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TrioCFD: code & numerical schemes

Erell Jamelot

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TrioCFD: code & numerical schemes

POEMs 2019

CIRM

E. Jamelot and TrioCFD team

Commissariat à l'Energie Atomique et
aux Energies Alternatives
Centre de Saclay

CEA/DEN/DANS/DM2S/STMF/LMSF

May 2, 2019

Outline

Industrial context

Numerical schemes

Work in progress

Numerical results

Future work

TrioCFD code (previously Trio_U code)

- Developed by the CEA/DEN since the early 1990s.
- Dedicated to unsteady, low Mach number, turbulent flows.
- C++, designed for HPC, open source since 2015.

- Cartesian and triangular/tetrahedral meshes.
- Explicit and semi-implicit time integration.
- Turbulence models: RANS, LES, Wall models.

- Participation in the FVCA 8 benchmark session.

- PhD student: MsFEM (Q. Feng, P. Omnes, G. Allaire).
- Post-docs: ALE (R. Pegonen, A. Puscas) and sensitivity analysis (C. Fiorini, A. Puscas).

- Website (under evolution): www-trio-u.cea.fr
Download: <https://sourceforge.net/projects/triocfd/>

- Our objectives: support polygonal/polyhedral meshes and high order methods.

Navier-Stokes equations

- Find $(\vec{u}, p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that:

$$\begin{cases} \partial_t \vec{u} - \nu \Delta \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{u} + \vec{\nabla} p & = \vec{f}, \\ \vec{\nabla} \cdot \vec{u} & = 0. \end{cases}$$

- Time discretization (semi-implicit scheme):

- Prediction step:** Compute U^* the solution of

$$\delta t^{-1} M U^* + A U^* + L(U^n) U^* + {}^t B P^n = F^{n+1} + \delta t^{-1} M U^n.$$

At this step, we may have $B U^* \neq 0$.

- Pressure computation step:** Compute P' the solution of

$$B M^{-1} {}^t B P' = \delta t^{-1} B U^*, \quad P^{n+1} = P' + P^n.$$

- Correction step:** Compute U^{n+1} the solution of

$$M U^{n+1} = M U^* - \delta t {}^t B P'.$$

Stokes Problem

- Find $(\vec{u}, p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{cases} -\nu \Delta \vec{u} + \vec{\nabla} p &= \vec{f}, \\ \vec{\nabla} \cdot \vec{u} &= 0. \end{cases}$$
- Find $(\vec{u}, p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{cases} \forall \vec{v} \in \vec{H}_0^1(\Omega), & a(\vec{u}, \vec{v}) + b(\vec{v}, p) &= \langle \vec{f}, \vec{v} \rangle, \\ \forall q \in L_0^2(\Omega), & b(\vec{u}, q) &= 0. \end{cases}$$

where $a(\vec{u}, \vec{v}) = (\nu \vec{\nabla} \vec{u}, \vec{\nabla} \vec{v})_0$ and $b(\vec{v}, p) = -(p, \vec{\nabla} \cdot \vec{v})_0$.

Well-posedness: a and b continuous, a coercive on $\vec{H}_0^1(\Omega)$ and the *inf-sup condition* holds

$$\forall q \in L_0^2(\Omega), \quad \sup_{\vec{v} \in \vec{H}_0^1(\Omega)} \frac{b(\vec{v}, q)}{\|\vec{v}\|_1} \geq \beta \|q\|_0.$$

Stokes Problem

$$(\vec{u}, p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega) \mid \begin{cases} \forall \vec{v} \in \vec{H}_0^1(\Omega), & a(\vec{u}, \vec{v}) + b(\vec{v}, p) = \langle \vec{f}, \vec{v} \rangle, \\ \forall q \in L_0^2(\Omega), & b(\vec{u}, q) = 0. \end{cases}$$

Well-posedness: a and b continuous, a coercive on $\vec{H}_0^1(\Omega)$ and the *inf-sup condition* holds

$$\forall q \in L_0^2(\Omega), \quad \sup_{\vec{v} \in \vec{H}_0^1(\Omega)} \frac{b(\vec{v}, q)}{\|\vec{v}\|_1} \geq \beta \|q\|_0.$$

$$\vec{H}_0^1(\Omega) = \vec{V} \oplus \vec{V}^\perp, \quad \begin{cases} \vec{V} & := \{ \vec{v} \in \vec{H}_0^1(\Omega); \quad \vec{\nabla} \cdot \vec{v} = 0 \}, \\ \vec{V}^\perp & = \{ (-\Delta)^{-1} \vec{\nabla} q; \quad q \in L^2(\Omega) \}. \end{cases}$$

Abstract tools: Girault-Raviart'86, chap I, cor. 2.4.

1°) The operator $\vec{\nabla}$ is an isomorphism of $L_0^2(\Omega)$ onto \vec{V}^0 such that

$$\vec{V}^0 := \left\{ \vec{y} \in \vec{H}^{-1}(\Omega); \quad \langle \vec{y}, \vec{\phi} \rangle = 0 \quad \forall \vec{\phi} \in \vec{V} \right\}.$$

2°) The operator $\vec{\nabla} \cdot$ is an isomorphism of \vec{V}^\perp onto $L_0^2(\Omega)$.

Space discretizations

- Let \mathcal{T}_h a conforming triangulation of Ω , let \mathcal{F}_h be the set of facets.
- Conforming discretization: $\vec{X}_h \subset \vec{H}_0^1(\Omega)$.

Find $(\vec{u}_h, p_h) \in \vec{X}_h \times M_h$ such that

$$\begin{cases} \forall \vec{v}_h \in \vec{X}_h, & a(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) = (\vec{f}, \vec{v}_h)_0, \\ \forall q_h \in M_h, & b(\vec{u}_h, q_h) = 0. \end{cases}$$

Nodal finite element method: Taylor-Hood, conforming Crouzeix-Raviart...

Advantage: Well-posedness of the discrete pb (check ISC).

Drawback: Couplings between the vertices.

- Non-conforming discretization: $\vec{X}_h \not\subset \vec{H}_0^1(\Omega)$.

Find $(\vec{u}_h, p_h) \in \vec{X}_h \times M_h$ such that

$$\begin{cases} \forall \vec{v}_h \in \vec{X}_h, & a_h(\vec{u}_h, \vec{v}_h) + b_h(\vec{v}_h, p_h) = (\vec{f}, \vec{v}_h)_0, \\ \forall q_h \in M_h, & b_h(\vec{u}_h, q_h) = 0. \end{cases}$$

Advantage: Couplings between the facets.

Drawback: Check continuity of a_h , b_h , ellipticity of a_h and ISC.

Space discretizations

- Find $(\vec{u}_h, p_h) \in \vec{X}_h \times M_h$ such that

$$\begin{cases} \forall \vec{v}_h \in \vec{X}_h, & a_{(h)}(\vec{u}_h, \vec{v}_h) + b_{(h)}(\vec{v}_h, p_h) = (\vec{f}, \vec{v}_h)_0, \\ \forall q_h \in M_h, & b_{(h)}(\vec{u}_h, q_h) = 0. \end{cases}$$

- In both cases, the main technical difficulty is that $\vec{V}_h \not\subset \vec{V}$ where:

$$\vec{V}_h := \{\vec{v}_h \in \vec{X}_h \mid \forall q_h \in M_h, \quad b_{(h)}(\vec{v}_h, q_h) = 0\}.$$

- Conforming discretization, a priori error estimates:

$$\begin{cases} \|\vec{u} - \vec{u}_h\|_1 \lesssim \inf_{\vec{v}_h \in \vec{X}_h} \|\vec{u} - \vec{v}_h\|_1 + \inf_{q_h \in M_h} \|p - q_h\|_0, \\ \|p - p_h\|_0 \lesssim \inf_{\vec{v}_h \in \vec{X}_h} \|\vec{u} - \vec{v}_h\|_1 + \inf_{q_h \in M_h} \|p - q_h\|_0. \end{cases}$$

- Non-conforming discretization, a priori error estimates:

$$\begin{cases} \|\vec{u} - \vec{u}_h\|_{\vec{X}_h} \lesssim \inf_{\vec{v}_h \in \vec{X}_h} \|\vec{u} - \vec{v}_h\|_{\vec{X}_h} + \inf_{q_h \in M_h} \|p - q_h\|_0, \\ \|p - p_h\|_{M_h} \lesssim \inf_{\vec{v}_h \in \vec{X}_h} \|\vec{u} - \vec{v}_h\|_{\vec{X}_h} + \inf_{q_h \in M_h} \|p - q_h\|_0 + \|r_h(u, p)\|_h^*. \end{cases}$$

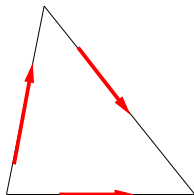
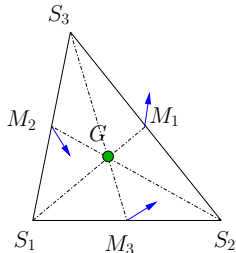
Non-conforming space discretization

- (\vec{P}_1^{NC}, P_0) proposed in [Crouzeix-Raviart'73](#) $\|\vec{u} - \vec{u}_h\|_0 \lesssim h (|\vec{u}|_1 + \|p\|_0)$.
 $\vec{X}_h := \vec{H}_0^1(\Omega) \cap \vec{P}_1^{NC}$, $M_h := L_0^2(\Omega) \cap P_0$:

$$\begin{aligned} \vec{P}_1^{NC} &:= \{ \vec{v}_h \mid \forall T \in \mathcal{T}_h \vec{v}_h|_T \in (P_1(T))^d \text{ and } \forall f \in \mathcal{F}_h [\vec{v}_h](\vec{x}_f) = 0 \}, \\ P_0 &:= \{ q_h \mid \forall T \in \mathcal{T}_h, q_h \in \mathbb{R} \}. \end{aligned}$$

$$a_h(\vec{u}_h, \vec{v}_h) = \sum_T (\nu \vec{\nabla} \vec{u}_h, \vec{\nabla} \vec{v}_h)_{0,T}, \quad b_h(\vec{v}_h, q_h) = - \sum_T (\vec{\nabla} \cdot \vec{v}_h, q_h)_{0,T}.$$

Drawback: Spurious discrete-divergence free vectors.



- $p_h \in L_0^2(\Omega) \cap P_0$
- $\vec{u}_h \in \vec{H}_0^1(\Omega) \cap \vec{P}_1^{NC}$

$$\vec{u}_h \cdot \vec{n}|_{\partial T} = 0 \Rightarrow \int_T \vec{\nabla} \cdot \vec{u}_h dT = 0.$$

Non-conforming space discretization

- Less degrees of freedom in \vec{X}_h Hecht'81.
- More degrees of freedom in M_h . In 2D:
- $(\vec{P}_1^{NC}, P_1 + P_b)$ Bernardi-Hecht'00 $M_h = L_0^2(\Omega) \cap P_1 + L_0^2(\Omega) \cap P_b$

$$(\vec{u}, p) \in \vec{H}^{s+1}(\Omega) \times H^s(\Omega) \quad \|\vec{u} - \vec{u}_h\|_{\vec{X}_h} \lesssim h^s (\|\vec{u}\|_{s+1} + \|p\|_s)$$

Drawback: Convergence of the discrete pressure ?

- (\vec{P}_1^{NC}, P_1) Heib'03 $M_h = L_0^2(\Omega) \cap P_1$

$$(\vec{u}, p) \in \vec{H}^{s+1}(\Omega) \times H^s(\Omega) \quad \|\vec{u} - \vec{u}_h\|_{\vec{X}_h} + \|p - p_h\|_0 \lesssim h^s (\|\vec{u}\|_{s+1} + \|p\|_s)$$

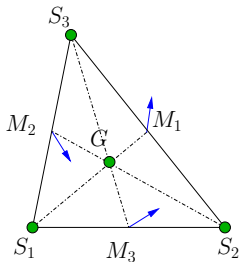
Drawback: Spurious discrete divergence-free vectors.

- $(\vec{P}_1^{NC}, P_0 + P_1)$ Heib'03 $M_h = L_0^2(\Omega) \cap P_0 + L_0^2(\Omega) \cap P_1$

$$(\vec{u}, p) \in \vec{H}^2(\Omega) \times H^1(\Omega) \quad \|\vec{u} - \vec{u}_h\|_{\vec{X}_h} + \|p - p_h\|_0 \lesssim h (\|\vec{u}\|_2 + \|p\|_1).$$

$(\vec{P}_1^{NC}, P_0 + P_1)$ discretization

- $M_h = L_0^2(\Omega) \cap P_0 + L_0^2(\Omega) \cap P_1$, $P_1 := \{q_h \in C^0(\Omega) \mid \forall T \in \mathcal{T}_h, q_h \in P_1(T)\}$
- $b_h(\vec{v}_h, q_{h,0} + q_{h,1}) = - \sum_T \left(\vec{\nabla} \cdot \vec{v}_h, q_{h,0} \right)_{0,T} + \sum_T \left(\vec{v}_h, \vec{\nabla} q_{h,1} \right)_{0,T}$.



- $p_h \in L_0^2(\Omega) \cap P_0 + L_0^2(\Omega) \cap P_1$
- $\vec{u}_h \in \vec{H}_0^1(\Omega) \cap \vec{P}_1^{NC}$

$\forall \vec{v}_h \in \vec{V}_h :$

$$\vec{\nabla} \cdot \vec{v}_h|_T = 0,$$

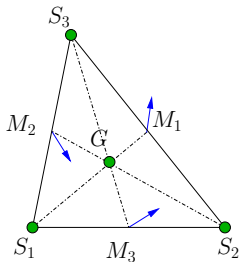
$$\sum_T \left(\vec{\nabla} q_{h,2}, \vec{v}_h \right)_{0,T} = 0 \quad \forall q_{h,2} \in P_2,$$

$$\sum_T \int_{\partial T} q_{h,2} \vec{v}_h \cdot \vec{n} \, d\sigma = 0 \quad \forall q_{h,2} \in P_2.$$

$$P_2 := \{q_{h,2} \in C^0(\Omega) \mid \forall T \in \mathcal{T}_h, q_{h,2} \in P_2(T)\}$$

$(\vec{P}_1^{NC}, P_0 + P_1)$ discretization

- $M_h = L_0^2(\Omega) \cap P_0 + L_0^2(\Omega) \cap P_1$, $P_1 := \{q_h \in C^0(\Omega) \mid \forall T \in \mathcal{T}_h, q_h \in P_1(T)\}$
- $b_h(\vec{v}_h, q_{h,0} + q_{h,1}) = - \sum_T (\vec{\nabla} \cdot \vec{v}_h, q_{h,0})_{0,T} + \sum_T (\vec{v}_h, \vec{\nabla} q_{h,1})_{0,T}$.



Proof based on Simpson's rule,
exact for order 3 polynomials on a segment:

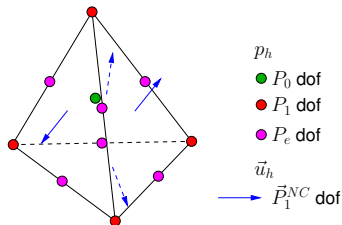
$$\int_e q_{h,3} d\sigma = \frac{|e|}{6} (q_{h,3}(S_{1,e}) + 4q_{h,3}(M_e) + q_{h,3}(S_{2,e}))$$

- $p_h \in L_0^2(\Omega) \cap P_0 + L_0^2(\Omega) \cap P_1$
- $\vec{u}_h \in \vec{H}_0^1(\Omega) \cap \vec{P}_1^{NC}$

Non-conforming space discretization

- More degrees of freedom in M_h . In 3D:

▷ $(\vec{P}_1^{NC}, (P_0 + P_1 + P_e))$ T. Fortin'06.



- ▷ Numerical integration scheme on a triangle, exact for order 3 polynomials:

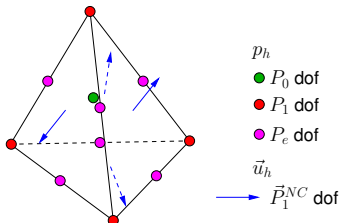
$$\int_T q_{h,3} dT = \frac{|T|}{5} \left(\sum_{i=1}^3 \frac{1}{4} q_{h,3}(S_{i,T}) + \frac{2}{3} \sum_{i=1}^3 q_{h,3}(M_{i,T}) + \frac{3}{4} q_{h,3}(G_T) \right)$$

- ▷ Numerous degrees of freedom.

Non-conforming space discretization

- More degrees of freedom in M_h . In 3D:

▷ $(\vec{P}_1^{NC}, (P_0 + P_1 + P_e))$ T. Fortin'06.



▷ $M_h^e := \{q_h \in C^0(\Omega) \mid \forall T \in \mathcal{T}_h, q_h \in P_e(T)\}$.

For $e = [S_i S_j]$, $\gamma_{e|T} := 4 \lambda_{i,T} \lambda_{j,T}$.

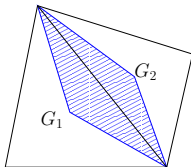
▷ $M_h = ((M_h^e / M_h^{1,p} + M_h^1) \cap L_0^2(\Omega) + M_h^0 \cap L_0^2(\Omega))$

▷ For Stokes, if $\vec{f} = \vec{\nabla} p$, where $p \in H^4(\Omega) \cap L_0^2(\Omega)$ and $|p|_4 \neq 0$, then

$$\|\vec{u}_h\|_{\vec{X}_h} \lesssim h^5 \quad \text{and} \quad \|p_h - p\|_0 \lesssim h^2.$$

TrioCFD: a *finite volume-element* code.

- Control volumes of the FVM. [Emonot'92](#)



- FVM: the mass matrix is diagonal in $2D$ and $3D$.
FEM: the mass matrix is diagonal in $2D$ only.
In TrioCFD code, the mass matrix is diagonal in $2D$ and $3D$ as in FVM.
- FVM and FEM: Diffusion matrix is the same.
- FVM and FEM: Coupling matrix is the same.
- FVM or FEM: Convection schemes (centered, upwind, muscl, stab)
- FVM and FEM: Source term is different ($\approx h^2$).
In TrioCFD code, the discrete source term is coded as in FEM.

Helmholtz decomposition

- $\vec{L}^2(\Omega)$ -Helmholtz decomposition and $\vec{L}^2(\Omega)$ -Helmholtz projector (Ω connected):

$$\forall \vec{f} \in \vec{L}^2(\Omega), \quad \exists (\psi, \vec{w}) \in H^1(\Omega) \times \vec{H}_0(\text{div}0, \Omega) \mid \vec{f} = \vec{\nabla} \psi + \vec{w},$$

$$\vec{L}^2(\Omega) = \vec{H}^\perp \oplus \vec{H} : \quad (\vec{\nabla} \psi, \vec{w})_0 = -(\psi, \vec{\nabla} \cdot \vec{w})_0 = 0.$$

$$\mathbb{P} : \vec{L}^2(\Omega) \rightarrow \vec{L}^2(\Omega), \quad \mathbb{P}(\vec{f}) = \vec{w}.$$

- Navier-Stokes variational formulation:

$$\text{Find } (\vec{u}, p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega) \text{ s.t. } \forall (\vec{v}, q) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega)$$

$$\begin{aligned} \frac{d}{dt}(\vec{u}, \vec{v})_0 + (\nu \vec{\nabla} \vec{u}, \vec{\nabla} \vec{v})_0 + \left((\vec{u} \cdot \vec{\nabla}) \vec{u}, \vec{v} \right)_0 - (p, \vec{\nabla} \cdot \vec{v})_0 &= (\vec{f}, \vec{v})_0, \\ (q, \vec{\nabla} \cdot \vec{u})_0 &= 0. \end{aligned}$$

- $\vec{u} = \mathbb{P}(\vec{u}), \quad \mathbb{P}(\vec{\nabla} \vec{u}) = 0.$

Helmholtz decomposition

- $\vec{L}^2(\Omega)$ -Helmholtz decomposition and $\vec{L}^2(\Omega)$ -Helmholtz projector (Ω connected):

$$\forall \vec{f} \in \vec{L}^2(\Omega), \quad \exists (\psi, \vec{w}) \in H^1(\Omega) \times \vec{H}_0(\text{div}0, \Omega) \mid \vec{f} = \vec{\nabla} \psi + \mathbb{P}(\vec{f}),$$

$$\vec{L}^2(\Omega) = \vec{H}^\perp \oplus \vec{H} : \quad \left(\vec{\nabla} \psi, \mathbb{P}(\vec{f}) \right)_0 = - \left(\psi, \mathbb{P}(\vec{f}) \right)_0 = 0.$$

- Divergence-free momentum balance: $\forall \vec{v} \in \vec{V}$,

$$\begin{aligned} \frac{d}{dt} (\vec{u}, \vec{v})_0 + \left(\nu \vec{\nabla} \vec{u}, \vec{\nabla} \vec{v} \right)_0 + \left(\mathbb{P}((\vec{\nabla} \cdot \vec{u}) \vec{u}), \vec{v} \right)_0 &= \left(\mathbb{P}(\vec{f}), \vec{v} \right)_0, \\ &= \left(\mathbb{P}(\vec{f} + \vec{\nabla} \phi), \vec{v} \right)_0, \end{aligned}$$

- Irrotational* momentum balance: $\forall \vec{v}^\perp \in \vec{V}^\perp$,

$$\frac{d}{dt} (\vec{u}, \vec{v}^\perp)_0 - \left(p, \vec{\nabla} \cdot \vec{v}^\perp \right)_0 + \left((\vec{\nabla} \cdot \vec{u}) \vec{u}, \vec{v}^\perp \right)_0 = \left(\vec{f}, \vec{v}^\perp \right)_0.$$

- Possible discrete biorthogonalization.

Raviart-Thomas FE

- $\vec{H}(\text{div}, \Omega)$ -conforming FE [Raviart-Thomas'77](#):

$$\vec{X}_h^{RT_0} := \{ \vec{v}_h \in \vec{H}(\text{div}, \Omega) : \forall T \in \mathcal{T}_h, \vec{v}_h|_T(\vec{x}) = \vec{a}_T + b_T \vec{x} \}$$

- Raviart-Thomas projection operator (lowest order):

$$\Pi^{RT_0}(\vec{v}_h|_T)(\overrightarrow{OM}_i) \cdot \vec{n}_i = |F_i|^{-1} \int_{F_i} \vec{v}_h \cdot \vec{n}_i \, d\vec{x}, \quad \forall i \in \{1, \dots, d+1\}.$$

- $\forall \vec{v}_h \in \vec{X}_h, \quad \vec{\nabla} \cdot \vec{v}_h|_T = \vec{\nabla} \cdot (\Pi^{RT_0} \vec{v}_h|_T).$
- Let $(\vec{\phi}_i^\alpha)_{i,\alpha}$ be the basis of \vec{X}_h :

For $i \in \{1, \dots, N_f\}, \alpha \in \{1, \dots, d\}, \vec{\phi}_i^\alpha|_T := (1 - d\lambda_{i|T})\vec{e}_\alpha$. One can show that:

$$\Pi^{RT_0}(\vec{\phi}_i^\alpha|_T) = \frac{\vec{S}_{i|T} \cdot \vec{e}_\alpha}{d|T|} (\vec{x} - \overrightarrow{OS}_{i|T}).$$

Raviart-Thomas projection

- *Towards a better consistency* Linke et al: Use Raviart-Thomas projection.
- Source term:

$$\forall \vec{v}_h \in \vec{P}_1^{NC} : \begin{cases} (\vec{f}, \vec{v}_h)_0 \neq -\sum_T (\psi, \vec{\nabla} \cdot \vec{v}_h)_{0,T} + (\vec{w}, \vec{v}_h)_0, \\ (\vec{f}, \Pi^{RT_0} \vec{v}_h)_0 = -\sum_T (\psi, \vec{\nabla} \cdot \vec{v}_h)_{0,T} + (\vec{w}, \Pi^{RT_0} \vec{v}_h)_0. \end{cases}$$

- Convection term: $(\vec{u} \cdot \vec{\nabla}) \vec{u} = (\vec{\nabla} \times \vec{u}) \times \vec{u} + \frac{1}{2} \vec{\nabla} (|\vec{u}|^2)$

$$\left((\vec{u}^{n-1} \cdot \vec{\nabla}) \vec{u}^n, \vec{v} \right)_0 = \left(\vec{\nabla} \times \vec{u}^n, \vec{u}^{n-1} \times \vec{v} \right)_0 - \frac{1}{2} \left(\vec{u}^n \cdot \vec{u}^{n-1}, \vec{\nabla} \cdot \vec{v} \right)_0.$$

$$\sum_T \vec{\nabla} \times \vec{u}_{h|T}^n \cdot \int_T \vec{u}_h^{n-1} \times \vec{v}_h dT \rightarrow \sum_T \vec{\nabla} \times \vec{u}_{h|T}^n \cdot \int_T \Pi^{RT_0} \vec{u}_h^{n-1} \times \Pi^{RT_0} \vec{v}_h dT$$

- Mass term: $(\vec{u}_h, \vec{v}_h)_0 \rightarrow \left(\Pi^{RT_0} \vec{u}_h, \Pi^{RT_0} \vec{v}_h \right)_0.$

Numerical results in 2D: Stokes equation, no flow

- $\Omega = (0, 1)^2$, $\nu = 1$, $\vec{f} = {}^t(0, Ra(1 - y + 3y^2))$, $Ra > 0$.

$$\vec{u} = \vec{0}, \quad p = Ra \left(y^3 - \frac{y^2}{2} + y - \frac{7}{12} \right)$$

- Values of $\|\vec{u}_h\|_0$ for $Ra = 1$

$1/h$	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_1$	$\vec{P}_1^{NC} - P_0 P_1$
20	$2.74 e - 04$	$9.01 e - 13$	$1.21 e - 04$	$1.12 e - 08$
40	$6.53 e - 05$	$1.30 e - 13$	$1.39 e - 05$	$6.97 e - 10$
80	$1.66 e - 05$	$4.44 e - 13$	$1.77 e - 06$	$4.56 e - 11$
160	$4.13 e - 06$	$4.41 e - 13$	$2.21 e - 07$	$2.82 e - 12$
τ	2	—	3	4

- Values of $\|\vec{u}_h\|_0$ for $Ra = 100$

$1/h$	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_1$	$\vec{P}_1^{NC} - P_0 P_1$
20	$2.74 e - 02$	$9.01 e - 13$	$1.21 e - 06$	$1.12 e - 06$
40	$6.53 e - 03$	$1.30 e - 13$	$1.39 e - 07$	$6.97 e - 08$
80	$1.66 e - 03$	$4.44 e - 13$	$1.77 e - 08$	$4.56 e - 09$
160	$4.13 e - 04$	$4.41 e - 13$	$2.21 e - 09$	$2.82 e - 10$
τ	2	—	3	4

Numerical results in 2D: Navier-Stokes equation, stationary vortex

- $\Omega = (-1, 1)^2$, $\nu = 1$, $\vec{f} = (Re - 1)^t(x, y)$, $Re > 0$.

$$\vec{u} = {}^t(-y, x), \quad p = Re \left(\frac{x^2 + y^2}{2} - \frac{4}{3} \right)$$

- Values of $\|\vec{u} - \vec{u}_h\|_0$ for $Re = 1$

$1/h$	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_0 P_1$
20	$1.32 e - 04$	$8.80 e - 12$	$3.36 e - 05$
40	$3.00 e - 05$	$1.27 e - 11$	$7.52 e - 06$
80	$7.86 e - 06$	$3.70 e - 11$	$2.17 e - 06$
160	$1.92 e - 06$	$1.25 e - 11$	$5.04 e - 07$
τ	2	—	4

- Values of $\|\vec{u} - \vec{u}_h\|_0$ for $Re = 10$

$1/h$	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_0 P_1$
20	$8.97 e - 04$	$8.77 e - 11$	$3.36 e - 05$
40	$2.01 e - 04$	$1.26 e - 10$	$7.52 e - 06$
80	$5.29 e - 05$	$3.70 e - 10$	$2.17 e - 06$
160	$1.12 e - 05$	$1.25 e - 10$	$5.04 e - 07$
τ	2	—	4

Numerical results in 3D: Navier-Stokes equation, stationary vortex

- $\Omega = (0, 1)^3$, $\nu = 1$, $\vec{f} = (Re - 1)^t(0, 2y - 1, 2z - 1)$.

$$\vec{u} = {}^t(0, -2z + 1, 2y - 1), \quad p = Re \left(\frac{(2y - 1)^2 + (2z - 1)^2}{2} - \frac{1}{3} \right)$$

- Values of $\|\vec{u} - \vec{u}_h\|_0$ for $Re = 1$

# DoFs	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_0 P_1$
$7.71e + 03$	$1.00 e - 03$	$3.22 e - 05$	$6.08 e - 03$
$1.53e + 04$	$5.62 e - 03$	$1.77 e - 05$	$3.80 e - 03$
$3.05e + 04$	$3.16 e - 04$	$8.77 e - 06$	$2.45 e - 03$
$6.11e + 04$	$1.78 e - 04$	$4.42 e - 06$	$1.72 e - 03$
τ	2.5	2.8	1.7

- Values of $\|\vec{u} - \vec{u}_h\|_0$ for $Re = 100$

# DoFs	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_0 P_1$
$7.71e + 03$	$1.02 e - 01$	$3.19 e - 05$	$6.10 e - 03$
$1.53e + 04$	$5.58 e - 01$	$1.75 e - 05$	$3.83 e - 03$
$3.05e + 04$	$3.15 e - 02$	$8.75 e - 06$	$2.43 e - 03$
$6.11e + 04$	$1.76 e - 02$	$4.39 e - 06$	$1.74 e - 03$
τ	2.5	2.8	1.7

Fortin-Soulie with Π^{RT_1}

- Matlab tests (Stokes, 2D) and Fortin-Soulie'86 with M. Rihani (M2 internship last year).
- No flow in $\Omega = (0, 1)^2$ with $\vec{f} = \vec{\nabla}(x^3)$ ($h = 0.05$, $N_{\vec{u}} = 6486$, $N_p = 3152$):

	$\ \vec{u} - \vec{u}_h - h\ _0$	$\ \vec{u} - \vec{u}_h\ _h$	$\ p - p_h\ _h$
$F - S$	$2.6e - 07$	$3.7e - 05$	$2.9e - 03$
$F - S + \Pi^{RT_1}$	$9.7e - 19$	$3.3e - 17$	$2.9e - 03$

- Vortex in $\Omega = (0, 1)^2$ with varying ν ($h = 0.1$, $N_{\vec{u}} = 1534$, $N_p = 726$):
 $\vec{u} = \overrightarrow{\text{curl}}(x^2(1-x)^2y^2(1-y^2))$, $p = x^3 + y^3 - 1/2$, $\vec{f} = -\nu\Delta\vec{u} + \vec{\nabla}p$.

	ν	$\ \vec{u} - \vec{u}_h - h\ _0$	$\ \vec{u} - \vec{u}_h\ _h$	$\ p - p_h\ _h$
$F - S$	$1.0e - 0$	$1.5e - 5$	$9.2e - 4$	$2.1e - 3$
	$1.0e - 2$	$2.8e - 4$	$2.0e - 2$	$1.9e - 3$
	$1.0e - 4$	$9.4e - 2$	$1.9e - 0$	$1.9e - 3$

	ν	$\ \vec{u} - \vec{u}_h - h\ _0$	$\ \vec{u} - \vec{u}_h\ _h$	$\ p - p_h\ _h$
$F - S + \Pi^{RT_1}$	$1.0e - 0$	$1.7e - 5$	$1.1e - 4$	$2.8e - 3$
	\vdots	\vdots	\vdots	\vdots
	$1.0e - 8$	$1.7e - 5$	$1.1e - 4$	$2.8e - 3$

Outlook

- Ongoing work:

Using Raviart-Thomas projection:

- ▷ Convective form: $\left((\Pi^{RT_0}(\vec{u}_h^{n+1}) \cdot \vec{\nabla}) \vec{u}_h^n, \Pi^{RT_0}(\vec{v}_h) \right)_0$.
- ▷ Unsteady state with $\Pi^{RT_0}(M)$.

Splines for INS using GeoPDEs with L. Dray, M2 student.

- Future work:

Handle polyhedral meshes in particular prism or hexahedra and tetrahedra.

- ▷ PolyMAC code (A. Gershenfeld et al).

Order 2 method.

- ▷ Preliminary work with M. Rihani, former M2 student.

Matlab tests for Stokes in $2D$ and Fortin-Soulie'86 with Π^{RT_1} .

Thank you for your attention!