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Cities are systems with a large number of constituents and agents interacting with each other and can be considered as emblematic of complex systems. Modeling these systems is a real challenge and triggered the interest of many disciplines such as quantitative geography, spatial economics, geomatics and urbanism, and more recently physics. (Statistical) Physics plays a major role by bringing tools and concepts able to bridge theory and empirical results, and we will illustrate this on some fundamental aspects of cities: the growth of their surface area and their population, their spatial organization, and the spatial distribution of activities. We will present state-of-the-art results and models but also open problems for which we still have a partial understanding and where physics approaches could be particularly helpful. We will end this short review with a discussion about the possibility of constructing a science of cities.

I. INTRODUCTION

Modeling cities and getting reliable quantitative predictions about their behavior have become major challenges for our modern world. Concentration of humans in a relatively small spatial area is essentially the result of economical considerations but spatial localization also leads to an increase of housing costs, traffic congestion, urban sprawl, pollution and other environmental problems. Organizing the city, building new infrastructures and new transportation means rely then a lot on our understanding of their effects on the collective behavior and their potential impact on the city, its functioning and its economical activity. This is particularly important at a time when we expect the number of megacities (with population larger than 10 millions) to be around 30 in 2050 [1] (see also [2] for a more thorough discussion), and emergent countries see their cities exploding in size: it is estimated that by 2050 the world population will add 2.5 billion people to urban areas, with 90% of this increase that will take place in Africa and Asia [1].

From a theoretical point of view, cities comprise all the difficulties inherent to complex systems: a large variety of agents interacting with each other, a large variety of temporal and spatial scales, and a large number of processes that modify the spatial structure and organization of these systems. The always growing availability of urban data will hopefully help us to construct robust models and thereby to improve our knowledge of the main mechanisms at play during the evolution of urban systems. This fundamental level is necessary in order to construct robust computational models for describing more realistic or specific situations. Indeed, simulating the behavior of a urban system by 'simply' adding together all the possible processes and gathering as much information as possible, is neither a guarantee of its robustness, nor of the correctness of its predictions.

Economics and regional science were the first disciplines to tackle this problem of modeling cities. Most of the urban economics is built on the model proposed in the 60s by Alonso, Muth and Mills (see for example the textbook [3] and references therein). This model relies on different assumptions. The first important one is that all individuals behave in the same way and maximize the same utility function subject to the same budget constraint. The second assumption is the monocentric city structure organized around a unique central business district (CBD). The information about transport infrastructure is here absent and space is considered as homogeneous and isotropic. All individuals are assumed to work at this CBD and the transportation cost depends on the distance from home to it (here we can see the impact of the first model proposed by Von Thunen [4] with a unique market in an isolated state, where space is homogeneous and isotropic, and transportation cost depend only on the euclidean distance to the central market). Later, Fujita and Ogawa [5] considered a general model where both individuals and firms are finding their optimal location in the city. In particular they introduced an agglomeration effect that describes the benefit for companies to be close to each other. These approaches constitute the basis of urban economics and we refer the interested reader to the books [3, 6].

Agent-based models and large simulations also played an important role for understanding cities and were primarily developped by geographers in the 50-60s. The primary goal was to forecast the effects of planning policies and several formalisms were used [7, 8]. The complexity of these simulations, the large number of parameters and processes however make it difficult to test thoroughly these models. In addition to these numerical simulations, it thus seems important to develop parsimonious models that help identifying the main mechanisms and to construct robust models. In this respect physics and more precisely statistical physics with its ability to connect microscopic behaviors to the emergence of collective and non-trivial behavior seems to be a particularly well-suited tool. In fact, statistical physics came into play later and connections with cities appeared first in the

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FIG. 2: Evolution of the built area of various cities. These curves demonstrate an evolution faster than t (typically as t^2 for most cities). Data from [25].

FIG. 3: Area versus population for different regions in the world. (Left) Area of the 494 world cities with population larger than 1 million inhabitant. Both the linear and nonlinear fit are acceptable and very close to each other (data from various sources compiled by Wikipedia). (Right) US Urban areas with population over $50,000$ (data from the 2010 Census population). Here also both linear and nonlinear fits are acceptable and a more thorough statistical analysis is needed (such as in [26] for example).

the evolution of the cities surface area versus the population living in it, a long standing problem in the field ([12]). Here also, we don't have a clear picture yet. For cities in various countries, we can fit the data with a linear function of the form

$$A = aP \tag{1}$$

where a is the inverse of the average density and represents the typical area per inhabitant. This corresponds to the intuitive expectation that cities evolve in such a way that their average population density $\rho = P/A$ remains constant. The quantity a is the average surface occupied by each individual (the assumption of a constant density is equivalent to a constant average surface per capita). We can probably expect this behavior to hold for relatively small cities where there is an constant increase of the built-up area when the population is increasing. However, this regime is certainly limited as a city cannot extend indefinitely and we expect a later behavior characterized by a slower increase of the area. In some other cases, we can fit the data by a nonlinear function of the form

$$A = aP^\tau \tag{2}$$

where τ is an exponent usually slightly smaller than 1. In the linear case the density is contant $\rho = P/A = 1/a$ and in the nonlinear case the density increases with population $\rho \sim P^{1-\tau}$: larger cities are also denser.

Empirically, we do observe that cities around the world display very different behaviors as shown in Fig. 3. When we mix together different countries of the world by taking urban areas with population larger than 1million (Fig. 3(left)), the nonlinear fit predicts an exponent slighly less than one ($\tau \approx 0.95$, $r^2 = 0.66$) and with this dataset it seems not possible to conclude between a constant density or a slightly increasing function of population (with exponent of order 0.05). At a smaller scale, in the case of the US for example (Fig. 3(right)),

the nonlinear behavior is also undistinguishable from the linear one. The linear fit predicts an average area of order $660m^2$ which is large, while the nonlinear fit gives an exponent $\beta \approx 0.87$. The linear fit predicts thus a constant density of order $1,500\text{hab}/km^2$, which is correct for many cities, but we actually see deviations for large cities with density much larger for proper cities (and not for urban areas that mixes heterogeneous zones). For example for NYC, we have a density of order $10,000\text{hab}/km^2$ or $6,600\text{hab}/km^2$ for San Francisco. This behavior is however completely different for a country such as Japan for example (Fig. 4) where both linear and nonlinear fits are not good. The noise in this case cannot allow to conclude and the data suggests that the area is not a function of population only.

FIG. 4: Area versus population for urban areas in Japan (data from the UN, Demographic Yearbook 2012).

After this little tour of empirical results, the absence of a clear stylized fact leaves us in the dark, and theoretical models, even very simplified ones, could shed some light on this problem. In order to illustrate how models could help us to understand the data, we will discuss here

two different approaches to this problem: scaling and a statistical model based on dispersal ideas.

B. Scaling approach

Bettencourt [27] proposed a phenomenological approach accounting for the value of the exponents governing the evolution of various quantities versus the population. In this approach the quantity of interest Y can be any 'urban socioeconomic output' such as income, etc. Empirically we observe in general [28, 29] a power law

$$Y \sim P^{\beta} \qquad (3)$$

where β is an exponent that can be different from 1. It was measured in [29] for various quantities and we can distinguish 3 groups according to its value. For social-related quantities (such as the number of patents, number of serious crimes, etc.), we observe $\beta > 1$ possibly rooted in the fact that interactions in cities grow very fast with population, typically as P^2. In contrast, we also observe values $\beta < 1$, which indicates an economy of scale (road surface, length of electric cables, etc.), and the last category with exponent $\beta = 1$ comprises essentially quantities that do not depend on the size of the city (water consumption or other human-dependent quantities for example).

In order to estimate these exponents (and to obtain as a by-product the behavior of area with population), Bettencourt assumes that the economic output per capita Y/P is proportional to the average number of interactions $Y/P \sim g\langle k \rangle$. The quantity g is assumed to be constant and the average number of interactions is assumed to be $\langle k \rangle \sim \rho\sigma$ where σ is the 'cross-section' of individuals and $\rho = P/A$ the average population density. This implies that

$$Y/P = G\frac{P}{A} \qquad (4)$$

where G is a constant. In addition, Bettencourt assumes that Y/P is of the order of the cost of transport which is also proportional to the average distance ℓ travelled in the city. The average distance ℓ depends on the fractal dimension H of the transportation network and Bettencourt writes $\ell \sim A^{H/d}$ (where usually $d = 2$ is the dimension of the embedding space). We then have

$$Y/P \sim \frac{P}{A} \sim A^{H/d} \qquad (5)$$

$$\Rightarrow A \sim P^{\frac{d}{H+d}} \qquad (6)$$

Usually $d = 2$, and in the simplest case of a linear transportation network $H = 1$, we obtain $\beta = 2/3$. This simple approach thus confirms a sublinear behavior with population but is at this point not able to explain the large fluctuations and the variety of behavior observed in real world cities.

C. A dispersal model

We are left with the question of constructing a simple model that is able to explain some of the features observed for the growth of the surface area of cities. A natural approach is to adapt dispersal models used in theoretical ecology. These models were developed to describe the proliferation of animal colonies [30, 31] and also as simplified models for cancerous tumor growth [32, 33]. The main feature 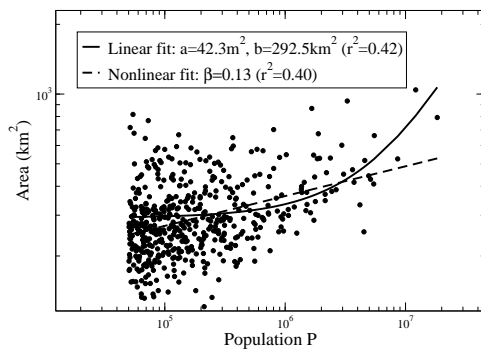of dispersal models is the concomitant existence of two growth mechanisms. The first process is the growth of the main ('primary') colony, which occurs typically via a reaction-diffusion process (for example as described by a FKK-like equation [34, 35]) and leads to a constant growth with a velocity c that depends on the details of the system. In the case of urban systems, this process would correspond to new buildings constructed at the fringe of the city and which are in turn triggering the construction of new buildings. The second ingredient is *random dispersal* from the primary colony, which represents the emergence of secondary settlements. In the urban sprawl case, this second process corresponds to the creation of small towns in the periphery of large cities. In real-world systems, dispersion is not isotropic and is governed mainly by transportation systems (blood vessels in the case of metastatic tumors, winds and rivers in ecological examples), but in this first approach we will neglect these effects. We assume that these secondary colonies grow also at the velocity c and eventually will coalesce with the primary colony. This image of a growing city, that eventually coalesces with neighboring small towns is consistent with our knowledge of urban growth (see for example the case of West London in [36]).

We will follow here the approach introduced by Kawasaki and Shigesada in [30, 35] in order to study this simplified model. We will consider that the primary colony grows at radial velocity c and emits a secondary colony at a rate λ and at a fixed distance ℓ from its border (long-range dispersal). Besides, we assume that each secondary colony also grows with the same radial speed c and does not emit tertiary colonies. The dependence of the emission rate on the colony size is taken into account by the functional form

$$\lambda(r) = \lambda_0 r^{\theta} , \qquad (7)$$

r being the radius of the primary colony and $\theta \geq 0$. When $\theta = 0$ the growth rate is independent from the primary colony size, for $\theta = 1$ it is proportional to its perimeter and for $\theta = 2$ to its area. For cities, we can imagine that at least the perimeter or the surface are the relevant variables for triggering new towns in its surrounding and that $\theta \geq 1$ is therefore probably the most relevant for urban systems.

We consider two variants of this process [37] - in a first version (model M_0) we assume that the primary colony remains circular after the coalescence with a secondary colony. In contrast, we can consider a modified version of the process (model M_1) where after the coalescence

with a secondary colony, the shape of the primary colony does not remain circular. This important difference is illustrated in the Fig. 5.
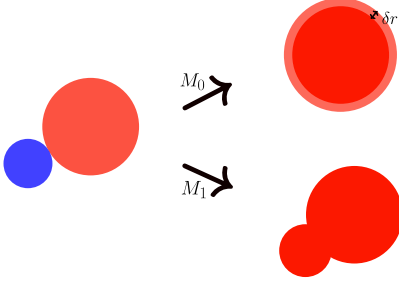

FIG. 5: Example of coalescence in models M_0 and M_1. In M_0, the primary colony (in red) remains circular and the area of the secondary colony is evenly distributed on the rim; in M_1, the shapes are simply 'concatenated'. Figure taken from [37].

If we denote by $r(t)$ the radius of the primary colony at time t and by $x(t)$ the radius of the colony absorbed at time t, we can show [37] that the equations governing the evolution of these quantities are [35, 37]

$$\begin{cases} \frac{dr}{dt} = c + \frac{\lambda_0 r\left(t - \frac{x(t)}{c}\right)^\theta}{2\pi r(t)}\left(1 - \frac{\dot{x}(t)}{c}\right)\pi x(t)^2 , \\ \ell = r(t) - r\left(t - \frac{x(t)}{c}\right) + x(t) . \end{cases} \quad (8)$$

These nonlinear differential equations capture the physics of the coalescence and allows us to extract the large time behavior of the main quantities of interest in this problem. In particular, assuming scaling laws at large times, $r(t) \sim at^\beta$ and $x(t) \sim dt^{-\alpha}$, we obtain

$$\beta = \frac{3}{4 - \theta} , \qquad \alpha = \beta - 1 . \quad (9)$$

Note that for $\theta \to 4$, we have $\beta \to \infty$, the radius grows faster than a power law and explodes exponentially. For $\theta = 1$, we obtain $\alpha = 0, \beta = 1$ which means that we have $x(t) = x^*$ independent of t and a linear behavior of $r(t)$. In this case the effective radial velocity c' of the primary colony is given by

$$c' = c + \frac{\lambda_0}{2}x^{*2} \quad (10)$$

and the value of x^* can be obtained by solving Eq. (8) that can be written as

$$\frac{\lambda_0}{2c}x^{*3} + 2x^* - \ell = 0 . \quad (11)$$

This result, for the specific case of $\theta = 1$, was first obtained by Shigesada and Kawasaki [30]. More generally, we can test our prediction for $\theta > 1$ on numerical simulations, and in Fig. 6, we plot the values of the exponents β obtained by power law fits and compare it with the theoretical prediction Eq. (9). We observe a good agreement with some deviations for higher values of θ which

FIG. 6: Plot of the exponent β as a function of θ, obtained from a power law fit on $r(t)$ versus t. The theoretical prediction Eq. (9) is shown in red. See [37] for details.

are probably due to finite size effect. Even if this model is very simple and is more 'metaphorical' than realistic [38], it suggests that the simple mechanism of dispersal implies that the growth of the citie's area (for $\theta \geq 1$) is scaling at least as t^2 in agreement with the empirical observations discussed above.

This simple Shigesada-Kawasaki coalescing model is based on the circular approximation and if we drop this assumption, analytical calculations seem out of reach. We can however investigate this problem numerically [37] and assume that the area A and the perimeter L obey to a power law scaling of the form

$$A(t) \sim t^\mu \qquad L(t) \sim t^\nu . \quad (12)$$

In the case of a constant emission rate $\lambda(r) = \lambda_0$, numerical results seem to show that $\mu \approx 2$ and $\nu \approx 1$. The dominant behavior are then $A(t) \sim \pi c^2 t^2$ and $L(t) \sim 2\pi ct$. This shows that for $\theta = 0$ and for large values of t, the circular approximation is valid: the city grows isotropically and absorbs neighboring towns. We can also consider the case where the emission rate λ behaves as

$$\lambda(t) = \lambda_0 L(t) , \quad (13)$$

where $L(t)$ is the total perimeter of the primary colony at time t. This case corresponds to $\theta = 1$ in the model M_0. The simulations results for the area $A(t)$ and the perimeter $L(t)$ of the primary colony suggest that we still have $\mu \approx 2$ and $\nu \approx 1$ as in the M_0 model. We can go further and investigate the prefactors of $A(t)$ and $L(t)$. We recall that for the M_0 model with $\theta = 1$, the radius of the primary colony increases with an effective radial velocity $c' \neq c$. We can then study the quantities $A(t)/\pi c'^2 t^2 - 1$ and $L(t)/2\pi c't - 1$; if the prefactor is the same of the M_0 model we should find (as we did for $\theta = 0$) that these quantities tend to zero for large values of t. In fact we observe that these two quantities tend to a constant that depends on ℓ. These results show that the circular approximation is not appropriate in the case described by

Eq. 13 (see Fig. 7). When the emission rate is propor-

FIG. 7: Example of shapes obtained for the model M_1. (Left) Constant emission rate and $\ell = 10$. (Right) Emission rate proportional to the perimeter (Eq. 13), and $\ell = 2$. Figures taken from [37].

tional to the perimeter, the circular approximation thus breaks down and the roughness of the primary colony can not be discarded, thus modifying the scaling exponents. For cities, we expect that the rate of creation of new towns depends on the economical activity of the city and is at least proportional to its perimeter. These results show then that – independently from the anisotropy of transport networks – cities will in general not grow in an anisotropic way due to the coalescence with neighboring towns.

This model is of course very simplified and we are still far from a quantitative description of cities and their surface area, but we believe that this type of approach could serve as a basis for more elaborate ones and eventually to construct the equation governing the evolution of the area of cities.

III. MODELING THE POPULATION DYNAMICS

The population of cities varies over several order of magnitudes: from small towns with hundreds of inhabitants to megacities with more than 10 million inhabitants. This large disparity of sizes has been noticed for some time already and Zipf uncovered in the 40s the 'universal' behavior of the form (see [39])

$$P(R) \sim R^{-\nu} \qquad (14)$$

where $P(R)$ is the population of the city at rank R (cities are sorted in decreasing order according to their population). Interestingly enough, this result is robust and valid for different periods in time: even if there is a no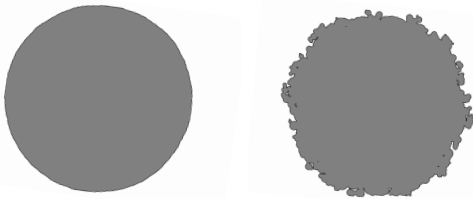n-trivial microdynamics with the rank of cities changing all the time, Zipf's law remains stable [40]. The exponent ν is usually close to 1 for most countries which implies that the population distribution is close to $\rho(P) \sim 1/P^2$ (in general the population size distribution is a power law with exponent $\kappa = 1 + 1/\nu$). More recent empirical evidences seem however to show that there are non-negligible fluctuations of this exponent (see [41] for an extensive study over 73 countries). The Zipf law has

a number of interesting consequences: the ratio of the largest to second-largest city is given by $P_1/P_2 = 2^{1/\nu}$; for $\nu < 1$ the largest city will have a population scaling as $P_{max} \sim N^{1/\nu}$; and the total population of a country scales as $W \sim N^{1/\nu}$ where N is the number of cities.

Zipf's law for cities triggered a huge number of studies in economics and also in physics and we discuss here some of the most important approaches together with an interesting connection with a classical model of statistical physics. First, Gibrat proposed in 1931 a simple rule stating that the growth rate of a firm is independent from its size [42], and applied to the growth of city populations, gives the following equation

$$P_i(t+1) = \eta_i(t) P_i(t) \qquad (15)$$

where $P_i(t)$ is the population of city i at time t (usually the year). This equation basically describes the randomness of births and deaths by introducing an effective random growth rate $\eta_i(t)$ which is assumed to be independent from a city to another and without any time correlations. This simple equation with multiplicative noise naturally leads to a lognormal population distribution in contrast with empirical results, implying that the growth of urban areas is not consistent with Gibrat's law. Several other approaches were then proposed in order to understand Zipf's law [43, 44]. In particular, Gabaix [44] proposed an alternative approach that is so far considered as the best explanation for Zipf's law. It is based on the Gibrat model with the constraint that small cities cannot shrink to zero. In other words, the process considered by Gabaix is a random walk with a lower reflecting barrier which can, in some conditions, produce a power law distribution of populations. In addition to the reflecting barrier, a necessary condition is a drift towards the barrier and it is the combination of these two ingredients that can give rise to a power law [45]. The 'regularization' of the Gibrat model proposed by Gabaix relies on the assumption of a minimum size of cities and predicts the value $\nu = 1$, but cities can actually disappear and this model is not completely satisfying. Ideally, we would like an approach based on reasonable mechanisms, linking micromotives with the large-scale behavior described by Zipf's law. An interesting step towards such a description was proposed by Bouchaud and Mézard [46] (in the context of the wealth distribution) who discussed the diffusion equation with noise. This model has implications far beyond cities such as finance, directed polymers and the KPZ equation, etc. (see [46] and references therein). The diffusion equation with noise for the population $P_i(t)$ of city i reads in the continuous time limit as

$$\frac{dP_i}{dt} = (\eta_i(t) - 1)P_i(t) + \sum_j J_{ij}P_j(t) - J_{ji}P_i(t) \qquad (16)$$

where the first term represents the 'internal' growth as given by Gibrat's law and where the two last terms represent migrations between cities. We note that in contrast with some other models, the ingredients needed here are

reasonable and correspond to a process that does occur in the real world. The random variables $\eta_i(t)$ are assumed to be identically independent Gaussian variables with the same mean and a variance given by $2\sigma^2$. The flow (per unit time) from city i to j is denoted by J_{ji} and for a general form of these couplings we are unable to solve this equation. In the mean-field limit where all cities are exchanging individuals with each other ($J_{ij} = J/N$ where N is the number of cities), Bouchaud and Mézard could show that the stationary distribution of the normalized population $w_i = P_i/\overline{P}$ (where $\overline{P}$ is the average population) is

$$\rho(w) \sim w^{-(1+\kappa)}e^{-(\kappa-1)/w} \qquad (17)$$

where the exponent is

$$\kappa = 1 + \frac{J}{\sigma^2} \qquad (18)$$

When the migration term J is nonzero, this regularization changes the lognormal distribution to a power law for large w and the exponent is between 1 and 2 (for $J/\sigma^2 < 1$). The exponent κ in this approach is not universal and depends on the details of the system providing a possible explanation for the diversity of values observed empirically [41]. This diffusion model suggests an interesting connection between a central model in statistical physics and the old problem of the urban population distribution. It also shows that Zipf's law finds its origin in the interplay between internal random growth and exchanges between different cities. An important consequence of this result is that increasing inter-urban mobility should actually increase κ and therefore reduces the heterogeneity of the city size distribution. Other empirical tests are needed at this point, and from a theoretical point of view the mean-field assumption is not obvious. We expect that in general the couplings J_{ij} are not constant and depend on the distance between cities, which could dramatically alter the results.

IV. SPATIAL ORGANIZATION OF CITIES

In this section, we will focus in the intra-urban scale. In particular, we will discuss approaches for understanding the spatial organization of a city. It is a problem of crucial importance as the location of residences and of the economic activity govern commuting flows and mobility patterns, a vital ingredient for assessing the efficiency of infrastructures and for planning. It is thus important to understand how households and companies choose a certain location and what are the main driving factors. We will first discuss here the main model used in urban economics and upon which many variants are constructed. We will then discuss another approach proposed by Krugman and which is a good example of a non-equilibrium approach to the city structure. Finally we will discuss a more recent model proposed in a statistical physics spirit.

A. The Alonso-Muth-Mills model of urban economics

The Alonso-Muth-Mills (AMM) model is a pillar of urban economics and constitute the basis on which most economic models are constructed. In this respect we believe that it is important to know this model and we describe its main lines here. The first ingredient in this model is a utility function U that describes the preferences of individuals (or households – they are usually considered to be the same in this simple approach) that are considered to be all equivalent. This function U is usually assumed to depend on the land consumption s (which corresponds to the surface area of apartments) and on the composite commodity z (which corresponds to the money left when rent and transportation costs are substracted from the income)

$$U = U(z, s) \qquad (19)$$

This utility has to satisfy the general constraints

$$\frac{\partial U}{\partial s} > 0 \ , \ \ \frac{\partial U}{\partial z} > 0 \qquad (20)$$

which means that households in general prefer larger apartment and smaller costs. The budget constraint is given by

$$Y = z + T(x) + R(x)s \qquad (21)$$

where Y is the income of an household, $T(x)$ the transportation cost to work when living at location x and $R(x)$ the renting cost per unit area at x. In this simplest version of the AMM model all households are renting their apartments and all landlords are living out of the city. The problem is then to optimize the utility subject to this budget constraint

$$\max_{z,s} U(z, s) \ \text{subject to} \ Y = z + T(x) + R(x)s \qquad (22)$$

This is a classical problem which can be solved with Lagrange multipliers, but we will here discuss a faster way to obtain general results. We introduce the constraint with $z = Y - sR - T$ and we maximize $U(Y - R(x)s - T(x), s)$ with respect to s

$$\frac{\partial U}{\partial s} = 0 = \partial_1 U(-R(x)) + \partial_2 U \qquad (23)$$

where $\partial_i U$ denotes the derivative of U with respect to the ith variable. From this equation, we obtain the renting cost under the form

$$R(x) = \frac{\partial_2 U}{\partial_1 U} \qquad (24)$$

An additional requirement is that the maximum utility U^* should be independent from x. If it is not, then individuals could choose another better location and we

wouldn't be at equilibrium. We thus have to write

$$\frac{dU^*}{dx} = 0 = \partial_1 U \left(-s\frac{dR}{dx} - R\frac{ds}{dx} - T'(x) \right) + \partial_2 U \frac{\partial s}{\partial x} \tag{25}$$

where the functions $s(x)$ and $R(x)$ are computed at equilibrium. Combining Eqs. (24) and (25) which are valid for all x, we then obtain the central result for the AMM model (see for example [47])

$$\frac{dR}{dx} = -\frac{T'(x)}{s(x)} \tag{26}$$

This relation Eq. (26) allows us to discuss the location of individuals in the city. For example, for discussing the impact of income, we assume that transportation cost are linear in x and then $T'(x) = t$ and that we have two income categories, rich and poor characterized by their (fixed) land consumption s_R and s_P, and transportation costs t_R and t_P. The category of individuals that lives in a given area of the city is then the one that is willing to pay more for the rent at this location. The condition for the poor living in teh center can be shown to be

$$\frac{t_P}{s_P} > \frac{t_R}{s_R} \tag{27}$$

In the opposite case, rich individuals will live in the center as they are willing to pay more than the poor for this location (see [48] for a detailed discussion about this point in the context of the AMM model).

We refer the interesting reader to the large litterature on the subject and in particular to the books [3] and [6], and we will just make a few remarks about this approach. First, it assumes that cities are in equilibrium and their structure optimizes some objective function. Given the large variety of temporal (and spatial) scales, of processes and interactions, this assumption seems difficult to accept for cities. Even if we accept it, we are left with the difficult problem that some features of the city (such as the population profile for example) will actually depend on the precise form of the utility function. This poses the problem of the choice of the utility function and how to test it empirically. Also for this model, or for the more involved Fujita-Ogawa model (see [5] and next sections), the theoretical predictions are usually not thoroughly tested against data. This is often true for classical studies of urban economics: theories and assumptions are not tested against data and serve as conceptual guides to understand some phenomena but usually with no clear guarantee of their validity.

B. Krugman's model: The Edge-City model

A non-equilibrium model for the spatial structure of cities and in particular how the economic activity clusters in specific regions was proposed by Krugman [49]. This approach is a good example of a minimal model that could probably be built upon and amenable to predictions that can be tested. The most important aspect here is the presence of interactions between firms. These interactions can lead to a polycentric organization of the city in which businesses are concentrated in spatially separated clusters. We follow the discussion proposed by [49] and consider that the city is one-dimensional and the density of businesses is described by the function $\rho(x)$ whose integral is assumed to be constant. We assume that all locations are initially equivalent (which means in particular that there is relatively uniform transport infrastructure network) and the attractiveness of a location will depend on the spatial distribution of businesses. In order to describe this mathematically, Krugman introduces a quantity $\Pi(x)$ 'a market potential' which describes the level of attractivity of location x and is given by

$$\Pi(x) = \int K(x - z)\rho(z)\mathrm{d}z \tag{28}$$

where the kernel is chosen as $K(x) = A(x) - B(x)$ with functions A and B that are both decreasing with the distance. These functions represent the positive and negative spillovers and how they vary with distance. Businesses will have an incentive to come to certain location depending on the level of attractiveness Π and the simplest assumption is to compare $\Pi(x)$ to the average spatial level $\overline{\Pi} = \int \Pi(x)\rho(x)\mathrm{d}x$ and to write

$$\frac{d\rho(x)}{dt} = \gamma \left[\Pi(x) - \overline{\Pi} \right] \tag{29}$$

The density will then increase at locations where $\Pi(x) > \overline{\Pi}$ and decrease otherwise. This equation describes in a simple way the evolution of the business density with time and provide an explanation for the self-organized nature of cities. Note that since $\Pi(x)$ depends on ρ, the quantity $\overline{\Pi} = \int \Pi(x)\rho(x)\mathrm{d}x$ is nonlinear in ρ. Numerical simulations indicate that this nonlinear system indeed leads to various situations with multiple centers at different locations, depending on the initial conditions. Essentially, if positive spillovers are larger we will observe bigger clusters of businesses. We also see that this concentration has a reinforcement effect: regions with a large market potential will be more attractive and will therefore grow.

At a more quantitative level, we denote by λ the size of spatial fluctuations and by r_1, r_2 the range of positive and negative spillovers, respectively. For large frequencies, $\lambda \ll r_1, r_2$ there is basically a compensation effect of positive and negative spillover and we do not expect the growth rate to be large. Also, for very low frequency fluctuations such as for example around a maximum of $\rho(x)$ which decays slowly around x, and for strong enough negative spillover, the growth rate at x will be negative. Both low and high frequencies have thus negligible growth rate and it is natural to expect a frequency with a maximal growth rate, well-tuned to the spatial decay of positive and negative spillovers, leading to a specific spatial pattern in this city. In order to get a quick analytical

insight, we can linearize the equation for $\rho(x)$ around the flat city $\rho(x) = \rho_0 + \delta\rho(x)$ and find

$$\frac{d\delta\rho}{dt} \simeq \gamma \int K(x-z)\delta\rho(z)\mathrm{d}z + \mathcal{O}(\delta\rho^2) \qquad (30)$$

where we choose the normalization $\int \rho(x) = 1$. Using the Fourier transform

$$\delta\rho(k) = \int \mathrm{e}^{ikx}\delta\rho(x)\mathrm{d}x \qquad (31)$$

we obtain by integrating the linear differential equation

$$\delta\rho(k) \sim \mathrm{e}^{\gamma\hat{K}(k)} \qquad (32)$$

where $\hat{K}(k)$ is the Fourier transform of the kernel $K(x)$. This expression shows that the Fourier mode k^* for which $\hat{K}(k)$ is maximum will develop faster and will lead to the appearance of a spatial pattern characterized by k^*. We can obtain explicit expressions with the choice $K(x) = A\mathrm{e}^{-|x|/r_1} - B\mathrm{e}^{-|x|/r_2}$ which leads to

$$\frac{d\delta\rho(k,t)}{dt} = 2\gamma\left[\frac{Ar_1}{1+(r_1k)^2} - \frac{Br_2}{1+(r_2k)^2}\right]\delta\rho(k,t) \quad (33)$$

We thus have for each mode a growth rate proportional to

$$\Lambda(k) = \frac{Ar_1}{1+(r_1k)^2} - \frac{Br_2}{1+(r_2k)^2} \qquad (34)$$

which satisfies $\Lambda(0) = A - B$ which is negative for large enough B and $\Lambda(k) \to 0$ for k large. We thus expect in general a maximum Λ^* for a value k^* that can be easily computed. This value k^* depends here on r_1, r_2, A and B and is thus finite and independent from the city size. For a one dimensional city of size L, the number of business clsuters is then given by

$$H \sim Lk^* \qquad (35)$$

This simple model thus predicts a linear increase of the number of activity centers (or 'hostpots') with city size, but doesn't explain in particular how this quantity scales with population. Indeed, new datasources such as cell phone networks, employment data, smart card transactions and taxi GPS trajectories provide a lot of information about cities and their structure. In particular, it has been shown [50, 51] that the number H of activity centers (or 'hotspots') is scaling with population as

$$H \sim P^\sigma \qquad (36)$$

where the exponent σ is found to be around $0.5 - 0.6$. The number of these hotspots thus scales sublinearly with the population size, a result that will serve as a guide for constructing a theoretical model. We thus see that Krugman's model is not able to explain this behavior. Although it is interesting to see how simple nonlinear effects give rise to a nontrivial spatial pattern, this model doesn't produce at this stage predictions that are directly testable on empirical data.

C. A variant of Fujita-Ogawa

The Krugman model discussed above is unable to explain the empirical behavior Eq. (36). We will show here how a simplified variant of the Fujita-Ogawa model [5] can actually help us to understand this empirical result. We start from the standard economical assumption that an invidivual will choose a residence located at x and to work at location y such that the quantity

$$Z(x,y) = W(y) - C_R(x) - T(x,y) \qquad (37)$$

is maximum. The quantity $W(y)$ is the typical wage earned at location y, $C_R(x)$ is the rent cost at x, and $T(x,y)$ is the transportation cost to go from x to y and is usually taken to be proportional to the time $\tau(x,y)$ spent to cover this distance. In their original paper, Fujita and Ogawa [5] couldn't find the general solution, but tested the stability of some specific urban forms. For example, by neglecting congestion and writing $\tau(x,y) = d(x,y)/\overline{v}$ (where $d(x,y)$ is the euclidean distance between x and y and $\overline{v}$ is the free flow average velocity), they could show that if the transportation cost for the typical interaction distance between companies becomes too large, the monocentric organization with a central business district surrounded by residential areas is unstable. However, this formalism does not allow to predict the resulting urban structure, and we have to simplify it in order to reach testable predictions. First, we assume that each agent has a residence located at random. Second, a quantity as complex as the wage results from a large number of interactions and factors, and it is tempting – in the spirit of random Hamiltonians for heavy ions [52] – to replace this complex quantity by a random number $\phi(y)$: $W(y) = s\phi(y)$ where s sets the salary scale (and with a certain distribution of ϕ which is irrelevant at this point). Last but not least, we assume that most of the displacements are made by car and we include congestion effect which implies that the time $\tau(x,y)$ depends on the traffic $Q(x,y)$ as described by the simple Bureau of Public Roads function (see for example [53])

$$\tau(x,y) = \frac{d(x,y)}{\overline{v}}\left[1 + \left(\frac{Q(x,y)}{C}\right)^\mu\right] \qquad (38)$$

where C is the road system capacity, and μ an exponent, usually between 2 and 5. These ingredients put together allow for a simple mean-field analysis showing that the monocentric organization is unstable for a value of the population larger than a threshold P^* (which depends on the details of the city) and that the number H of distinct activity centers is given by

$$H \sim \left(\frac{P}{P^*}\right)^{\frac{\mu}{\mu+1}} \qquad (39)$$

(see Fig. 8). This simplified model thus predicts a sublinear behavior for the number of activity centers with an exponent given by $\sigma = \mu/(\mu+1)$ (for $\mu \approx 2$, we then recover the empirical value $\sigma \approx 0.6$, and for $\mu = 0$ we have

$H \sim \mathcal{O}(1)$ in the absence of congestion). We observe that whatever the value of μ, the behavior is sublinear, and congestion appears as a critical factor that shapes the structure of the city and favors the appearance of new activity centers.
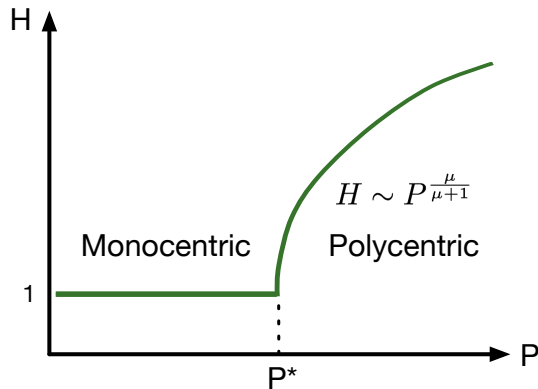

FIG. 8: Number of activity centers versus the population for the model discussed in the text: we observe a monocentric-polycentric transition for a value P^* that depends on the city.

Finally, we note that knowing both the residence and workplace locations allows to discuss multiple aspects of the commuting to work [54]. In particular, we can estimate the total commuting time and the quantity of CO_2 emitted by cars (see also [55]).

V. AVERAGING OVER MANY CITIES ?

Various studies of cities such as scaling [29] use data from different cities at different times and plot some quantity versus population. For many quantities (denoted Y here), we observe a power law of the form $Y \sim P^\beta$ where P is the population of the city [29]. As we already discussed above, we can distinguish different behavior according to the value of β, in particular if it is larger than one or not. Once we have measured this scaling form, we could in principle use it for predicting the behavior of an individual city when its population changes. In order to illustrate the possible problems of such an approach we consider the particular case of delays due to traffic congestion and analyze a dataset for 101 US cities in the time range 1982-2014 [56]. This is a particularly interesting dataset as it is both transversal – it contains many cities – and longitudinal – for each city we have the temporal evolution of the delay.

The scaling form obtained by agglomerating all the available data for different cities and for different years displays a nonlinear behavior, seemingly in agreement with general empirical results about scaling [29]. More precisely, we can first compute the scaling exponent by mixing together all cities but for a given year. We can also compute this exponent for each year and we observe an exponent whose value varies in the range [1.2, 1.4] for
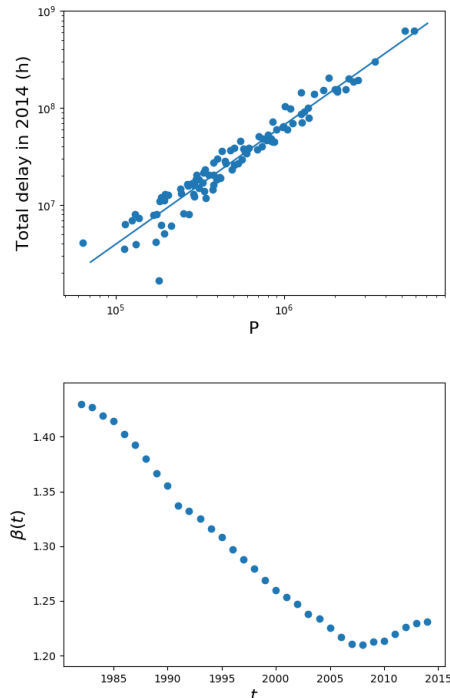

FIG. 9: (Top) Plot of the annual delay $\delta\tau$ versus the number of drivers P for all cities in 2014. The straight line is a power law fit in this loglog representation and gives an exponent value $\beta \approx 1.23$ (and $R^2 = 0.93$). (Bottom) Scaling exponent $\beta(t)$ for the delay computed for each year separately, from 1982 to 2014. All these values are consistent with a superlinear behavior found in [57]. Figure taken from [56].

years from 1982 to 2014 (see Fig. 9). Finally, we can consider all cities and for all years and we then obain a delay in traffic jams scaling as

$$\delta\tau \sim P^\beta \tag{40}$$

with $\beta \approx 1.36$ consistent with a superlinear behavior.

However, if we don't average over the different cities, we observe the behaviors shown by different colors in the Fig. 10. The different cities display then a variety of behaviors ranging from pure power laws with exponents larger than one, or two power laws, etc. (see [56] for details). The scaling obtained by averaging over all cities appears then completely unrelated to the dynamics of individual cities when their population grow. There seems to be no simple scaling at the individual city level but a variety of behaviors. In the language of statistical physics, the delay is not a state function determined by the population only, and displays some sort of aging effect where it depends not on the population only but also on time, and probably on the whole history of the city. This idea of path-dependency is natural for many complex systems, and in particular in statistical physics (such as spin-glasses [58] for example). The delay is not a simple function of population as it is usually assumed
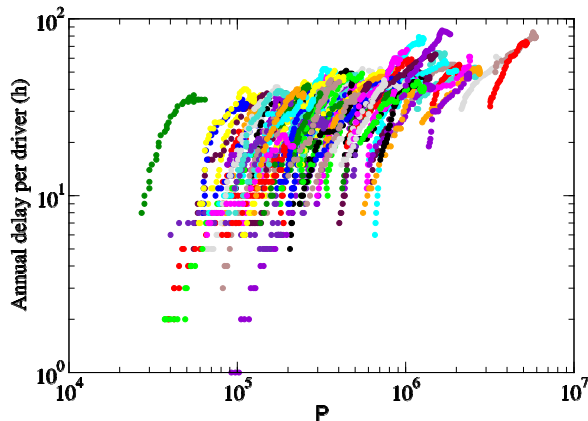
temporal series for individual cities. This is a fundamental problem that needs to be clarified when looking for generic properties of cities.

FIG. 10: Annual delay per capita $\delta\tau/P$ versus P for all the 101 cities and for all years (1982 – 2014). The points are colored according to the city they describe (one color per city). As we discuss in the text there is no obvious relation between the global power-law scaling and the individual behavior of cities. Figure taken from [56].

for the scaling approach for cities, and if we reflect upon this idea, it doesnt make sense in general to compare two cities having the same population but at very different dates: both central Paris and Phoenix (AZ) had a population of about 1 million inhabitants, the former in 1840 and the latter in 1990. It is very likely that the dynamics for many quantities from 1840 in Paris will be very different from the one starting in 1990 in Phoenix, implying that the usual scaling form does not apply in general. This discussion on congestion induced delays highlights the risk of agglomerating data for different cities and to consider that cities are scaled-up versions of each other: there are strong constraints for being allowed to do that such as path-independence, which is apparently not satisfied in the case of congestion and which should be checked in each case. Beyond scaling, these results also pose the challenging problem of using transversal data (ie. for different cities) in order to get some information about the

VI. PERSPECTIVES

Many aspects and studies about cities were not addressed in this paper: the evolution of infrastructure networks [59, 60], the coupling between networks [61], multimodality [62, 63], heat islands and urban forms [64], etc. An important point here was to show through various examples how a combination of empirical results, economical ingredients, and statistical physics tools can lead to parsimonious models with predictions in agreement with observations. In particular, describing individual actions by stochastic processes and replacing complex quantities resulting from the interactions of several agents by random variables, seem to be a good way to construct models for understanding the evolution of cities and in agreement with empirical observations.

We however still have to think about the best approach for understanding complex systems such as cities. Is it possible to construct a generic model of cities that can predict general trends and behaviors ? In this perspective, a particular city would then just be described by this generic model subject to local spatial and historical constraints. In other words, this assumption means that all cities belong to the same species but each individual city evolved on a different substrate. As we saw in this short review, this possibility and the existence of generic behaviors is not empirically clear. We do observe generic properties such as the Zipf law and mixing together different cities doesn't seem to pose a problem in this case, in contrast with the exemple of congestion induced delays. There is however the risk that this debate becomes quickly outdated as machine learning displays impressive results in practical applications. At least, we could always hope that parsimonious models have the advantage to provide a simple language for making sense of the vast amount of data and to identify critical factors for the evolution of these systems.

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