), (2.3). The distribution $f(\ell)$ in (2.2) is the probability that the first return to zero of the random walk starting at the origin occurs at time ℓ. Note that both ℓ and L are even‖. Asymptotically, for ℓ large, this distribution has a power-law tail with index θ equal to $1/2$, and tail parameter $c = 1/\sqrt{2\pi}$,

$$f(\ell) \approx \frac{1}{\sqrt{2\pi}\ell^{3/2}}. \tag{4.1}$$

§ The first version of [1] already predicted (1.4) for the critical correlation function of the same model *with open boundary conditions* [13]. This result was questioned by the present Author since it contradicted (1.5) (in particular by missing its x dependence) [14]. In the published version [1], the boundary conditions were changed to periodic, the derivation of (1.4) was left unchanged.

‖ This induces an additional factor 2 in the denominator of the continuum expression (4.1).

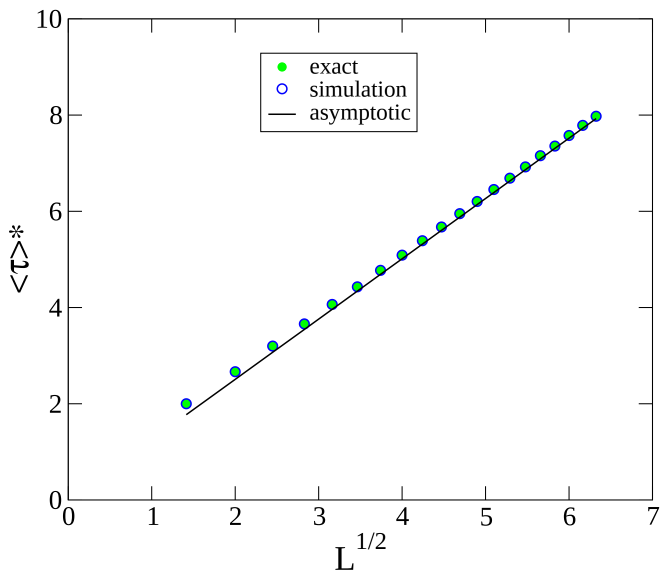

Figure 2. Mean time interval $\langle\tau\rangle^\star$ between two successive returns to the origin of a tied-down random walk, against $\sqrt{L}$, where L is the time of ultimate return to the origin, $L = 2, 4, \ldots, 40$ (see text). Comparison of the exact expression (4.2) (green dots), simulation (blue empty circles), and asymptotic prediction (4.3) (continuous line).

The denominator $Z(L)$ in (2.3) is the probability of return of the walk to the origin at time L. It has the explicit expression

$$Z(L) = \frac{1}{2^L}\binom{L}{L/2} \approx \sqrt{\frac{2}{\pi L}}.$$

It can be shown that [9]

$$\langle\tau\rangle^\star = \frac{1}{Z(L)}, \tag{4.2}$$

hence, asymptotically,

$$\langle\tau\rangle^\star \approx \sqrt{\frac{\pi L}{2}}, \tag{4.3}$$

which is precisely (3.4) with $c = 1/\sqrt{2\pi}$ and $\theta = 1/2$.

Furthermore, for the Brownian bridge, which is the continuum limit of the tied-down random walk, the analytical form of the full correlation function (of the sign of the position of the walker) is known [2]

$$C(x, r, L) = 1 - \frac{2}{\pi}\arccos\sqrt{\frac{x(L-x-r)}{(x+r)(L-x)}}. \tag{4.4}$$

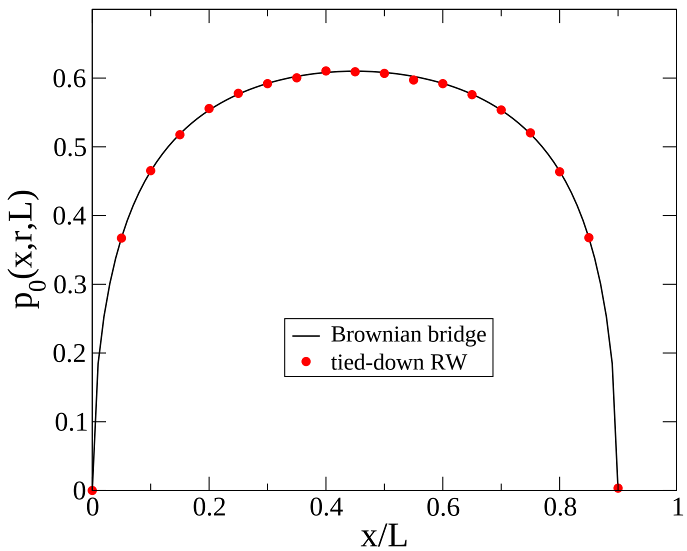

Figure 3. Probability for an interval of size $r = 2000$ to be empty as a function of x/L, for a tied-down random walk of $L = 20\,000$ steps, measured on 10^5 samples (red dots). This probability is compared to the analytical expression of the correlation function of the sign of the position for the Brownian bridge (continuous curve).

This compact expression encompasses all regimes of interest. In particular, in the regime of short separations ($1 \ll r \ll x \sim L$), it reads

$$C(x, r, L) \approx 1 - \frac{2}{\pi}\left[\frac{x}{L}\left(1 - \frac{x}{L}\right)\right]^{-1/2}\left(\frac{r}{L}\right)^{1/2}, \tag{4.5}$$

which is (1.5), with $\theta = 1/2$. Averaging this expression on x yields

$$C(r, L) \approx 1 - 2\left(\frac{r}{L}\right)^{1/2}, \tag{4.6}$$

which is (1.6) with $\theta = 1/2$. Actually, the average of (4.4) on x has the exact form

$$C(r, L) = \left[1 - \left(\frac{r}{L}\right)^{1/2}\right]^2 = 1 - 2\left(\frac{r}{L}\right)^{1/2} + \frac{r}{L}, \tag{4.7}$$

where the first two terms in the right side reproduce (4.6), for r/L small, and the last term gives the explicit correction to this expression.

Finally, we can complement our study by performing numerical simulations of the tied-down random walk.

Figure 2 depicts a comparison between the measured mean interval $\langle\tau\rangle^\star$, its theoretical prediction (4.2), and its asymptotic expression (4.3), showing excellent agreement. We also measured $p_0(x, r, L)$, for $0 < x < L - r$, on 10^5 samples of the tied-down random walk, with $L = 20\,000$ and $r = 2000$. For $x > L - r$, $p_0(x, r, L)$ vanishes because there is always the point at L inside the interval. The result is depicted in figure 3 together with the analytical expression (4.4) of the correlation function for the Brownian bridge. Comparison between the two sets of data demonstrates the adequacy between the two processes, and confirms also that $p_0(x, r, L)$ accounts faithfully for the full correlation function.

5. Conclusion

Tied-down renewal processes generated by a power-law distribution of intervals with tail index $\theta < 1$, of which the critical TIDSI model (a spin model with long-range order) is a particular example, are genuinely non stationary. This holds for the infinite system as well as for the finite one, when the intervals are conditioned by the value of their sum. For such processes, the independent interval approximation put forward in [1] for the computation of the spin-spin correlation function is not the proper approach, even if periodic boundary conditions are imposed on the system, entailing translation invariance. In the case at hand, translation invariance only implies tracing the non-stationary correlation function $C(x, r, L)$ on x.

References

[1] Barma M, Majumdar S N and Mukamel D 2019 *J. Phys. A* **52** 254001
[2] Godrèche C 2017 *J. Stat. Mech.* P073205
[3] Thouless D 1969 *Phys. Rev.* **187** 732
[4] Anderson P W, Yuval G and Hamann D 1970 *Phys. Rev. B* **1** 4464
[5] Aizenman M, Chayes J, Chayes L and Newman C 1988 *J. Stat. Phys.* **50** 1
[6] Bar A and Mukamel D 2014 *Phys. Rev. Lett.* **112** 015701
[7] Bar A and Mukamel D 2014 *J. Stat. Mech.* P11001
[8] Bar A, Majumdar S N, Schehr G and Mukamel D 2016 *Phys. Rev.* **93** 052130
[9] Godrèche C 2017 *J. Phys. A* **50** 195003
[10] Wendel J G 1964 *Math. Scand.* **14** 21
[11] Godrèche C and Luck J M 2001 *J. Stat. Phys.* **104** 489
[12] Bouchaud J P and Dean D S 1995 J. Phys. France **5** 265
[13] Barma M, Majumdar S N and Mukamel D 2019 arXiv:1902.06416v1 [cond-mat.stat-mech]
[14] Private correspondence with M Barma