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Probabilistic response of an elastic perfectly plastic oscillator under Gaussian white noise

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Abstract

The response of an elastic perfectly plastic oscillator under zero mean Gaussian white noise excitation is studied in this paper. Considering the works of previous studies, a closed form expression of the mean maximum of the plastic drift is given assuming that the plastic process is equivalent to a Brownian motion. In order to better describe the plastic drift a probabilistic model is proposed for the yield increments which occur in clumps. To estimate the input parameters of this model, three methods, based on numerical computations of some relevant integrals, are presented. Alternatively, these parameters can be estimated, more conveniently, according to the results obtained more recently in the literature with the Slepian model approach. The results of numerical simulations show a quite satisfactory agreement with theoretical predictions.
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Keywords: Nonlinear oscillator; White noise; Brownian motion; Reliability function; Slepian model

1. Introduction

One of the major concerns of a seismic design is to estimate the ductility demand of structures subject to earthquake excitations. Thus, the question of the value of the mean maximum of the nonlinear response under random excitation arises. The purpose of this paper is to contribute to the answer of this question by considering the study of the behaviour of an elastic perfectly plastic single degree of freedom (SDOF) system under zero mean Gaussian white noise excitation. This is the simplest structural model exhibiting hysteretic behaviour and many physical systems can be approximated in this way. In consequence, some research effort has been devoted for over forty years to this problem. Caughey [1] was the first to obtain the stationary response statistics of this type of system to Gaussian white noise excitation, by using an equivalent linearization technique. The same technique has been utilised in several works (e.g. [2–6]), but the major drawback of the equivalent linearization technique is that, generally, it gives satisfactory results on very limited quantities of the response statistics such as the mean-square values. Therefore, alternative methods have been developed. A complete review of these methods can be found in references [7–9].

The present paper is focused on the original model introduced by Karnopp and Scharton [10] and developed by Vanmarcke and Veneziano [11] and more recently by Ditlevsen and Bognar [12]. The main idea proposed by Karnopp and Scharton [10] was to study the plastic-part (or plastic process) of the nonlinear response, knowing that between two plastic excursions the nonlinear system has a linear behaviour. They formulated the problem in terms of "idealized" yield increments which are estimated by assuming that the excess energy beyond the yield limit is completely dissipated by the yielding action. On the same bases Vanmarcke and Veneziano [11] developed an analytical model which gives the estimates of the first and the second moment of the plastic process, in the case of "rare" yielding events. More recently, based on the theory of Slepian processes, which describes the transient behaviour of Gaussian random processes, Ditlevsen and Bognar [12] proposed very satisfactory approximations of the yield increment statistics.

On the basis of the analytical model of Vanmarcke and Veneziano [11], and following an idea of Paparizos and Iwan [13] and Borsoi and Labbe [14] according to which the plastic process can be viewed as a Brownian motion, closed form expressions of the probability density function (PDF) of

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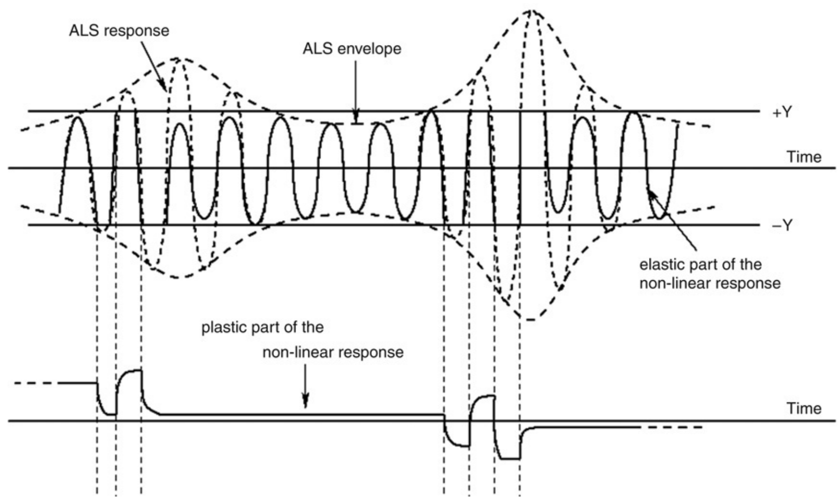

Fig. 1. Nonlinear response versus ALS response.

the absolute maximum and of the mean maximum of the plastic process are given. The corresponding mean displacement ductility demand of the nonlinear response is then proposed. These expressions require the knowledge of the variance of the clumps of plastic jumps. A closed form expression of this variance involving five parameters is then proposed by means of a probabilistic model which describes the local behaviour of the plastic excursions. Combining the previous studies found in the literature, three methods, based on the numerical computations of some relevant integrals, are briefly presented to estimate the above parameters. Nevertheless, for practical purposes, it is proposed to use the closed form results obtained by Ditlevsen and Bognar [12] with the Slepian model approach. These results are briefly summarised for completeness.

In comparison with the methods presented in the beginning, the advantage of the proposed approach results in the usefulness of the information given on the nonlinear response statistics, in a seismic engineering point of view.

As already mentioned, a nonlinear SDOF system is considered here. Its equation of motion, under zero mean Gaussian white noise $W(t)$, reads:

$$\ddot{X}_{nl} + 2\beta\omega_0\dot{X}_{nl} + F(X_{nl}) = -W, \tag{1}$$

where $X_{nl}(t)$ is the displacement of the mass, β is the damping ratio, $\omega_0 = 2\pi f_0$ is the circular frequency and $F(X_{nl})$ is the spring force which is bounded by $F(X_{nl}) = \pm\omega_0^2 Y$ if Y is the yield limit. If $D(t)$ denotes the plastic process (or plastic deformation) and $X(t)$ the elastic-part of the nonlinear response:

$$X_{nl}(t) = D(t) + X(t). \tag{2}$$

The values of $X(t)$ cannot exceed $\pm Y$. As in the cases discussed in the previous studies [10–12], this work is limited to configurations for which plastic jumps are "rare" events (high yield thresholds). Therefore, the consequences of yielding are limited to a few cycles of the response, so that $X(t)$ tends, between two successive clumps, towards the response of an associated linear system (ALS) whose equation of motion is:

$$\ddot{\tilde{X}} + 2\beta\omega_0\dot{\tilde{X}} + \omega_0^2\tilde{X} = -W. \tag{3}$$

2. Global characterization of the plastic process

2.1. Phenomenology

Fig. 1 illustrates the partitioning of the nonlinear response into elastic- and plastic-parts. The response of the ALS is also shown. It is generally observed that, for high yield thresholds, each clump of plastic jumps is associated with a yield threshold crossing by the ALS envelope with positive slope.

Thus, according to Vanmarcke and Veneziano [11], the expected frequency of occurrence of the clumps can be estimated by the mean frequency μ_Y of the threshold crossings by the ALS envelope. To study the statistical behaviour of the plastic process the authors consider that each clump may be replaced by the sum of its individual jumps and that the duration of each clump is infinitely small. The plastic process $D(t)$ is then simplified in a sum of individual independent idealized plastic jumps whose amplitudes d_i and number $N(t)$ are random variables during the time interval $[0, t]$:

$$D(t) = \sum_{i}^{N(t)} d_i. \tag{4}$$

$N(t)$ is assumed to have a Poisson distribution with a mean value equal to $\mu_Y t$ and d_i are zero mean random variables. Moreover, if the number of plastic jumps is relatively important ($t \gg 1/\mu_Y$), it can be argued, by virtue of the central limit theorem, that $D(t)$ has a Gaussian distribution whose expected value and variance are [11]:

$$E(D(t)) = \sum_{i}^{E[N(t)]=\mu_Y t} E(d_i) = \mu_Y t E(d_i) = 0 \tag{5}$$

$$\sigma^2_{D(t)} = \sum_{i}^{E[N(t)]=\mu_Y t} V(d_i) = \mu_Y t E(d_i^2). \tag{6}$$

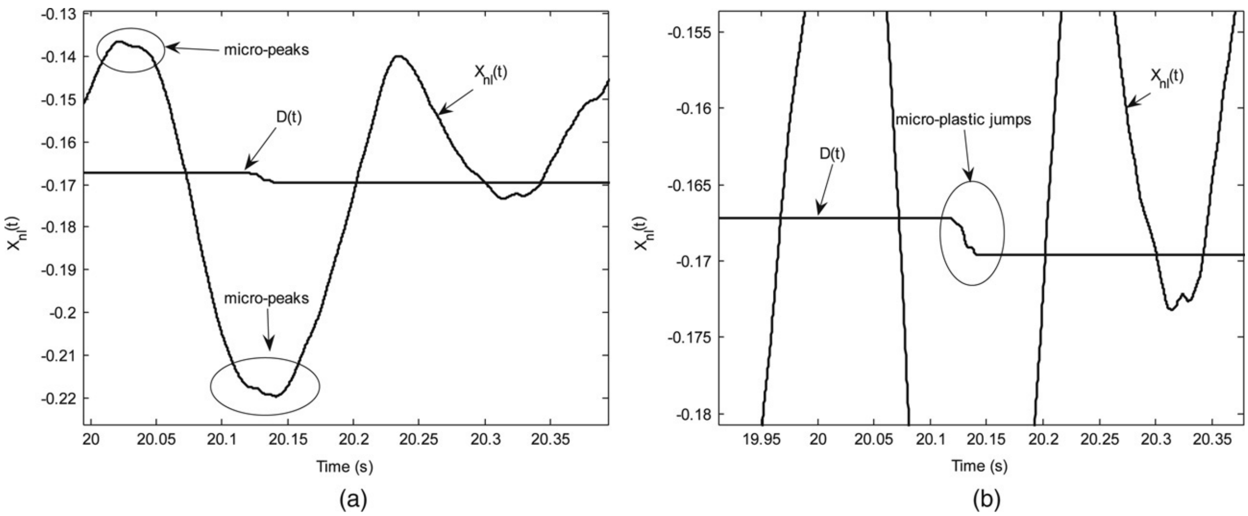

Fig. 2. Micro-peaks (a) and micro-plastic jumps (b).

Remark. We notice that under stationary Gaussian excitation, the average rate of peaks of a lightly damped linear oscillator (like the ALS) response increases without limit with increasing the excitation bandwidth [15]. Crandall showed that the peaks of the linear response occur in clusters of microscopic peaks associated with each macroscopic peak, as shown in Fig. 2(a). When the excitation bandwidth is such that the fourth spectral moment of the linear response becomes infinite, Crandall [15] and Ditlevsen and Bognar [12] showed that the PDF of the peaks is Gaussian. The nonlinear counterpart of this phenomenon is that each macro-plastic jump can be associated with several micro-plastic jumps as shown in Fig. 2(b).

In this study the sum of the microscopic and macroscopic plastic excursions is considered and the microscopic plastic jumps are neglected. Therefore, the peaks of the ALS response are assumed to have a Rayleigh distribution under Gaussian white noise excitation.

2.2. *Probability density function of the absolute maximum of $D(t)$*

The determination of the absolute maximum PDF of the plastic process can be achieved through the reliability function defined by:

$$W(b, T) = P(|D(t)| < b) = P((D(t) < b) \quad \text{and} \quad (D(t) > -b)), \tag{7}$$

where $W(b, T)$ is the probability that the process $|D(t)|$ remains below a given threshold b throughout the time interval $[0, T]$. The mean maximum $\overline{D}_m$ of $D(t)$ in $[0, T]$ reads:

$$\overline{D}_m = \int_0^{+\infty} b \underbrace{\frac{\partial W(b, T)}{\partial b}}_{\text{PDF of the absolute maximum in [0,T]}} \mathrm{d}b. \tag{8}$$

In order to determine $W(b, T)$, the statistics of the plastic process are assumed to be the same as those of a Brownian motion (see [13,14]). A complete study of this type of process can be found, in particular, in [17]. Since the two events $(D(t) < b)$ and $(D(t) > -b)$ are independent, $W(b, T)$ is given by:

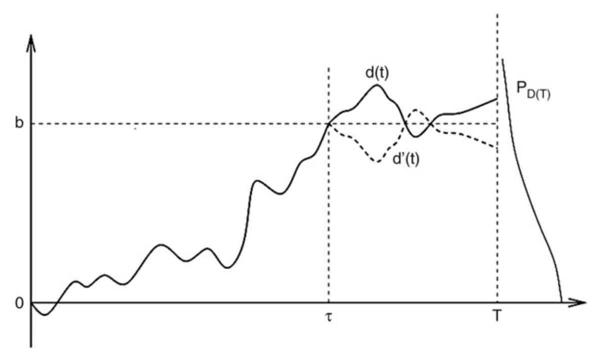

Fig. 3. Example of equiprobable time-histories leading to failure.

$$W(b, T) = (1 - P_f(b, T))^2. \tag{9}$$

$P_f(b, T)$ being the first crossing probability of level b. As shown in Fig. 3, to evaluate $P_f(b, T)$ we consider that each time-history which has an absolute maximum greater than or equal to the level b in the time interval $[0, T]$, leads to a necessary threshold crossing before time T. Thus:

$$P_f(b, T) = P(\max_{0 \le t \le T} D(t) \ge b) \tag{10}$$

is computed by using the "reflection principle" defined in [17]. Let us consider a sample function $d(t)$ of the process $D(t)$. If τ designates the moment of the first crossing of level b another sample function exists defined by (Fig. 3):

$$d'(t) = \begin{cases} d(t) & t \le \tau \\ b - [d(t) - b] & t > \tau. \end{cases} \tag{11}$$

The "reflection principle" is based on the independence between past and future. It reflects the concept of equiprobability of the two time-histories $d(t)$ and $d'(t)$ which both reach the level b at time τ, so that:

$$\max_{0 \le t \le T} d(t) \ge b \quad \text{and} \quad \max_{0 \le t \le T} d'(t) \ge b. \tag{12}$$

The probability of first crossing of level b is then given by [17]:

$$P_f(b,T) = P(\max_{0\le t\le T} D(t) \ge b) = 2P(D(T) \ge b) \quad (13)$$

and the reliability function reads:

$$W(b,T) = (1 - 2P(D(T) \ge b))^2. \quad (14)$$

As $D(t)$ has a Gaussian distribution, for $t \gg 1/\mu_Y$, $P(D(T) \ge b)$ is given by:

$$P(D(T) \ge b) = 1 - \Phi\left(\frac{b}{\sigma_{D(T)}}\right). \quad (15)$$

$\Phi(x)$ being the Gaussian (0, 1) cumulative distribution function and $\sigma_{D(T)}$ the standard deviation of $D(t)$ at time T (cf. Eq. (6)).

From Eq. (14), the expression of the PDF of the absolute maximum in $[0, T]$ reads:

$$\frac{\partial W(b,T)}{\partial b} = \frac{8}{\sigma_{D(T)}}\varphi\left(\frac{b}{\sigma_{D(T)}}\right)\left[\Phi\left(\frac{b}{\sigma_{D(T)}}\right) - \frac{1}{2}\right] \quad (16)$$

with $\varphi(x) = \mathrm{d}\Phi(x)/\mathrm{d}x$. Combining Eqs. (8), (16) and (6) the mean maximum of the plastic process is given by:

$$\overline{D}_m = 2\sqrt{\frac{\sigma^2_{D(T)}}{\pi}} = 2\sqrt{\frac{E(d_i^2)\mu_Y T}{\pi}}. \quad (17)$$

Consequently, if μ is the displacement ductility demand, which is defined by the ratio of the maximum nonlinear displacement to the yield level, its mean value can be estimated by:

$$\bar{\mu} = 1 + \frac{\overline{D}_m}{Y}. \quad (18)$$

3. Local characterization of the plastic process

Eq. (16) shows that the PDF of the absolute maximum in $[0, T]$ can be determined if the variance of the idealized plastic jumps is known. In the following, a model which enables an estimate of this variance is presented. This model is based on a detailed description of the clumps of plastic jumps.

3.1. Phenomenology

Once more, we point out that this study is limited to white noise excitation and to the cases for which plastic jumps are "rare" events (see Fig. 1). Consequently:

- the clumps of plastic jumps are independent,
- the displacement and the velocity at the onset of the first plastic jump of a clump may be considered as being those of the stationary ALS response,
- the amplitudes of the subsequent plastic jumps are not correlated. They come from a transient phase whose initial conditions are zero relative velocity and elastic-part displacement equal to $\pm Y$ (see [10] and Fig. 1).

Thus, a clump can be considered as the sum of independent plastic jumps with alternating sign and which have the same probability of occurrence. Moreover, the PDF of the first plastic jump in a clump is different from the one of the following jumps.

3.2. Mathematical model of the clumps

A clump can be described by a random variable d which is defined as the sum of N_c independent plastic jumps δ_i with opposite signs, N_c being a random variable. Two auxiliary random variables can be considered, $d^- = \sum_{i=1}^{N_c}(-1)^i\delta_i$ and $d^+ = \sum_{i=1}^{N_c}(-1)^{i+1}\delta_i$, which have the same probability of occurrence equal to 1/2. The variables d^- and d^+ have the same absolute mean value but with opposite signs, so that d is a zero mean random variable with the following variance (or mean-square):

$$V(d) = E[d^2] = \frac{1}{2}E[d^{-2}] + \frac{1}{2}E[d^{+2}] = E[d^{-2}]. \quad (19)$$

In the following, p is called the probability of occurrence of a jump δ_i in a clump. By definition, the probability of the first jump δ_1 of a clump is equal to 1. Thus:

$$\begin{aligned} &P(1\text{ jump}) = (1-p)\\ &P(2\text{ jumps}) = p(1-p)\\ &\vdots\\ &P(n\text{ jumps}) = p^{n-1}(1-p). \end{aligned} \quad (20)$$

The expected value of the clump size N_c is therefore given by:

$$\begin{aligned} E[N_c] &= (1-p)\sum_{n=1}^{\infty} np^{n-1}\\ &= (1-p)\frac{1}{(1-p)^2} = \frac{1}{(1-p)}. \end{aligned} \quad (21)$$

In the following, we do not make any assumption about the PDFs of the plastic jumps and:

- α_1 and α_2 are called the mean value and the mean-square of the first jump δ_1 respectively,
- β_1 and β_2 are called the mean value and the mean-square of the following jumps δ_i respectively.

The mean-square of d^- is defined by:

$$\begin{aligned} E[d^{-2}] &= E\left[\left(\sum_{i=1}^{N_c}(-1)^i\delta_i\right)^2\right]\\ &= E\left[\sum_{i=1}^{N_c}\delta_i^2\right] + 2E\left[\sum_{i=1}^{N_c-1}\sum_{j=i+1}^{N_c}(-1)^{i+j}\delta_i\delta_j\right]. \end{aligned} \quad (22)$$

After some algebra, this mean-square is given by:

$$\begin{aligned} E[d^{-2}] = \alpha_2 + \frac{p}{(1-p)}\beta_2 - 2\frac{p}{(1+p)}\alpha_1\beta_1\\ - 2\frac{p^2}{(1-p^2)}\beta_1^2 \end{aligned} \quad (23)$$

while the mean reads:

$$E[d^-] = -E[d^+] = E\left[\sum_{i=1}^{N_c} (-1)^i \delta_i\right] = -\alpha_1 + \frac{p}{(1+p)}\beta_1. \quad (24)$$

Remark. If the PDFs of the plastic jumps are exponential ($\alpha_2 = 2\alpha_1^2$ and $\beta_2 = 2\beta_1^2$) Eq. (23) reads (see also [14]):

$$E[d^{-2}] = 2\left(\alpha_1^2 + \frac{p}{(1-p^2)}\beta_1^2 - \frac{p}{(1+p)}\alpha_1\beta_1\right). \quad (25)$$

This probabilistic model shows that estimates of five parameters are necessary to determine the mean displacement ductility demand of the response of an elastic perfectly plastic SDOF system under zero mean Gaussian white noise excitation.

4. Estimates of the parameters

Many approaches proposed in the literature are based on the fact that between two plastic excursions, the nonlinear system behaves like a linear oscillator, for which the mathematical theory is well-established. Following these considerations, to obtain a simple estimate of a plastic jump δ_i, Karnopp and Scharton [10] assume that the kinetic energy at the threshold crossing ($\dot{x}^2/2$) is dissipated only by the yielding action into the plastic work ($\omega_0^2 Y \delta_i$), neglecting the work of the excitation force and of the damping force. An estimate of δ_i is then given by

$$\delta_i = \frac{\dot{x}^2}{2\omega_0^2 Y} \quad (26)$$

and an estimate of the distribution of δ_i can be obtained by means of the distribution of the linear system velocity at the threshold.

Alternatively, if e is a peak greater than the yield level, it can be assumed that the excess of potential energy ($\omega_0^2(e^2 - Y^2)/2$) is dissipated into the plastic work. Another estimate of δ_i is then given by:

$$\delta_i = \frac{1}{2Y}\left(e^2 - Y^2\right) \quad (27)$$

and another estimate of the PDF of δ_i can be obtained by means of the distribution of the peaks of the ALS response.

Four methods, inspired by the literature and proposed in the following subsections exploit these two models for the calculation of the parameters of Eq. (23).

4.1. Method M1

The method M1 can be used to obtain numerical estimates of β_1, β_2 and p. As already mentioned, Karnopp and Scharton [10] pointed out that, at the end of each plastic excursion, the system is in a transient phase exhibiting an elastic behaviour. This transient phase can be described by means of the joint PDF $p_{X\dot{X}}(x, \dot{x}, t/x_0, \dot{x}_0)$ of the elastic-part of the nonlinear response (see Appendix A), the initial conditions $(x_0, \dot{x}_0)$ being known exactly since the velocity and the elastic-part are respectively equal to zero and $\pm Y$ [10]. The probability of having only one plastic jump in the time interval $[0, T_0 = 1/f_0]$ can be computed classically by the first passage problem of the elastic-part of the nonlinear response, assuming that a threshold crossing corresponds to only one plastic jump within a cycle of the elastic response. Hence, the probability of occurrence p can be estimated by:

$$p = \int_0^{T_0}\int_0^{+\infty} \dot{x}\, p_{X\dot{X}}(Y, \dot{x}, t/x_0 = -Y, \dot{x}_0 = 0)\, \mathrm{d}\dot{x}\mathrm{d}t. \quad (28)$$

The joint PDF $p_{X\dot{X}}(x, \dot{x}, t/x_0, \dot{x}_0)$ can also be used to compute the crossing velocity PDF, which is defined in $[0, T_0]$ by:

$$p_{\dot{X}}(\dot{x}) = \frac{1}{p}\int_0^{T_0} \dot{x}\, p_{X\dot{X}}(Y, \dot{x}, t_f/x_0 = -Y, \dot{x}_0 = 0)\mathrm{d}t_f. \quad (29)$$

Using Eq. (26), the PDF p_{δ_i} of a plastic jump in a clump δ_i results in:

$$p_{\delta_i}(\delta_i) = \frac{p_{\dot{X}}(\dot{x})}{\dot{x}}\omega_0^2 Y. \quad (30)$$

4.2. Method M2

The method M2 can be used to obtain numerical estimates of α_1 and α_2. As previously mentioned, the first jump of a clump can be associated with the first threshold crossing in stationary conditions (Fig. 1). In order to estimate the PDF of the maximum after the first crossing, the joint PDF $p_E(e_{n-1}, e_n, \tau)$ of the ALS response envelope E can be used (see Appendix B). Indeed, $p_E(e_{n-1}, e_n, 1/2f_0)$ being the PDF of two successive peaks of $|\tilde{X}(t)|$, the PDF of the peak $e_n \geq Y$, since the previous one e_{n-1} is below the yield level, can be computed. Using Eq. (27), the first plastic jump PDF is then given by:

$$p_{\delta_1}(\delta_1) = \frac{Y}{\sqrt{Y^2 + 2Y\delta_1}} p_E\left(e_n \geq Y/e_{n-1} < Y\right). \quad (31)$$

4.3. Method M3

The method M3 can be used to obtain numerical estimates of β_1, β_2 and p using also the joint PDF of the ALS response envelope. In this case, we use an idea of Ditlevsen and Bognar [12] according to which, initial conditions such as $x_0 = \pm Y$ and $\dot{x}_0 = 0$, for the transient phase, are equivalent to the condition that the peak e_{n-1} is equal to the yield level. In consequence, another estimate of the probability of occurrence of a plastic jump in a clump is given by:

$$p = P\left(e_n \geq Y/e_{n-1} = Y\right) = \frac{\int_Y^\infty p_E\left(e_n, e_{n-1} = Y\right) de_n}{\int_0^\infty p_E\left(e_n, e_{n-1} = Y\right) de_n} \quad (32)$$

and another estimate of the PDF of a plastic jump δ_i is given by:

$$p_{\delta_i}(\delta_i) = \frac{Y}{\sqrt{Y^2 + 2Y\delta_i}} p_E\left(e_n \geq Y/e_{n-1} = Y\right). \quad (33)$$

$$p_{U_i}(u_i) = \frac{\eta}{(1-\nu^2)\varphi(\gamma\eta) + \sqrt{1-\nu^2}\nu\eta\Phi(-\gamma\eta)}\varphi\left(\frac{\sqrt{\eta^2+2\eta u_i}-\nu\eta}{\sqrt{1-\nu^2}}\right)$$

$$E[U_i] = \frac{2\gamma}{\eta(1+\gamma^2)^2}\left[\frac{\left(4\gamma^2+(1-\gamma^2)\eta^2\right)\varphi(\gamma\eta)+\gamma(1-\gamma^2)\eta(3-\eta^2)\Phi(-\gamma\eta)}{2\gamma\varphi(\gamma\eta)+(1-\gamma^2)\eta\Phi(-\gamma\eta)}\right].$$

Box I.

4.4. The method of Ditlevsen and Bognar

The previous three methods are based on the numerical computations of some integrals to estimate the five parameters of Eq. (23). For practical purposes, the closed form results obtained by Ditlevsen and Bognar [12], with the Slepian model approach, can also be used. For completeness, we remind their main results.

The closed form expression of p is given by:

$$p = \frac{2\gamma\varphi(\gamma\eta)+(1-\gamma^2)\eta\Phi(-\gamma\eta)}{2\gamma\varphi\left(\frac{1-\gamma^2}{2\gamma}\eta\right)+(1-\gamma^2)\eta\Phi\left(\frac{1-\gamma^2}{2\gamma}\eta\right)} \quad (34)$$

with $\gamma = (1-\nu)/\sqrt{1-\nu^2}$, $\alpha = \beta/\sqrt{(1-\beta^2)}$ and $\nu = \exp[-\alpha\pi]$. $\eta = Y/\sigma_{\tilde{X}}$ is the yielding displacement normalized with respect to the standard deviation $\sigma_{\tilde{X}}$ of the ALS response given by:

$$\sigma_{\tilde{X}}^2 = \frac{S_0}{8\beta\omega_0^3}, \quad (35)$$

where S_0 is the one-sided power spectral density of the white noise defined in the frequency domain.

The PDF $p_{U_1}(u_1)$ of the first normalized plastic jump $U_1 = \delta_1/\sigma_{\tilde{X}}$ and the closed form expression of its mean value are respectively given by [12]:

$$p_{U_1}(u_1) = \frac{\eta}{1-(1+\nu)\Phi(-\gamma\eta)} \times \Phi\left(\frac{\eta-\nu\sqrt{\eta^2+2\eta u_1}}{\sqrt{1-\nu^2}}\right)\exp[-\eta u_1] \quad (36)$$

$$E[U_1] = \frac{1}{\eta}\left[1-\frac{2\gamma(1-\gamma^2)}{(1+\gamma^2)^2}\frac{\gamma(1-\eta^2)\Phi(-\gamma\eta)+\eta\varphi(\gamma\eta)}{1+\gamma^2-2\Phi(-\gamma\eta)}\right]. \quad (37)$$

The closed form expressions of the PDF and the mean value of the subsequent normalized plastic jumps are respectively given by the equations in Box I [12].

Closed form expressions for p, α_1, β_1 are given by this method. The mean-squares α_2 and β_2 have to be computed numerically using the closed form expressions of the PDFs of the plastic jumps. In the following this method is noted method DM1.

Alternatively, exponential PDFs can be assumed for the plastic jumps to simply obtain closed form expressions of the second moments. This variant is noted method DM2.

4.5. The expected frequency of occurrence of the clumps

As previously mentioned, the expected frequency of occurrence of the clumps can simply be estimated by the mean frequency μ_Y of the threshold crossings by the ALS envelope [11]:

$$\mu_Y = 2f_0\exp[-\eta^2/2]\left(1-\exp\left[-\sqrt{\pi/2}\,\eta\delta\right]\right), \quad (38)$$

where δ is a spectral bandwidth measure which may be approximated, in the case of lightly damped linear oscillator ($\beta < 0.1$), by [16]:

$$\delta \cong \sqrt{\frac{4\beta}{\pi}}. \quad (39)$$

5. Numerical simulations

Combining Eqs. (6), (17), (18) and (23) an expression of the mean displacement ductility demand is obtained for the nonlinear response of an elastic perfectly plastic SDOF system under zero mean Gaussian white noise:

$$\bar{\mu} = 1 + \frac{2}{Y\sqrt{\pi}}\left[\left(\alpha_2+\frac{p}{(1-p)}\beta_2 - 2\frac{p}{(1+p)}\alpha_1\beta_1 - 2\frac{p^2}{(1-p^2)}\beta_1^2\right)\mu_Y t\right]^{1/2}. \quad (40)$$

This expression involves six parameters μ_Y, α_1, α_2, β_1, β_2 and p. The first one is given by Eq. (38). The others can be estimated by the different methods proposed in Section 4.

To assess the range of validity of the above methods, comparisons with numerical simulations are proposed herein. The damping ratio of the SDOF system considered is equal to 2%. The statistical results are obtained from 500 sample functions of the nonlinear response having a duration equivalent to 80 cycles of the ALS response.

Fig. 4(a)–(h) show the ratio of theoretical to numerical simulation results for the six relevant parameters. It can be seen that all the methods give better results for high yield thresholds ($\eta \geq 1.5$). When $\eta = 1$, the expected displacement ductility demand is globally underestimated by a factor close to 0.8. The variable exhibiting the more important discrepancy is the mean time between successive clumps ($\tau_c = 1/\mu_Y$). The method M3 gives better estimates than the method M1, except for the probability of occurrence of a jump in a clump. The method DM1 gives good estimates except for the probability of occurrence p when $\eta \to 1$. Consequently, the mean-square

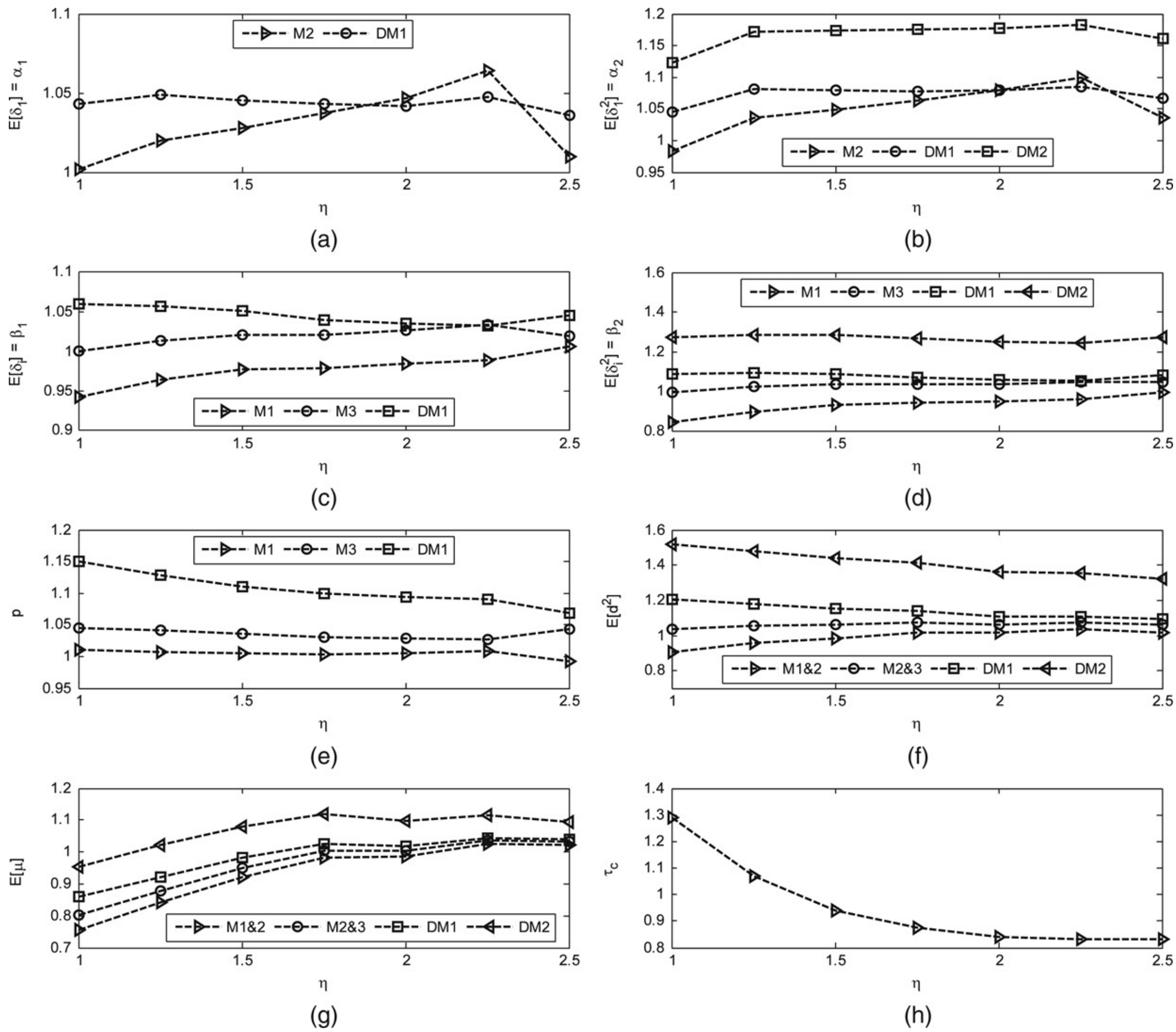

Fig. 4. Ratios of theoretical to numerical simulation results for $\beta = 0.02$ and various η: (a) α_1; (b) α_2; (c) β_1; (d) β_2; (e) p; (f) $E[d^2]$; (g) $E[\mu]$; and (h) $\tau_C = 1/\mu_Y$.

of the clumps is overestimated when $\eta \to 1$. The method DM2 leads to an overestimation of α_2 and β_2. It follows that the mean-square of the clumps is overestimated by more than 30% for $1 \leq \eta \leq 2.5$. This discrepancy decreases on increasing η because, in this case, the exponential PDF assumption is more satisfactory [12]. Nevertheless, the method DM2 gives a satisfactory estimate of the mean displacement ductility demand.

Fig. 5(a)–(d) show comparisons between the PDFs of the normalized plastic jumps U_i obtained by the methods M2, M3 and DM1 (solid lines) and by numerical simulations (dashed lines) respectively. These statistical results show a good agreement between the PDFs of the first jump determined theoretically and numerically. Concerning the subsequent plastic jumps of a clump, the method DM1 underestimates the PDF close to zero, nevertheless the agreement is better for high values of δ_i.

Comparisons done for $\beta = 1\%$ and $\beta = 5\%$ give quite similar results.

6. Conclusions

In this work we extended the analytical model developed by Vanmarcke and Veneziano [11], using the Brownian motion assumption for the statistical behaviour of the plastic process. We obtained closed form expressions of the PDF of the absolute maximum of the plastic process, of the mean maximum and of the ductility demand which involve five parameters (α_1, α_2, β_1, β_2 and p). To estimate these parameters, three methods, which combine previous studies found in the literature, are presented. Because these methods are based on numerical computations of some integrals, we propose, for practical purposes, to use the results obtained by Ditlevsen and Bognar [12] with the Slepian model approach, assuming that the plastic jumps have exponential PDFs.

The agreement between theoretical predictions and simulations shows that the main physical phenomena have been captured in this work. Giving similar probabilistic results as those classically obtained in the linear case (mean maximum of the response), this work is particularly interesting in seismic engi-

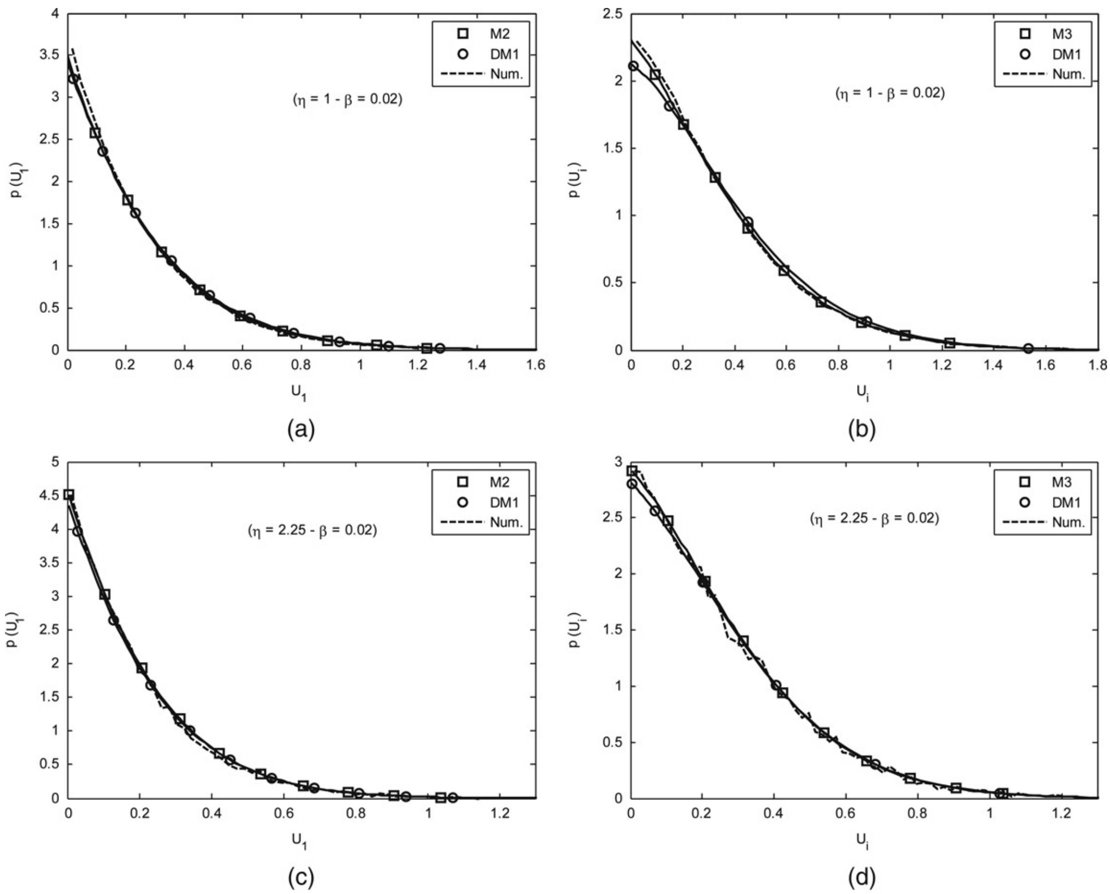

Fig. 5. PDFs of the normalized plastic jumps for $\beta = 0.02$: (a) U_1 for $\eta = 1$; (b) U_i for $\eta = 1$; (c) U_1 for $\eta = 2.25$; and (d) U_i for $\eta = 2.25$.

neering since seismic design in practice is based on linear response spectra, although its application is limited to the case of high yield thresholds, for an elastic perfectly plastic oscillator under zero mean Gaussian white noise. In order to take into account a more realistic frequency content of the input, a future work should consider the case of a stationary broadband non-white noise excitation. The main challenge would be to propose a new model for plastic clumps dealing with correlation between plastic jumps which are no more uncorrelated.

Appendix A

The joint PDF $p_{X\dot{X}}(x, \dot{x}, t/x_0, \dot{x}_0)$ of the elastic-part of the nonlinear response is given by [10]:

$$p_{X\dot{X}}(x, \dot{x}, t/x_0, \dot{x}_0) = \frac{\left(\sqrt{1-\rho(t)^2}\right)^{-1}}{2\pi\sigma_x(t)\sigma_{\dot{x}}(t)} \times \exp\left[-\frac{1}{2(1-\rho(t)^2)}\left(\frac{(x-\mu_x(t))^2}{\sigma_x^2(t)} - \frac{2\rho(t)(x-\mu_x(t))(\dot{x}-\mu_{\dot{x}}(t))}{\sigma_x(t)\sigma_{\dot{x}}(t)} + \frac{(\dot{x}-\mu_{\dot{x}}(t))^2}{\sigma_{\dot{x}}^2(t)}\right)\right], \quad (41)$$

where x and $\dot{x}$ are respectively the displacement and velocity values at time t, $(x_0, \dot{x}_0)$ are the initial conditions of the transient phase, $(\mu_x(t), \sigma_x^2(t))$ and $(\mu_{\dot{x}}(t), \sigma_{\dot{x}}^2(t))$ are respectively the mean and the variance of $X(t)$ and $\dot{X}(t)$. They are defined by:

$$E[X, t/x_0, \dot{x}_0] = \mu_X(t) = x_0 r(t) - \dot{x}_0 \dot{r}(t)/\omega_0^2 \quad (42)$$

$$E[\dot{X}, t/x_0, \dot{x}_0] = \mu_{\dot{X}}(t) = x_0 \dot{r}(t) - \dot{x}_0 \ddot{r}(t)/\omega_0^2 \quad (43)$$

$$\sigma_x^2(t) = \sigma_{\tilde{X}}^2(1 - A(t)) \quad (44)$$

$$\sigma_{\dot{x}}^2(t) = \sigma_{\dot{\tilde{X}}}^2(1 - B(t)) = \omega_0^2\sigma_{\tilde{X}}^2(1 - B(t)), \quad (45)$$

where $\sigma_{\tilde{X}}^2$ is given by Eq. (35). $\rho(t)$ is the correlation coefficient between $X(t)$ and $\dot{X}(t)$:

$$\rho(t) = \frac{2\beta\dot{r}^2(t)/\omega_0^2}{[(1-A(t))(1-B(t))]^{1/2}}. \quad (46)$$

$r(t)$ and $\dot{r}(t)$ are respectively defined by:

$$r(t) = \exp[-\alpha\omega|t|](\alpha\sin(\omega t) + \cos(\omega t)) = R(t)/\sigma_{\tilde{X}}^2 \quad (47)$$

$$\dot{r}(t) = -\frac{\omega_0^2}{\omega}\exp[-\alpha\omega|t|]\sin(\omega t) = \dot{R}(t)/\sigma_{\tilde{X}}^2, \quad (48)$$

where $\omega = \omega_0\sqrt{(1-\beta^2)}$, $\alpha = \beta/\sqrt{(1-\beta^2)}$ and

$$A(t) = \frac{\exp[-2\beta\omega_0 t]}{\omega^2} \times \left(\omega_0^2 - \omega_0^2\beta^2\cos(2\omega t) + \beta\omega_0\omega\sin(2\omega t)\right) \quad (49)$$

$$B(t) = \frac{\exp[-2\beta\omega_0 t]}{\omega^2} \times \left(\omega_0^2 - \omega_0^2\beta^2\cos(2\omega t) - \beta\omega_0\omega\sin(2\omega t)\right). \quad (50)$$

Appendix B

The joint PDF $p_E(e_{n-1}, e_n, \tau)$ of the ALS response envelope E is defined by [18]:

$$p_E(e_1, e_2, \tau) = \frac{e_1 e_2}{B} I_0\left[\frac{e_1 e_2}{B}\left(\mu_{13}^2 + \mu_{14}^2\right)^{1/2}\right] \times \exp\left[-\frac{\sigma_{\tilde{X}}^2}{2B}\left(e_1^2 + e_2^2\right)\right] \quad (51)$$

where e_i is an envelope value at time t_i, $\tau = t_2 - t_1$, $B = \sigma_{\tilde{X}}^4 - \mu_{13}^2 - \mu_{14}^2$, $\mu_{13} = R(\tau)$, $\mu_{14} = -\dot{R}(\tau)/\omega_0$ (see Appendix A) and I_0 is the Bessel function of order 0:

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \mathrm{e}^{-x\cos(t)}\mathrm{d}t. \quad (52)$$

When $\tau = 1/2f_0$, Eq. (51) is the joint PDF of two successive peaks of $\left|\tilde{X}(t)\right|$.

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