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# ARES: a crude but efficient approach to adaptive time integration for stiff Differential-Algebraic Equations



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# Outline

- 1 General context
- 2 **Adaptive Relaxed Euler Scheme**
- 3 Why it works in theory
- 4 Numerical results
- 5 A good rule of thumb
- 6 Conclusions

In many industrial or natural configurations, involving heat and reactive transport we face :

## Stiff Nonlinear Differential-Algebraic Equations

→ *Chemical processes, Nuclear PWR clogging, Radioactive waste recycling or disposal, Hydrogeology, CO<sub>2</sub> underground storage...*

**Stiffness** of a solution curve may result from chemical reactions depending on temperature, catalytic processes or complex fluxes etc...

When looking for the **steady state** of a system, through a transient simulation with stiff phenomena, **adaptive time integration** then becomes a necessity, to adapt the timestep to local stiffness.

2 broad categories of time integrators are often used :

- BDF (backwards differentiation formula) and MEBDF (modified extended backwards differentiation formula) approaches (Gear, DASSL or MEBDFDAE algorithms)
- Runge-Kutta approaches

**High order time discretization**  $\Rightarrow$  costly additional operations at each iteration of the nonlinear solver

What if you **only care for a precise evaluation of the steady state**, however precise the transient way to get there may be ?

$\rightarrow$  **ARES** may be of service.

*You are the god(dess) of war aiming at a swift victory  
over your nonlinear foe !*

→ Let  $F : (\mathbb{R}, \mathbb{R}^p, \mathbb{R}^p) \rightarrow \mathbb{R}$  be a continuously differentiable nonlinear function. A typical DAE solution  $y : \mathbb{R} \rightarrow \mathbb{R}^p$  writes :

$$F\left(t, y, \frac{dy}{dt}\right) = 0$$

$$\forall t, \forall i \in \llbracket 1, p \rrbracket, l_i \leq y_i(t) \leq u_i$$

→ Euler implicit time integration scheme means that we solve at every  $t_n$  :

$$F\left(t_n, y_n, \frac{y_n - y_{n-1}}{h_n}\right) = 0$$

→ Implicit Euler time schemes with an adaptive timestep differ mainly by the strategy governing the choice of the next timestep  $h_n = t_n - t_{n-1}$ .

→ As is done very often in adaptive time schemes, we use a predictor to propose a prediction of  $y_n$ , noted as  $y_n^*(o)$ , based upon a Taylor evaluation of order  $o = 1$  or  $o = 2$

$$\begin{cases} y_n^*(1) = y_{n-1} + d_{n-1}h_n & \text{with } d_{n-1} = \frac{y_{n-1} - y_{n-2}}{h_{n-1}} \\ y_n^*(2) = y_n^*(1) + \frac{1}{2}d_{n-1}^2h_n^2 & \text{with } d_{n-1}^2 = \frac{d_{n-1} - d_{n-2}}{h_{n-1}} \end{cases}$$

→  $y_n^*(o)$  is used as the initial guess for a classical Newton method.

→ We monitor the behaviour of the solution using the number of iterations used by the Newton method (noted henceforth  $I_n$ ) to converge towards the solution. This value is a direct product of the Newton solver and **needs no further calculation.**

*A very basic and simple idea :*

→ The Newton solver's number of iterations is viewed here as an indicator of the gap between  $y_n$  and the predicted value  $y_n^*(o)$ .

→ This gap between  $y_n$  and  $y_n^*(o)$  is informally related to the stiffness of the solution.

→ We allow the timestep to increase when this number is low, because the solution should be smooth enough between  $t_{n-1}$  and  $t_n$ . Conversely, we decrease the timestep when this number is high.

→ Since we aim only for the steady state :  $F(t, y, 0) = 0$ , high order time discretization of  $\frac{dy}{dt}$  is nullified in the end and order 1 is sufficient.

- $l_n$  : number of iterations needed to compute  $y_n$   
 $l_u$  : a user-defined "ideal" number of iterations  
 $\gamma_a > 1$  : acceleration rate  
 $0 < \gamma_d < 1$  : deceleration rate

At first, we take  $h_n = h_{n-1}$  and then we apply the following choices, naming  $h_n^*$  the previous value of  $h_n$  :

- Convergence fails  $\rightarrow h_n = \gamma_d h_n^*$
- Convergence succeeds in  $l_n$  iterations :
  - ▶  $l_n = l_u \rightarrow$  next timestep with  $h_{n+1} = h_n^*$
  - ▶  $l_n > l_u \rightarrow$  treated as failure
  - ▶  $l_n < l_u \rightarrow$  next timestep with  $h_{n+1} = \max(\gamma_a h_n^*, h_{max})$

*Delayed version :  $l_n > l_u$  is not treated as a failure, but the timestep is nevertheless reduced :  $h_{n+1} = \gamma_d h_n^*$ .*

ARES fully solves at each timestep the equation  $F(t_n, y_n, \frac{y_n - y_{n-1}}{h_n}) = 0$ , through the Newton method, at any requested precision.

Stationary time :  $y_n - y_{n-1} = 0 \implies F(t_n, y_n, 0) = 0$

$\rightarrow y_n$  solves the exact equation of the steady state equation (provided of course that this solution remains within the boundaries).

*ARES can also be used for a transient calculation, with  $h < h_{max}$  to ensure a minimal precision, since the order of the time scheme remains 1.*

Let  $G : \mathbb{R}^p \rightarrow \mathbb{R}^p$  be the following function :

$$G(y) = F(t_n, y, \frac{y - y_{n-1}}{h_n}) \quad (1)$$

At every timestep  $t_n$ , we search for  $y_n$  as the solution of  $G(y) = 0$  through the Newton method. Let  $J(y)$  be the jacobian of  $G$  for  $y$ . One Newton iteration (step  $k$ ) verifies :

$$J(y^k)(y^{k+1} - y^k) = -G(y^k) \quad (2)$$

If we decompose  $y = [y_A, \dots, y_p]$  and  $G(y) = [g_1(y), \dots, g_p(y)]$  accordingly for a  $p$ -sized system, we have :

$$\sum_{j=1}^p (y_j^{k+1} - y_j^k) \frac{\partial g_i}{\partial y_j}(y_A^k, \dots, y_p^k) = -g_i(y_A^k, \dots, y_p^k) \quad \forall i \in [1, p] \quad (3)$$

Let  $z : \mathbb{R} \rightarrow \mathbb{R}^p$  be solution of the following differential equation :

$$\frac{d}{dt}[g_i(z(t))] = -g_i(z(t)) \quad (4)$$

$\rightarrow \forall i \in [1, p]$  (3) can be seen as an **implicit Euler time discretization** of (4), with a constant timestep=1.

The solution  $z(t)$  ("Newton flow") verifies :

$$g_i(z(t)) = g_i(z(0)) \exp^{-t} \quad (5)$$

If the Newton iterations run by (3) converge towards the solution  $y_n$ , there is by construction a discret solution of (4)  $\forall i \in [1, p]$ .

The values of  $z(t)$  form then a trajectory towards  $y_n$ , with a constant timestep.

According to (5), the trajectory depends only on the values of  $g_i(z(0))$  or **the first iterate of the Newton method**.

→ Direct correspondence between the number of Newton iterations  $l_n$  and the overall closeness of  $G(y_n^*(o))$  to zero.

→ Preserving a low  $l_n$  ensures then a relative smoothness of the solution between  $t_{n-1}$  and  $t_n$ .

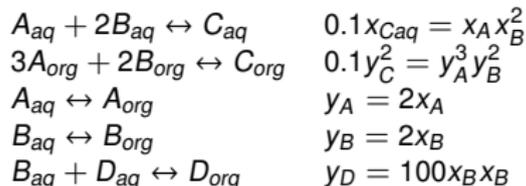
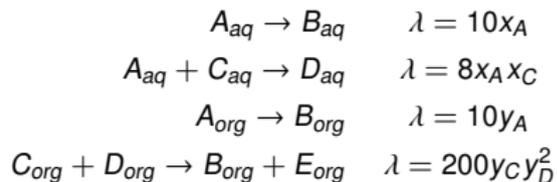
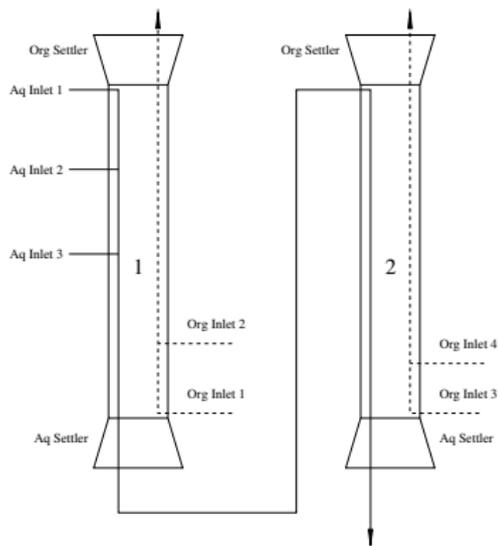
SIDES is a module of the open source platform TRUST, dedicated to thermohydraulics in the context of nuclear plant simulations :

- C++ platform providing many numerical tools, such as PETSc linear solvers, pre- and postprocessing tools etc...
- Basis for TrioCFD, Genepi+, Cathare...
- SIDES aggregates objects associated with numerical methods : being given a nonlinear function of multiple variables, its jacobian and a set of bounds, it may be used to solve any  $F\left(t, y, \frac{dy}{dt}\right) = 0$

→ ARES was used in SIDES for an industrial code dedicated to Liquid-Liquid-Extraction :

- reactive and heat transport between an organical and an aqueous phase countercurrently,
- both intra- and interphasic chemical reactions,
- chemical reactions under kinetics or at equilibrium,
- retroaction of chemistry on temperature.

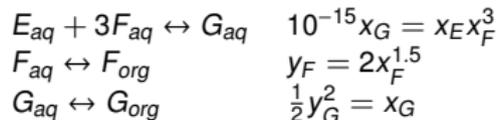
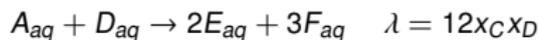
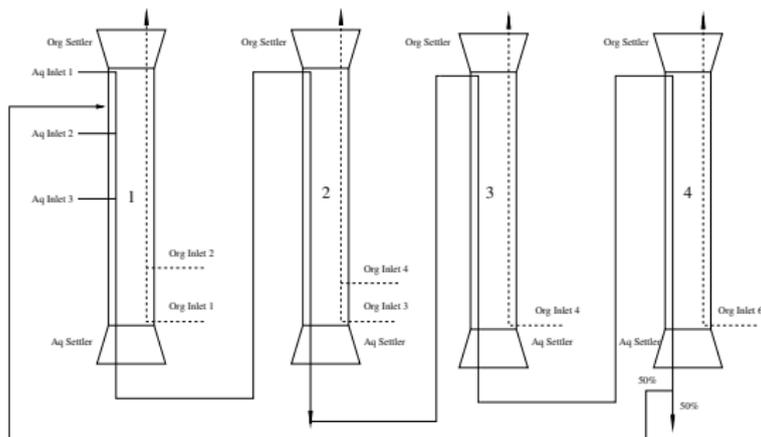
## 2 Pulsed Columns, 2 chemical phases with 11 fictitious species



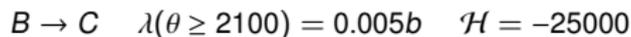
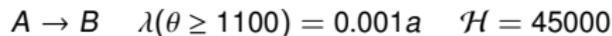
Density laws :  $\rho_{aq}, \rho_{org}$

Variable fluxes with phase swelling

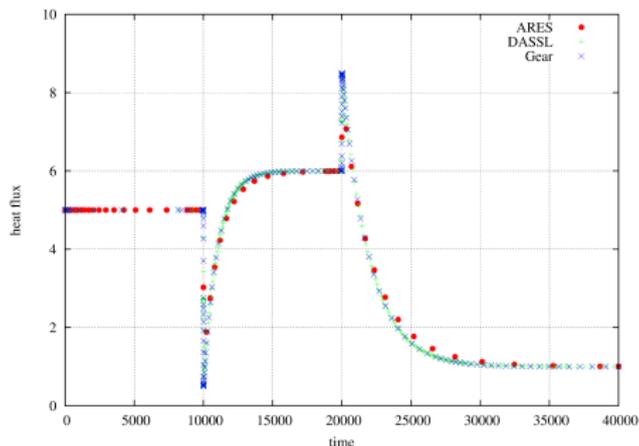
# 4 Pulsed Columns, 5 more fictitious species, 1 more kinetics and 3 more equilibria



Analytical case study : 2 fast reactions, with a threshold temperature.



Heating device :  $\theta(t) = 100 + 0.1t$ . We monitor the needed heat flux  $\Phi$  :



→ Steady plateaux  $\Phi = 5$ ,  $\Phi = 6$  and  $\Phi = 1$  accurately given by ARES, with much less simulation times.

→ As expected : Gear or DASSL more accurate on transient regimes.

method	Gear	DASSL	ARES	ARES delayed
case study 1	114	172	78	81
case study 2	294	492	295	189
case study 3	3.6	6	0.8	0.8

## How to choose an ideal number of iterations $l_u$ ?

A DAE system contains both algebraic and differential equations :

- For differential equations, decreasing the timestep hopefully means that the overall function giving  $y^{k+1}$  in relation to  $y^k$  may become **contractive**.
- However, the presence of purely algebraic equations, independent from time derivatives, invalidates or at least challenges this strategy.

Let us consider a DAE system containing only one algebraic equation :

$$\begin{bmatrix} y_A - y_A^{n-1} - h_n f_1(y) \\ \dots \\ y_{p-1} - y_{p-1}^{n-1} - h_n f_{p-1}(y) \\ f_p(y) \end{bmatrix} = 0 \quad (6)$$

If we decompose the jacobian  $J = J_1 + h_n J_2$ , where  $J_1$  gathers all terms independant of  $h_n$ , the Newton step writes :

$$\begin{bmatrix} y_A^k - y_A^{n-1} - h_n f_1(y^k) \\ \dots \\ y_{p-1}^k - y_{p-1}^{n-1} - h_n f_{p-1}(y^k) \\ f_p(y^k) \end{bmatrix} = \begin{bmatrix} (1 + \frac{\partial f_p}{\partial y_A}(y^k))(y_A^k - y_A^{k+1}) \\ \dots \\ (1 + \frac{\partial f_p}{\partial y_{p-1}}(y^k))(y_{p-1}^k - y_{p-1}^{k+1}) \\ \sum_{j=1}^p \frac{\partial f_p}{\partial y_j}(y^k)(y_j^k - y_j^{k+1}) \end{bmatrix} + h_n J_2 [y^k - y^{k+1}] \quad (7)$$

This simplifies as :

$$\begin{bmatrix} -y_A^{n-1} - h_n f_1(y^k) \\ \dots \\ -y_{p-1}^{n-1} - h_n f_{p-1}(y^k) \\ f_p(y^k) \end{bmatrix} = -y^{k+1} + \begin{bmatrix} \frac{\partial f_p}{\partial y_A}(y^k)(y_A^k - y_A^{k+1}) \\ \dots \\ \frac{\partial f_p}{\partial y_{p-1}}(y^k)(y_{p-1}^k - y_{p-1}^{k+1}) \\ y_p^{k+1} + \sum_{j=1}^p \frac{\partial f_p}{\partial y_j}(y^k)(y_j^k - y_j^{k+1}) \end{bmatrix} + h_n J_2 [y^k - y^{k+1}] \quad (8)$$

Convergence leading to  $y^n$  can be interpreted here as the fixed point of that function :

$$y^{k+1} = M^{-1} \left( \begin{array}{c} y_A^{n-1} + h_n f_1(y^k) \\ \dots \\ y_{p-1}^{n-1} + h_n f_{p-1}(y^k) \\ f_p(y^k) \end{array} \right) + \left[ \begin{array}{c} \frac{\partial f_p}{\partial y_A}(y^k) y_A^k \\ \dots \\ \frac{\partial f_p}{\partial y_{p-1}}(y^k) y_{p-1}^k \\ \sum_{j=1}^p \frac{\partial f_p}{\partial y_j}(y^k) y_j^k \end{array} \right] + h_n J_2 y^k \quad (9)$$

→ Strong assumptions on  $f_p$  and its derivatives to ensure contraction !  
Whereas, in the purely differential case ( $f_p$  and the  $p^{th}$  line disappear) :

$$y^{k+1} = (Id_p + h_n J_2)^{-1} \left( \begin{array}{c} y_A^{n-1} + h_n f_1(y^k) \\ \dots \\ y_{p-1}^{n-1} + h_n f_{p-1}(y^k) \end{array} \right) + h_n J_2 y^k \quad (10)$$

→  $M = Id_p + h_n J_2$  can be made contractive for a certain  $h_n$ , in the vicinity of the Newton search

In the general context of real chemical equilibria, the algebraic equations will come in many unpleasant shapes and impose (at best !) a **lower bound to the number of Newton iterations needed, independently of the value of  $h_n$ .**

→ A good rule of thumb is to give  $l_U$  a value close to that of the lower bound, which depends on the system.

→ A guess can be made from the behaviour of the system in the first timesteps :  $l_U = l_1$  and  $l_{max} = 3l_U$  ( $\gamma_a = 1.2$  and  $\gamma_d = 0.5$  for ARES and  $\gamma_a = 1.5$  and  $\gamma_d = 0.8$  for ARES delayed).

- ARES aims at solving a paradox : due to Stiff Nonlinear DAE, reactive transport simulations are usually very costly but it is common to have to run a transient calculation in order obtain a steady state in a robust way, although this costly transient calculation itself is not necessarily of interest.
- By allowing a relative loss of accuracy on a transient calculation, ARES may help in providing the desired result at a reduced cost, by proposing an intermediate approach between overly simple methods such as the constant timestep Euler scheme and complex high-order approaches.
- ARES is very easy to implement and to use, since it uses very few parameters.