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We consider a nonequilibrium reaction-diffusion model on a finite one dimensional lattice with bulk and boundary dynamics inspired by Glauber dynamics of the Ising model. We show that the model has a rich algebraic structure that we use to calculate its properties. In particular, we show that the Markov dynamics for a system of a given size can be embedded in the dynamics of systems of higher sizes. This remark leads us to devise a technique we call the *transfer matrix Ansatz* that allows us to determine the steady state distribution and correlation functions. Furthermore, we show that the disorder variables satisfy very simple properties and we give a conjecture for the characteristic polynomial of Markov matrices. Lastly, we compare the transfer matrix Ansatz used here with the matrix product representation of the steady state of one-dimensional stochastic models.

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I. INTRODUCTION

The study of systems far from equilibrium has been greatly helped by the discovery of exactly solvable models, because explicit computations for these models provide us with excellent testing grounds for general hypotheses about nonequilibrium statistical mechanics [1, 2].

One important difference between equilibrium and nonequilibrium behaviour is encoded in the detailed balance condition. This condition states that at equilibrium the total transition rate between two arbitrary micro-configurations vanishes identically [3]. Conversely, nonequilibrium steady states usually break detailed balance, which results in the existence of current loops in the configuration space of the system, leading to a non-zero macroscopic (physical) current that transports matter, momentum or energy from one region of the system to another. Because of this current, the boundaries of the system can affect its bulk and the modification of the boundary conditions through a control parameter can induce dynamical phase transitions even in one dimensional systems [4]. Such sensitiveness to the boundary conditions is well demonstrated by exact results obtained for the asymmetric exclusion process (ASEP) in one dimension, which is one of the simplest examples of a driven lattice gas and one of the most exhaustively investigated interacting particle systems [5–7]. For the ASEP on a periodic ring the steady state is uniform and all configurations have the same stationary weight; in contrast, for the ASEP on a finite lattice with open boundary conditions (that allow injection and removal of particles at the end sites) the measure is non-uniform and in the limit of large sizes, the system can exist in three different phases: maximal current, low density and high density (the last two phases being separated by a line of shocks). The exact expression of the stationary measure valid for any system size was first derived in [8], introducing a method now called the matrix product representation that has become an important technique for one dimensional interacting particle processes. In particular, this matrix representation has an interpretation in terms of discrete lattice paths that leads to exact combinatorial results for finite size systems (see [9] for an exhaustive and recent review). Thanks to this matrix product method and to more standard integrability techniques such as the Bethe Ansatz [10], an plethora of results have been derived for the ASEP [11].

Here, we study a reaction-diffusion model on a finite lattice of L sites with open boundaries in which hard-core particles perform asymmetric jumps and can undergo pairwise annihilation. This model is inspired by the Glauber dynamics [12] for the Ising model, in which the elementary excitations (or particles) are not the individual spins but rather the domain walls between sets of opposite spins. More precisely, we shall consider the totally asymmetric Glauber dynamics where each spin changes its orientation with a certain probability based strictly on the spin to its left. Hence, the domain walls, which are represented by particles, move only to the right and, if two of them collide, they annihilate each other. Reaction-diffusion models have been thoroughly studied in the nonequilibrium statistical physics literature (see e.g. [13, 14] for similar studies on the real line and [15–19] on the lattice) and their relation to non-Hermitian spin chains has been established [20, 21]. In particular, it has been shown that for certain values

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of the reaction-rates, they reduce to free fermion models that can be solved by using Jordan-Wigner fermionization techniques [22–28]. This method was applied to periodic boundary conditions where translation invariance allows the use of Fourier transform to diagonalize the free fermion Hamiltonian. More recently, similar techniques were used by Farago [29], and Farago and Pitard [30] to calculate large deviation functions of the time integrated injected power when a single spin is allowed to perform Poissonian flips.

We shall study the effect of open boundaries conditions on the long time dynamics of the model. At the boundary sites of the lattice, we allow creation and destruction of particles in such a manner that the boundary dynamics is compatible with the bulk dynamics. Here, the Jordan-Wigner transformation introduces non-local terms in the free fermion Hamiltonian and the lack of translation invariance precludes the use of the Fourier transform.

We shall define a one-parameter family of models with a nontrivial steady state that generalizes the totally asymmetric exclusion process (TASEP) on a finite lattice, first solved in [8]. For a special value of the parameter (which corresponds to the free fermion point), we calculate the exact nonequilibrium steady state measure by using a recursion between systems of different sizes. Here, the key to the solution is not a matrix product representation but a different approach: using a linear transform between configuration spaces of different dimensions, we shall prove that the Markov matrix of the system of size L can be embedded in the Markov matrix of the system of size $L+1$. The existence of such a rectangular (thus, non-invertible) similarity matrix, that will be called a 'transfer matrix Ansatz', will allow us to derive exact combinatorial expressions for the local density and for correlation functions of the model, and to extract their asymptotic behaviour in the infinite system size limit.

The organization of the paper is as follows: In Section II, we introduce the model. In Section III, we describe the transfer matrix Ansatz in general terms and apply it to our system. This leads us to a closed form expression for the 'partition function' that normalizes the stationary probabilities and we derive a formula for the joint occupation of the first k sites for all k. In Section IV, we define the disorder variables and calculate the associated one-point and two-point functions. We use these variables to calculate the density and the rate of evaporation; we also prove some general properties of higher correlation functions. In Section V, we conjecture some properties of the spectrum of the Markov matrices. Section VI is devoted to concluding remarks and some open problems. In Appendix A, we write explicit recursion relations for the steady state probabilities that allow us to compare the transfer matrix Ansatz with the matrix product representation.

II. THE MODEL

We consider a nonequilibrium system on a finite lattice with L sites labelled from 1 to L. We denote the boundary of the domain wall, that is the boundary between oppositely charged spins, by a particle using the standard notation 1. Empty sites are denoted by 0. The evolution rule in the bulk, biased Glauber dynamics, is thus given by

$$\begin{aligned} 10 &\to 01 \text{ with rate } 1\,, \\ 11 &\to 00 \text{ with rate } \lambda\,. \end{aligned} \tag{1}$$

We remark that the first rule represents the movement of the domain wall to the right (eg. $++\,\big|\,--\to+++\big|-$) and the second, the annihilation of two domain walls (eg. $++\,\big|\,-\,\big|+\to++++$).

The evolution of the first site is given by

$$\begin{aligned} 0 &\to 1 \text{ with rate } \alpha, \\ 1 &\to 0 \text{ with rate } \alpha\lambda. \end{aligned} \tag{2}$$

Particles can exit from the last site according to

$$1 \to 0 \text{ with rate } \beta. \tag{3}$$

The rules (2) and (3) were constructed by considering the finite lattice to be a 'window' of the infinite one-dimensional lattice. Suppose that there is a virtual site labelled 0 to the left of the first site and a virtual site labelled $L+1$ to the right of the last site. The left boundary conditions are deduced from the bulk rules (1) by looking at the second component of the bond $0-1$ and the right boundary conditions are obtained by looking at the first component of the bond $L-(L+1)$; α and β are free control parameters.

The rules (1), (2) and (3) fully define the nonequilibrium model by allowing one to construct explicitly its Markov matrix of dimension 2^L. Notice that we use terminology directly from the original TASEP problem for the bulk and boundary rates and that we have ensured the consistency of the bulk and the boundary rates by fixing the rates appropriately. Indeed, the value $\lambda = 0$ corresponds exactly to the TASEP solved in [8].

The model can be discussed using an approximate mean field argument. The evolution equation for the density at site $i > 1$ is given by

$$\frac{d}{dt}\langle\eta_i\rangle = \langle\eta_{i-1}(1-(\lambda+1)\eta_i)\rangle - \langle\eta_i(1-(\lambda-1)\eta_{i+1})\rangle\,, \tag{4}$$

where η_i represents the occupation variable at site i. For $\lambda = 1$, these equations become simpler because the state of a site depends only on the preceding sites. Then, using the mean-field assumption, we obtain the following recursion valid in the stationary state:

$$\langle\eta_i\rangle = \frac{\langle\eta_{i-1}\rangle}{1+2\langle\eta_{i-1}\rangle}. \tag{5}$$

The stationary density at the first site can be obtained exactly by writing the equation

$$0 = \frac{d}{dt}\langle\eta_1\rangle = \alpha\langle(1-2\eta_i)\rangle - \langle\eta_1\rangle, \tag{6}$$

and therefore one obtains the general formula,

$$\langle\eta_k\rangle = \frac{\alpha}{1+2k\alpha}. \tag{7}$$

This implies that the density falls off like k^{-1} for large k. This mean-field result is wrong: in one-dimension, the actual exponent is $-1/2$. Note that the exit rate β did not enter the calculation.

The fact that the value $\lambda = 1$ plays a special role can readily be understood from the dynamical rules: indeed the exit rate from any site i is equal to 1 whether site $i+1$ is occupied or not. Besides, $\lambda = 1$ corresponds to the free fermion point of the associated spin chain (see Section VI for a more detailed explanation of this fact).

Remark 1 *More generally, for $\lambda = 1$, this model has the following important property. The correlation function $\langle\eta_{i_1}\dots\eta_{i_n}\rangle$ does not depend on the state of η_{i_n+1} simply because the exit rate from any configuration that contributes to this expectation value is equal to 1 whether site i_n+1 is occupied or not (by (1)). In particular, for a system of size L, all correlation functions which depend on sites other than the last one are going to be strictly independent of β and moreover, any correlation function of the form $\langle\eta_{i_1}\dots\eta_{i_n}\rangle$ is independent of L as long as $L \geq i_n + 1$.*

For the rest of the paper, we will take $\lambda = 1$ as that is the only case for which we can derive explicit combinatorial formulae.

III. SEMI-SIMILARITY BETWEEN MARKOV PROCESSES

A. The Transfer Matrix Ansatz

Let M_L denote the Markov matrix for a system with L sites. Typically the size of the matrix will be exponential in L. We first give a general definition of the "transfer matrix Ansatz" and then apply it to the specific case of the problem defined above.

Let us consider a family of Markov processes defined by Markov matrices $\{M_L\}$ of increasing sizes (in most physical applications, L is the size of the system). We shall see that for certain systems there exists a natural embedding of M_L into M_{L+1}.

Definition 1 *We say that a family of Markov processes satisfies the* Transfer Matrix Ansatz *if there exist matrices $T_{L,L+1}$ for all sizes L such that*

$$M_{L+1}T_{L,L+1} = T_{L,L+1}M_L\,. \tag{8}$$

We also impose that this equality is nontrivial in the sense that

$$M_{L+1}T_{L,L+1} \neq 0. \tag{9}$$

The rectangular transfer matrices $T_{L,L+1}$ can be interpreted as *semi-similarity transformations* connecting Markov matrices of different sizes. Another way to view the transfer matrices is that that the following diagram

$$\begin{array}{ccc} \Omega_L & \xrightarrow{M_L} & \Omega_L \\ {\scriptstyle T_{L,L+1}}\downarrow & & \downarrow{\scriptstyle T_{L,L+1}} \\ \Omega_{L+1} & \xrightarrow{M_{L+1}} & \Omega_{L+1} \end{array} \tag{10}$$

commutes, where Ω_L is the space of 2^L configurations of size L.

We first explain the importance of the last condition. For the nonequilibrium systems that we are interested in, there is usually one unique steady state, which means the multiplicity of the zero eigenvalue is one. If $|v_L\rangle$ is a non-zero vector in the kernel of M_L and $\langle 1_L| = (1, 1, \ldots, 1)$, the matrix $V_{L,L+1} = |v_{L+1}\rangle\langle 1_L|$ satisfies (8) since the Markov matrices satisfy the condition $\langle 1_L|M_L = 0$. However, we have $M_{L+1}V_{L,L+1} = V_{L,L+1}M_L = 0$, and the condition (9) is violated: therefore this trivial solution is excluded.

The above definition leads immediately to some important observations. First we have

$$0 = T_{L,L+1}M_L|v_L\rangle = M_{L+1}T_{L,L+1}|v_L\rangle, \tag{11}$$

which, assuming $T_{L,L+1}|v_L\rangle \neq 0$, and taking into account the uniqueness of the steady state, allows us to define $|v_{L+1}\rangle$ so that

$$T_{L,L+1}|v_L\rangle = |v_{L+1}\rangle. \tag{12}$$

The other important consequence is related to the eigenvalues of M_L. Let $|w\rangle$ be any eigenvector of M_L with eigenvalue μ. Then

$$\mu T_{L,L+1}|w\rangle = T_{L,L+1}M_L|w\rangle = M_{L+1}T_{L,L+1}|w\rangle, \tag{13}$$

which, again assuming $T_{L,L+1}|w\rangle \neq 0$, means that $T_{L,L+1}|w\rangle$ is an eigenvector of M_{L+1} with the same eigenvalue μ. In other words, an eigenvector of M_L that is not in the kernel $T_{L,L+1}$ is also an eigenvector of M_{L+1} with the same eigenvalue (if the matrix M_L is not diagonalizable, then we cannot immediately make a statement about the multiplicities). Conversely, if M_L and M_{L+1} have a common eigenvalue μ, then the rank-one rectangular matrix $|\mu_{L+1}\rangle\langle\mu_L|$ is a transfer matrix; but the image of the steady state $|v_L\rangle$ of M_L by this matrix vanishes. Hence, $|\mu_{L+1}\rangle\langle\mu_L|$ cannot be used to construct the steady state of M_{L+1} knowing $|v_L\rangle$. Therefore, in order to study the stationary state, we must look for transfer matrices that satisfy the additional condition: $T_{L,L+1}|v_L\rangle \neq 0$.

B. The case of the asymmetric annihilation process

For the system we consider here, the Markov matrices are of size 2^L. It is most convenient for us to take the naturally ordered basis of binary sequences of size L. For example, when $L = 2$, the ordered list is $(00, 01, 10, 11)$. The first important observation is that there is a recursion of order one among the Markov matrices.

Theorem 1 *Let σ denote the matrix*

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{14}$$

and $\mathbb{1}_L$ denote the identity matrix of size 2^L. Then

$$M_L = \begin{pmatrix} M_{L-1} - \alpha(\sigma \otimes \mathbb{1}_{L-2}) & \alpha\mathbb{1}_{L-1} + (\sigma \otimes \mathbb{1}_{L-2}) \\ \alpha\mathbb{1}_{L-1} & M_{L-1} - \mathbb{1}_{L-1} - \alpha(\sigma \otimes \mathbb{1}_{L-2}) \end{pmatrix}, \tag{15}$$

where M_L is written as a 2×2 block matrix with each block made up of matrices of size 2^{L-1}. The initial matrix for $L = 1$ is given by

$$M_1 = \begin{pmatrix} -\alpha & \alpha + \beta \\ \alpha & -\alpha - \beta \end{pmatrix}. \tag{16}$$

Equation (15) is proved as follows. Let v, v' denote binary vectors of length L and w, w' denote vectors of length $L-1$. The 2^L binary configurations of the system of size L are listed increasingly from $00\ldots00$ to $11\ldots11$. If v_j and v_i are two binary configurations the (i,j)th entry of the Markov matrix M_L is the rate of the process $v_j \to v_i$ for $i \neq j$ and the diagonal entries are given by $-\sum_{j\neq i}(M_L)_{j,i}$. Decompose the Markov matrix of size 2^L in four blocks of size 2^{L-1} according to the first bit,

$$M_L = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}. \tag{17}$$

First let us consider M_{10}. This encodes all transitions $v = 0w \to v' = 1w'$. The only allowed transition is $0w \to 1w$ with rate α. Hence $M_{10} = \alpha \mathbb{1}_{L-1}$. Similarly, M_{00} encodes transition of the form $v = 0w \to v' = 0w'$. If $w \neq w'$ and the first bits of both w and w' are the same, the transition is either in the bulk or on the right boundary, which is encoded completely in the matrix for $w \to w'$, namely M_{L-1}. There is, however, an additional transition which is present in the $L-1$ system but is not present in the L system. Suppose that the first bit w_1 of w and the first bit w'_1 of w' satisfy $w_1 = 1 - w'_1$ and that all the remaining bits are the same, then $w \to w'$ with rate α because of (2). We therefore have to subtract these transitions from M_{00}. Thus $M_{00} = M_{L-1} - \alpha(\sigma \otimes \mathbb{1}_{L-2})$. Note that the sum of the elements of any of the first 2^{L-1} columns of M_L is indeed equal to zero, because in each column we have once added and once subtracted α and therefore the diagonal terms are unchanged.

For M_{01}, we consider transitions of the form $v = 1w \to v' = 0w'$. If $w = w'$, this transition occurs with rate α. One other transition depends on the first bits of w, w' and assumes that all the other bits are the same. If $w_1 = 1 - w'_1$, then $w \to w'$ with rate 1 in the system of size L because of the transitions (1). Thus $M_{01} = \alpha \mathbb{1}_{L-1} + (\sigma \otimes \mathbb{1}_{L-2})$. Lastly M_{11} encodes transitions $v = 1w \to v' = 1w'$. The argument is now similar to that of the M_{00} case. If the first bits of w and w' are the same, then all the other transitions are encoded by M_{L-1}. Additionally, if $w_1 = 1 - w'_1$ and the other bits are the same, then there is a transition given by (2) for the system of size $L-1$ which is not present in the system of size L and we have to subtract this contribution. Besides, in M_{01} we have added $1 + \alpha$ to each column and we have subtracted α in M_{11}, therefore, to ensure probability conservation, we must subtract 1 from the diagonal in M_{11}, which results in $M_{11} = M_{L-1} - \mathbb{1}_{L-1} - \alpha(\sigma \otimes \mathbb{1}_{L-2})$.

This system satisfies the transfer matrix Ansatz: one special solution can be constructed recursively as we now explain. In fact from the study of small systems (up to size $L = 6$), we found 2^L independent solutions of (8). This reflects the fact that the spectrum of M_L is fully embedded in that of M_{L+1}.

Theorem 2 *There exists a transfer matrix for the model which can be expressed by a recursion of order one. If one writes the transfer matrix from size $L-1$ to size L by a block decomposition of matrices of size $2^{L-1} \times 2^{L-1}$ as*

$$T_{L-1,L} = \begin{pmatrix} T_1^{(L-1)} \\ T_2^{(L-1)} \end{pmatrix}, \tag{18}$$

then the matrix $T_{L,L+1}$ can be written as

$$T_{L,L+1} = \begin{pmatrix} T_1^{(L)} \\ T_2^{(L)} \end{pmatrix} \tag{19}$$

with

$$T_1^{(L)} = \begin{pmatrix} T_1^{(L-1)} + \dfrac{1}{\alpha} T_2^{(L-1)} & 2T_2^{(L-1)} + \dfrac{1}{\alpha} T_2^{(L-1)}(\sigma \otimes \mathbb{1}_{L-2}) \\ T_2^{(L-1)} & \dfrac{1}{\alpha} T_2^{(L-1)} \end{pmatrix} \quad \textit{and} \quad T_2^{(L)} = \begin{pmatrix} 2T_2^{(L-1)} & T_2^{(L-1)}(\sigma \otimes \mathbb{1}_{L-2}) \\ 0 & T_2^{(L-1)} \end{pmatrix}. \tag{20}$$

This, along with the initial condition

$$T_{1,2} = \begin{pmatrix} 1 + \beta + \alpha\beta & \alpha + \beta + \alpha\beta \\ \alpha & 1 \\ \alpha + \alpha\beta & \alpha\beta \\ 0 & \alpha \end{pmatrix}, \tag{21}$$

determines recursively a family of transfer matrices.

The proof is carried out by induction. We first suppose that equation (8) is satisfied for the transfer $L-1 \to L$. We then prove (8) for the transfer $L \to L+1$ using the conjectured formula (20) for the transfer matrices. For convenience, we omit the subscripts denoting the dimension of the identity matrices. Using the decomposition (18) and (15) in the equation $M_L T_{L-1,L} = T_{L-1,L} M_{L-1}$ yields the following two identities:

$$M_{L-1}T_1 - \alpha(\sigma\otimes\mathbb{1})T_1 + \alpha T_2 + (\sigma\otimes\mathbb{1})T_2 = T_1 M_{L-1}, \tag{22}$$

and

$$\alpha T_1 + M_{L-1}T_2 - T_2 - \alpha(\sigma\otimes\mathbb{1})T_2 = T_2 M_{L-1}. \tag{23}$$

(Here, to simplify the notations we have written T_1 instead of $T_1^{(L-1)}$, T_2 instead of $T_2^{(L-1)}$ and $\mathbb{1}$ instead of $\mathbb{1}_{L-2}$). We shall need two other identities satisfied by the transfer matrices defined by the recursion (20). These identities will be proved by induction.

The third identity is the following:

$$\alpha T_2(\sigma\otimes\mathbb{1}) = \alpha T_1 - T_2. \tag{24}$$

Assume that this is true for the blocks T_1, T_2 of the transfer matrix $T_{L-1,L}$ (18). Now, using (20), we have for the corresponding matrices in $T_{L,L+1}$

$$\begin{aligned}\alpha T_1^{(L)} - T_2^{(L)} =& \alpha\begin{pmatrix} T_1 + T_2/\alpha & 2T_2 + T_2(\sigma\otimes\mathbb{1}_{L-2})/\alpha \\ T_2 & T_2/\alpha \end{pmatrix} - \begin{pmatrix} 2T_2 & T_2(\sigma\otimes\mathbb{1}_{L-2}) \\ 0 & T_2 \end{pmatrix}, \\ =& \alpha\begin{pmatrix} T_2(\sigma\otimes\mathbb{1}_{L-2}) & 2T_2 \\ T_2 & 0 \end{pmatrix} = \alpha T_2^{(L)}(\sigma\otimes\mathbb{1}_{L-1}),\end{aligned} \tag{25}$$

which proves (24).

The fourth identity is

$$T_2[M_{L-1}, (\sigma\otimes\mathbb{1})] = [T_2, (\sigma\otimes\mathbb{1})]. \tag{26}$$

This equation is proved by noting from (15) that

$$[M_{L-1}, (\sigma\otimes\mathbb{1})] = \begin{pmatrix} (\sigma\otimes\mathbb{1}) & \mathbb{1} \\ -\mathbb{1} & -(\sigma\otimes\mathbb{1}) \end{pmatrix}. \tag{27}$$

Then, using the induction hypothesis, we find that the left hand side of (26) is given by

$$\begin{pmatrix} 2T_2 & T_2(\sigma\otimes\mathbb{1}) \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} \sigma\otimes\mathbb{1} & \mathbb{1} \\ -\mathbb{1} & -\sigma\otimes\mathbb{1} \end{pmatrix} = \begin{pmatrix} T_2(\sigma\otimes\mathbb{1}) & T_2 \\ -T_2 & -T_2(\sigma\otimes\mathbb{1}) \end{pmatrix}, \tag{28}$$

which is easily verified to be equal to the right hand side of (26) using again the induction hypothesis.

Finally, using (20) and (15), we calculate explicitly $T_{L,L+1}M_L$ in square blocks of size 2^{L-1} as

$$\left(\begin{array}{c|c} \begin{matrix} T_1M_{L-1} + T_2M_{L-1}/\alpha \\ -\alpha T_1(\sigma\otimes\mathbb{1}) + 2\alpha T_2 \end{matrix} & \begin{matrix} \alpha T_1 + 2T_2M_{L-1} - 2T_2 + T_1(\sigma\otimes\mathbb{1}) \\ -2\alpha T_2(\sigma\otimes\mathbb{1}) + T_2(\sigma\otimes\mathbb{1})M_{L-1}/\alpha \end{matrix} \\ \hline \begin{matrix} T_2M_{L-1} + T_2 \\ -\alpha T_2(\sigma\otimes\mathbb{1}) \end{matrix} & \begin{matrix} \alpha T_2 - T_2/\alpha \\ +T_2M_{L-1}/\alpha \end{matrix} \\ \hline 2T_2M_{L-1} - \alpha T_2(\sigma\otimes\mathbb{1}) & \alpha T_2 + T_2(\sigma\otimes\mathbb{1}) + T_2(\sigma\otimes\mathbb{1})M_{L-1} \\ \hline \alpha T_2 & T_2M_{L-1} - T_2 - \alpha T_2(\sigma\otimes\mathbb{1}) \end{array}\right), \tag{29}$$

and similarly we write $M_{L+1}T_{L,L+1}$ as

$$\left(\begin{array}{c|c} \begin{matrix} M_{L-1}T_1 + M_{L-1}T_2/\alpha \\ -\alpha(\sigma\otimes\mathbb{1})T_1 + 2\alpha T_2 \end{matrix} & \begin{matrix} T_2 + 2M_{L-1}T_2 - \alpha(\sigma\otimes\mathbb{1})T_2 \\ +(\sigma\otimes\mathbb{1})T_2/\alpha + M_{L-1}T_2(\sigma\otimes\mathbb{1})/\alpha \\ -(\sigma\otimes\mathbb{1})T_2(\sigma\otimes\mathbb{1}) \end{matrix} \\ \hline \begin{matrix} M_{L-1}T_2 + T_2 \\ -\alpha(\sigma\otimes\mathbb{1})T_2 \end{matrix} & \begin{matrix} \alpha T_2 + M_{L-1}T_2/\alpha \\ -T_2/\alpha - (\sigma\otimes\mathbb{1})T_2 + T_2(\sigma\otimes\mathbb{1}) \end{matrix} \\ \hline \begin{matrix} \alpha T_1 + 2M_{L-1}T_2 - T_2 \\ -2\alpha(\sigma\otimes\mathbb{1})T_2 \end{matrix} & \begin{matrix} 2\alpha T_2 + M_{L-1}T_2(\sigma\otimes\mathbb{1}) \\ +(\sigma\otimes\mathbb{1})T_2 - \alpha(\sigma\otimes\mathbb{1})T_2(\sigma\otimes\mathbb{1}) \end{matrix} \\ \hline \alpha T_2 & M_{L-1}T_2 - T_2 - \alpha(\sigma\otimes\mathbb{1})T_2 \end{array}\right). \tag{30}$$

(Horizontal and vertical lines were inserted to separate the different blocks that compose the matrices).

The fact that the matrices (29) and (30) are equal is a direct consequence of the four identities (22), (23), (24) and (26). This completes the proof of Theorem 2.

There are many properties of the matrix $T_{L,L+1}$ which can be used to prove statements about the steady state distribution of the system. In particular, we can use (12) and the recursion (20) to compute the kernel $|v_L\rangle$. The entries in v_L are necessarily polynomials if we start with the vector $|v_1\rangle = \begin{pmatrix}\alpha+\beta\\ \alpha\end{pmatrix}$. Because the submatrix T_2 of T_L is always proportional to α, it is easy to check that the value of v_L for a configuration with k 1's is always proportional to α^k. In particular, the last entry of v_L which corresponds to the configuration with all sites occupied is equal to α^L.

We shall call the sum Z_L of the entries in v_L, the *partition function*. This allows us to define the steady state probabilities as the vector $|p_L\rangle = \frac{1}{Z_L}|v_L\rangle$. Therefore the probability of the configuration with all sites occupied in a system of size L is given by α^L/Z_L. This corresponds to the smallest probability. Note that the polynomial $Z_L = Z_L(\alpha,\beta)$ is the least common multiple of the denominators of the entries of the kernel p_L of M_L provided that the greatest common divisor of the numerators of the entries is one. For example, the system of size one has

$$|v_1\rangle = \begin{pmatrix}\alpha+\beta\\ \alpha\end{pmatrix}, \tag{31}$$

whence $Z_1 = \alpha+2\beta$. We find a remarkable property of the partition function of the system, namely its super-extensive growth with the size of the system.

Corollary 3 *The partition function of the system of size L is given by*

$$Z_L = 2^{\binom{L-1}{2}}(1+2\alpha)^{L-1}(1+\beta)^{L-1}(2\alpha+\beta). \tag{32}$$

This formula will be proved by induction. By definition of the partition function we have

$$Z_{L+1} = \langle 1_{L+1}|v_{L+1}\rangle = \langle 1_{L+1}|T_{L,L+1}|v_L\rangle, \tag{33}$$

where we have defined $\langle 1_K| = \langle 1,\ldots,1|$ to be the line-vector of length 2^K with all entries equal to 1, K being an arbitrary integer. We now prove by induction that $\langle 1_K|$ is a left-eigenvector of $T_1^{(K)}$ and $T_2^{(K)}$. More precisely, we show that

$$\begin{aligned}\langle 1_K|T_1^{(K)} &= 2^{K-1}(1+\alpha)(1+\beta)\langle 1_K|,\\ \langle 1_K|T_2^{(K)} &= 2^{K-1}\alpha(1+\beta)\langle 1_K|.\end{aligned} \tag{34}$$

For the initial condition in (21), we have $\langle 1,1|T_1^{(1)} = (1+\alpha)(1+\beta)\langle 1,1|$ and $\langle 1,1|T_2^{(1)} = \alpha(1+\beta)\langle 1,1|$. Then, using (20) we express $T_2^{(K)}$ in terms of $T_2^{(K-1)}$. This leads to

$$\langle 1_K|T_2^{(K)} = 2\langle 1_{K-1}|T_2^{(K-1)}. \tag{35}$$

(Here, we have used the fact that $\langle 1_{K-1}|$ is an eigenvector of $T_2^{(K-1)}$ which in turn implies that $\langle 1_{K-1}|T_2^{(K-1)} = \langle 1_{K-1}|T_2^{(K-1)}(\sigma\otimes \mathbb{1}_{K-2})$). Next, we use (24) to obtain $\langle 1_K|T_1^{(K)} = (1+1/\alpha)\langle 1_K|T_2^{(K)}$, thus proving the recurrence (34). Finally, we have

$$\begin{aligned}Z_{L+1} &= \Big(2^{L-1}(1+\alpha)(1+\beta)\langle 1_L| + 2^{L-1}\alpha(1+\beta)\langle 1_L|\Big)v_L\rangle,\\ &= 2^{L-1}(1+2\alpha)(1+\beta)\langle 1_L|v_L\rangle\\ &= 2^{L-1}(1+2\alpha)(1+\beta)Z_L,\end{aligned} \tag{36}$$

which, knowing Z_1, proves the desired formula (32).

We now proceed by calculating some correlation functions in the model:

Corollary 4 *For a system of size greater than $k+1$, the probability of the first k sites being occupied is*

$$\langle \eta_1\ldots\eta_k\rangle = \frac{\alpha^k}{2^{\binom{k}{2}}(1+2\alpha)^k}. \tag{37}$$

To prove this relation, we only have to consider a system of size $k+1$ as follows from Remark 1. In this case, only two configurations contribute to this expectation value — either the last (i.e.the $k+1$th) site is occupied or it is not. In the basis in which we have written our Markov matrix, this corresponds to the last two entries of $|v_{k+1}\rangle$.

To obtain the sum of these two entries, we have to multiply the vector $|v_{k+1}\rangle$ on the left by the vector $w_{k+1} = \langle 0,\dots,0,1,1|$ with $2^{k+1}-2$ zeros. Using (12), we have to calculate the action of $\langle w_{k+1}|$ on the transfer matrix $T_{L-1,L}$. Note that w couples only with the last two rows of the transfer matrix. From the recursion (20) and the initial condition (21), we see that the only nonzero entries in the last two rows are from the 2×2 block at the bottom right. Notice also that the column-wise sum of these two blocks is the same and is equal to $\alpha(1+\beta)$. Therefore, we obtain

$$\begin{aligned}\langle \eta_1 \dots \eta_k\rangle &= \frac{\langle w_{k+1}|T_{k+1,k}|v_k\rangle}{Z_{k+1}} = \frac{\alpha(1+\beta)\langle 0,\dots,0,1,1|v_k\rangle}{Z_{k+1}} = \frac{\alpha(1+\beta)Z_k\langle \eta_1\dots\eta_{k-1}\rangle}{Z_{k+1}} \\ &= \frac{\alpha}{2^{k-1}(1+2\alpha)}\langle \eta_1\dots\eta_{k-1}\rangle;\end{aligned} \tag{38}$$

this recursion, along with the initial condition $\langle \eta_1\rangle = \frac{\alpha}{1+2\alpha}$, proves (37).

IV. CALCULATION OF ONE-POINT AND TWO-POINT CORRELATIONS

A. Disorder Variables

We now introduce a set of non-local variables that will allow us to calculate exact properties of the model. These are similar to the so-called interparticle distribution functions in [14]. Consider the random variable

$$\xi_i = (-1)^{\#\{1\le j\le i|\eta_j=1\}} \tag{39}$$

which takes values ± 1 depending on whether the number of occupied sites between the first site and the ith site is even or odd respectively. We call ξ_i the *disorder variable* at site i. This definition is inspired from the theory of spin chains, where the Jordan-Wigner transformation allows us to write fermionic creation and annihilation operators from spin operators. More precisely, let S_j^+, S_j^-, S_j^z represent operators which raise, lower and measure the spin at site j respectively. They satisfy the relations $\{S_j^+, S_j^-\} = \mathbb{1}$ and $[S_j^+, S_k^-] = 0$. From these operators one can construct fermionic operators $f_j, f_j^\dagger$ which satisfy the standard anticommutation relations

$$\begin{aligned} f_j^\dagger &= S_j^+(-1)^{\phi(j)}, \\ f_j &= (-1)^{\phi(j)}S_j^-, \end{aligned} \tag{40}$$

where $\phi(j) = \sum_{k=1}^{j-1}(1/2 + S_k^z)$ measures the number of up-spins to the left of the site j. Our variable ξ_i corresponds to the variable $\phi(j)$ in the theory of spin chains.

For an exclusion process like the one we consider here, ξ_i has a more convenient representation,

$$\xi_i = \prod_{j=1}^{i}(1-2\eta_j). \tag{41}$$

Note that these random variables satisfy $\xi_i^2 = 1$ since the occupation variables satisfy $\eta_i^2 = \eta_i$. The steady state expectation values of the ξ_i variables satisfy some remarkable properties, which we will state in this section. These expectation values will be extremely useful in proving results for the density and the evaporation rate.

Theorem 5 *For a system with size L and $i < L$,*

$$\langle \xi_i\rangle = \frac{1}{(1+2\alpha)^i}\,, \tag{42}$$

and

$$\langle \xi_L\rangle = \frac{\beta}{(\beta+2\alpha)(1+2\alpha)^{L-1}}\,. \tag{43}$$

From (41), we have for all $i, 1 \leq i \leq L$,

$$\xi_i = (1 - 2\eta_i)\xi_{i-1}, \tag{44}$$

from which we also have

$$\eta_i \xi_i = -\eta_i \xi_{i-1}, \tag{45}$$

which substituting back in (44) gives

$$\xi_i = \xi_{i-1} + 2\eta_i \xi_i. \tag{46}$$

We now write down the evolution equation for ξ_i in the bulk. The main idea, and one of the reasons this variable is useful, is that a transition taking place strictly between 1 and i does not affect the value of $\langle \xi_i \rangle$. The first bulk transition in (1) does not change the number of particles, and the second reduces it by two, and hence does not change the parity. We will first write down the equation and then explain each term.

$$\begin{aligned} 0 = \frac{d}{dt}\langle \xi_i \rangle &= +\alpha \langle -\xi_i \rangle + \langle \eta_i(-\xi_i) \rangle - \alpha \langle \xi_i \rangle + \langle \eta_i \xi_i \rangle, \\ &= -2\alpha \langle \xi_i \rangle - 2\langle \eta_i \xi_i \rangle. \end{aligned} \tag{47}$$

The first two terms describe ways of entering the configuration contributing to $\langle \xi_i \rangle$. If one starts with a configuration contributing to $-\langle \xi_i \rangle$ and a particle enters or leaves site 1 the parity will change because of (2). Similarly if one starts from configuration contributing to $-\langle \xi_i \rangle$ and if the ith site is occupied and that particle leaves that site, then the parity changes. The last two terms describe ways of exiting the configuration contributing to $\langle \xi_i \rangle$: when a particle enters from the left reservoir or when a particle occupying site i leaves it, we end up with a configuration contributing to $-\langle \xi_i \rangle$. Using (47) in (46), one ends up with

$$\langle \xi_i \rangle = \frac{\langle \xi_{i-1} \rangle}{1 + 2\alpha}, \tag{48}$$

which along with the initial condition, $\langle \xi_0 \rangle = 1$ leads us to (42). For the last site, the balance equation is similar,

$$0 = \frac{d}{dt}\langle \xi_L \rangle = -2\alpha \langle \xi_L \rangle - 2\beta \langle \eta_L \xi_L \rangle \tag{49}$$

because the last site exits with rate β instead of rate one. And using (46), one obtains

$$\langle \xi_L \rangle = \frac{\beta \langle \xi_{L-1} \rangle}{\beta + 2\alpha}, \tag{50}$$

which leads to (43).

We now calculate correlations among the disorder variables. These are directly related to the quantities of interest in the model. For example,

$$\langle \xi_i \xi_{i-1} \rangle = \langle 1 - 2\eta_i \rangle, \tag{51}$$

will give us the density. We will also see the relation between this problem and a random walk in two dimensions. For convenience, we introduce a new variable

$$\xi_{m,n} = \frac{1 - \xi_m \xi_n}{2}, \tag{52}$$

and define the expression n_+ to be the positive part of n. Namely $n_+ = n$ if $n \geq 0$ and 0 otherwise.

Theorem 6 *For a system with size L and $0 \leq n \leq m < L$,*

$$\langle \xi_{m,n} \rangle = \frac{\alpha}{2^{(m+n-2)_+}(1+2\alpha)^m} \sum_{j=0}^{m-1} (1+2\alpha)^j 2^{(m-j-2)_+} \sum_{k=1}^{m-n} \binom{(m+n-2)_+ - (m-j-2)_+}{m-k}, \tag{53}$$

and if $L > n > m$, then $\xi_{m,n} = \xi_{n,m}$.

We first write down the balance equation for the correlation function for $m, n \geq 1$,

$$\begin{aligned} 0 = \frac{d}{dt}\langle \xi_m \xi_n \rangle &= -2\langle \eta_m \xi_m \xi_n \rangle - 2\langle \eta_n \xi_m \xi_n \rangle, \\ &= \langle \xi_{m-1} \xi_n \rangle + \langle \xi_m \xi_{n-1} \rangle - 2\langle \xi_m \xi_n \rangle, \end{aligned} \tag{54}$$

where we have used ideas very similar to (47) in the first line and (46) in the second line. In this expression, ξ_0 is set identically to 1. The simplicity of the expression is due to the fact that the product $\xi_m \xi_n$ is affected only by transitions taking place between the mth and the nth sites. Any other transition either changes the signs of both ξ_m and ξ_n (e.g, first site) or does not change the sign of either of them (e.g, last site). From equation (54), we deduce the recursion relation

$$\langle \xi_{m,n} \rangle = \frac{\langle \xi_{m-1,n} \rangle + \langle \xi_{m,n-1} \rangle}{2}. \tag{55}$$

We now prove (53) by showing that it verifies this recursion and the boundary conditions. Since the formula (53) is valid in the triangular region $0 \leq n \leq m$, we have to check that it is valid for $m = n$ and for $n = 0$. The case $m = n$ is easily done: substituting $n = m$ in (53) gives zero (because the sum is empty) and from (52), we also obtain $\xi_{m,m} = 0$. For the case $n = 0$, we have

$$\begin{aligned} \langle \xi_{m,0} \rangle =& \frac{\alpha}{2^{(m-2)_+}(1+2\alpha)^m} \sum_{j=0}^{m-1} (1+2\alpha)^j 2^{(m-j-2)_+} \sum_{k=1}^{m} \binom{(m-2)_+ - (m-j-2)_+}{m-k}, \\ =& \frac{\alpha}{2^{m-2}(1+2\alpha)^m} \left((1+2\alpha)^{m-1} \sum_{k=2}^{m} \binom{m-2}{m-k} + \sum_{j=0}^{m-2} (1+2\alpha)^j 2^{m-j-2} \sum_{k=m-j}^{m} \binom{j}{m-k} \right), \end{aligned} \tag{56}$$

where we have split the sum according to whether $j = m-1$ or not, and changed the limits of k to count only the nonzero summands. The summation on k is easily done in both terms to give

$$\langle \xi_{m,0} \rangle = \alpha \left(\frac{1}{1+2\alpha} + \sum_{j=0}^{m-2} (1+2\alpha)^{j-m} \right) = \frac{(1+2\alpha)^m - 1}{2(1+2\alpha)^m}, \tag{57}$$

which is the correct answer knowing $\xi_{m,0} = (1 - \xi_m)/2$ and the expectation value $\langle \xi_m \rangle$ from Theorem 5. Thus we have verified the boundary conditions. We now need to verify the recursion (54). We can take $m > n \geq 1$ which implies $m + n - 3 \geq 0$. Then,

$$\begin{aligned} \frac{\langle \xi_{m,n-1} \rangle + \langle \xi_{m-1,n} \rangle}{2} =& \frac{\alpha}{2 \cdot 2^{m+n-3}} \sum_{j=0}^{m-1} (1+2\alpha)^{j-m} 2^{(m-j-2)_+} \sum_{k=1}^{m-n+1} \binom{m+n-3-(m-j-2)_+}{m-k} \\ &+ \frac{\alpha}{2 \cdot 2^{m+n-3}} \sum_{j=0}^{m-2} (1+2\alpha)^{j-m+1} 2^{(m-j-3)_+} \sum_{k=1}^{m-n-1} \binom{m+n-3-(m-j-3)_+}{m-k-1}. \end{aligned} \tag{58}$$

We replace $j \to j+1$ in the second sum to get the summands in both these terms to be exactly the same except for the binomial coefficients, in which the lower index differs by one. One can thus combine both these sums using the usual addition formula for binomial coefficients in the common range to get

$$\begin{aligned} &\frac{\langle \xi_{m-1,n} \rangle + \langle \xi_{m,n-1} \rangle}{2} = \\ &\frac{\alpha}{2^{m+n-2}} \sum_{j=1}^{m-1} (1+2\alpha)^{j-m} 2^{(m-j-2)_+} \left(\sum_{k=1}^{m-n-1} \binom{m+n-2-(m-j-2)_+}{m-k} + \sum_{k=m-n}^{m-n+1} \binom{m+n-3-(m-j-2)_+}{m-k} \right) \\ &+ \frac{\alpha}{2^{m+n-2}} (1+2\alpha)^{-m} 2^{m-2} \sum_{k=1}^{m-n+1} \binom{n-1}{m-k}. \end{aligned} \tag{59}$$

The second sum over k, which comes from the contribution of $\langle \xi_{m,n-1} \rangle$, involves only two terms which can again be summed using the addition formula for binomial coefficients and the result can be included in the first sum as the $k = m - n$ term. The third sum over k (which is also an untouched term from $\langle \xi_{m,n-1} \rangle$ corresponding to $j = 0$) is equal to one because the binomial coefficient is nonzero if $n - 1 \geq m - k$, which is satisfied only for the upper limit, $k = m - n + 1$, where the value is one. Finally, this third sum over k adds to the first sum as the $j = 0$ term, giving the right hand side of (53) and confirming the recurrence relation.

B. Density

We know from Remark 1 that the density at site k is a fixed quantity for sites $L > k$ and depends only on α. From (51), it is clear that $\eta_m = \xi_{m,m-1}$. Using Theorem 6 we arrive at an explicit expression for the expectation value of the density,

$$\langle \eta_m \rangle = \frac{\alpha}{2^{(2m-3)_+}(1+2\alpha)^m} \sum_{j=0}^{m-1} (1+2\alpha)^j 2^{(m-j-2)_+} \binom{(2m-3)_+ - (m-j-2)_+}{m-1}. \tag{60}$$

When $m = 1$, we only have the $j = 0$ term and the answer matches that given by Corollary 4 with $k = 1$. For $m \geq 2$, we split the sum according to whether $j = m-1$ or not,

$$\langle \eta_m \rangle = \frac{\alpha}{2^{2m-3}(1+2\alpha)^m} \left((1+2\alpha)^{m-1} \binom{2m-3}{m-1} + \sum_{j=0}^{m-2} (1+2\alpha)^j 2^{m-j-2} \binom{m-1+j}{m-1} \right). \tag{61}$$

We now multiply and divide this formula by two: the denominator in the prefactor becomes 2^{2m-2} and the two terms inside the brackets are multiplied by two. For the first term, we get

$$2\binom{2m-3}{m-1} = \binom{2m-3}{m-2} + \binom{2m-3}{m-1} = \binom{2m-2}{m-1}, \tag{62}$$

and for the second term, the power of two inside the summation becomes 2^{m-j-1}. We can then combine both these terms and write the density explicitly as

$$\langle \eta_m \rangle = \frac{\alpha}{2^{2(m-1)}(1+2\alpha)^m} \sum_{j=0}^{m-1} \binom{m-1+j}{j} (1+2\alpha)^j 2^{m-1-j}. \tag{63}$$

One can analyze the asymptotics of the density using Stirling's formula and noting that the largest summand is the one where $j = m-1$. One finds that

$$\langle \eta_m \rangle \sim \frac{1}{2\sqrt{\pi m}}, \tag{64}$$

independent of α. This asymptotic result was obtained by Lebowitz, Neuhauser and Ravishankar using stochastic coupling methods (see Theorem 2 of [31]). The problem studied by these authors corresponds to the case $\alpha = 1$ and was inspired by earlier studies of the Toom model [32] on the semi-infinite lattice.

C. Evaporation Rate

Since, unlike the TASEP, particles are *not* conserved in the bulk, a quantity like the current that is constant across all bonds does not exist in the present model. But we can define an evaporation rate at the bond $(m-1, m)$, as the correlation $\langle \eta_{m-1}\eta_m \rangle$: this quantity does not depend on the size L of the system as long as $L > m$ because of Remark 1. We can calculate the evaporation rate using the expansion

$$\langle \xi_{m,m-2} \rangle = \langle 1 - 2\eta_m - 2\eta_{m-1} + 4\eta_{m-1}\eta_m \rangle, \tag{65}$$

from which we have

$$\langle \eta_{m-1}\eta_m \rangle = \frac{\langle \xi_{m,m-2} \rangle - 1 + 2\langle \eta_m \rangle + 2\langle \eta_{m-1} \rangle}{4}. \tag{66}$$

An alternative method is to use the balance equation for the density

$$\frac{d}{dt}\langle \eta_m \rangle = \langle \eta_{m-1} \rangle - \langle \eta_m \rangle - 2\langle \eta_{m-1}\eta_m \rangle = 0, \tag{67}$$

to obtain

$$\langle \eta_{m-1}\eta_m \rangle = \frac{\alpha^2}{2^{2m-3}(1+2\alpha)^m} \sum_{i=0}^{m-2} \frac{m-1-i}{m-1} \binom{m-2+i}{i} \gamma^i 2^{m-2-i}. \tag{68}$$

The asymptotics for the evaporation rate are given by

$$\langle \eta_{m-1}\eta_m \rangle \sim \frac{1}{8\sqrt{\pi m^3}}. \tag{69}$$

One can similarly compute higher order correlations of a bunch of consecutive sites $\langle \eta_i \dots \eta_{i+k} \rangle$ using Theorem 6. The fact that the asymptotic evaporation rate in (69) is given (upto a factor of 2) by the derivative of the asymptotic density (64) can probably be explained by looking at the hydrodynamic equation for this model.

D. Structure of higher correlation functions

We outline some general properties of correlation functions in the model. The reason one can make strong statements about correlations in this model is explained in Remark 1. We will comment on correlations for both order and disorder variables.

1. Density correlations

We earlier used the balance equation for the density (67) in computing the evaporation rate. We also found simple closed form expressions for the correlations of sites $1, \dots, k$ being occupied in Corollary 4 and for correlations of the form $\langle \prod_{j=1}^{k}(1-2\eta_j)\rangle$ in Theorem 5. In Theorem 6 we found a more complicated expression for correlations of the form $\langle \prod_{j=l}^{k}(1-2\eta_j)\rangle$, where $l \neq 1$ and $l < k$. Here, we will say more about the balance equations obeyed by more general order variables.

Consider a general correlation function of order n, written as $\langle \eta_{p_1} \cdots \eta_{p_n} \rangle$. In general we know no closed form expression for such a correlation function, but we can write down the balance equation satisfied by this object. We group the positions $i_1, \cdots i_n$ according to blocks of consecutive sites. The total number of blocks is given by j and the length of the lth block is denoted by $k_l + 1$. The blocks are thus labelled as $(\eta_{i_1}, \cdots, \eta_{i_1+k_1})$ up to $(\eta_{i_j}, \cdots, \eta_{i_j+k_j})$. Note that neighbouring blocks are separated by at least one site and that the size of a block could be one.

Theorem 7 *The steady state equation satisfied by the correlation function of sites $i_1, \cdots, i_1+k_1, \cdots, i_j, \cdots, i_j+k_j$ where $i_1 > 1$ is given by*

$$\begin{aligned} n \langle \eta_{p_1} \dots \eta_{p_n} \rangle = \sum_{l=1}^{j} \Bigg\langle & \left(\prod_{m=1}^{l-1} \eta_{i_m} \dots \eta_{i_m+k_m} \right) \\ & \eta_{i_l-1}(1-2\eta_{i_l})\eta_{i_l+1} \dots \eta_{i_l+k_l} \left(\prod_{m=l+1}^{j} \eta_{i_m} \dots \eta_{i_m+k_m} \right) \Bigg\rangle. \end{aligned} \tag{70}$$

The main idea is to consider what happens for a single block made of consecutive n sites $i, \dots, i+n-1$. The balance condition for such a block can be written easily,

$$\begin{aligned} 0 = \frac{d}{dt}\langle \eta_i \dots \eta_{i+n-1} \rangle =& \langle \eta_{i-1}(1-\eta_i)\eta_{i+1} \dots \eta_{i+n-1} \rangle \\ & - \langle \eta_{i-1}\eta_i\eta_{i+1} \dots \eta_{i+n-1} \rangle - n\langle \eta_i \dots \eta_{i+n-1} \rangle, \\ =& \langle \eta_{i-1}(1-2\eta_i)\eta_{i+1} \dots \eta_{i+n-1} \rangle \\ & - n\langle \eta_i \dots \eta_{i+n-1} \rangle, \end{aligned} \tag{71}$$

because the configuration with all the sites $i, \dots, i+n-1$ occupied can only be reached if the $(i-1)$th site is occupied, the ith site is empty and the remainder are occupied; we can exit the configuration if the $(i-1)$th site is occupied along with all the others from i to $i+n-1$ with rate one, and in n different ways (each site between i and $i+n-1$ could jump) otherwise. The expression (71) is precisely (70) for $j = 1$. Two blocks are separated by at least one site and this argument applies independently to each block. From each block l we get a contribution of $k_l + 1$ times $\langle \eta_{p_1} \dots \eta_{p_n} \rangle$ all of which add to give the factor n on the left hand side of (70) whereas on the right hand side the first factor η_{i_l} of each block l is successively replaced by a factor $\eta_{i_l-1}(1-2\eta_{i_l})$ whereas the other blocks are left unchanged.

2. Disorder variables correlations

A nontrivial observation about one-point and two-point correlation functions of the disorder variables ξ_i is that their evolution equations (47) and (54), supplemented by the recursion (46) are *closed* in the sense that they involve only other one-point and two-point disorder correlations respectively. In general, one would expect a hierarchy where the equations for lower correlations would necessarily involve higher correlations making the problem extremely difficult to solve for generic n-point correlations. As we will show below, this property of decoupling, which is very specific to this model, holds for all n.

Theorem 8 *In the steady state, the correlation function of n disorder variables $\langle \xi_{p_1} \dots \xi_{p_n} \rangle$ satisfies the equation*

$$n \langle \xi_{p_1} \dots \xi_{p_n} \rangle = \sum_{k=1}^{n} \langle \xi_{p_1} \dots \xi_{p_{k-1}} \xi_{p_k - 1} \xi_{p_{k+1}} \dots \xi_{p_n} \rangle, \tag{72}$$

if n is even, and

$$(n + 2\alpha) \langle \xi_{p_1} \dots \xi_{p_n} \rangle = \sum_{k=1}^{n} \langle \xi_{p_1} \dots \xi_{p_{k-1}} \xi_{p_k - 1} \xi_{p_{k+1}} \dots \xi_{p_n} \rangle, \tag{73}$$

if n is odd.

To prove these identities, one simply writes down the evolution equation, which for even n is given by

$$\frac{d}{dt} \langle \xi_{p_1} \dots \xi_{p_n} \rangle = -2 \sum_{k=1}^{n} \langle \xi_{p_1} \dots \xi_{p_{k-1}} \xi_{p_k} \eta_{p_k} \xi_{p_{k+1}} \dots \xi_{p_n} \rangle, \tag{74}$$

and for odd n is given by

$$\frac{d}{dt} \langle \xi_{p_1} \dots \xi_{p_n} \rangle = -2\alpha \langle \xi_{p_1} \dots \xi_{p_n} \rangle - 2 \sum_{k=1}^{n} \langle \xi_{p_1} \dots \xi_{p_{k-1}} \xi_{p_k} \eta_{p_k} \xi_{p_{k+1}} \dots \xi_{p_n} \rangle \,. \tag{75}$$

One has to treat separately the even and the odd cases because a particle entering from the left reservoir makes a difference to the product $\xi_{p_1} \dots \xi_{p_n}$ only if n is odd. The terms inside the sum itself are easily explained analogous to (47) and (54); each factor η_{p_k} occurs because the jump of the particle at site p_k changes only the sign of ξ_{p_k} and keeps all the others intact. Then we use the recursion (46) for each term in the sum to prove the result. We emphasize that no problem arises if any of the p_k's are consecutive. If $p_{k-1} = p_k - 1$ for a particular value of k, a factor $\xi^2_{p_{k-1}} = 1$ appears in the expectation value and drops out from the product. As an example, the correlation function for three consecutive sites satisfies: $(3 + 2\alpha)\langle \xi_i \xi_{i+1} \xi_{i+2} \rangle = \langle \xi_{i-1} \xi_{i+1} \xi_{i+2} \rangle + \langle \xi_i \rangle + \langle \xi_{i+2} \rangle$.

V. SPECTRUM OF THE MARKOV MATRICES

In this section, we mention some spectral properties of the Markov matrix of the asymmetric annihilation model. We observed that its characteristic polynomials factorize into linear factors and can be written explicitly. We first define the polynomials $A_L(x)$ and $B_L(x)$ as

$$\begin{aligned} A_L(x) &= \prod_{k=0}^{\lceil L/2 \rceil} (x + 2k)^{\binom{L-1}{2k}}, \\ B_L(x) &= \prod_{k=0}^{\lfloor L/2 \rfloor} (x + 2k + 1)^{\binom{L-1}{2k+1}}. \end{aligned} \tag{76}$$

Conjecture 1 *The characteristic polynomial $P_L(x)$ of M_L is given by*

$$P_L(x) = A_L(x) A_L(x + 2\alpha + \beta) B_L(x + \beta) B_L(x + 2\alpha), \tag{77}$$

and successive ratios of characteristic polynomials are given by

$$\frac{P_{L+1}(x)}{P_L(x)} = B_L(x + 1) B_L(x + 2\alpha + \beta + 1) A_L(x + \beta + 1) A_L(x + 2\alpha + 1). \tag{78}$$

If this is true, the Markov matrix has only $2L$ distinct eigenvalues. In particular, the negative of all the factors in the denominator Z_L are roots of the characteristic polynomial; for example, the denominator Z_3 according to Corollary 3 is $2(1+2\alpha)^2(1+\beta)^2(2\alpha+\beta)$ whereas $P_3(x)$ is given by

$$x\,(x+2)\,(\beta+x+1)^2\,(2\,\alpha+x+1)^2\,(2\,\alpha+2+x+\beta)\,(2\,\alpha+x+\beta)\,. \tag{79}$$

We also conjecture that the Markov matrices are *maximally undiagonalizable* in the sense that each eigenvalue seems to have exactly one eigenvector independent of the number of times the eigenvalue appears (this implies that the Jordan blocks are of the maximum possible size). In other words, the degeneracy seems to be so strong that the Markov matrix M_L which is of size 2^L has only $2L$ eigenvectors. We can also write $P_L(x)$ in a different manner. We define $n_1(m)$, for any integer m, as (-1) raised to the number of ones in the binary expansion of m. For example, $n_1(1) = n_1(2) = -1, n_1(3) = 1$.

Conjecture 2 *Let us index the rows and columns from 0 to 2^L-1. The characteristic polynomial of M_L is then given by*

$$P_L(x) = \prod_{i=0}^{2^L-1} (x - (M_L)_{(i,i)} - \alpha n_1(i)). \tag{80}$$

Equation (77) has been checked for systems of length ≤ 7. The proof of this conjecture seems to be a nontrivial problem in determinant evaluation [34].

We have been able to prove a much weaker result. Consider the matrix $\widetilde{M}_L = M_L - \alpha(\sigma \otimes \mathbb{1})$ written in the block diagonal decomposition as

$$\left(\begin{array}{c|c} M_{L-1} - \alpha(\sigma \otimes \mathbb{1}_{L-2}) & (\sigma \otimes \mathbb{1}_{L-2}) \\ 0 & M_{L-1} - \mathbb{1}_{L-1} - \alpha(\sigma \otimes \mathbb{1}_{L-2}) \end{array} \right), \tag{81}$$

using (15). Notice that the two diagonal blocks are $\widetilde{M}_{L-1}$ and $\widetilde{M}_{L-1} - \mathbb{1}_{L-1}$. Using (16), one can see that $\widetilde{M}_1$ is upper triangular. Therefore $\widetilde{M}_L$ is also upper triangular for all L. Therefore the eigenvalues of $\widetilde{M}_L$ are simply the elements on the diagonal. Using the special nature of the diagonal blocks, one can easily prove by induction that the characteristic polynomial of $\widetilde{M}_L$ is given by

$$\widetilde{P}_L(x) = \prod_{k=0}^{L-1} (x+\alpha+k)^{\binom{L-1}{k}} (x+\alpha+\beta+k)^{\binom{L-1}{k}}. \tag{82}$$

Note the close similarity with the characteristic polynomial of M_L in (77).

VI. DISCUSSION AND CONCLUSION

In this work we have studied a nonequilibrium system on a finite size lattice with open boundaries in which particles diffuse and interact through hard-core exclusion and pairwise annihilation. The breaking of detailed balance in the bulk of the system is ensured by the asymmetric hopping rules and by the absence of pair-creation process. Besides, the difference in the chemical potentials of the left and the right reservoirs is also a source of nonequilibrium behaviour. The bulk dynamics is characterized by a single dimensionless parameter λ which represents the ratio between evaporation and hopping rates (1). For $\lambda = 0$, the model is identical to the totally asymmetric exclusion process with open boundaries [8]. In the present work we have derived exact results for the case $\lambda = 1$. The fact that $\lambda = 1$ is a special point can be understood if one writes the Markov matrix M_L of this stochastic process as a non-Hermitian spin chain operator using the Pauli matrices [22, 33]. For general values of λ we obtain

$$M_L = \sum_{i=1}^{L-1} M_{i,i+1} + R + L\,, \tag{83}$$

where

$$\begin{aligned} M_{i,i+1} &= S_i^+ S_{i+1}^- + \lambda S_i^+ S_{i+1}^+ + \frac{1-\lambda}{4} S_i^z S_{i+1}^z + \frac{1+\lambda}{4} S_i^z - \frac{1-\lambda}{4} S_{i+1}^z - \frac{1+\lambda}{4}\,, \\ L &= \alpha \left(S_1^- + \lambda S_1^+ - \frac{1-\lambda}{2} S_1^z - \frac{1+\lambda}{2} \right), \\ R &= \beta \left(S_L^+ - \frac{1-S_L^z}{2} \right). \end{aligned} \tag{84}$$

We recall that the spin operators are given by: $S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $S^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For $\lambda = 1$, we observe that all the interaction terms of the type $S_i^z S_{i+1}^z$ disappear from the spin chain operator: this corresponds to the *free fermion point.* This absence of interaction gives an alternative explanation for the simplicity of the model at the special value $\lambda = 1$. However, we remark that at this special point, the boundary terms give a nonlocal contribution because of lack of periodicity and therefore the usual strategies for diagonalizing the quadratic Hamiltonian on a periodic ring (i.e. Fourier Transform or Bogoliubov transformation) do not seem to apply here. The transfer matrix technique allows us to by-pass these difficulties. For $\lambda = 0$ the spin chain operator (84) represents the totally asymmetric exclusion process which is also exactly solvable. It would therefore be of interest to explore the integrability properties of this system for general values of λ taking into account the boundary conditions. Another possible extension is to allow backward hopping of the particles (which corresponds to the partially asymmetric case) and to formulate the boundary conditions so that the model remains solvable.

For this asymmetric annihilation process, we have been able to derive exact combinatorial expressions for correlation functions such as local densities and evaporation rates. The use of disorder variables has been helpful. However, the main tool that we have introduced is a recursion relation between systems of two consecutive sizes. This recursion is encoded in a semi-similarity operator (the transfer matrix Ansatz) that conjugates the Markov matrices for systems of sizes L and $L+1$. The model studied here admits a transfer matrix Ansatz whereas the TASEP does not (as we explicitly checked on small systems). Conversely, the TASEP can be solved using a quadratic matrix product representation (which allows the calculation of physical observables such as the density and two-point correlation functions) whereas the steady state weights of the asymmetric annihilation process cannot be written easily as a simple matrix product (see Appendix A). Indeed, the fact that the normalization Z_L grows super-extensively as $2^{L(L-1)/2}$ (see (32)) implies that the matrices for the stationary weights should involve $L(L-1)/2$ tensor products as in the case of multi-species exclusion processes [35, 36] and the calculation of physical observables would be a true challenge.

To summarize, the method for solving this model differs considerably from that used for the exclusion process. The transfer matrix Ansatz encodes in a particularly efficient way recurrence relations for the steady state probabilities and it allows one to deduce information about the steady state in a rather elementary manner. We believe that the existence of a transfer matrix Ansatz is rather general: it applies to the multi-species exclusion process on a ring [37] and perhaps in a non-obvious manner to the ASEP with open boundaries.

Of a more fundamental interest is the following question. The system is out of equilibrium and therefore its steady state violates detailed balance: this implies the existence of elementary currents between microscopic configurations. However, due to the evaporation of particles, there is no obvious way of defining a conserved current in the system. It would be interesting to find an observable that demonstrates at the macroscopic level the breaking of detailed balance.

Finally, we remarked above that the normalization Z_L grows super-exponentially with the size of the system for $\lambda = 1$ whereas for $\lambda = 0$, $\log Z_L$ is extensive in L. Although Z_L has no direct physical interpretation, it grows with L roughly as the ratio of the most probable configuration to the least probable one. It would be of interest to see when this transition from exponential to super-exponential growth occurs as λ varies and to interpret it physically. More generally, a challenging problem is to calculate for arbitrary values of λ the scaling function $f(\lambda)$ defined by $Z_L \sim \exp(f(\lambda)L^2)$. Related questions were addressed in rice-pile models by P. Pradhan and D. Dhar [38] and in the so-called Raise and Peel model by de Gier and collaborators [39].

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APPENDIX A: RECURSIONS FOR THE STEADY STATE PROBABILITIES

In this appendix, we give an algorithm that allows to calculate recursively the steady state weights of configurations in a system of size L knowing the ones for size $L-1$. These recursions are based on numerical observations and are conjectural. They are stated in this appendix in order to be compared with the recurrences for the TASEP. They could also be used as a tool to build a matrix product representation.

We define the *pushing operator* P that acts on a block of 1's followed by a block of 0's. $P(1^j0^k)$ is given by a linear combination of all possible $\binom{j+k}{k}$ binary words with j 1's and k 0's. The coefficient of a given word is a power of 2 as follows: each move of the 1 which began at position j (i.e. the rightmost 1) contributes a factor of 2; each move of the 1 which began at position $j-1$ contributes a factor of $1/2$; each move of the 1 which began at position $j-2$ contributes a factor of 2 and so on. We give some examples,

$$\begin{aligned} P(100) &= (100) + 2(010) + 4(001), \\ P(1100) &= (1100) + 2(1010) + 4(1001) + (0110) + 2(0101) + (0011). \end{aligned} \tag{A1}$$

In the first example, there was only one 1 and we gained a factor of two for every push of the 1. In the second example, we gained a factor of two for each push of the rightmost 1 but lost a factor of two for each push of the other 1.

We shall now construct an operator N which acts on words and generates a linear combination of words of size one less. The operator N will be defined in two stages: first we specify the action of N on special words that we call *sub-configurations* (which are building blocks for general words), then we explain how to extend this action on arbitrary words by concatenation.

A sub-configuration is a word that contains a single block of empty sites. In a *bulk sub-configuration,* of the type 10^k1 with $k > 0$, the block is surrounded by an occupied site on the left and on the right. In a *right (left) sub-configuration,* of the type 10^k (0^k1) with $k > 0$, there is a single occupied site on the left (right). The configuration with all empty sites will be treated separately. We shall distinguish three cases:

- Bulk sub-configuration:

$$N(10^k1) = \sum_{j=0}^{\lfloor k/2 \rfloor} 2^j P(1^{2j+1}0^{k-2j}). \tag{A2}$$

- Left sub-configuration: the relation involves α,

$$N(0^k1) = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} 2^j P(1^{2j+1}0^{k-1-2j}) + \alpha \sum_{j=0}^{\lfloor k/2 \rfloor} 2^j P(1^{2j}0^{k-2j}). \tag{A3}$$

- Right sub-configuration: the relation now involves β,

$$N(10^k) = \beta \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} 2^j P(1^{2j+1}0^{k-1-2j}) + (1+\beta) \sum_{j=0}^{\lfloor k/2 \rfloor} 2^{k-1-j} P(1^{2j}0^{k-2j}). \tag{A4}$$

We are now in a position to construct the operator N for any configuration except the one with all empty sites. Any nonempty configuration can be written as $\tau = 0^{k_0}1^{l_1}0^{k_1}\dots1^{l_m}0^{k_m}$. In this notation, a bulk sub-configuration is $10^{k_i}1$ for $i = 1, \dots, m-1$, the left sub-configuration is $0^{k_0}1$ and the right sub-configuration is 10^{k_m} assuming k_0 and k_m to be nonzero. If either of them is zero, the corresponding boundary sub-configuration does not exist. The idea is to use the algorithms for each of these sub-configurations as defined in (A2), (A3) and (A4) and concatenate. The actual formula depends on whether k_0 is positive or not:

$$N(\tau) = \begin{cases} (0^{k_0}1) \oplus \bigoplus_{i=1}^{m-1} \left(1^{l_i-1} \oplus (10^{k_i}1)\right) \oplus 1^{l_m-1} \oplus (10^{k_m}), & \\ & \text{if } k_0 > 0, \\ \alpha \bigoplus_{i=1}^{m-1} \left(1^{l_i-1} \oplus (10^{k_i}1)\right) \oplus 1^{l_m-1} \oplus (10^{k_m}), & \\ & \text{if } k_0 = 0, \end{cases} \tag{A5}$$

where the symbol $\oplus$ denotes concatenation and where we use the summation formulae for the sub-configurations defined previously. We emphasize that the size of the configuration is exactly one less than τ. Note also that the right sub-configuration term $N(10^{k_m})$ is empty if $k_m = 0$. For example,

$$\begin{aligned} N(010) &= ((1) + \alpha(0)) \oplus (\beta(1) + (1+\beta)(0)), \\ &= \beta(11) + \alpha\beta(01) + (1+\beta)(10) + \alpha(1+\beta)(00). \end{aligned} \tag{A6}$$

We have thus defined the operator N for all words except those consisting only of 0's. The operator N encodes recursions which are precisely those given by the transfer matrix Ansatz as we checked on system of size ≤ 7.

The configuration with all empty sites requires a slightly different algorithm: this is the only case where the recursion is different from the one provided by the transfer matrix in (20). In fact, $N(0^L)$ involves all 2^{L-1} configurations of size $L-1$ and the coefficients depend on whether these smaller configurations end in a one or a zero. We need here a related but different pushing operator, P', which we define presently. $P'(1^j0^k)$ is again a sum of all possible $\binom{j+k}{k}$ configurations but the coefficient (which is again a power of two) is assigned differently. We first assign 2^k to the configuration 1^j0^k and this time, we divide by two for every push of the rightmost 1, multiply by two for every push of the second-from-right 1, divide by two for the third-from-right 1 and so on. For example,

$$\begin{aligned} P'(110) &= 2(110) + (101) + 2(011) \\ P'(1100) &= 4(1100) + 2(1010) + (1001) + 4(0110) + 2(0101) + 4(0011). \end{aligned} \tag{A7}$$

(Note the similarities and differences between (A1) and (A7).) The recursion for the configuration with all zeros is given by

$$\begin{aligned} N(0^L) = \beta &\left(\sum_{j=0}^{\lfloor (L-3)/2 \rfloor} 2^j P(1^{2j+1}0^{L-3-2j}) \oplus 1 + \alpha \sum_{j=0}^{\lfloor (L-2)/2 \rfloor} 2^j P(1^{2j}0^{L-2-2j}) \oplus 1 \right) \\ + (1+\beta) &\left(\alpha \sum_{j=0}^{\lfloor (L-3)/2 \rfloor} 2^j P'(1^{2j+1}0^{L-3-2j}) \oplus 0 + \sum_{j=1}^{\lfloor (L-2)/2 \rfloor} 2^{j-1} P'(1^{2j}0^{L-2-2j}) \oplus 0 \right) \\ + 2^{L-2}&(1+\alpha)(1+\beta)(0^{L-1}). \end{aligned} \tag{A8}$$

We remark that the powers of two are governed by the two pushing operators P and P'. Besides, except for the exceptional term with all zeros, there is a prefactor β if the configuration ends in one and a prefactor $1+\beta$ if it ends in zero.

Equations (A5) and (A8) fully define the action of N on an arbitrary binary word. The operator N acting on a word of size L returns a linear combination of words of size $L-1$. If $\mathcal{C}$ is a configuration of the system of size L, then its unnormalized weight $W(\mathcal{C})$ is defined as

$$W(\mathcal{C}) = \langle \mathcal{C} | v_L \rangle \,, \tag{A9}$$

where v_L is the steady state vector constructed using the transfer matrix Ansatz (11). We claim that $W(\mathcal{C})$ can be calculated knowing the weights of the configurations of size $L-1$ from the following formula:

$$W(\mathcal{C}) = W(N(\mathcal{C})) \,. \tag{A10}$$

Namely, the weight of $\mathcal{C}$ is the linear combination of the weights of the configurations of smaller size generated by applying the operator N to $\mathcal{C}$. For example, from (A6), we deduce

$$W(010) = \beta W(11) + \alpha\beta W(01) + (1+\beta) W(10) + \alpha(1+\beta) W(00). \tag{A11}$$

A challenging problem would be to find a matrix representation (or more generally an algebra) that embodies the reduction rules (A5) and (A8). We believe that such an algebra does exist as in the case of ASEP. However, we emphasize that we have bypassed the matrix product representation altogether thanks to the transfer matrix Ansatz defined in (8).

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