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Kirone Mallick

To cite this version:

Kirone Mallick. Random oscillator with general Gaussian noise. Physica A: Statistical Mechanics and its Applications, 2007, 384 (1), pp.64-68. 10.1016/j.physa.2007.04.070 . cea-02927301

HAL Id: cea-02927301

https://cea.hal.science/cea-02927301v1

Submitted on 1 Sep 2020

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Anomalous diffusion in a random nonlinear oscillator due to high frequencies of the noise

K. Mallick
Service de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette, France
(Dated: November 21, 2018)

We study the long time behaviour of a nonlinear oscillator subject to a random multiplicative noise with a spectral density (or power-spectrum) that decays as a power law at high frequencies. When the dissipation is negligible, physical observables, such as the amplitude, the velocity and the energy of the oscillator grow as power-laws with time. We calculate the associated scaling exponents and we show that their values depend on the asymptotic behaviour of the external potential and on the high frequencies of the noise. Our results are generalized to include dissipative effects and additive noise.

PACS numbers: 05.10.Gg,05.40.-a,05.45.-a

arXiv:0710.4063v1 [nlin.CD] 22 Oct 2007

The interplay of randomness with nonlinearity gives rise to a variety of phenomena both of practical and theoretical significance that have been the subject of many studies since the pioneering work of Stratonovich [1, 2, 3, 4] For example, electronic and Josephson junctions subject to thermal noise [5, 6], the Faraday instability of surface waves [7] and even human rythmic movements [8] can be modelized by nonlinear oscillators subject to various sources of noise (for a general reference on this subject, see e.g., [9]). In the regime where the dissipation of the effective nonlinear oscillator can be neglected, the random perturbation injects energy into the system and physical observables such as the oscillator's mechanical energy, root-mean-square position and velocity, grow algebraically with time [10, 11, 12]. This instability saturates when dissipative effects (that scale typically as the square-root of the energy) become large enough to overcome the injection rate and the system reaches an non-equilibrium steady state. In series of recent papers [13, 14, 15], we have derived analytical results for quasi-Hamiltonian nonlinear random oscillators subject to external or internal random perturbation that is either a Gaussian white noise or an Ornstein-Uhlenbeck process. In particular, we have calculated the scaling exponents characterize the algebraic growth of the energy, the position and the velocity of the oscillator.

In the present work, we consider the general case a nonlinear oscillator subject to a multiplicative noise with an arbitrary correlation function. We show that the physical observables display a scaling behaviour in the long time limit and we calculate the associated scaling exponents. The formulae obtained are valid for any type of noise whose power-spectrum decays as a power-law at large frequencies; in particular, the results previously derived for white and Ornstein-Uhlenbeck noises appear as special cases of the general expressions that are derived in the present work. Finally, we briefly explain how our results can be generalized to include dissipative effects, additive noise, and how the scaling behaviour is modified when the noise power-spectrum decays decays exponentially at large frequencies.

I. THE DYNAMICAL EQUATIONS IN ACTION-ANGLE VARIABLES

We consider a nondissipative oscillator of amplitude $x(t)$, trapped in a nonlinear confining potential $\mathcal{U}(x)$ and subject to a multiplicative noise $\xi(t)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = -\frac{\partial \mathcal{U}(x)}{\partial x} + x(t)\,\xi(t)\,. \tag{1}$$

Our aim is to study the effect of the statistical properties of $\xi(t)$ on the long time properties of the dynamical variable $x(t)$.

We restrict our analysis to the case where the dominant term in the potential $\mathcal{U}$ when $|x| \to \infty$ is an even power of x. A suitable rescaling of x allows us to write

$$\mathcal{U} \sim \frac{x^{2n}}{2n} \quad \text{with} \quad n \geq 2\,. \tag{2}$$

As the amplitude $x(t)$ of the oscillator grows with time, the leading behavior of $\mathcal{U}(x)$ for $|x| \to \infty$ only is relevant and Eq.(1) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + x(t)^{2n-1} = x(t)\,\xi(t)\,. \tag{3}$$

We shall analyse the motion of the nonlinear stochastic oscillator following the method explained in [13]. Defining the energy and the angle variables,

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2n}x^{2n} \quad , \quad \text{and} \quad \phi = \frac{\sqrt{n}}{(2n)^{1/2n}} \int_0^{x/E^{1/2n}} \frac{\mathrm{d}u}{\sqrt{1-\frac{u^{2n}}{2n}}} \, , \tag{4}$$

we transform the coordinates in phase space from position and velocity to energy and angle:

$$x = E^{\frac{1}{2n}} \, s_n(\phi) \, , \tag{5}$$

$$\dot{x} = (2n)^{\frac{n-1}{2n}} E^{\frac{1}{2}} \, s_n'(\phi) \, , \tag{6}$$

where the hyperelliptic function s_n is defined by the relation $s_n(\phi) = x/E^{\frac{1}{2n}}$. Using the auxiliary variable Ω, that satisfies

$$\Omega = (2n)^{\frac{n+1}{2n}} \, E^{\frac{n-1}{2n}} \, , \tag{7}$$

equation (3) is rewritten [13, 14] as a system of two coupled stochastic differential equations

$$\dot{\Omega} = (n-1) \, s_n(\phi) s_n'(\phi) \, \xi(t) \, , \tag{8}$$

$$\dot{\phi} = \frac{\Omega}{(2n)^{\frac{1}{n}}} - \frac{s_n(\phi)^2}{\Omega} \, \xi(t) \, . \tag{9}$$

This system is rigorously equivalent to the original problem (Eq. 3) and has been derived without any hypothesis on the random perturbation term $\xi(t)$. It appears clearly in this formulation that the angle ϕ is a fast variable whereas the action Ω is a slow variable : an effective stochastic equation for Ω can thus be derived by adiabatic averaging over the angular variable.

We now specify the statistical properties of the random perturbation $\xi(t)$. We shall consider a stationary Gaussian noise of zero mean value. A Gaussian process is entirely characterized by its auto-correlation function, defined as

$$\mathcal{S}(t'-t) = \langle \xi(t')\xi(t) \rangle \, . \tag{10}$$

In Fourier space, the power-spectrum of the noise is given by

$$\hat{\mathcal{S}}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}t \exp(i\omega t) \mathcal{S}(t) = \int_{-\infty}^{+\infty} \mathrm{d}t \exp(i\omega t) \langle \xi(t)\xi(0) \rangle \, . \tag{11}$$

For instance, if $\xi(t)$ is a white noise of amplitude $\mathcal{D}$, we have $\hat{\mathcal{S}}(\omega) = \mathcal{D}$. When $\xi(t)$ is an Ornstein-Uhlenbeck process of amplitude $\mathcal{D}$ and autocorrelation time τ, we have

$$\mathcal{S}(t'-t) = \frac{\mathcal{D}}{2\,\tau} \, \mathrm{e}^{-|t-t'|/\tau} \quad \text{and} \quad \hat{\mathcal{S}}(\omega) = \frac{\mathcal{D}}{1+\omega^2\tau^2} \, . \tag{12}$$

In the white noise case, we derived in [13] the following scaling relations, valid when $t \to \infty$

$$E \sim (\mathcal{D}t)^{\frac{n}{n-1}} \, , \quad x \sim (\mathcal{D}t)^{\frac{1}{2(n-1)}} \, , \quad \dot{x} \sim (\mathcal{D}t)^{\frac{n}{2(n-1)}} \, . \tag{13}$$

In the case when $\xi(t)$ is an Ornstein-Uhlenbeck process, we found in [14] that, in the long time limit, the energy, the amplitude and the velocity of the oscillator grow as :

$$E \sim \left(\frac{\mathcal{D}t}{\tau^2}\right)^{\frac{n}{2(n-1)}} \, , \quad x \sim \left(\frac{\mathcal{D}t}{\tau^2}\right)^{\frac{1}{4(n-1)}} \, , \quad \dot{x} \sim \left(\frac{\mathcal{D}t}{\tau^2}\right)^{\frac{n}{4(n-1)}} \, . \tag{14}$$

These colored noise scalings are always observed when $t \to \infty$, even if the correlation time τ is arbitrarily small. We remark in equations (13) and (14) that the scaling exponents only depend on the asymptotic growth rate of the potential. However, the exponents are reduced by a factor 2 for Ornstein-Uhlenbeck noise as compared to their value for white noise; this means physically that the energy transfer from the noise to the oscillator is less efficient when the noise is correlated in time. In [14], we derived an analytic expression for the probability distribution function (PDF) in the long time limit that allowed us to derive not only the scaling laws but also to calculate all the prefactors. However, we did not find any satisfactory explanation that would explain the origin of the factor 2 reduction in of the scaling exponents. We show in the next section by analyzing the general case that the scaling exponents not only depend on the potential at infinity but also on the asymptotic behaviour of the spectral density of the noise at high frequencies.

II. SCALING BEHAVIOUR IN PRESENCE OF ARBITRARY NOISE

We consider now the more general case of a Gaussian noise which can be generated from the white noise by solving a linear differential equation of order σ. Such a noise has a power-spectrum that decays as follows at high frequencies :

$$\hat{\mathcal{S}}(\omega) \sim \mathcal{D}(\omega\tau)^{-2\sigma} \quad \text{when} \quad |\omega| \to \infty\,. \tag{15}$$

The amplitude $\mathcal{D}$ of the noise and the correlation-time τ are defined by dimensional analogy with equation (12). The exponent σ characterizes the high frequency behaviour of the the power-spectrum of the noise. We shall show in this section that, in the long time limit, the scaling behaviour of the nonlinear oscillator in presence of a noise with a power-spectrum that satisfies equation (15), is given by

$$\begin{aligned} E &\sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\frac{n}{(\sigma+1)(n-1)}}, \\ x &\sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\frac{1}{2(\sigma+1)(n-1)}}, \\ \dot{x} &\sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\frac{n}{2(\sigma+1)(n-1)}}. \end{aligned} \tag{16}$$

For $\sigma = 0$ (respectively $\sigma = 1$) we recover the scalings given in equation (13) for a white noise (respectively the scalings given in equation (14) for Ornstein-Uhlenbeck noise). The formulae (16) provide us with the explicit dependence of the growth exponents on the confining potential and on the noise spectral density. In particular, we observe that when the relative weight of the large frequencies is decreased (by increasing the value of σ) the effective diffusion in the oscillator phase space becomes slower.

We now derive the power laws (16). One method to obtain these scalings is to apply the recursive adiabatic averaging procedure developped in [14, 15]. This advantage of this technique is to provide explicit expressions for the prefactors in the scaling law. The basic idea is the following : the naive averaging procedure applied to the system (8,9) fails if the correlations between the angle ϕ and the noise $\xi(t)$ are discarded. However, by taking the derivatives of the equations (8,9) repeatedly, a white noise contribution will appear (after σ derivatives) and will provide the leading term in long-time behaviour of the oscillator. However, in order to apply this procedure, we need an explicit differential equation that relates the noise $\xi(t)$ to the white noise. Furthermore, this technique, though quantitatively precise, leads to very intricate calculations and is suitable for simple noises such as the Ornstein-Uhlenbeck process or the harmonic noise.

We shall therefore derive here the power laws (16) from a simpler argument. First, because we only want to extract the leading scaling behaviour when $t \to \infty$, we simplify the equations (8) and (9), as we did in [14] : we keep only the dominant terms, replace the hyperelliptic function $s_n(\phi)$ by the circular function $\sin\phi$ and put all the numerical factors to 1. We thus obtain the following system

$$\dot{\Omega} = \xi(t)\sin\phi \quad \text{and} \quad \dot{\phi} = \Omega\,, \tag{17}$$

which describes a pendulum with random frequency [17]. Now, the second order cumulant expansion [16] of the stochastic Liouville equation for the PDF $P_t(\Omega,\phi)$ associated with equations (17) is given by

$$\frac{\partial P_t}{\partial t} = \mathbf{L}_0 P_t + \int_0^t d\theta \langle \mathbf{L}_1(t)\exp(\mathbf{L}_0\theta)\mathbf{L}_1(t-\theta)\exp(-\mathbf{L}_0\theta)\rangle P_t \tag{18}$$

where the differential operators are defined as

$$\mathbf{L}_0 P_t = -\frac{\partial}{\partial\phi}(\Omega P_t)\,, \tag{19}$$

$$\mathbf{L}_1(t) P_t = -\frac{\partial}{\partial\Omega}(\xi(t)\sin\phi\, P_t)\,. \tag{20}$$

Following the method developed in [18], we evaluate the right hand side of equation (18) by using to the formula

$$\exp(A)B\exp(-A) = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots\,, \tag{21}$$

with $A = \mathbf{L}_0$ and $B = \mathbf{L}_1(t-\theta)$. We find by induction the expression of the n-th commutator

$$[\mathbf{L}_0, [\ldots, [\mathbf{L}_0, [\mathbf{L}_0, \mathbf{L}_1(t-\theta)]]\ldots] = \xi(t-\theta)\left(\frac{\partial}{\partial\Omega}H_1^{(n)}(\Omega,\phi) + \frac{\partial}{\partial\phi}H_2^{(n)}(\Omega,\phi)\right), \tag{22}$$

where the functions $H_1^{(n)}$ and $H_2^{(n)}$ are given by

$$H_1^{(n)}(\Omega,\phi) = (-1)^{n-1}\Omega^n \sin(\phi + n\frac{\pi}{2}), \tag{23}$$

$$H_2^{(n)}(\Omega,\phi) = (-1)^{n-1}n\Omega^{n-1}\cos(\phi + n\frac{\pi}{2}). \tag{24}$$

Substituting the equations (21-24) in equation (18), we obtain, after taking the average over the noise and integrating over the variable θ :

$$\int_0^t d\theta \langle \mathbf{L}_1(t)\exp(\mathbf{L}_0\theta)\mathbf{L}_1(t-\theta)\exp(-\mathbf{L}_0\theta)\rangle P_t = -\frac{\partial}{\partial\Omega}\sin\phi\left(\frac{\partial}{\partial\Omega}\mathcal{H}_1(\Omega,\phi,t)P_t + \frac{\partial}{\partial\phi}\mathcal{H}_2(\Omega,\phi,t)P_t\right), \tag{25}$$

with

$$\mathcal{H}_1(\Omega,\phi,t) = \sum_{n=0}^{\infty}\frac{H_1^{(n)}(\Omega,\phi)}{n!}\int_0^t \mathrm{d}\theta\,\theta^n\mathcal{S}(\theta) = -\int_0^t \sin(\Omega\theta+\phi)\mathcal{S}(\theta)\mathrm{d}\theta, \tag{26}$$

$$\mathcal{H}_2(\Omega,\phi,t) = \sum_{n=0}^{\infty}\frac{H_2^{(n)}(\Omega,\phi)}{n!}\int_0^t \mathrm{d}\theta\,\theta^n\mathcal{S}(\theta) = \int_0^t \sin(\Omega\theta-\phi)\theta\mathcal{S}(\theta)\mathrm{d}\theta. \tag{27}$$

In the limit $t\to\infty$, we average the differential operator (25) over the fast variable ϕ and we deduce that

$$\begin{aligned}\int\frac{d\phi}{2\pi}\int_0^\infty d\theta\langle \mathbf{L}_1(t)\exp(\mathbf{L}_0\theta)\mathbf{L}_1(t-\theta)\exp(-\mathbf{L}_0\theta)\rangle &= \frac{1}{2}\frac{\partial^2}{\partial\Omega^2}\int_0^\infty\cos(\Omega\theta)\mathcal{S}(\theta)\mathrm{d}\theta - \frac{1}{2}\frac{\partial}{\partial\Omega}\int_0^\infty\sin(\Omega\theta)\theta\mathcal{S}(\theta)\mathrm{d}\theta \\ &= \frac{1}{4}\frac{\partial^2}{\partial\Omega^2}\hat{\mathcal{S}}(\Omega) - \frac{1}{4}\frac{\partial}{\partial\Omega}\hat{\mathcal{S}}'(\Omega),\end{aligned} \tag{28}$$

where $\hat{\mathcal{S}}'(\Omega)$ represents the derivative of the power-spectrum with respect to the frequency. Finally, we apply this averaged operator to $\tilde{P}_t(\Omega)$, the probability distribution function of the slow variable Ω and we derive, in the long time limit, an effective averaged Fokker-Planck equation for $\tilde{P}_t(\Omega)$:

$$\frac{\partial\tilde{P}_t}{\partial t} = \frac{1}{4}\frac{\partial}{\partial\Omega}\Big(\hat{\mathcal{S}}(\Omega)\frac{\partial\tilde{P}_t}{\partial\Omega}\Big). \tag{29}$$

This equation only involves the power spectrum of the noise. If we substitute in this equation the large frequency behaviour (15) of $\hat{\mathcal{S}}$, we conclude by elementary dimensional analysis that :

$$\Omega^{2(\sigma+1)} \sim \frac{\mathcal{D}t}{\tau^{2\sigma}}. \tag{30}$$

By using the definition (7) of Ω and the relations (5) and (6) between the amplitude x, the velocity $\dot{x}$ and the energy E, the scaling relations (16) are derived.

Our calculation can be generalized to the case of a weakly dissipative system, by including a linear friction term in equation (1) with dissipation rate γ. In this case, the oscillator's amplitude does not grow without bounds but saturates after a time of the order $1/\gamma$. The system reaches a non-equilibrium steady state described by a stationary probability, which is in general not a Gibbs-Boltzmann distribution. In the case of a vanishingly small rate, the saturation time is large and the averaging technique can be still be applied. It can be shown, using the same method as above, that the stationary probability behaves as

$$P_{\mathrm{stat}}(E) \sim \exp\Big(-C\frac{\gamma\tau^{2\sigma}E^{\frac{(\sigma+1)(n-1)}{n}}}{\mathcal{D}}\Big) \quad \text{when } E\to\infty, \tag{31}$$

where C is a constant. The analytical expression of C and of the prefactors in $P_{\rm stat}$ can not be obtained by the method used here (a recursive adiabatic averaging similar to the one used for white and Ornstein-Uhlenbeck noise would be needed to go beyond scaling).

We have asuumed hitherto that the noise $\xi(t)$ is generated from white noise by solving a linear differential equation of order σ. Therefore, $\xi(t)$ is Gaussian and its power-spectrum $\hat{\mathcal{S}}(\omega)$ decreases as a power-law at larges frequencies. However, the derivation of the effective Fokker-Planck equation (29) does not rely on these assumptions and can still be performed when the correlation function of $\xi(t)$ has long-tails, for example, when $\hat{\mathcal{S}}(\omega) \sim \exp(-\tau|\omega|)$ (Lorentzian power-spectrum). We then obtain that

$$\Omega \sim \frac{1}{\tau} \log \frac{t}{\tau}\,, \tag{32}$$

i.e., the amplitude of the system grows logarithmically with time. Similarly, if the power-spectrum of the noise is Gaussian, *i.e.*, $\hat{\mathcal{S}}(\omega) \sim \exp(-(\tau\omega)^2)$ at high frequencies, the action variable Ω^2 - increases logarithmically with time. If the power-spectrum has a high-frequency cut-off at ω_0, *i.e.*, if $\hat{\mathcal{S}}(\omega) = 0$ for $\omega > \omega_0$ we conjecture Ω saturates at large times. This fact is true, in particular, when the noise is a deterministic circular function for example in the case $\xi(t) = \cos(\omega_0 t)$.

Finally, we emphasize that the method we have used here can also be applied to a nonlinear oscillator subject to an additive noise, described by the following Langevin equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) = -\frac{\partial \mathcal{U}(x)}{\partial x} + \xi(t)\,, \tag{33}$$

where the power-spectrum of the noise $\xi(t)$ satisfies the equation (15). Then, by calculations similar to those described above, the following scaling behaviour is derived (an alternative method that leads to the same results would be to use the Markovian approximation for the energy dynamics in the low friction limit, developed in [19, 20]) :

$$E \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\frac{n}{\sigma(n-1)+n}}\,, \quad x \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\frac{1}{2\sigma(n-1)+2n}}\,, \quad \dot{x} \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\frac{n}{2\sigma(n-1)+2n}}\,. \tag{34}$$

In particular, when the noise is white, we recover the fact that the energy grows linearly with time.

III. CONCLUDING REMARKS

A particle trapped in a nonlinear confining potential and subject to a random noise undergoes anomalous diffusion in phase space. In this work, we have found analytical expressions for the associated diffusion exponents, by using an action-angle representation of the equations of motion that allowed us to separate the slow variable from the fast variable and to derive an effective Langevin dynamics for the slow variable. Our results are derived for an arbitrary correlation function of the noise and are valid for multiplicative noise as well as for additive noise. We have shown that the high frequency components in the noise spectrum play a crucial role : the fastest the power spectrum of the noise decays at high frequencies, the slower is the diffusion. This fact has the following physical interpretation : the energy transfer from the external driving to the oscillator is optimal when the driving frequency is of the order of the natural frequency of the oscillator (*e.g.*, parametric resonance is maximal for driving at twice the natural frequency). However, because of the nonlinearity, the period T of the underlying deterministic oscillator decreases with the amplitude. Therefore, at large amplitudes, the energy transfer between the noise and the system involves higher and higher frequencies. If these frequencies have a small weight in the noise power spectrum, the amplification of the oscillator is less efficient. Our calculations put this intuitive reasoning on a quantitative basis and can be generalized to include dissipative effects. In presence of dissipation, the system reaches a nonequilibrium steady state in which the stationary PDF is not the canonical Gibbs-Boltzmann distribution but a streched exponential.

Acknowledgments

I am thankful to E. Bogomolny'i for a question that was at the origin of this work.

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