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Probability density of determinants of random matrices.

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Abstract. In this brief note the probability density of a random real, complex and quaternion determinant is rederived using the singular values. The behaviour of suitably rescaled random determinants is studied in the limit of infinite order of the matrices.

1 Introduction and results.

We consider $n \times n$ matrices whose elements are either real, or complex or quaternions (in what follows, the four components of the quaternions will always be real); the real parameters entering these elements are independent gaussian random variables with mean zero and the same variance. The number of real parameters needed to characterize an $n \times n$ matrix is thus βn^2 , where β is 1, 2 or 4 according as the matrix elements are real, complex or quaternions. We will derive the probability density of their determinant.

The determinant of random real matrices is an old subject [1], that of random complex matrices and of random hermitian complex matrices was studied some time back [2], that of random quaternion matrices presents some peculiar features due to the non-commutative multiplication, as we will see below while the case of real symmetric matrices has been settled recently for odd n [3].

The method we will use here is to start with the joint probability density of the singular values rather than that of the eigenvalues. As the absolute value of the determinant is the product of all the singular values, we can find its probability density by calculating its Mellin transform. This gives new proofs of the known results for random real and complex matrices and of a partial result for quaternion matrices.

For quaternions, multiplication being not commutative, it is not possible to define a determinant having the usual three properties [4]; namely, (i) $\det A = 0$ if and only if $Ax = 0$ has a non-zero solution $x \neq 0$, (ii) $\det(AB) = \det A \cdot \det B$, (iii) $\det A$ is multi-linear in the rows of A . So the definition of a determinant varies according to which of the property or properties one wants to keep. We will adopt the following definition due to Dieudonné or Artin [5].

Any matrix A is either singular (i.e. $Ax = yA = 0$ have non-zero solutions) or has an inverse (i.e. $AB = BA = I$) [4]. If A is singular, define $\det A = 0$. If A has an inverse, define $\det A$ by recurrence on n as follows. If $n = 1$, define $\det A = |a_{11}|$, where $|x|$ means the norm of (the quaternion) x . If $n > 1$, then let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by removing the i -th row and the j -th column of A . The matrix elements of B , the inverse of A , are written as b_{ij} . Not all b_{ij} are zero. One shows [5] that whenever $b_{ij} = 0$, $\det A_{ji} = 0$, and whenever $b_{ij} \neq 0$, $\det A_{ji} \neq 0$ and $|b_{ij}^{-1} \det A_{ji}|$ is independent of i or j . One then defines $\det A = |b_{ij}^{-1} \det A_{ji}|$. Thus for a quaternion matrix A , $\det A$ is a non-negative real number. (For real or complex A this definition also gives a non-negative real number, the absolute value of the usual ordinary determinant.) Note that this determinant is not linear in the rows of A , but has the other two properties [5]. Also that a quaternion matrix A may be singular while its transpose has an inverse [4].

The eigenvalues and eigenvectors of a matrix A are defined as the solutions of $A\varphi = \varphi x$, where φ is an $n \times 1$ matrix and x is a number. For a real or complex A one can eliminate φ to get $\det(A - xI) = 0$, where I is the unit matrix. For quaternion A , if x is an eigenvalue with the eigenvector φ and μ any constant quaternion, then $\mu^{-1}x\mu$ is an eigenvalue with the eigenvector $\varphi\mu$. Thus x and $\mu^{-1}x\mu$ are not essentially distinct as eigenvalues. It is not evident that an $n \times n$ quaternion matrix should have n (quaternion) eigenvalues, but it has [6]. One can actually put them in correspondance with complex numbers [7]. Here we will only note that the norm of the product of eigenvalues gives the determinant defined above.

If all the eigenvalues of A are essentially distinct, then one can diagonalise A by a non-singular matrix. To make things clearer, we give an example:

$$\begin{bmatrix} 1 & e_2 \\ e_1 & e_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ e_2 & e_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e_2 & e_1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 - e_3 \end{bmatrix}, \quad (1.1)$$

$$\begin{bmatrix} 1 & e_1 \\ e_2 & e_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}, \quad (1.2)$$

with

$$a = \frac{1}{2}(1 - \sqrt{3})(e_1 - e_2) \quad , \quad b = \frac{1}{2}(1 + \sqrt{3})(e_1 - e_2), \quad (1.3)$$

$$x_1 = \frac{1}{2}(1 + \sqrt{3}) - \frac{1}{2}(1 - \sqrt{3})e_3 \quad , \quad x_2 = \frac{1}{2}(1 - \sqrt{3}) - \frac{1}{2}(1 + \sqrt{3})e_3, \quad (1.4)$$

showing that the eigenvalues of $\begin{bmatrix} 1 & e_2 \\ e_1 & e_3 \end{bmatrix}$ are 0 and $1 - e_3$, while those of its transpose $\begin{bmatrix} 1 & e_1 \\ e_2 & e_3 \end{bmatrix}$ are x_1 and x_2 . Their determinants are respectively 0 and 2.

If all the eigenvalues x_i are real and positive (respectively, real and non-negative), one says that A is positive definite (respectively, positive semi-definite). Denote by A^\dagger the transpose, hermitian conjugate or the dual of A according as A is real, complex or quaternion. For any matrix A , the product AA^\dagger (or $A^\dagger A$) is positive semi-definite, its eigenvalues are real and non-negative. The positive square roots of the eigenvalues of AA^\dagger (or of $A^\dagger A$, they are the same) are known as the singular values [8] of A . The eigenvalues and singular values of A have, in general, nothing in common, except that

$$\prod_{i=1}^n \lambda_i^2 = \det(AA^\dagger) = \det A \cdot \det A^\dagger = |\det A|^2 = \prod_{i=1}^n |x_i|^2, \quad (1.5)$$

where λ_i are the singular values and x_i are the eigenvalues of A .

In section 2 we start with the joint probability density of the singular values and calculate the Mellin transform of the probability density $p(|y|)$ of the (absolute value) of the determinant y of a random matrix A . From symmetry, when A is real, y is real and $p(y)$ is even in y ; when A is complex, y is complex and $p(y)$ depends only on $|y|$. When A is quaternion, y is by definition real and non-negative. One can therefore recover $p(y)$ from $p(|y|)$ when A is real or complex. Our results, confirming those in the known cases, are as follows.

$$p_1(y) = \prod_{j=1}^n [\Gamma(j/2)]^{-1} G_{0,n}^{n,0} \left(y^2 \left| 0, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \dots, \frac{n-1}{2} \right. \right), \quad y \text{ real}, \quad (1.6)$$

$$p_2(y) = \frac{1}{\pi} \prod_{j=1}^n [\Gamma(j)]^{-1} G_{0,n}^{n,0} (|y|^2 \mid 0, 1, 2, \dots, n-1), \quad y \text{ complex}, \quad (1.7)$$

$$p_4(y) = 2 \prod_{j=1}^n [\Gamma(2j)]^{-1} G_{0,n}^{n,0} \left(y^2 \left| \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \dots, 2n - \frac{1}{2} \right. \right), \quad y \text{ real non negative}. \quad (1.8)$$

Here $G_{0,n}^{n,0}$ is a Meijer G-function. In the above results the gaussian probability distribution $P(A)$ for the matrix A was taken $P(A) \propto e^{-a \operatorname{tr} A^\dagger A}$, with $a = 1$. Next we show that the probability density of the random variable $y = [\det A^\dagger A]^{1/n}$ converges to $\delta(y - 1/e)$ in the large n limit, with $P(A) \propto e^{-a \operatorname{tr} A^\dagger A}$ and $a = \beta n/2$. In section 3 we study the large n limit for a non-gaussian random complex matrix and show that the random variable $y = [\det A^\dagger A]^{1/n}$ converges in the large n limit to a constant, whose value, depending on the parameters in the non-gaussian probability distribution, is different in the two phases of

the model. Finally for a hermitian complex random matrix H with probability density $P(H) \propto e^{-n \text{tr} H^2}$, we show that in the large n limit $|\det H|^{1/n}$ tends to the constant $1/\sqrt{2e}$.

Some of these results are probably known to some experts, since analogous results appear in the literature [9].

2 Gaussian matrices.

The joint probability density of the singular values can conveniently be derived in two steps from the two observations [10],[11]

(i) Any matrix A can almost uniquely be written as $U\Lambda V$ where Λ is a diagonal matrix with real non-negative diagonal elements, while U and V are real orthogonal, complex unitary or quaternion symplectic matrices according as A is real, complex or quaternion; “almost uniquely” referring to the fact that either U or V is undetermined up to multiplication by a diagonal matrix.

(ii) Any positive semi-definite matrix $H = AA^\dagger$ can be written uniquely as $H = TT^\dagger$, where T is a triangular matrix with real non-negative diagonal elements.

As a result the gaussian joint probability density $\exp(-a \text{tr} AA^\dagger)$ for the matrix elements of A gets transformed to

$$F(\Lambda) \equiv F(\lambda_1, \dots, \lambda_n) = \text{const.} \exp\left(-a \sum_{j=1}^n \lambda_j^2\right) |\Delta(\lambda^2)|^\beta \prod_{j=1}^n \lambda_j^{\beta-1} \quad (2.1)$$

where $\lambda_1, \dots, \lambda_n$ are the singular values of A , Δ is the product of differences

$$\Delta(\lambda^2) = \prod_{1 \leq j < k \leq n} (\lambda_k^2 - \lambda_j^2) \quad (2.2)$$

and $\beta = 1, 2$ or 4 according as A is real, complex or quaternion.

The Mellin transform of the product of the λ 's is

$$\begin{aligned} \mathcal{M}_n(s) &= \text{const.} \int_0^\infty \eta^{s-1} \delta(\eta - \lambda_1 \dots \lambda_n) F(\Lambda) d\lambda_1 \dots d\lambda_n d\eta \\ &= \text{const.} \int_0^\infty \exp\left(-a \sum_{j=1}^n \lambda_j^2\right) |\Delta(\lambda^2)|^\beta \prod_{j=1}^n \lambda_j^{\beta+s-2} d\lambda_j \\ &= \text{const.} \int_0^\infty \exp\left(-a \sum_{j=1}^n t_j\right) |\Delta(t)|^\beta \prod_{j=1}^n t_j^{(\beta+s-3)/2} dt_j \end{aligned}$$

$$= a^{-n(s-1)/2} \prod_{j=1}^n \left[\frac{\Gamma\left(\frac{s-1}{2} + \frac{j\beta}{2}\right)}{\Gamma\left(\frac{j\beta}{2}\right)} \right] \quad (2.3)$$

In the last line we have used a result derived from Selberg's integral [12]. The constant, independent of s , is fixed from the requirement that $\mathcal{M}_n(1) = 1$.

The inverse Mellin transform of the expression (2.3) is a Meijer G-function [13]

$$p_\beta(|y|) = 2 a^{n/2} \prod_{j=1}^n [\Gamma(j\beta/2)]^{-1} G_{0,n}^{n,0} \left(a^n |y|^2 \left| \frac{\beta-1}{2}, \frac{2\beta-1}{2}, \dots, \frac{n\beta-1}{2} \right. \right) \quad (2.4)$$

When $\beta = 1$, the matrix A is real, its determinant y is real, from symmetry the probability density $p_1(y)$ is an even function of y and we have

$$p_1(y) = \frac{1}{2} p_1(|y|) \quad (2.5)$$

giving equation (1.6) with $a = 1$. When $\beta = 2$, A is complex, y is complex, from symmetry $p_2(y)$ depends only on the absolute value $|y|$ of y , and one has for $a = 1$,

$$\begin{aligned} p_2(y) &= \frac{1}{2\pi|y|} p_2(|y|) \\ &= \frac{1}{\pi} \prod_{j=1}^n [\Gamma(j)]^{-1} \frac{1}{|y|} G_{0,n}^{n,0} \left(|y|^2 \left| \frac{1}{2}, \frac{3}{2}, \dots, n - \frac{1}{2} \right. \right) \\ &= \frac{1}{\pi} \prod_{j=1}^n [\Gamma(j)]^{-1} G_{0,n}^{n,0}(|y|^2 | 0, 1, \dots, n-1) \end{aligned} \quad (2.6)$$

which is equation (1.7). When $\beta = 4$, A is quaternion, y is, by definition, real positive, and $p_4(y) = p_4(|y|)$, giving equation (1.8).

In reference 2, appendix A.5, we somewhat conventionally mapped the (quaternion) eigenvalues on to the essentially equal eigenvalues having the scalar part and only one other component at most, accounting for a factor y^2 in the probability density. Moreover, equation (A.43) there has a misprint, $\Gamma((s+2j+1)/2)$ there should read $\Gamma((s/2) + 2j + 1)$. Thus equation (1.8) tallies with equation (A.43) of reference 2.

We now evaluate the large n behaviour of the moments $\langle y^k \rangle$ of the random variable $y = [\det A^\dagger A]^{1/n}$ which show that it converges to a constant in the $n \rightarrow \infty$ limit. Next the same result is obtained by the saddle point method.

In the study of large n , the proper choice of the parameter a for the gaussian ensembles is $a = \beta n/2$. Then eq.(2.3) implies

$$\left\langle \left(\det A^\dagger A \right)^k \right\rangle = \left(\frac{\beta n}{2} \right)^{-nk} \prod_{j=1}^n \left[\frac{\Gamma(k + \frac{j\beta}{2})}{\Gamma(\frac{j\beta}{2})} \right] \quad (2.7)$$

that is

$$\begin{aligned} \log \left\langle \left(\det A^\dagger A \right)^k \right\rangle &= -nk \log \frac{\beta n}{2} + \sum_{j=1}^n \log \frac{\Gamma(k + \frac{j\beta}{2})}{\Gamma(\frac{j\beta}{2})} \\ &\approx -nk \log \frac{\beta n}{2} + n \int_0^1 dx \log \frac{\Gamma(k + \frac{\beta}{2} + \frac{nx\beta}{2})}{\Gamma(\frac{\beta}{2} + \frac{nx\beta}{2})} + O(\log n) \\ &\approx -nk + O(\log n) \end{aligned} \quad (2.8)$$

where the Euler-Maclaurin formula has been used to estimate the large n asymptotics. When x is near 0, the integrand is a constant and its contribution is negligible. When x is not small, one can ignore other terms compared to $nx\beta/2$.

Replacing $k \rightarrow k/n$ in eqs.(2.7), (2.8), one gets

$$\lim_{n \rightarrow \infty} \log \left\langle \left(\det A^\dagger A \right)^{k/n} \right\rangle = -k \quad (2.9)$$

From the knowledge of all the moments (2.9), we conclude that the random variable $y = (\det A^\dagger A)^{1/n}$ converges in the large n limit to the constant $1/e$.

It is convenient to evaluate the above large n limit also by a saddle point approximation because this is easy to generalize to different probability distributions. Let us recall the asymptotic density of squared singular values (see [14] eq.(10) after setting $L = 1$, $m^2 = 1$, $g = 0$, hence $A = 0$, $B = 4$)

$$\rho(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_j \langle \delta(t - t_j) \rangle = \frac{1}{2\pi} \sqrt{\frac{4-t}{t}} \quad , \quad 0 < t \leq 4 \quad (2.10)$$

Then it is easy to evaluate

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \left\langle \log \left(\det A^\dagger A \right)^k \right\rangle &= \lim_{n \rightarrow \infty} \frac{1}{n} \left\langle k \sum_{j=1}^n \log t_j \right\rangle \\ &= k \int_0^4 \log t \rho(t) dt = -k \end{aligned} \quad (2.11)$$

confirming eq.(2.9).

3 Large n for non-gaussian complex matrices.

Let us now consider an example of non-gaussian probability distribution such that in the large n limit two different spectral densities for the singular values exist. For simplicity, we consider the ensemble of $n \times n$ complex matrices A with the non-gaussian probability distribution

$$P(A) \propto e^{-n(a \operatorname{tr} A^\dagger A + 2b \operatorname{tr} A^\dagger A A^\dagger A)} \quad , \quad b > 0 \quad , \quad a \text{ real} \quad (3.1)$$

The analogous ensemble of real matrices would need only trivial changes.

Again the evaluation of all the moments of $y = [\det A^\dagger A]^{1/n}$ may be performed in terms of t_j , the squared singular values of the matrix A

$$\langle \left(\det A^\dagger A \right)^k \rangle = \frac{\int \dots \int_0^\infty \Delta^2(t) \prod_{j=1}^n t_j^k e^{-n(at_j + 2bt_j^2)} dt_j}{\int \dots \int_0^\infty \Delta^2(t) \prod_{j=1}^n e^{-n(at_j + 2bt_j^2)} dt_j} \quad (3.2)$$

Since $\prod_{j=1}^n (t_j)^{k/n} = \exp\left(\frac{k}{n} \sum_1^n \log t_j\right)$, the large n limit for all the moments $\langle y^k \rangle$ are easily evaluated by the saddle point approximation

$$\lim_{n \rightarrow \infty} \langle y^k \rangle = \lim_{n \rightarrow \infty} \langle \exp\left(\frac{k}{n} \sum_1^n \log t_j\right) \rangle = \exp\left(k \int dt \rho(t) \log t\right) \quad (3.3)$$

where $\rho(t)$ is the solution of the saddle point equation

$$\frac{a}{2} + 2bt = \int \frac{\rho(y)}{t-y} dy \quad (3.4)$$

The solution of eq.(3.4) has two different forms, $\rho_1(t)$, $\rho_2(t)$ (see [14] eq.(10) after setting $L = 1$, $m^2 = a$, $g = b$), depending on the values of the real number a/\sqrt{b} being larger or smaller than the critical value $a/\sqrt{b} = -4$. At the critical value, $\rho_1(t) = \rho_2(t)$.

$$\begin{aligned} \rho_1(t) &= \frac{1}{\pi} \sqrt{\frac{C-t}{t}} \left[2bt + \frac{2a + \sqrt{a^2 + 48b}}{6} \right] \quad , \quad 0 < t \leq C \quad , \\ C &= \frac{\sqrt{a^2 + 48b} - a}{6b} \quad , \quad a/\sqrt{b} \geq -4 \quad ; \end{aligned} \quad (3.5)$$

The definite integrals related to eq.(3.3) for the spectral function $\rho_1(t)$ are known in closed form and

$$\int_0^C \rho_1(t) \log t dt = \log \frac{C}{4} - \frac{1}{2} - \frac{aC}{8} \quad (3.6)$$

Therefore in the ‘‘perturbative phase’’, that is for $a/\sqrt{b} \geq -4$, the random variable y converges in the large n limit to a constant

$$\lim_{n \rightarrow \infty} [\det A^\dagger A]^{1/n} = \frac{C}{4} e^{-\frac{1}{2} - \frac{aC}{8}} \quad (3.7)$$

For $a/\sqrt{b} \leq -4$ we have

$$\begin{aligned} \rho_2(t) &= \frac{2b}{\pi} \sqrt{(B-t)(t-A)} \quad , \quad A \leq t \leq B \quad , \\ A+B &= -\frac{a}{2b} \quad , \quad B-A = \frac{2}{\sqrt{b}} \quad , \quad a/\sqrt{b} \leq -4 \end{aligned} \quad (3.8)$$

The definite integral related to eq.(3.3) for the spectral function $\rho_2(t)$ may still be evaluated

$$\begin{aligned} I(a,b) &= \int_A^B \rho_2(t) \log t \, dt = \frac{2b}{\pi} \int_A^B \sqrt{(B-t)(t-A)} \log t \, dt \\ &= \frac{b}{8} (\sqrt{B} - \sqrt{A})^4 + \frac{b}{2} (B-A)^2 \log \left(\frac{\sqrt{B} + \sqrt{A}}{2} \right) \end{aligned} \quad (3.9)$$

One may still conclude that also in the “non-perturbative phase”, that is for $a/\sqrt{b} \leq -4$, the random variable y converges in the large n limit to a constant

$$\lim_{n \rightarrow \infty} [\det A^\dagger A]^{1/n} = e^{I(a,b)} \quad , \quad a/\sqrt{b} \leq -4 \quad (3.10)$$

We remark that other functions of the determinant, such as $w = \frac{1}{n^{n^2}} [\det A^\dagger A]^n$, may not have a finite limiting probability distribution, in the large n limit, though their large n behaviour may still be evaluated.

Hermitian matrices. The large n limit of the absolute value of the determinant of gaussian hermitian matrices may be evaluated from the exact finite n moments given in [2]. Let us consider an ensemble of $n \times n$ hermitian matrices H with probability distribution $P(H) \propto e^{-n \operatorname{tr} H^2}$ and the random variable y

$$y = |\det H|^{1/n} . \quad (3.11)$$

Let x_j , $j = 1, \dots, n$ be the eigenvalues of the matrix H , then

$$\langle y^k \rangle = \frac{\int \dots \int_{-\infty}^{\infty} \Delta^2(x) \prod_{j=1}^n |x_j|^{k/n} e^{-nx_j^2} dx_j}{\int \dots \int_{-\infty}^{\infty} \Delta^2(x) \prod_{j=1}^n e^{-nx_j^2} dx_j} \quad (3.12)$$

These moments are known from [2].

$$\langle y^k \rangle = n^{-k/2} \prod_{j=1}^n \frac{\Gamma(\frac{1}{2} + \frac{k}{2n} + b_j^+)}{\Gamma(\frac{1}{2} + b_j^+)} \quad , \quad b_j^+ = \left[\frac{j}{2} \right] \quad (3.13)$$

$$\log \langle y^k \rangle \approx -\frac{k}{2} \log(2e) + O\left(\frac{\log n}{n}\right) . \quad (3.14)$$

Thus in the large n limit $|\det H|^{1/n}$ tends to the constant $1/\sqrt{2e}$.

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Providence, R.I., (1963), Chapter 3.4. In the last formula there is a misprint; the eigenvalues of H being λ_j^2 , and not λ_j , the last formula should read as

$$\begin{aligned}\dot{Z} &= 2^{-n^2} D^2(\lambda_1^2, \dots, \lambda_n^2) d\lambda_1^2 \dots d\lambda_n^2 \cdot \dot{U}[\dot{U}] \\ &= 2^{-n^2+n} D^2(\lambda_1^2, \dots, \lambda_n^2) \prod_{j=1}^n \lambda_j d\lambda_j \cdot \dot{U}[\dot{U}]\end{aligned}$$

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