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A low-order reduced model for the long range propagation of infrasound in the atmosphere

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Abstract

6 This paper considers a class of low-order, range-dependent propagation models obtained
7 from the normal mode decomposition of infrasounds in complex atmospheres. The classical
8 normal mode method requires calculating eigenvalues for large matrices making the compu-
9 tation expensive, even though some modes have little influence on the numerically obtained
10 results. By decomposing atmospheric perturbations into a wavelet basis, it is shown that
11 the most sensitive eigenvalues provide the best reduced model for infrasound propagation.
12 These eigenvalues lie on specific curves in the complex plane that can be directly deduced
13 from atmospheric data through a WKB approach. The computation cost can be reduced
14 by computing the invariant subspace associated with the most sensitive eigenvalues. The
15 reduction method is illustrated in the case of the Fukushima explosion (12 March 2011).
16 The implicitly-restarted Arnoldi algorithm is used to compute the 3 most sensitive modes
17 and the correct tropospheric arrival is found with a cost of 2 % of the total run time. The
18 cost can be further reduced by using a stationary phase technique. Finally, it is shown that
19 adding uncertainties triggers a stratospheric arrival even though the classical criteria, based
20 on the ratio of stratospheric sound speed to that at ground level, is not satisfied.

I. Introduction

The usual approach in studying infrasound propagation involves computing the acoustic component superimposed to a given atmospheric state, which is classically obtained from global empirical models (*Drob et al.*¹⁶, *Hedin et al.*²⁰) as well as operational numerical weather prediction centers (e.g., NOAA global forecast system or ECMWF integrated forecast system). The justification of such an approach is questionable given that unresolved large-scale gravity waves, turbulence and matching conditions of the various available data may affect the waveforms in such a way that some arrivals may literally appear/disappear. While a significant amount of data has become available on the temperature and wind structure of the atmosphere, it is not clear whether classical propagation techniques (ray tracing, PE method) are able to *statistically* capture the effects of small-scale uncertainties (*Kulichkov et al.*²⁵, *Chunchuzov et al.*¹³, *Hedlin et al.*²¹).

The lack of knowledge concerning the atmosphere leads to consider the atmospheric structures as being generated by random gravity wave sources, an approach that was recently popularized by *Drob et al.*¹⁷. As far as statistics are concerned, the question of the computational cost is critical. According to the law of large numbers, the more runs we make, the closer the sample mean of the results should be to the true mean. This law, however, does not describe how the convergence takes place. Indeed, the number of simulation runs necessary to attain a given confidence level depends on random atmospheric structures

40 and thus, a statistical analysis may be computationally expensive.

41 This paper considers a class of low order propagation models obtained from the normal
42 mode decomposition of infrasounds in complex atmospheres. While the ray theory cannot
43 predict some important refracted paths (*Waxler, Gilbert and Talmadge*⁴¹), the normal mode
44 technique can be used to simulate signals within environments that cover a wide range of
45 vertical scales. The complete acoustic field is constructed by summing up contributions
46 of modes, which can be obtained by seeking the roots of the so-called dispersion relation.
47 In most cases the poles are complex, and thereby requires the use of a two-dimensional
48 search algorithm. In addition to the difficulties inherent in searching in two dimensions, it
49 is important to point out that the wave equation in a finite domain has an infinite set of
50 discrete eigenvalues and thus, the number of numerically obtained modes is proportional to
51 the number of grid points used. Then, for a fixed number of grid points per wavelength,
52 the numerical cost increases with the frequency. Subsequently, *Waxler*³⁹ has emphasized
53 that the imaginary parts (the attenuation rates) can be obtained through a perturbative
54 approach, by solving a fixed point equation with a Newton-like method. Such an approach,
55 however, is restricted to poles that are close to the real axis. In this paper we are interested
56 in the manner in which the modes provide the best reduced model (finding the dominant
57 modes) and more generally we are interested in the behavior of the selected poles in the
58 complex plane as the background atmospheric state is varied.

59 This work is an outgrowth of the authors' effort to find the simplest mathematical
60 model in order to estimate signal statistics. It is often the case that first-principles model-
61 ing or system identification result in unnecessarily high-dimensional mathematical models.
62 Model (order) reduction is about systematic approximation of such models. There are many
63 advantages in working with models with low-dimensional state space since low-dimensional
64 models are often easier to analyze and much faster to simulate. As pointed out by *Antoulas*³
65 in his survey on reduction methods, we can identify two sets of methods which are currently
66 in use, namely the SVD (Singular Value Decomposition) based methods and the moment
67 matching methods.

68 In fluid dynamics applications, SVD based methods are widely used, the most common
69 approach being the Proper Orthogonal Decomposition (POD) method and its variants such
70 as the BPOD (Balanced Proper Orthogonal Decomposition) (*Rowley*³⁴). POD methods
71 are based on the Galerkin projection of the dynamical model on a subspace obtained with
72 a SVD analysis of experimental or numerical datasets. It is known that only a few POD
73 modes can summarize the flow organization in many cases (*Delville et al.*¹⁵, *Galletti*¹⁸). It
74 also appears that reduced-order models generally lose accuracy for flow parameters different
75 from those used to generate the POD basis, even if a local improvement can be made with
76 a sensitivity analysis of the modes (*Hay*¹⁹). For control optimization problems, instead of
77 using a POD basis, some authors directly use the global eigenmodes as a projection basis,

78 even if it appears that it does not show sufficient robustness (*Barbagallo*⁵). For instance,
79 the optimal growth in a separated boundary-layer flow can be studied with the sum of the
80 non-normal global modes (that are the most sensitive to any flow perturbation) (*Akervik*²).

81 The other set of methods, the moment matching approach, is not commonly used for
82 fluid dynamics model reduction and is mostly found in structural dynamics or circuit simu-
83 lations. It consists in using Krylov subspace methods to get a low-order model that shows
84 a good approximation of the transfer function, and more precisely of the first coefficients
85 of its Laurent expansion (*Bai*⁴, *Srinivasan Puri*³⁶). Arnoldi and Lanczos algorithms that
86 perform iterative projections on Krylov subspaces (*Watkins*³⁸) are modified in order to guide
87 the reduction to the best approximation of the transfer function. Progress in other Krylov
88 subspace techniques allow the reduction of a large variety of dynamical systems, including
89 second-order systems (*Salimbahrami*³⁵) and nonlinear systems (*Bai*⁴). With regards to the
90 wave equation, it is easy to apply the Krylov-Arnoldi algorithm to compute a few of the
91 smallest or largest modes of a large matrix obtained from the discretized problem. But there
92 is no universal way in which the modes may be sorted, and indeed much of the emphasis
93 in the present work is to show that the modes lie in the vicinity of regions in the complex
94 plane, for each of which there are different features in the vertical sound speed profile.

95 The paper is organized as follows: the derivation of the range-dependent normal-mode
96 method is described in section 2. A sensitivity analysis is discussed in section 3 and it is shown

97 that, to a first approximation, the most sensitive modes to waveguide perturbations provide
98 the best low-order reduced model for long-range infrasound propagation. In section 4, we use
99 a WKB approach to find the most sensitive regions in the complex plane and an implicitly
100 restarted Arnoldi procedure is used to obtain the most sensitive modes. We show that the
101 contribution of each mode is confined to a narrow range of frequencies so that when the
102 stationary phase method is limited to the first most sensitive mode, it gives a very good
103 approximation of the waveform. We illustrate this in the case of the Fukushima explosion
104 that was detected at the Japanese infrasound station I30JP (12 March 2011), located 243 km
105 away from the source. Finally, in section 5, we show that adding stratospheric uncertainties
106 may trigger a stratospheric arrival even though the classical criteria, based on the ratio of
107 stratospheric sound speed to that at ground level, is not satisfied.

108 **II. Normal mode decomposition of the pressure field**

109 **A. The mean atmosphere**

110 There is no consensus on how mean atmospheric specifications can be obtained from
111 operational numerical weather predictions or atmospheric climate reanalysis. In this work,
112 the wind and temperature profiles are extracted from the atmospheric database provided by
113 the European Centre for Medium-Range Weather Forecasts (ECMWF). Following *Millet et*
114 *al.*³⁰, the profiles are obtained from the ERA-Interim and 91-level datasets for altitudes less

115 than 50 km and 80 km, respectively, and matched to statistical data for higher altitudes. The
116 statistical data is obtained using empirical reference models, known as HWM-93 (Horizontal
117 Wind Model) and MSIS-90 (Mass Spectrometer and Incoherent Radar Model). Although
118 these models represent a compromise between the original data sources, they are known to
119 present systematic differences, particularly near the mesopause (*Hedin et al.*²⁰).

120 In this study, we consider the explosion of the Fukushima Daiichi nuclear power plant
121 (suspected to be caused by hydrogen gas) that occurred on 12 March 2011. The recorded
122 infrasound signal at the Japanese station I30JP is shown in figure 1 as well as the effective
123 sound speed profiles. The profiles obtained from the 91-level dataset and the ERA-interim
124 dataset are shown in figure 1, above the source and above the infrasound station I30JP.
125 Recent simulations by *Bertin et al.*⁹ have shown that the signal is composed of two arrivals:
126 a strong tropospheric arrival, 12-13 minutes after the explosion, followed by a weaker strato-
127 spheric ‘tail’. As part of their study, *Bertin et al.*⁹ have suggested on numerical grounds
128 that, due to the vertical profile between the source and the infrasound station, the amplitude
129 of the first arrival can change by several orders of magnitude. It is therefore reasonable to
130 conclude that the range-independent assumption breaks down for this tropospheric arrival,
131 even though the waveform shape does not seem to be affected. In the present work, due to
132 its strong dependence to the sound speed profile, we will first focus on the first arrival using
133 a range-dependent approach.

134 It is not clear how small changes in the profile along the source-to-receiver path affect
135 the infrasound propagation. The atmosphere is usually described as a thin layer of fluid in
136 which any vertical structure has its counterpart in the horizontal direction. Following the
137 method of multiple scales, the effective sound speed may be represented analytically in the
138 form of the function $c(z, \epsilon r)$, where r and z are the distance from the source and altitude,
139 respectively. Numerically, ϵ is a small parameter that can be interpreted either as a measure
140 of a ‘slow’ coordinate ϵr (*Bender and Orszag*⁸) or a perturbation of a given profile. The first
141 order Taylor series expansion of c about $\epsilon = 0$ can be written as

$$c(z, \epsilon r) \simeq c_0(z) + \epsilon c_1(z), \quad (1)$$

142 where $c_0(z) = c(z, 0)$ is the profile above the source. Hence, the degree of horizontal inho-
143 mogeneity of the atmosphere can be considered as a perturbation. Here, ϵ will be deemed a
144 small parameter and will be used as such later.

145 **B. The vertical structure equation**

146 As our starting point, we use the wave equation in cylindrical coordinates (r, θ, z) .
147 The acoustic wave is assumed to be initiated by a ground-based localized excitation of
148 frequency ω .

149 The most elementary approach consists in entirely neglecting the horizontal variability
150 of the medium: if the atmosphere is considered as uniform in the horizontal direction, the

151 vertical structure $\phi(z)$ of the pressure field satisfies the Helmholtz equation

$$\frac{d^2\phi}{dz^2} + \left[\frac{\omega^2}{c^2(z)} - k^2 \right] \phi = -\frac{\delta(z)}{2\pi}, \quad (2)$$

152 which, for unbounded atmospheres, requires a boundedness condition as $z \rightarrow \infty$. In the
153 above equation $c(z)$ is the effective sound speed, k is the horizontal wavenumber and δ is
154 the Dirac delta function which mimics a point source at $z = 0$.

155 In the present work, we use the effective sound speed approximation and substitute the
156 moving fluid by a motionless fluid with the effective sound speed, as proposed by *Rayleigh*³³.
157 The density effect can easily be considered by defining $\phi = p/\sqrt{\bar{\rho}}$, where p and $\bar{\rho}$ are the
158 pressure and the mean density, respectively. This transformation allows to reduce the original
159 Helmholtz equation to the normal mode equation (2), provided the acoustic wavelength is
160 not too large. Indeed, upon substituting ϕ into the original Helmholtz equation when the
161 density depends on altitude, we obtain an additional term $\frac{1}{4}\phi H^{-2}$, where H is the atmospheric
162 height scale. This term, however, can be neglected for frequencies larger than 0.1 Hz and
163 the pressure field can readily be obtained from $p = \phi\sqrt{\bar{\rho}}$.

164 The solution of (2) is, within a constant of proportionality, the Green's function of the
165 wave equation. Following the classical approach (e.g. *Jensen et al.*²³), the spatial part p of
166 the wave field can be obtained from the discrete-spectrum contribution, which is composed

167 of the sum of the residues evaluated at the (simple) roots of the dispersion relation $D(k, \omega)$

$$p(r, z; \omega) = \frac{i}{2} \sum_{j=0}^N \frac{\phi_j(z; \omega) \phi_j(0; \omega)}{\partial_k D(k_j(\omega), \omega)} k_j(\omega) H_0^{(1)}(k_j(\omega)r), \quad (3)$$

168 where $H_0^{(1)}$ is the Hankel function of the first kind (*Abramowitz and Stegun*¹) and ϕ_j is the

169 eigenfunction associated with the spatial eigenvalue $k_j(\omega)$. Equation (3) can be simplified

170 by properly scaling ϕ_j . Taking

$$\int_0^\infty \phi_j^2(z; \omega) dz = 1, \quad (4)$$

171 a straightforward calculation (*Jensen et al.*²³) gives $\partial_k D(k_j, \omega) = 2k_j(\omega)$ so that the pressure

172 field at ground level ($z = 0$) reads

$$p(r, 0; \omega) = \frac{i}{4} \sum_{j=0}^N \phi_j^2(0; \omega) H_0^{(1)}(k_j(\omega)r), \quad (5)$$

173 where N is the number of modes involved in waveguide propagation. Here we follow *Waxler*³⁹

174 and use the approximation $\sqrt{\bar{\rho}(0)} = 1$ in the calculation of ground-based signals. Note that,

175 for unbounded atmospheres, ϕ_j is in general non-analytic in k and the spatial part $p(r, z; \omega)$

176 involves a continuous-spectrum contribution that arises from the integral along the branch

177 cuts. In practice, however, this contribution can be neglected, provided we are sufficiently

178 far from the source (*Waxler*^{40, 39}).

179 The poles are readily identified as the zeroes of $D(k, \omega)$, *i.e.* the spatial eigenvalues

180 $k_j(\omega)$. Zeroes of $D(k, \omega)$ may also be sought in terms of temporal eigenvalues $\omega_j(k)$ when

181 the wavenumber is assumed to be real. In this work, we use a boundary value (or implicit)

182 method for solving the eigenvalue problem. The wave equation (2) is reduced to a linear
 183 algebraic equation using a pseudo-spectral technique (*Candelier et al.*^{12,11}). The global
 184 eigenvalues are obtained by applying a QR algorithm to the generalized eigenvalue problem.
 185 While care must be taken to preclude spurious eigenvalues, it is worth mentioning that
 186 such an approach can be applied to problems in which the eigenvalue appears nonlinearly
 187 (*Bridges and Morris*¹⁰) so that the absorption can be considered within the same numerical
 188 framework.

189 The leading behaviour of the temporal response is then obtained by applying the inverse
 190 Fourier transform to (5) to obtain

$$p(r, 0; t) \sim \frac{i}{8\pi} \sum_{j=0}^N \int_F \phi_j^2(0; \omega) H_0^{(1)}(k_j(\omega)r) e^{-i\omega t} d\omega, \quad (6)$$

191 where the integration is performed along the path F in the complex ω -plane. One recognizes
 192 in expression (6) a wave packet composed of freely evolving spatial modes generated by a
 193 localized pulsed disturbance, at $r = 0$. The contour cannot be chosen arbitrarily since both
 194 the convergence of (6) and the *causality condition*, $p(r, z; t) = 0$ for $t < 0$, should be ensured.
 195 As pointed out by *Batchelor et al.*²², the contour F has to be chosen to lie above all temporal
 196 eigenvalues $\omega_j(k)$ as k travels along the real axis. Thus, to avoid confusion, the frequency
 197 parameter ω is treated as a Fourier-Laplace transform variable, $\omega = \omega_r + i\omega_i$, with a small
 198 positive imaginary part $\omega_i \ll 1$. For this reason, the eigenvalues $k_j(\omega)$ lie in the upper half
 199 of the complex k -plane, as shown in figure 2. Note that in figure 2, we introduce the phase

200 velocity $c_r = \omega_r/k_r$, where k_r is the real part of k .

201 Further insight into the nature of the modes can be gained by examining the manner
202 in which they evolve along some path in the complex ω -plane. For a fixed, real ω , the
203 problem (2) on a finite domain is self-adjoint and thus, it has a discrete spectrum of the
204 form $k_1 < k_2 < \dots < k_N$ and every eigenspace is one-dimensional. These eigenvalues are
205 real-valued wavenumbers of the guided-wave modes of the medium. For a complex frequency,
206 the eigenvalues $k_j(\omega)$ are continued in the complex k -plane: the larger the imaginary part
207 of ω the stronger the attenuation (given by the imaginary part of k) of the guided-wave
208 modes. The reason that these modes are decaying along the r -direction is directly tied
209 to the causality requirement which forces the F -contour to be placed above the real axis.
210 Thus, even though a medium would support growing (i.e. unstable) responses, its localized
211 excitation by a source that grows sufficiently fast in time produce only decaying responses
212 in space. In addition, the decaying rate depends on vertical inhomogeneities covered by
213 the corresponding eigenfunction. A very interesting aspect of using a complex frequency is
214 that the imaginary parts of eigenvalues $k_j(\omega)$ may be arranged so that the modes can be
215 associated with specific regions in the atmosphere and thus, the problem can be projected
216 onto the invariant subspace corresponding to a subset of these modes.

217 **C. Normal modes for range-dependent atmospheres**

218 Most environments of interest are spatially non uniform in the horizontal direction r .

219 The previous notions can then be taken to apply locally as long as the non-uniformities of
 220 the medium are small over a typical acoustic wavelength. Thus, the horizontal variation of
 221 the medium is characterized by a slow space variable $R = \epsilon r$, where ϵ is a small parameter of
 222 the same order of magnitude as the scaled nonuniformities. Decomposition into local normal
 223 modes leads to a dispersion relation of the form $D(k, \omega; R) = 0$. The eigenvalue problem is
 224 therefore solved for a discrete set of ranges R_1, R_2, \dots, R_N .

225 In going from one range to another, we assume that the modes couple adiabatically, i.e.
 226 without any transfer of energy into other modes. This approximation, that was introduced in
 227 the study of underwater acoustics by *Pierce*³², is a reasonable compromise between accuracy
 228 and run time. The WKB approximation then reads

$$p(z; \omega, R) \simeq \frac{ie^{-i\pi/4}}{2\sqrt{2\pi r}} \sum_{j=0}^N \frac{\phi_j(0; \omega, R)\phi_j(z; \omega, R)}{\sqrt{k_j(\omega, R)}} \exp \left\{ \frac{i}{\epsilon} \int_0^R k_j(\omega, s) ds \right\}, \quad (7)$$

229 where $\phi_j(z; \omega, R)$, which is the eigenfunction associated with the local wavenumber $k_j(\omega, R)$
 230 (for a given ω), satisfies the condition

$$\int_0^\infty \phi_j(z; \omega, R) \frac{\partial \phi_j}{\partial R}(z; \omega, R) dz = 0, \quad (8)$$

231 where the derivative can be approximated through a finite difference.

232 Figure 2 shows the trajectories of the local eigenvalues k_j as R increases from the source
 233 $R = 0$ to the infrasound station I30JP. While most eigenvalues are not significantly affected
 234 by horizontal variations of the vertical profile, the eigenvalues which are close to a cusp at

235 $c_r \simeq 335 \text{ m.s}^{-1}$ trace the longest trajectories in the k -plane (essentially along the imaginary
236 axis). Such eigenvalues are strongly sensitive to horizontal variations and correspond to
237 waves travelling with exactly the phase velocity at a local maximum of $c(z)$ (which is also
238 a double turning point in the limit $\omega \rightarrow \infty$). This maximum is associated with a low
239 altitude tropospheric waveguide (see figure 1), below 5 km altitude. Hence, waves of this
240 kind strongly depend on large-scales, as a result of the weak coupling between horizontal
241 and vertical low-altitude atmospheric structures, through the small parameter ϵ . A closer
242 examination of the trajectories in the complex k -plane shows that the path lengths of the
243 most sensitive modes are smaller for the 91-level dataset (figures 2c,d). In spite of this,
244 the phase velocity change (along the real axis) is larger and thus, the arrival time of the
245 corresponding waveform is more affected.

246 The cusp may be exploited to track the most sensitive eigenvalues: the appearance of
247 a cusp in the spectrum at a particular c_r is an indication that the eigenvalues are on the
248 verge of being critically dependent on small variations in the vertical profile. This criterion,
249 which is completely general, may be used to reduce the complexity (and computation time)
250 of any range-dependent infrasound propagation problem.

251 **III. Sensitivity of modes to the mean atmosphere**

252 This work focuses on the underlying assumption that the mean atmospheric state is fixed
253 and is given by operational numerical weather predictions or atmospheric climate reanalysis.

254 In these data, the major setback is that important subgrid-scale phenomena are filtered out
 255 during the data assimilation, thereby implying a degree of uncertainty associated with the
 256 mean atmospheric state.

257 Assuming that deviations of a given amplitude from the reference state may occur, we
 258 seek to determine the resultant effect on the computed signal at I30JP without computing
 259 the whole eigensystem. The sensitivity of the eigenvalues to modifications of the medium can
 260 be assessed through a perturbative approach of the normal modes (*Kato*²⁴, *Trefethen*³⁷).
 261 In order to carry out a comprehensive study of the influence in which every atmospheric
 262 structure has on the propagation, we decompose the perturbation in (1) into a wavelet basis

$$c_1(z) = c_0(z) \sum_m \sum_n c_{mn} \Psi_{mn}(z), \quad (9)$$

263 where each wavelet Ψ_{mn} is generated by translation and dilatation of a function $\Psi_{mn}(z) =$
 264 $a^{-m/2} \Psi(a^{-m}z - nb)$, with $a > 1$ and $b > 0$, so that the basis depicts every position and every
 265 scale.

266 The sensitivity of the eigenvalue k_j is obtained by substituting (1) into (2) and then
 267 differentiating with respect to ϵ . Taking an inner product with the eigenfunction ϕ_j leads to

$$\frac{\partial k_j}{\partial \epsilon} = -\frac{\omega^2}{k_j} \int_0^\infty \frac{\phi_j^2(z) c_1(z)}{c_0^3(z)} dz. \quad (10)$$

268 Upon substitution of (9) into (10), we obtain

$$\frac{\partial k_j}{\partial \epsilon} = \sum_m \sum_n c_{mn} K_{jmn}, \quad (11)$$

269 where K_{jmn} is given by

$$K_{jmn} = -\frac{\omega^2}{k_j} \int_0^\infty \frac{\phi_j^2(z)}{c_0^2(z)} \Psi_{mn}(z) dz. \quad (12)$$

270 It is clear from (12) that the components K_{jmn} are nothing but the (complex) sensitivities
 271 of k_j to each wavelet Ψ_{mn} . In practice, these components are easily computed through
 272 the multiresolution discrete wavelet transform of ϕ_j^2/c_0^2 with the Mallat algorithm that only
 273 involves recursive filterings (*Mallat*²⁸).

274 Now, to obtain a relation between the sensitivity of a given mode and its relative
 275 contribution to the sum of the residues (5), we use the completeness and the orthonormality
 276 of the wavelet basis to deduce from (12) that

$$\phi_j^2(z) = -\frac{k_j c_0^2(z)}{\omega^2} \sum_m \sum_n K_{jmn} \Psi_{mn}(z). \quad (13)$$

277 Upon substituting (13) into (5), the ground-based pressure field becomes

$$p(r, 0; \omega) \sim -\frac{ic_0(0)^2}{4\pi} \sum_j \frac{k_j(\omega)}{\omega^2} \sum_{(m,n)} K_{jmn}(\omega) H_0^{(1)}(k_j(\omega)r) \Psi_{mn}(0). \quad (14)$$

278 For a given mode, the above formula presents the advantage of determining which wavelets
 279 are relevant to the ground-based pressure field. We conclude, as expected intuitively, that
 280 only the wavelets that connect the perturbation at a given altitude to the source ($\Psi_{mn}(0) \neq 0$)
 281 play a role in the sum (14). For these wavelets, the larger the sensitivity K_{jmn} , the larger
 282 the contribution of the wavelet to the signal.

283 Figure 3 gives the sensitivities K_{jmn} of the three most sensitive modes to D10 wavelets
 284 (*Daubechies*¹⁴) as functions of the frequency ω_r . The results are shown for the wavelet that
 285 maximizes the sensitivity, among 2046 wavelets that cover the altitudes in the range 0-120 km
 286 and the wavelengths from 117 m to 60 km. As ω_r is increased, a subset of normal modes
 287 becomes highly sensitive to low altitude wavelets (figure 3c) and correspondingly their con-
 288 tribution to the ground-based pressure field increases, as shown in figure 4. Indeed, for each
 289 mode, the sensitivity can be described by a few of the wavelets (m, n) whose contributions
 290 are maximum over a narrow range of frequencies. These wavelets provide the vertical struc-
 291 tures that are responsible for the transition from the cusp to a region of constant imaginary
 292 part, as shown in figure 3b. Note that the region where the most sensitive modes converge
 293 (as $\omega_r \rightarrow \infty$) can be obtained through a WKB analysis of (2). The WKB modes are given
 294 by dashed lines in figure 3. This holds despite the fact that such an approximation fails to
 295 describe the transition.

296 It is important to point out that the family of (most sensitive) modes issuing from the
 297 cusp gives the leading-order contribution of (5) over frequencies as large as 1 Hz ($\omega_r = 2\pi$).
 298 Since the integration in k is performed first, we observe that the integral can be replaced
 299 by the sum of contribution of three poles. Figure 4 shows the truncation of (5) to the third
 300 most sensitive modes as a function of ω_r and z , for $r = 243$ km. The first of these modes
 301 gives the leading order behavior up to 0.5 Hz (figure 4a), whereas the third mode is required

302 as it describes the high frequency content close to the ground (figure 4c). Given that ω is
303 a continuous variable, there are an infinite number of such contributions, which have to be
304 included by the integration over ω . However, if the Fourier transform of the source is zero
305 outside a finite-length interval, usually taken to be $0 \leq \omega \leq \omega_0$, the contribution of high
306 frequencies $\omega > \omega_0$ turns out to be zero, as a result of the convolution theorem.

307 The previous results can be interpreted within the setting of the range-dependent for-
308 mulation presented in section II.C. Provided the sound speed profile is slowly varying in the
309 horizontal direction, the expansion (1) is the first order Taylor series of $c(z)$ about $R = 0$
310 and ϵ plays the same role as in the section II.C. When considering the acoustic propagation
311 downstream, $k_j(R; \omega)$ traces out a trajectory issuing from the cusp. The associated contri-
312 bution as given by (5) becomes dominant over a frequency range that depends on vertical
313 structures of $c_1(z)$ and thus, a lower bound for the frequency range can be obtained from
314 the sensitivities. For each R , this lower bound plays the role of a cut-off frequency $\omega_0(R)$.
315 On the other hand, it can be shown that the Hankel function in (6) vanishes as $\omega_r \rightarrow \infty$.

316 **IV. A low-order reduced model**

317 The basic principle in the reduction methods consists in computing a small number of
318 relevant modes (a part of the spectrum) through specific methods. If a matrix is really large,
319 the computation of its complete spectrum is out of reach. However, as noted in the previous
320 section, we only require the computation of the few of the eigenvalues that are closest to

321 some specified target value. Eigenvalue methods for large problems are designed to perform
 322 tasks like these by computing the invariant subspace associated with the desired eigenvalues.

323 **A. A WKB analysis of the spectrum**

324 A close examination of figure 2 shows that the eigenvalues lie on interconnected curves
 325 that are related to specific regions in the effective sound speed profile. The global structure of
 326 these curves can be obtained using an analysis which is similar to that of *Bender and Orszag*⁸,
 327 except that the eigenvalues are complex. The main assumption, besides the exclusion of
 328 range-dependence, is that the frequency is sufficiently large, so that we have $K = \epsilon k = O(1)$
 329 and $\Omega = \epsilon \omega = O(1)$, where ϵ is a small parameter. Using this approximation in (2) gives

$$\epsilon^2 \frac{d^2 p}{dz^2} = Q(z)p, \quad (15)$$

330 where Q is given by

$$Q(z) = K^2 - \frac{\Omega^2}{c^2(z)}. \quad (16)$$

331 For a real frequency, the WKB analysis performed in the limit $\epsilon \rightarrow 0$ for a vertical profile
 332 $c(z)$ with the typical shape displayed in figure 5 shows that the modes of the “bounded states”
 333 (i.e. with $K_1 < K < K_2$) are given by the well-known *Bohr-Sommerfeld* quantization rule
 334 (*Bender and Orszag*⁸)

$$\frac{1}{\epsilon} \int_{z_1}^{z_2} [-Q(z)]^{\frac{1}{2}} dz = 2\pi (n + 1/2), \quad (17)$$

335 where z_1 and $z_2 > z_1$ are first-order turning points and $\text{Re}[Q(z)] < 0$ for $z_1 < z < z_2$. For
 336 the general case in which Ω is a complex frequency, we note that z_1 and z_2 are not zeros
 337 of the complex function Q and thus, equation (17) no longer holds. New conditions can be
 338 derived from matching conditions of the approximations to $p(z)$. These matching conditions
 339 translate into constraints on both the real and imaginary parts of Q . The details of the
 340 algebra are given in the appendix for an arbitrary profile $c(z)$. For the sake of simplicity,
 341 we restrict the discussion here to the profile displayed in figure 5. Denoting z_1 and z_2 two
 342 consecutive zeros of $\text{Re}(Q)$, the constraint on the real part of Q is given by equation (17) in
 343 which Q must be replaced by $\text{Re}(Q)$. The constraint on the imaginary part of Q is

$$\frac{1}{\epsilon} \int_{z_1}^{z_2} \text{Im} [-Q(z)]^{1/2} dz = \frac{2}{3\epsilon} \text{Re} \left[\frac{Q^{3/2}(z_2)}{Q'(z_2)} - \frac{Q^{3/2}(z_1)}{Q'(z_1)} \right] + O(\epsilon), \quad (18)$$

344 which reduces to

$$\frac{1}{\epsilon} \int_0^{z_0} \text{Im} (-Q(z))^{1/2} dz = \frac{2}{3\epsilon} \text{Re} \left[\frac{Q^{3/2}(z_0)}{Q'(z_0)} \right] + O(\epsilon), \quad (19)$$

345 when the oscillating region extends to the ground; i.e. $\text{Re}[Q(z)] < 0$ for $0 \leq z < z_0$. Through
 346 algebraic manipulations, we can generalize the condition (18) to an arbitrary number of zeros
 347 of $\text{Re}(Q)$. It is worth mentioning that equation (18) is based on the fact that each zero is
 348 simple and thus, this approach fails for altitudes z at which two oscillating regions merge.
 349 This is a familiar restriction which excludes a small neighborhood of cusp regions in the
 350 complex K -plane.

351 Following the above reasoning, the real turning points $z(K)$ and the oscillating regions
352 are first obtained by solving $Q(z; K, \Omega) = 0$, for a fixed frequency Ω . As the mapping of a
353 given K into the complex z -plane is generally multivalued, we denote the various images of
354 K that lie along the real axis by $z_m(K)$. The mapping of a contour in the complex K -plane
355 into the z -axis may be rendered single-valued by constructing a multisheeted K -plane with
356 n sheets, each corresponding to a single oscillating region $z_m < z < z_{m+1}$. In some sense, the
357 profile $c(z)$ can be seen as a potential, each of its oscillating regions being associated with one
358 curve γ_n (lying on a single sheet) in the complex K -plane on which eigenvalues are distributed
359 according to the real part of the Bohr-Sommerfeld condition (17). The appearance of a cusp
360 in the spectrum structure (which is a branch point of the square root in (17)) at a particular
361 phase velocity serves as a warning signal that the partition into two disconnected sheets is
362 not possible. Such a cusp corresponds to a local maximum (a double turning point) of $c(z)$
363 at which two oscillating regions merge and, thus, two sheets are connected in the complex
364 plane, as illustrated in figure 5.

365 Equation (18), while hardly trivial, is readily solved (for a fixed Ω) by standard methods
366 for complex root determination. Here we use a Newton-Raphson method for finding K ,
367 subject to the constraint $z_m < z < z_{m+1}$. Figure 5 shows that typical curves γ_n ($n = 1, 2, 3$)
368 merge at $k_r = \omega_r/c$, where $c = 320 \text{ m.s}^{-1}$. Similar results are shown in figures 6 and 7
369 for both the ERA-Interim and the 91-level profiles. As can be seen from figures 6 and 7,

370 the most sensitive modes issuing from the cusp at $c_r \simeq 335 \text{ m.s}^{-1}$ converge to the path
371 *in which the imaginary part k_i (orange curve) is minimal*, so that, in view of the results in
372 section III, the eigenvalue computation can be restricted to the invariant subspace associated
373 with the neighborhood of this curve, which will be called γ_1 . Specifically, an implicitly
374 restarted Krylov-Arnoldi algorithm (*Watkins*³⁸) is used with a shift-and-invert method, or
375 a Cayley transform (*Lehoucq*²⁶, *Meerbergen et al.*²⁹), centered in the middle of γ_1 . Each
376 restart gives a better approximation to the desired invariant subspace. Repeated restarts
377 lead to convergence. Since Krylov subspace methods have good convergence properties when
378 combined with Tchebychev collocation points (*Beattie*⁶), we find a good agreement between
379 both the QR algorithm (LAPACK Fortran library) and the Arnoldi algorithm (ARPACK
380 Fortran library); the relative error of the eigenvalues being below 10^{-6} . Figure 8 shows
381 results for the profile $c_{1a}(z)$. In terms of computational cost, there is a huge benefit in doing
382 so, especially when the frequency is high (figure 8b). While the QR algorithm needs more
383 than 7 h to compute a signal up to 1 Hz (with 512 frequency samples) the application of the
384 Arnoldi algorithm in the vicinity of γ_1 leads to a total computation time as short as several
385 minutes (2 %) on a single processor.

386 It should be stressed that the main difficulties in applying the Krylov-Arnoldi algorithm
387 are to determine (1) the dimension m that warrants the accuracy of the required eigenvalues
388 and (2) the number n of desired eigenvalues ($n < m$). Given a vector x and a matrix

389 A , the Krylov space of A is the m -dimensional subspace spanned by $x, Ax, \dots, A^{m-1}x$.
390 The basic idea in *Krylov subspace* or *projection* methods is to construct eigenpairs (the
391 Ritz pairs) in the Krylov subspace \mathcal{K}_m so that the component orthogonal to that space
392 is sufficiently small for the Ritz pairs to be good approximations to eigenpairs of A . A
393 convenient way to do this is given by the Arnoldi factorization (*Watkins*³⁸) which provides
394 the eigenpairs of A from the eigenpairs of smaller matrices. Such a method, however, is
395 entirely dependent on the choice of the starting vector x . Here, we require x to be rich in
396 the subspace spanned by the eigenvectors corresponding to eigenvalues that are nearest the
397 curve γ_1 with very small components in the direction of the other eigenvectors. We follow
398 the implicitly restarted Arnoldi method (*Watkins*³⁸) and adaptively refine x to be a linear
399 combination of the n eigenvalues without explicitly computing a new Arnoldi factorization.
400 The larger m is, the better is our chance that the space \mathcal{K}_m contains good approximations to
401 desired eigenvectors. However, the convergence of the implicitly restarted Arnoldi method
402 is not uniform (*Lehoucq*²⁷) and there is no known a priori value of m leading to optimal
403 convergence. Even though convergence techniques can considerably decrease the numerical
404 cost of the Krylov approach, in the present study, however, the dimension of the Krylov space
405 is fixed to $m = 0.01N$, where N is the matrix size (typically $m = 20$). The number n of
406 eigenvalues plays an essential role since the conditioning of the $m - n$ unwanted eigenvalues
407 strongly affects the convergence of the implicitly restarted algorithm (*Beattie*⁷.) In this

408 study, we used a QR computation at the frequency of 1 Hz to estimate the number of
 409 eigenvalues lying on the curve γ_1 as $\omega_r \rightarrow 2\pi$. Based on the numerical results, we have 3 to
 410 5 eigenvalues, depending on the profile (figure 11a). Then, for each frequency ω , the Arnoldi
 411 algorithm was applied to find the $n = 10$ closest eigenvalues to γ_1 .

412 Recall that for the range-dependent approach, the sensitivity of eigenvalues strongly
 413 depends on the local vertical profile $c(z; R)$ so that, modes may become dominant over a
 414 fairly narrow range of distances or new modes (issuing from a single or several new paths) may
 415 literally appear/disappear. The higher the horizontal variation, the larger the probability
 416 is to have a significant contribution from these modes to the ground-based signal. That is
 417 the main reason for which the sound speed profile evolution $c(z; R)$ has to be studied before
 418 proceeding with our reduction method. Once the relevant curves $\gamma_n(\omega, R)$ (and plausible
 419 cusps) have been computed over the frequency-range domain, a Krylov-Arnoldi algorithm
 420 can be used to compute the invariant subspace associated with the most sensitive eigenvalues.
 421 The eigenvalues can finally be analyzed as R varies with a Newton-Raphson algorithm or a
 422 step-by-step Krylov-Arnoldi algorithm.

423 **B. Reduced models for range-dependent media**

424 The above reduction technique is based on the eigenvalue sensitivities which means
 425 that the reduced model preserves the overall structure under sufficiently small perturbations
 426 of the vertical profile. The duality between range-dependence and perturbation (revealed

427 by equation (1)) can be used to extend the validity domain of the reduced model to range-
 428 dependent media. From a numerical standpoint, this task reduces in following the selected
 429 eigenvalues as the vertical profile slowly varies along the source-receiver path.

430 Recall that the spatial eigenvalues $k_j(\omega)$ have been introduced as solutions of the disper-
 431 sion relation when the contour F in the complex ω -plane differs from the real axis. Although
 432 a straightforward interpretation is missing, the temporal branches $k_j(\omega)$ are objects that are
 433 naturally involved whenever the initial contour is gradually displaced upward from the real
 434 axis. According to the frequency shifting property of the Fourier-Laplace transform, the
 435 ground-based signal can be obtained from the classical FFT algorithm (for $\omega = \omega_r$) together
 436 with the mapping $p(t) \mapsto p(t)e^{\omega_r t}$. A trapezoidal rule was used to compute the phase in
 437 (7), using 12 intermediate ranges R between the source and the station. While the complete
 438 computation involves 580 modes, the implicitly restarted Krylov-Arnoldi method is used to
 439 find the 3 most sensitive modes that correspond to the tropospheric waveguide, with a cost
 440 of 2 % of the total run time, as previously indicated (figure 8). Here, we use a simplified
 441 source model defined by

$$s(t) = \left[1 - \cos \frac{\pi t}{2} \right] \sin \left(\pi t + \frac{\pi}{2} \right), \quad (20)$$

442 together with $s(t) = 0$ for $t < 0$ s and $t > 4$ s. Hence, the power spectra at the station I30JP
 443 ($R=243$ km) is simply the product of the Fourier Transform of (20) and (7).

444 It is important to emphasize that the tropospheric sheet in the complex k -plane (fig-

445 ure 5) contains all the necessary information for waveform synthesis. This sheet is an at-
 446 tracting sector for the mode trajectories (as the frequency increases) that are most sensitive
 447 to low altitude perturbations thereby leading to a physical interpretation in terms of atmo-
 448 spheric structures. Futhermore, the modes emerge from a cusp at a specific cutoff frequency
 449 ω_0 (0.09 s^{-1} , 0.39 s^{-1} and 0.71 s^{-1} for the first, second and third mode, respectively). On
 450 the other hand, since $\omega_i > 0$ these modes lie in the upper half of the complex k -plane and
 451 thus, the contribution of the various poles vanishes as $\omega_r \rightarrow +\infty$, as shown in figure 4. We
 452 can therefore take advantage of these results to obtain an approximation of the temporal
 453 Fourier transform.

454 To develop the leading-order term of the inverse Fourier transform, we first rewrite (7)
 455 along an arbitrary fixed spatio-temporal ray $r/(t - \tau) = v$, where $\tau\omega$ is the phase shift of
 456 the source term (20). Then, we take its Fourier transform so as to obtain

$$p(r, 0; r/v) = \frac{ie^{-i\frac{\pi}{4}}}{4\pi\sqrt{2\pi r}} \sum_{j \in J} e^{\omega_i(\frac{r}{v} + \tau)} \int_{-\infty}^{+\infty} p_j(\omega_r, r) \exp\left\{\frac{i}{\epsilon}\Phi_j(\omega_r, R)\right\} d\omega_r, \quad (21)$$

457 where J is the subset of most sensitive modes and

$$p_j(\omega_r, r) = \frac{\phi_j^2(0; \omega, R)}{\sqrt{k_j(\omega, R)}} |\hat{s}(\omega_r)|, \quad (22)$$

458 where \hat{s} is the Fourier transform of (20). In the limit $\epsilon \rightarrow 0$ with R fixed, the integral may be
 459 evaluated asymptotically by the well-known stationary phase method. The phase function

460 of (21) is

$$\Phi_j(\omega_r) = \int_0^R k_j(s, \omega_r) ds - \frac{\omega_r R}{v}. \quad (23)$$

461 The points of stationary phase are given by the roots of $\Phi'_j(\omega_r) = 0$ or, from (23),

$$\int_0^R \frac{\partial k_j}{\partial \omega} ds - \frac{R}{v} = 0, \quad (24)$$

462 subject to the constraint $\omega_0 < \omega < 2\pi$, where ω_0 is the cut-off frequency ($\omega_0=0.09 \text{ s}^{-1}$ for
463 the first most sensitive mode).

464 Two distinct behaviors are possible. Whenever $v_1 < v < v_2$, (24) has a unique
465 real solution that we denote by $\bar{\omega} = \bar{\omega}(R, v)$, as shown in figure 9. Thus, among all the
466 frequencies contained in the source, the atmosphere filters out, along each ray $x/t = v$ one
467 particular frequency $\bar{\omega}$ given by (24). In the opposite case, where the velocity v is greater
468 than v_2 or less than v_1 , there is no solution to (24) and thus, the integral vanishes (*Bender*
469 *and Orszag*⁸). Indeed, once the velocity front v_1 has reached the infrasound station, the
470 atmosphere returns to the rest state.

471 The leading-order contribution arising from the first dominant mode ($j = 1$) in expres-
472 sion (21) can be evaluated according to the general formulas given in *Bender and Orszag*⁸.

473 As $\epsilon \rightarrow 0$, the impulse response associated with this mode reduces to

$$p(r, v) \sim \frac{i\bar{\phi}_1^{-2}(R, v)e^{i\frac{\bar{\Phi}_1(R, v)}{\epsilon} + \omega_i(\frac{r}{v} + \tau)}}{4\pi\sqrt{r} \left[\frac{\bar{k}_1(R, v)}{\epsilon} \int_0^R \frac{\partial^2 k_1}{\partial \omega^2}(s, v) ds \right]^{\frac{1}{2}}} \left[1 + \frac{\alpha(R, v)}{\epsilon} + \dots \right] \quad (25)$$

474 where

$$\alpha = \frac{5ip_1[\Phi_1^{(3)}]^2 + 12ip_1^{(2)}\Phi_1^{(2)^2} - 12ip_1^{(1)}\Phi_1^{(3)}\Phi_1^{(2)} - 3ip_1\Phi_1^{(4)}\Phi_1^{(2)}}{12p_1[\Phi_1^{(2)}]^3}, \quad (26)$$

evaluated at $\bar{\omega}$, with the notation

$$\Phi_1^{(l)} = \frac{\partial^{(l)}\Phi_1}{\partial\omega^{(l)}}(R, \omega), \quad p_1^{(l)} = \frac{\partial^{(l)}p_1}{\partial\omega^{(l)}}(R, \omega) \quad \text{and} \quad \phi_1 = \phi_1(0; R, \omega).$$

475 The response takes the form of a wavepacket in the (r, t) plane, as shown in figure 9. The
 476 wave packet is confined within a wedge bounded by the two rays $r/t = 328 \text{ m.s}^{-1}$ and
 477 $r/t = 335 \text{ m.s}^{-1}$. Inside the packet, the contribution is dominated by the stationary phase.
 478 For $\alpha = 0$, the wave packet (25) is compared to the FFT with 580 modes in figure 10.
 479 We find a very good overall approximation of the waveform with a single mode (the first
 480 most sensitive mode issuing from the cusp at $c_r \simeq 335 \text{ m.s}^{-1}$) evaluated at the frequency
 481 $\bar{\omega}$. In practice, we apply the Krylov-Arnoldi algorithm to find the eigenvalues that are
 482 nearest the curve γ_1 in the complex k -plane. Once the eigenvalues k_j are obtained for a
 483 fixed frequency, a Newton-Raphson method is used to solve $\Phi_1' = 0$ for a given v , without
 484 significant supplementary CPU time. In striking contrast to other numerical methods, the
 485 complex waveform can be computed from a single mode evaluated at the frequency $\bar{\omega}$, with
 486 a CPU time of typically 1-10 seconds.

487 The question of how relevant the stationary phase approximation is to practical applica-
 488 tions is intimately related to confidence intervals. Through algebraic manipulations, we can

489 find a higher-order asymptotic expansion to (21). However, the expansion requires higher-
490 order derivatives $k_j^{(3)}, k_j^{(4)}, \dots$ (see (26)) and thus, accurate means of solving the dispersion
491 relation in the complex k -plane. Such an approach would require higher computer time. It
492 led us to believe that our approach, which gives an approximate waveform at a considerably
493 lower CPU time, promises to be more fruitful than seeking the maximum precision for a
494 fixed atmospheric state. In other words, we hypothesize that, the higher the sampling size,
495 the better the obtained statistics, which is the ultimate goal of many infrasound studies.
496 In the meantime, the present paper has shown how the CPU time can be considerably re-
497 duced, through the analytic continuation of the range-dependent normal mode method in
498 the complex k -plane, which provides relevant statistics for applications.

499 **V. On the role of atmospheric uncertainties**

500 According to the discussion of section IV, the propagation mechanisms that occur just
501 after the Fukushima explosion are dominated by three modes that evolve on a slow length
502 scale ϵ^{-1} , where ϵ is defined by equation (1). These modes trigger a tropospheric arrival
503 which is in good *qualitative* agreement with the first arrival of the recorded signal (see figure
504 1). However, the direct computation predicts also a stratospheric ‘tail’ which is several orders
505 of magnitude weaker than the tropospheric arrival (blue signal, figure 12). This is clearly
506 not what was recorded at the Japanese station.

507 An interesting aspect of using a complex frequency is that the imaginary parts of eigen-

508 values may be arranged so that the tropospherically-ducted modes lie along the path γ_1 in
509 the complex k -plane as $\omega_r \rightarrow \infty$. Since this path can be obtained at very low numerical cost
510 through a WKB analysis, the propagation problem can be projected onto an invariant sub-
511 space corresponding to these modes. Hence, we obtain a reduced model for the tropospheric
512 arrival by seeking eigenvalues that are close to γ_1 with a Krylov-Arnoldi algorithm. One
513 value of the reduced model lies in the reduction of the order (the number of modes) which is
514 based on the sensitivities of the eigenvalues over a given frequency range. By rearranging the
515 modes by decreasing sensitivities (and decreasing contribution to the ground-based signal),
516 the reduced model preserves the overall structure of waveforms under perturbations of the
517 troposphere. In a sense, the requirement for the reduced model to be reasonably robust
518 to variations of the profile $c(z)$ is similar to the so-called structural stability property in
519 mathematics.

520 Figure 11e-g shows the ground based signals at I30JP obtained from three synthetic
521 atmospheric conditions, with an effective sound speed at the local tropospheric maximum
522 significantly smaller, equal or significantly larger than that at ground level. These profiles
523 are obtained from c_{1a} perturbed by a Gaussian envelope centered at the altitude of maxi-
524 mum effective sound speed, as shown in figure 11a. The relevant eigenvalues (close to γ_1)
525 are computed with a Krylov-Arnoldi algorithm. We obtain 2, 3 or 4 eigenvalues at 1 Hz
526 (figure 11b-d), depending on the profile $c(z)$. The associated ground-based signals obtained

527 with the reduced model (truncated to 2, 3 and 4 modes) are in excellent agreement with
528 that obtained with the full model (i.e. with 580 modes). Being able to represent the signals
529 with a small number of modes offers an argument that the reduced model may be used to
530 derive statistical results for a wide range of tropospheric conditions.

531 In the above reduced model, it was explicitly assumed that the vertical sound speed pro-
532 files are fixed and obtained from the European Centre for Medium-Range Weather Forecasts
533 (ECMWF). This assumption, although providing a good approximation of the tropospheric
534 wavepacket, cannot be used to compute the stratospheric arrival. Indeed, it is now generally
535 recognized that small-scale gravity waves are filtered out of the ECMWF fields. At leading
536 order, the amplitude of the upward propagating waves grows in altitude as $1/\sqrt{\rho}$, where ρ
537 is the density of the atmosphere. Above 60 km, the amplitude of the unresolved waves can
538 reach 10% of the sound speed (*Drob et al.*¹⁷). Even though in the lower atmosphere the
539 amplitude of these waves represents a small fraction of the overall variations of the average
540 background state, a quantitative theory of waveform changes due to gravity waves is not
541 available.

542 Figure 12 provides further calculations, using as the input profile a slightly perturbed
543 version of c_{1a} (figure 1). For simplicity, the perturbation is modelled by a single upward
544 gravity wave modulated by a Gaussian envelope that mimics the gravity wave breaking.
545 The main effect of the localized perturbation is to slightly redistribute the sensitivities so

546 that a second subset of sensitive eigenvalues arises from another cusp region, as shown in
547 figure 12b. The associated modes trigger a stratospheric arrival (green signal) that appears
548 to be dominant for specific vertical wavelengths, even though the perturbed sound speed
549 remains smaller than the value at ground level. The effect is, however, non-uniform being
550 largest for a vertical wavelength of about 1 km.

551 Following the method described in section IV, the two arrivals can be obtained by
552 applying the Krylov-Arnoldi algorithm (figure 12c) twice and adding the new eigenvalues
553 to the reduced model. While the perturbation amplitude is limited to less than 1% of
554 the effective sound speed, it is evident from figures 12 and 1 that there is a considerably
555 improved overall agreement between calculated and recorded signals. Finally, note that the
556 main difficulty is to determine which phase dominates in the computed signal. *A priori* this
557 requires the detailed analysis of the subsets containing the most sensitive eigenvalues in the
558 complex k -plane.

559 VI. Conclusion

560 We have examined the pressure field generated by a point source explosion within
561 a background atmospheric state given by a combination of ECMWF data and empirical
562 reference models. The impact on long-range infrasound propagation of small departures
563 from a given profile was studied through perturbation theory. In practice, these imperfections
564 may stem from atmospheric structures that are not considered (e.g. gravity waves) or they

565 may simply be the result of horizontal variation. While a consensus seems to have emerged
566 that gravity waves with length scales of less than about 100 km are filtered out in available
567 atmospheric specifications, it is not clear whether classical high frequency techniques are
568 able to capture the effects of these scales.

569 In this paper, we used a range-dependent normal mode approach with no constraint
570 on the frequency. The analysis of the vertical structure of the wave equation in the com-
571 plex wavenumber plane (the so-called k -plane) indicates that only a few eigenvalues are
572 involved in long-range propagation of infrasounds. These modes, which are sensitive to key
573 atmospheric features, can be computed with a high degree of accuracy through a Krylov
574 subspace process that uses a shift-and-invert strategy. Sensitivity arguments indicate that
575 these modes are close to curves in the complex k -plane that may be obtained through the
576 analytic continuation of the classical WKB analysis. The process time of our reduced model
577 is two orders of magnitude less than the computation of the complete spectrum with a QR
578 algorithm. It therefore appears that a low order reduced model based on a few suitable
579 eigenvalues can provide the impulse response over large ranges of frequencies. This reduced
580 model maintains full generality and can be applied to range-dependent media so that wave-
581 forms can be recovered by using a stationary phase method. In this case the process time is
582 no longer than several seconds on a single processor.

583 Our reduced model gives insights into the role of atmospheric structures in the wave-

584 form features. By means of a wavelet-based decomposition of the profile perturbation, we
585 have shown that the components leading to the most sensitive eigenvalues give regions and
586 wavelengths of atmospheric structures involved in the propagation. Other regions of the
587 background atmosphere are not relevant to the computed waveform. Indeed, our reduced
588 model allows sensitivity analysis and statistical studies at minimum CPU time.

589 **I. WKB approximation**

590 We consider here the wave equation

$$\epsilon^2 \frac{d^2 p}{dz^2} = Q(z; K, \Omega) p, \quad (27)$$

where ϵ is a small parameter, with

$$Q(z; K, \Omega) = K^2 - \Omega^2/c^2(z),$$

591 where K and Ω are complex. Furthermore, we make the assumption that $\text{Im}[Q] \ll \text{Re}[Q]$ so
 592 that (27) refers to the complex perturbation of the real problem fully treated by *Bender and*
 593 *Orszag*⁸. By limiting the asymptotic expansion of the phase to order 1, the classical approach
 594 leads us to express the solution as a linear combination of two independent solutions:

$$p(z) = Q^{-1/4}(z) \left[A e^{\frac{1}{\epsilon}\phi(z)} + B e^{-\frac{1}{\epsilon}\phi(z)} \right], \quad (28)$$

595 where $\phi' = Q^{1/2}$ and A and B are scalar constants depending on boundary conditions. Since
 596 $Q(z)$ is close to the real axis, the solution clearly has two distinct behaviors along the z -axis:
 597 the solution is dominated by a real exponential for $\text{Re}[Q(z)] > 0$ and is mostly oscillating
 598 in regions where $\text{Re}[Q(z)] < 0$.

599 In the vicinity of points z^* such that $\text{Re}[Q(z^*)] = 0$, the asymptotic expansion is no
 600 longer valid. Nevertheless, a first order Taylor series expansion can be obtained through
 601 a change of variable $z \rightarrow Z(z; \epsilon)$, thereby leading to the Airy equation $d^2/dZ^2 p = Zp$.

602 The solution in this critical layer can be expressed as a linear combination of Airy functions.
 603 Matching arguments provide a direct linear relation between the coefficients of (28), (A_u, B_u)
 604 and (A_l, B_l) above and below the critical layer respectively;

$$\begin{bmatrix} A_u \\ B_u \end{bmatrix} = P \begin{bmatrix} A_l \\ B_l \end{bmatrix}. \quad (29)$$

605 Let the zeros of $\text{Re}(Q)$ be sorted so that $0 < z_1 < z_2 < \dots < z_N$, and P_1, P_2, \dots, P_N
 606 be the corresponding *passing* matrices. From an iterative process, we can deduce from (29)
 607 the relation

$$\begin{bmatrix} A_N \\ B_N \end{bmatrix} = M \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}. \quad (30)$$

608 where $M = P_N \cdots P_2 P_1$.

609 Finally, we take into account the boundary conditions at $z \rightarrow \infty$ and $z = 0$ to obtain
 610 conditions on Q . The boundary conditions can be written as linear relations involving the
 611 coefficients $\alpha_1 A_1 + \beta_1 B_1 = 0$ for $z = 0$, and $\alpha_N A_N + \beta_N B_N = 0$ for $z \rightarrow \infty$. Adding these
 612 conditions to the system (30) provides a unique relation that is a necessary and sufficient
 613 condition for the existence of a solution for the problem (27)

$$\beta_1 (\alpha_N M_{11} + \beta_N M_{21}) = \alpha_1 (\alpha_N M_{12} + \beta_N M_{22}). \quad (31)$$

614 Real and imaginary parts of the above relation lead to two expressions. First, we find
 615 the *Bohr-Sommerfeld* quantization rule (17) where z_1 and z_2 correspond to ordered pairs

616 surrounding an oscillating area ($\text{Re}[Q] < 0$). Then, since $\text{Im}[Q] \neq 0$, there is also the
617 necessary condition (18) or (19) in the case $\text{Re}[Q] < 0$ near the ground ($0 < z < z_1$). Note
618 that (18) and (19) are given for homogeneous boundary conditions. In other words, the
619 solution vanishes as $z \rightarrow \infty$ and there is a perfect reflection at $z = 0$ ($p_1'(0) = 0$).

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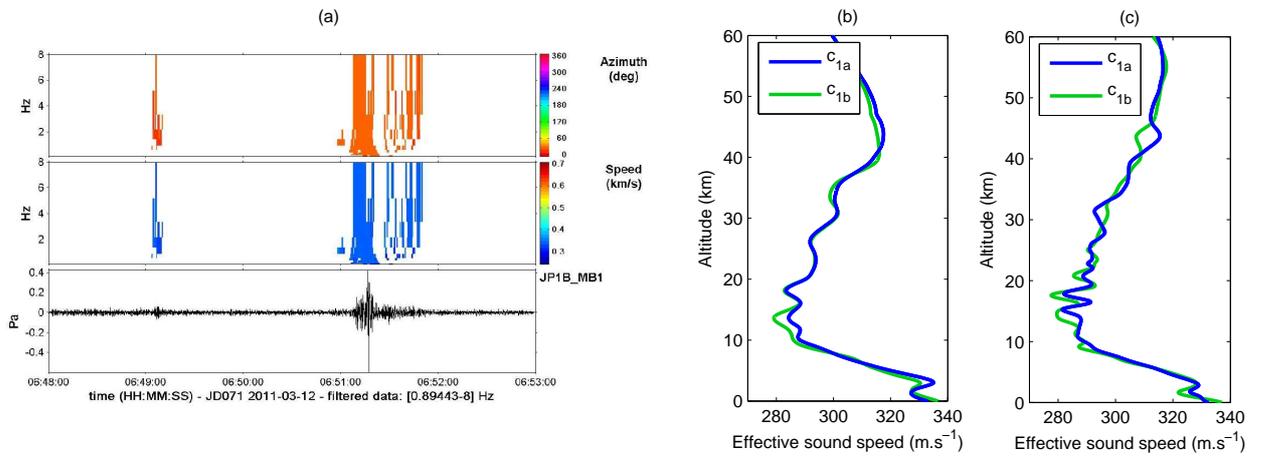


Figure 1: Data. (a): recorded signal at the Japanese infrasound station I30JP; effective sound speed profiles obtained from the ECMWF ERA-Interim dataset (b) and the ECMWF 91-level dataset (c) for locations that correspond to the Fukushima nuclear power plant (c_{1a} , c_{2a}) and I30JP (c_{1b} , c_{2b}).

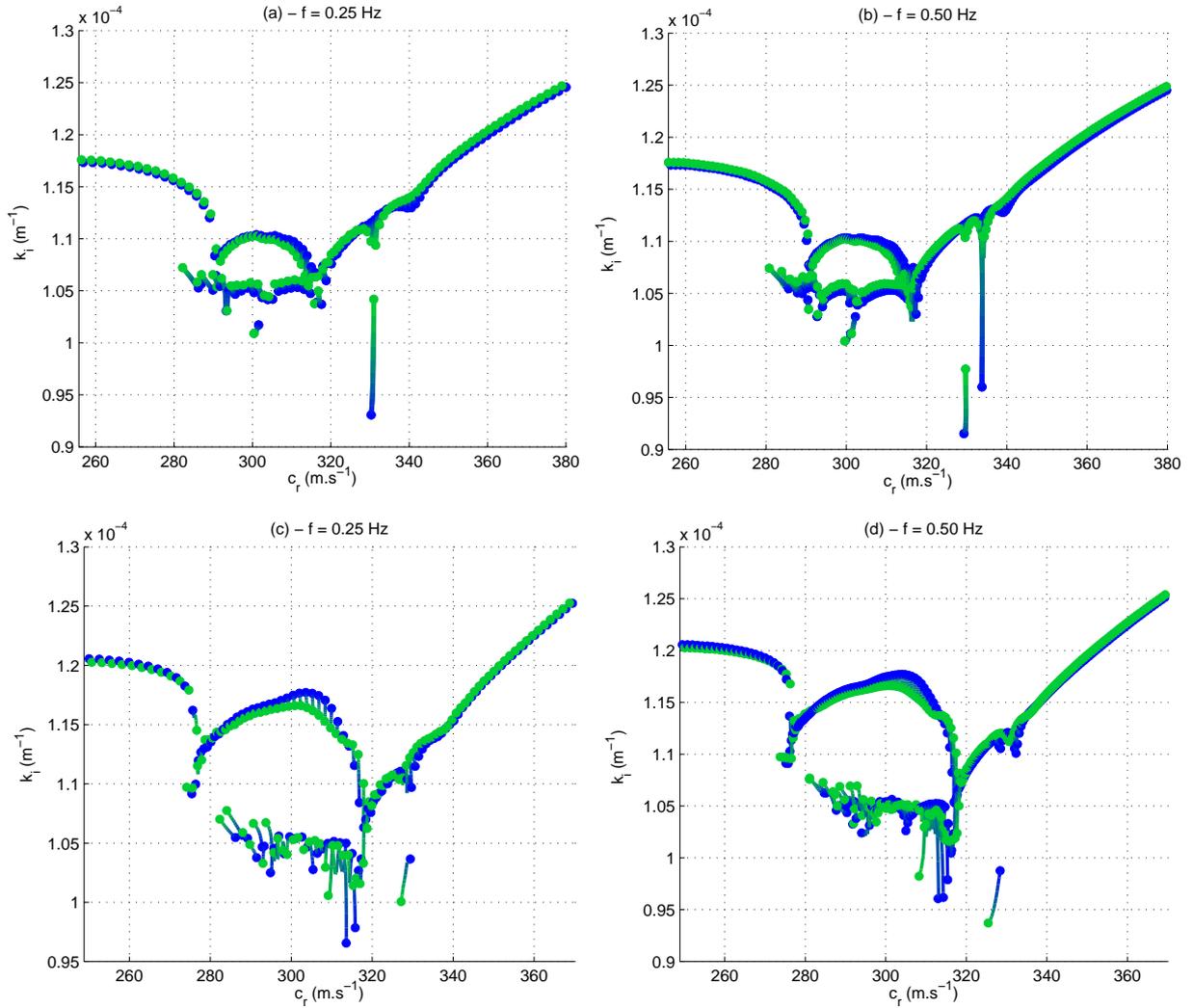


Figure 2: The complex k -plane showing the trajectories of the local eigenvalues as R varies for the ERA dataset (top) and the 91-level dataset (bottom). Blue: $r = 0$ km; green: $r = 243$ km (I30JP).

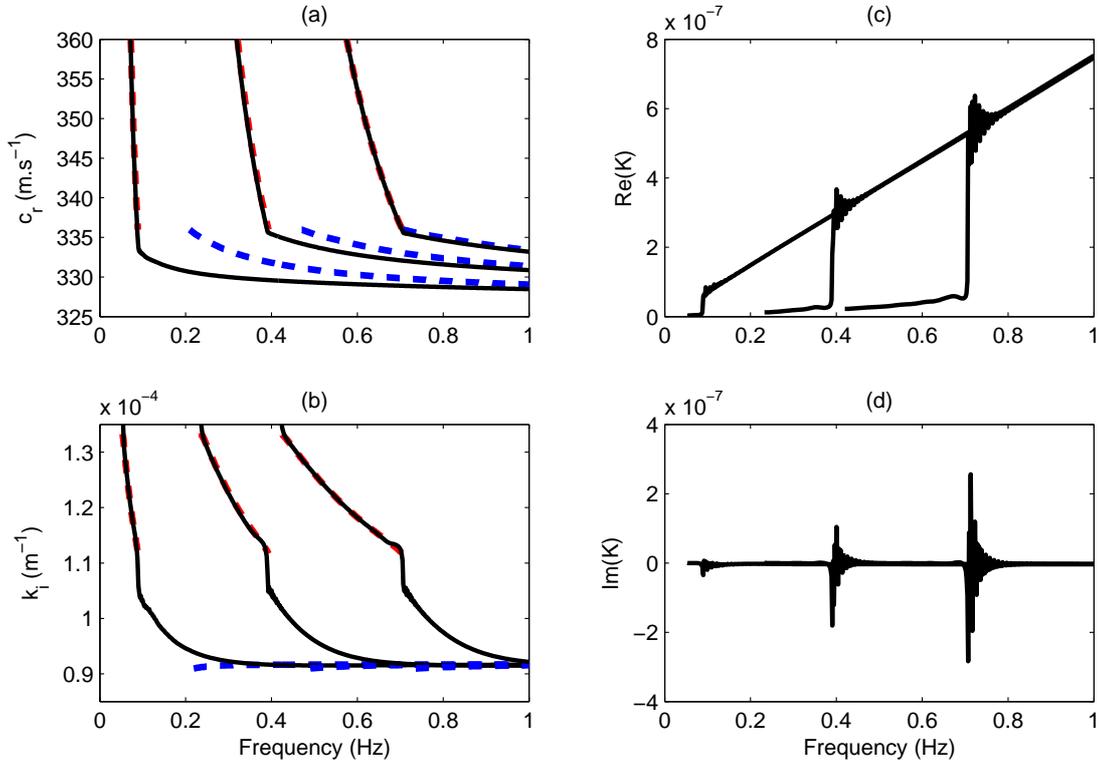


Figure 3: Phase velocity ω_r/k_r (a) and imaginary part k_i (b) of the 3 most sensitive eigenvalues, as functions of the frequency ω_r . Dashed lines represent the WKB approximations. Real (c) and imaginary (d) parts of sensitivities K_{jmn} associated with the most sensitive eigenvalues for D10 wavelets $a = 2$, $b = 1$, $m = 10$ and $n = 1$ (see equation 9). Eigenvalues are computed for frequencies with a constant imaginary part of $\omega_i=0.03 \text{ .s}^{-1}$.

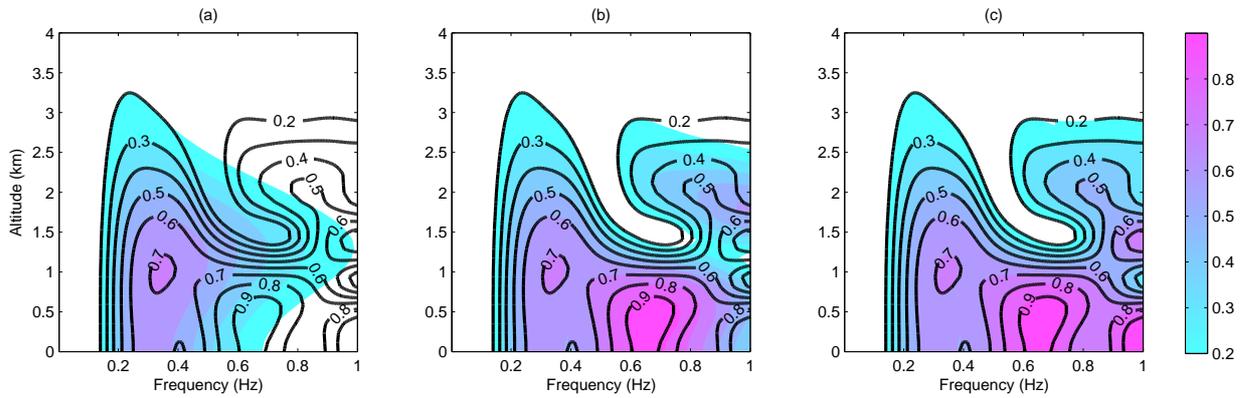


Figure 4: Pressure field contours in the ω_r - z plane at $r = 243$ km, for the three most sensitive modes of figure 3. (a) first mode; (b) two modes and (c) three modes. Black levels give the pressure field $|\hat{p}/\hat{p}_{\max}|$ computed with 580 modes. Eigenvalues are computed for frequencies with an imaginary part $\omega_i = 0.03 \text{ s}^{-1}$.

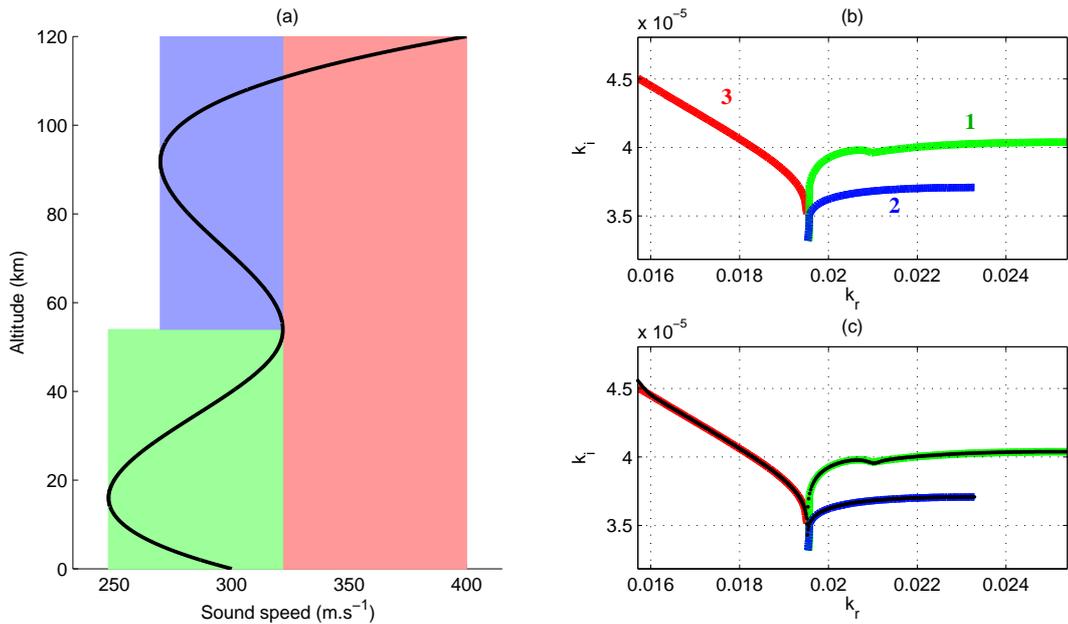


Figure 5: Typical effective sound speed profile (a) and various solutions of (17) (b) for $\omega = 2\pi + 0.03i$. The black circles (c) show the corresponding discrete spectrum (eigenvalues), as obtained with a QR algorithm.

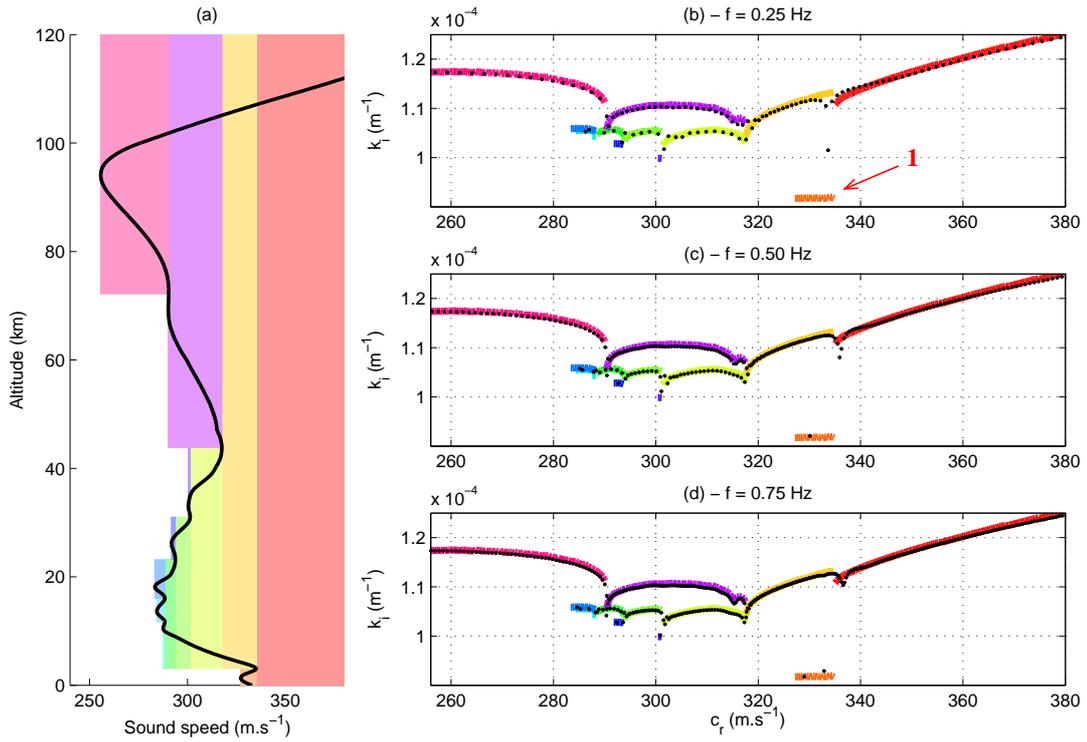


Figure 6: Various curves γ_n in the complex k -plane for $\omega_i = 0.03$ and the associated regions of the vertical profile $c(z)$. ERA-Interim data.

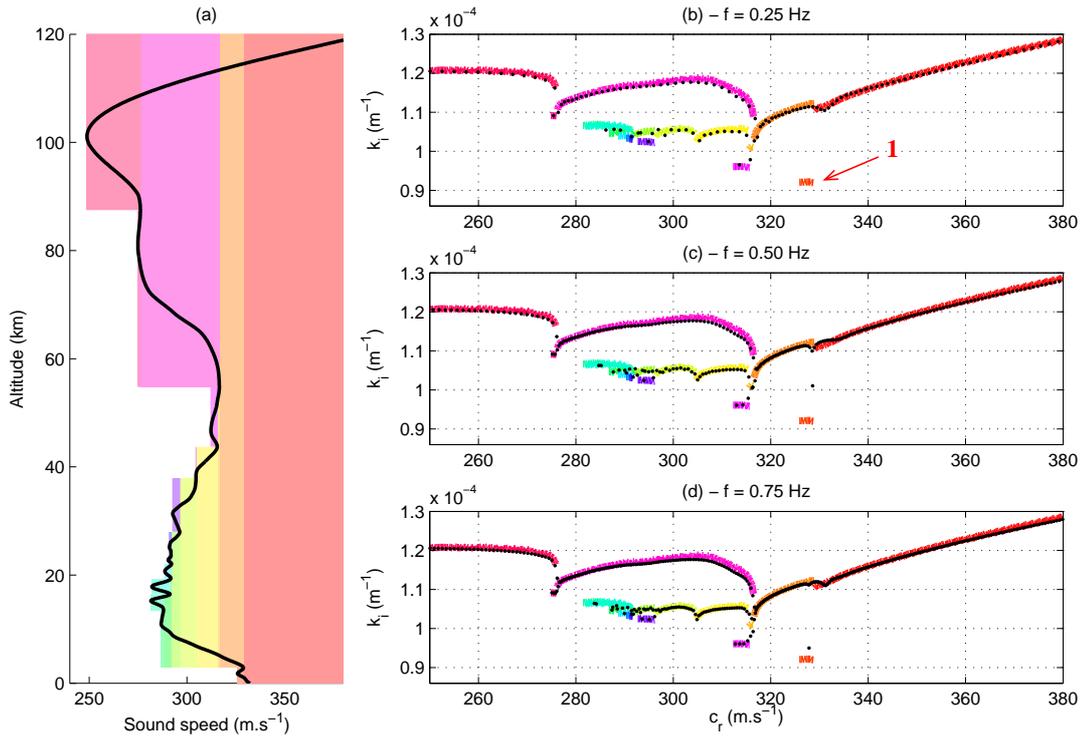


Figure 7: Various curves γ_n in the complex k -plane for $\omega_i = 0.03$ and the associated regions of the vertical profile $c(z)$. 91-levels data.

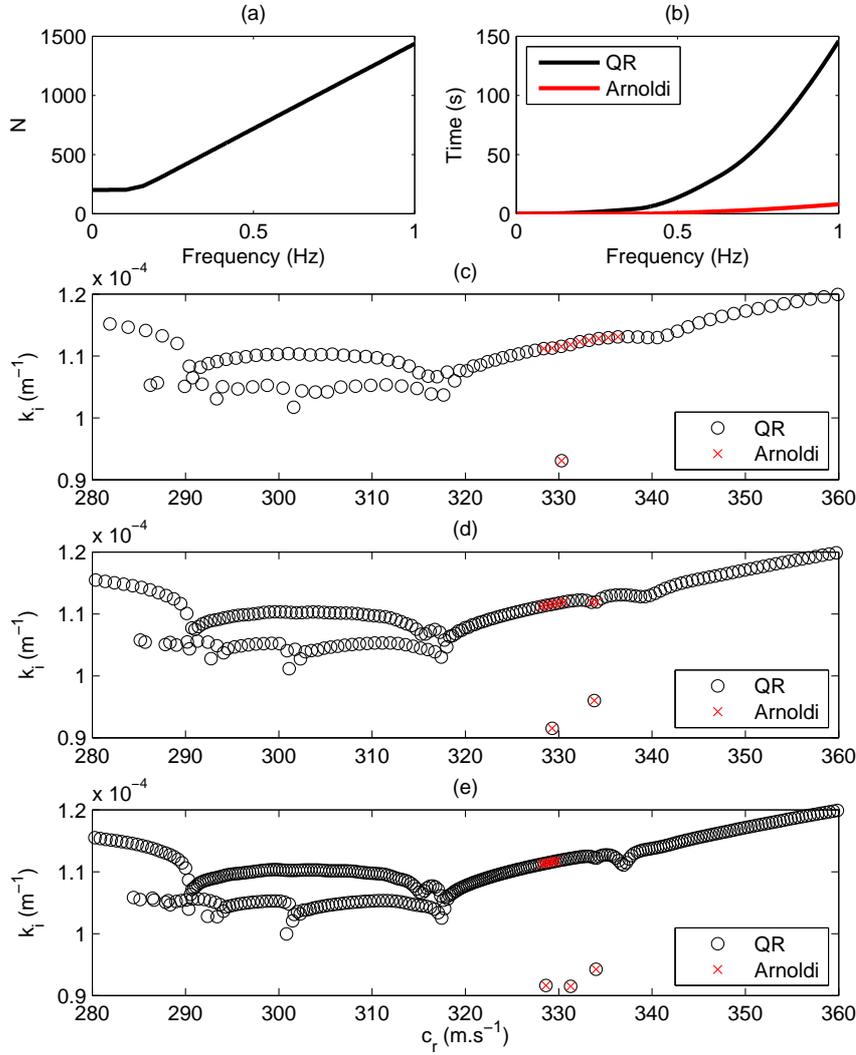


Figure 8: (a) Size N of the matrix discretization for the profile $c_{1a}(z)$ (figure 1) as a function of the frequency. (b) Computation time for both the QR algorithm (LAPACK) and the implicitly restarted Krylov-Arnoldi algorithm (ARPACK). (c),(d),(e) Eigenvalues computed with both the QR algorithm (whole spectrum) and the Krylov-Arnoldi algorithm (10 Ritz pairs) for frequencies 0.3 Hz, 0.6 Hz and 0.9 Hz, respectively.

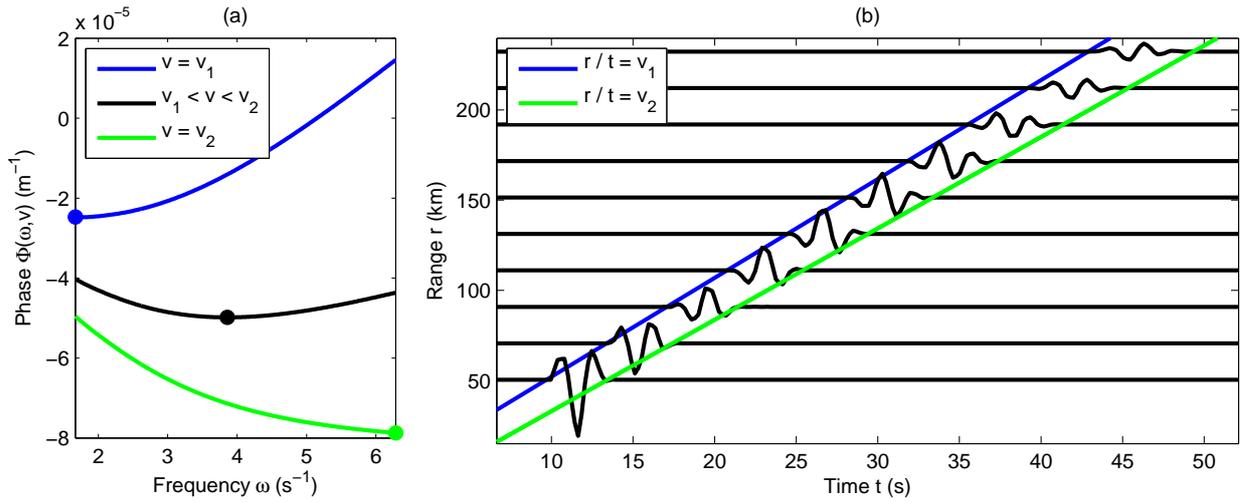


Figure 9: Phase Φ associated with the first dominant mode as a function of the real frequency ω_r for distinct values of the velocity v ($v_1=335 \text{ m.s}^{-1}$ and $v_2=328 \text{ m.s}^{-1}$). Circles show the point of stationary phase $\bar{\omega}$. Ground-based signals obtained with the stationary phase method for the first most sensitive mode. Signals are shown with a shifted time: $t' = t - r/c_0$ ($c_0 = 350 \text{ m.s}^{-1}$).

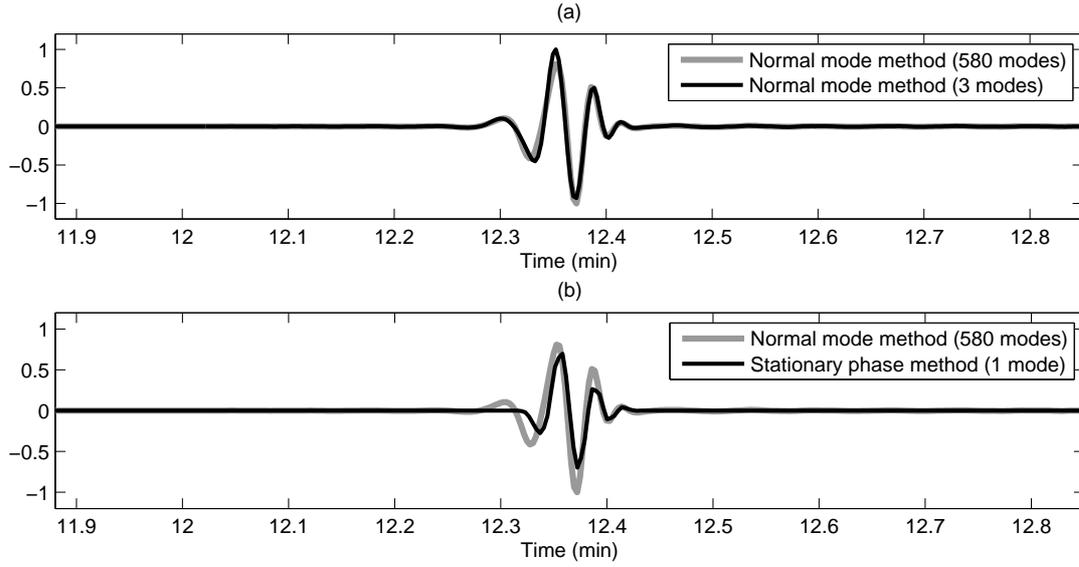


Figure 10: Ground-based signals computed at the station I30JP with the range-dependent normal mode approach. The reference signal (gray), obtained with 580 modes using the FFT algorithm (with 512 frequency samples), is compared to that obtained with the three most sensitive modes (top) and the stationary phase approximation for the first mode (bottom).

The source function is given by (20) with a multiplicative constant of 2.65×10^5 .

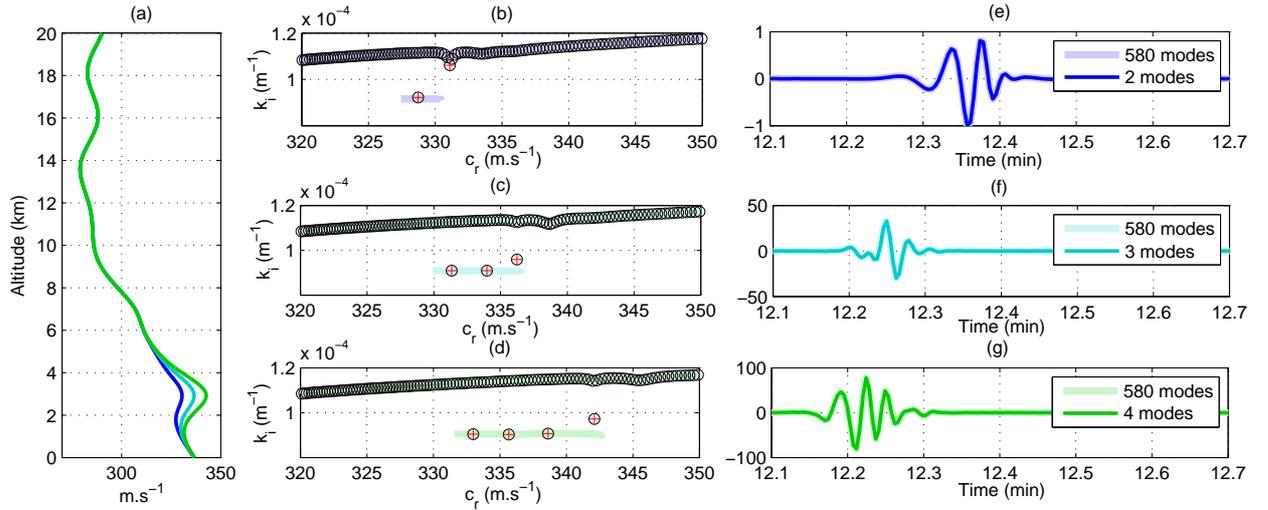


Figure 11: Ground-based signals (e-g) computed at the station I30JP for three vertical profiles (a). The profile c_{2a} (blue) is perturbed by a Gaussian envelope centered at the altitude of maximum effective sound speed. (b-d) Most sensitive eigenvalues lying on γ_1 as $\omega_r \rightarrow 2\pi$ (1 Hz). The eigenvalues obtained with the Krylov-Arnoldi algorithm for $\omega_r = 2\pi$ are given by red crosses.

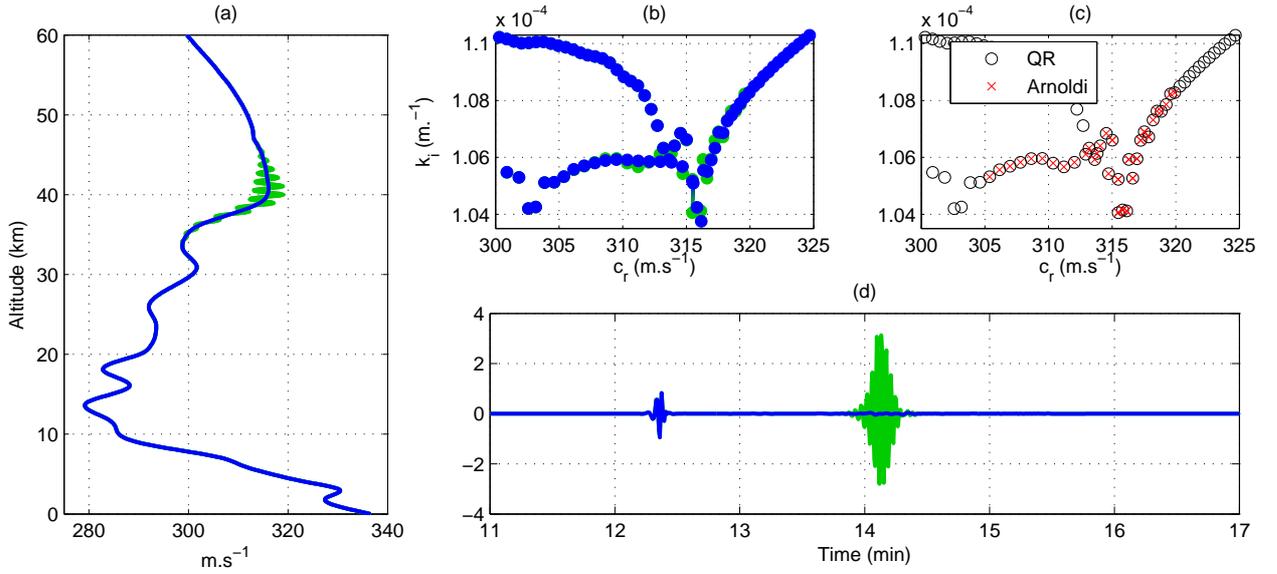


Figure 12: Ground-based signals (d) computed at the station I30JP for two vertical profiles (a). The profile c_{2a} (blue) is perturbed by a localized upward propagating gravity wave (green profile). The most sensitive eigenvalues (b) are obtained both with a QR algorithm and a Krylov-Arnoldi algorithm (c). *The source function is given by (20) with a multiplicative constant of 2.65×10^5 .*