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Abstract

We consider a model of D-dimensional tethered manifold interacting by excluded volume in $\mathbb{R}^d$ with a single point. By use of intrinsic distance geometry, we first provide a rigorous definition of the analytic continuation of its perturbative expansion for arbitrary D, $0 < D < 2$. We then construct explicitly a renormalization operation $\mathbf{R}$, ensuring renormalizability to all orders. This is the first example of mathematical construction and renormalization for an interacting extended object with continuous internal dimension, encompassing field theory.

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The Statistical Mechanics of random surfaces and membranes, or more generally of extended objects, poses fundamental problems [1]. Among those, the study of *polymerized* membranes, which are simple generalizations of linear polymers [2,3] to two-dimensionally connected networks, is prominent, with a number of possible experimental realizations [4,5]. From a theoretical point of view, a clear challenge is to understand self-avoidance (SA) effects in membranes. Recently, a model was proposed [6,7] which aimed to incorporate the advances made in polymer theory by Renormalization Group methods into the field of polymerized, or tethered membranes. These extended objects, a priori two-dimensional in nature, are generalized for theoretical purposes to intrinsically D- *dimensional manifolds* with internal points $x \in \mathbb{R}^D$, embedded in external d-dimensional space with position vector $\vec{\mathbf{r}}(x) \in \mathbb{R}^d$. The associated continuum Hamiltonian $\mathcal{H}$ generalizes that of Edwards for polymers [2]:

$$\beta\mathcal{H} = \frac{1}{2}\int d^D x \left(\nabla_x \vec{\mathbf{r}}(x)\right)^2 + \frac{b}{2}\int d^D x \int d^D x' \; \delta^d\big(\vec{\mathbf{r}}(x) - \vec{\mathbf{r}}(x')\big) \; , \tag{1}$$

with an elastic Gaussian term and a self-avoidance two-body δ-potential with interaction parameter $b > 0$. For $0 < D < 2$, the Gaussian manifold ($b = 0$) is *crumpled* with a finite Hausdorff dimension $d_H = 2D/(2-D)$; and the finiteness of the upper critical dimension $d^\star = 2d_H$ for the SA-interaction allows for an ϵ-expansion about $d^\star$ [6–8], performed via a direct renormalization method adapted from that of des Cloizeaux in polymer theory [9].

It should be stressed however that only the polymer case, with an *integer* internal dimension $D = 1$, can be mapped, following de Gennes [10], onto a standard field theory, namely a $(\mathbf{\Phi}^2)^2$ theory for a field $\mathbf{\Phi}$ with $n \to 0$ components. This is instrumental to show that the direct renormalization method for polymers is mathematically sound [11], and equivalent to rigorous renormalization schemes in standard local field theory, such as the landmark Bogoliubov Parasiuk Hepp Zimmermann (BPHZ) construction [12]. For manifold theory, we have to deal with *non-integer* internal dimension D, $D \neq 1$, where no such mapping exists. Therefore, two outstanding problems remain in the theory of interacting manifolds: (a) the mathematical meaning of a *continuous* internal dimension D; (b) the actual *renormalizability* of the perturbative expansion of a manifold model like (1), implying scaling as expected on physical grounds.

A first answer was brought up in [13], where a simpler model of a crumpled manifold interacting by excluded volume with a fixed Euclidean subspace of $\mathbb{R}^d$ was proposed. The direct resummation of leading divergences of the perturbation series indeed validates there *one-loop* renormalization, a result later extended to the Edwards model (1) [14].

In this Letter, we announce the results of an extensive study of these questions [15]. We first propose a mathematical construction of the D-dimensional internal measure $d^D x$ via distance geometry within the elastic manifold, with expressions for manifold Feynman integrals which generalize the α-parameter representation of field theory. In the case of the manifold model of [13] , we then describe the essential properties which make it indeed *renormalizable to all orders* by a renormalization of the coupling constant, and we directly construct a renormalization operation, generalizing the BPHZ construction to manifolds.

The simplified model Hamiltonian introduced in [13] reads:

$$\beta\mathcal{H} = \frac{1}{2}\int d^D x \left(\nabla_x \vec{\mathbf{r}}(x)\right)^2 + b\int d^D x\ \delta^d\big(\vec{\mathbf{r}}(x)\big)\ , \tag{2}$$

with now a pointwise interaction of the Gaussian manifold with the origin. Notice that this Hamiltonian also represents interactions of a fluctuating (possibly directed) manifold with a nonfluctuating D'- Euclidean subspace of $\mathbb{R}^{d+D'}$, $\vec{\mathbf{r}}$ then standing for the coordinates transverse to this subspace. The excluded volume case ($b > 0$) parallels that of the Edwards model (1) for SA-manifolds, while an attractive interaction ($b < 0$) is also possible, describing pinning phenomena. The dimensions of $\vec{\mathbf{r}}$ and b are respectively $[\vec{\mathbf{r}}] = [x^\nu]$ with a size exponent $\nu \equiv (2-D)/2$, and $[b] = [x^{-\epsilon}]$ with $\epsilon \equiv D - \nu d$. For fixed D and ν, the parameter d (or equivalently ϵ) controls the relevance of the interaction, with the exclusion of a point only effective for $d \leq d^\star = D/\nu$.

The model is described by its (connected) partition function $\mathcal{Z} = \mathcal{V}^{-1}\int \mathcal{D}[\vec{\mathbf{r}}]\exp(-\beta\mathcal{H})$ (here $\mathcal{V}$ is the internal volume of the manifold) and, for instance, by its one-point vertex function $\mathcal{Z}^{(0)}(\vec{\mathbf{k}})/\mathcal{Z} = \int d^D x_0 \langle e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}(x_0)}\rangle$, where the (connected) average $\langle\cdots\rangle$ is performed with (2). Those functions are formally defined via their perturbative expansions in the coupling constant b: $\mathcal{Z} = \sum_{N=1}^{\infty}\frac{(-b)^N}{N!}\mathcal{Z}_N$ and a similar equation for $\mathcal{Z}^{(0)}$ with coefficients $\mathcal{Z}_N^{(0)}$. The term of order N, $\mathcal{Z}_N$, is a ($b = 0$) Gaussian average involving N interaction points x_i. This average is expressed solely in terms of the Green function $G(x,y) = -\frac{1}{2}A_D|x-y|^{2\nu}$, solution of $-\Delta_x G(x,y) = \delta^D(x-y)$, with A_D a suitable normalization, hereafter omitted. In the following, it is important to preserve the condition $0 < \nu < 1$ (*i.e.* $0 < D < 2$), corresponding to the actual case of a crumpled manifold, and where $(-G)$ is positive and ultraviolet (UV) finite. A direct evaluation of $\mathcal{Z}_N$ then leads to its integral representation in terms of the normalized $G_{ij} \equiv -\frac{1}{2}|x_i - x_j|^{2\nu}$ [13]:

$$\mathcal{Z}_N\ =\ \frac{1}{\mathcal{V}}\int\prod_{i=1}^{N} d^D x_i\ \Big(\det\left[\Pi_{ij}\right]_{1\leq i,j\leq N-1}\Big)^{-\frac{d}{2}} \tag{3}$$

where the matrix $[\Pi_{ij}]$ is simply defined as $\Pi_{ij} \equiv G_{ij} - G_{Nj} - G_{iN}$ with a reference point, x_N, the symmetry between the N points being restored in the determinant. The integral representation of $\mathcal{Z}_N^{(0)}$ is obtained from that of $\mathcal{Z}_N$ by multiplying the integrand in (3) by $\exp(-\frac{1}{2}\vec{\mathbf{k}}^2\Delta^{(0)})$ with :

$$\Delta^{(0)} \equiv \frac{\det[\Pi_{ij}]_{0\leq i,j\leq N-1}}{\det[\Pi_{ij}]_{1\leq i,j\leq N-1}} \ , \tag{4}$$

and integrating over one more position, x_0. The resulting expression is quite similar to that of the manifold Edwards model [14].

Analytic continuation in D of the Euclidean measure. Integrals like (3) are *a priori* meaningful only for integer D. Still, an analytic continuation in D can be performed by use of *distance geometry*. The key idea is to substitute to the internal Euclidean coordinates x_i the set of all mutual (squared) distances $a_{ij} = (x_i - x_j)^2$. This is possible for integrands invariant under the group of Euclidean motions (as in (3) and (4)). For N integration points, it also requires D large enough, *i.e.* $D \geq N-1$, such that $N-1$ relative vectors spanning these points are linearly independent. We define the graph $\mathcal{G}$ as the set $\mathcal{G} = \{1, \ldots, N\}$ labelling the interaction points. Vertices $i \in \mathcal{G}$ will be remnants of the original Euclidean points after analytic continuation, and index the distance matrix $[a_{ij}]$. The change of variables $\{x_i\}_{i\in\mathcal{G}} \to a \equiv [a_{ij}]_{\substack{i<j \\ i,j\in\mathcal{G}}}$ reads explicitly [15]:

$$\frac{1}{\mathcal{V}} \int_{\mathbb{R}^D} \prod_{i\in\mathcal{G}} d^D x_i \ \cdots \ = \ \int_{\mathcal{A}_\mathcal{G}} d\mu_\mathcal{G}^{(D)}(a) \ \cdots \ , \tag{5}$$

with the measure

$$d\mu_\mathcal{G}^{(D)}(a) \equiv \prod_{\substack{i<j \\ i,j\in\mathcal{G}}} da_{ij} \ \Omega_N^{(D)} \left(P_\mathcal{G}(a)\right)^{\frac{D-N}{2}} \ , \tag{6}$$

where $N = |\mathcal{G}|$, $\Omega_N^{(D)} \equiv \prod_{K=0}^{N-2} \frac{S_{D-K}}{2^{K+1}}$ (here $S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is the volume of the unit sphere in $\mathbb{R}^D$), and

$$P_\mathcal{G}(a) \equiv \frac{(-1)^N}{2^{N-1}} \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & a_{12} & \ldots & a_{1N} \\ 1 & a_{12} & 0 & \ldots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{1N} & a_{2N} & \ldots & 0 \end{vmatrix} \ . \tag{7}$$

The factor $\Omega_N^{(D)}$ is the volume of the rotation group of the rigid simplex spanning the points x_i. The "Cayley-Menger determinant" [16] $P_\mathcal{G}(a)$ is proportional to the squared Euclidean volume of this simplex, a polynomial of degree $N-1$ in the a_{ij}. The set a of squared

distances has to fulfill the triangular inequalities and their generalizations: $P_{\mathcal{K}}(a) \geq 0$ for all subgraphs $\mathcal{K} \subset \mathcal{G}$, which defines the domain of integration $\mathcal{A}_{\mathcal{G}}$ in (5). For real $D > |\mathcal{G}| - 2$, $d\mu_{\mathcal{G}}^{(D)}(a)$ is a positive measure on $\mathcal{A}_{\mathcal{G}}$, analytic in D. It is remarkable that, as a distribution, it can be extended to $0 \leq D \leq |\mathcal{G}| - 2$ [15]. For integer $D \leq |\mathcal{G}| - 2$, although the change of variables from x_i to a_{ij} no longer exists, Eq.(6) still reconstructs the correct measure, concentrated on D-dimensional submanifolds of $\mathbb{R}^{N-1}$, *i.e.* $P_{\mathcal{K}} = 0$ if $D \leq |\mathcal{K}| - 2$ [15]. For example, when $D \to 1$ for $N = 3$ vertices, we have, denoting the distances $|ij| = \sqrt{a_{ij}}$:

$$\frac{d\mu_{\{1,2,3\}}^{(D\to 1)}(a)}{d|12|d|13|d|23|} = 2\,\delta\big(|12|+|23|-|13|\big) + \text{perm}$$

which indeed describes nested intervals in $\mathbb{R}$.

Another nice feature of this formalism is that the interaction determinants in (3) and (4) are themselves Cayley-Menger determinants. We have indeed $\det\left[\Pi_{ij}\right]_{1\leq i,j\leq N-1} = P_{\mathcal{G}}(a^\nu)$ where $a^\nu \equiv [a_{ij}^\nu]_{\substack{i<j \\ i,j\in\mathcal{G}}}$ is obtained by simply raising each squared distance to the power ν. We arrive at the representation of "Feynman diagrams" in distance geometry:

$$\begin{aligned} \mathcal{Z}_N &= \int_{\mathcal{A}_{\mathcal{G}}} d\mu_{\mathcal{G}}^{(D)}\, I_{\mathcal{G}}\ , \quad I_{\mathcal{G}} = \big(P_{\mathcal{G}}(a^\nu)\big)^{-\frac{d}{2}} \\ \mathcal{Z}_N^{(0)} &= \int_{\mathcal{A}_{\mathcal{G}\cup\{0\}}} d\mu_{\mathcal{G}\cup\{0\}}^{(D)}\, I_{\mathcal{G}}^{(0)}\ , \\ I_{\mathcal{G}}^{(0)} &= I_{\mathcal{G}}\ \exp\left(-\frac{1}{2}\vec{\mathbf{k}}^2 \frac{P_{\mathcal{G}\cup\{0\}}(a^\nu)}{P_{\mathcal{G}}(a^\nu)}\right), \end{aligned} \tag{8}$$

which are D-dimensional extensions of the Schwinger α-parameter representation. We now have to study the actual convergence of these integrals and, possibly, their renormalization.

Analysis of divergences. Large distance infrared (IR) divergences occur for manifolds of infinite size. One can keep a finite size, preserve symmetries and avoid boundary effects by choosing as a manifold the D-dimensional sphere $\mathcal{S}_D$ of radius R in $\mathbb{R}^{D+1}$. This amounts [15] in distance geometry to substituting to $P_{\mathcal{G}}(a)$ the "spherical" polynomial $P_{\mathcal{G}}^{\mathcal{S}}(a) \equiv P_{\mathcal{G}}(a) + \frac{1}{R^2}\det(-\frac{1}{2}a)$, the second term providing an IR cut-off, such that $a_{ij} \leq 4R^2$. In the following, this regularization will be simply ignored when dealing with short distance properties, where $P_{\mathcal{G}}^{\mathcal{S}} \sim P_{\mathcal{G}}$.

Schoenberg's theorem. This result of geometry [16] states that *for $0 < \nu < 1$, the set $a^\nu = [a_{ij}^\nu]_{\substack{i<j \\ i,j\in\mathcal{G}}}$ can be realized as the set of squared distances of a transformed simplex*

in $\mathbb{R}^{N-1}$, *whose volume* $P_{\mathcal{G}}(a^\nu)$ *is positive and vanishes if and only if at least one of the mutual original distances itself vanishes,* $a_{ij} = 0$. This ensures that, as in field theory, the only source of divergences in $I_{\mathcal{G}}$ and $I^{(0)}_{\mathcal{G}}$ is at *short distances*. Whether these UV singularities are integrable or not will depend on whether the external space dimension $d < d^\star = D/\nu$ or $d > d^\star$.

Factorizations. The key to convergence and renormalization is the following short distance *factorization* property of $P_{\mathcal{G}}(a^\nu)$. Let us consider a subgraph $\mathcal{P} \subset \mathcal{G}$, with at least two vertices, in which we distinguish an element, the *root* p of $\mathcal{P}$, and let us denote by $\mathcal{G}_{\diagup p}\mathcal{P} \equiv (\mathcal{G} \setminus \mathcal{P}) \cup \{p\}$ the subgraph obtained by replacing in $\mathcal{G}$ the whole subset $\mathcal{P}$ by its root p. In the original Euclidean formulation, the analysis of short distance properties amounts to that of contractions of points x_i, labeled by such a subset $\mathcal{P}$, toward the point x_p, according to: $x_i(\rho) = x_p + \rho(x_i - x_p)$ if $i \in \mathcal{P}$, where $\rho \to 0^+$ is the dilation factor, and $x_i(\rho) = x_i$ if $i \notin \mathcal{P}$. This transformation has an immediate correspondent in terms of mutual distances: $a_{ij} \to a_{ij}(\rho)$, depending on both $\mathcal{P}$ and p. Under this transformation, the interaction polynomial $P_{\mathcal{G}}(a^\nu)$ factorizes into [15]:

$$\begin{aligned} P_{\mathcal{G}}(a^\nu(\rho)) = P_{\mathcal{P}}(a^\nu(\rho))\, P_{\mathcal{G}_{\diagup p}\mathcal{P}}(a^\nu) \\ \times \left\{1 + \mathcal{O}(\rho^{2\delta})\right\} \quad . \end{aligned} \tag{9}$$

with $\delta = \min(\nu, 1-\nu) > 0$ and where, by homogeneity, $P_{\mathcal{P}}(a^\nu(\rho)) = \rho^{2\nu(|\mathcal{P}|-1)}\, P_{\mathcal{P}}(a^\nu)$.

Fig. 1: *Factorization property (9).*

The geometrical interpretation of (9) is quite simple: the contribution of the set $\mathcal{G}$ splits into that of the contracting subgraph $\mathcal{P}$ multiplied by that of the whole set $\mathcal{G}$ where $\mathcal{P}$ has been replaced by its root p (fig. 1), all correlation distances between these subsets being

suppressed. This is just, in this model, the rigorous expression of an *operator product expansion* [15].

The factorization property (9) does not hold for $\nu = 1$, preventing a factorization of the measure (6) $d\mu_{\mathcal{G}}^{(D)}(a)$ itself. Still, the integral of the measure, when applied to a factorized integrand, factorizes as:

$$\int_{\mathcal{A}_{\mathcal{G}}} d\mu_{\mathcal{G}}^{(D)} \cdots = \int_{\mathcal{A}_{\mathcal{P}}} d\mu_{\mathcal{P}}^{(D)} \cdots \int_{\mathcal{A}_{(\mathcal{G}/\!\!/_p\mathcal{P})}} d\mu_{(\mathcal{G}/\!\!/_p\mathcal{P})}^{(D)} \cdots \ . \tag{10}$$

This fact, explicit for integer D with a readily factorized measure $\prod_i d^D x_i$, is preserved [15] by analytic continuation only after integration over relative distances between the two "complementary" subsets $\mathcal{P}$ and $\mathcal{G}/\!\!/_p\mathcal{P}$.

Renormalization. A first consequence of factorizations (9) and (10) is the absolute convergence of $\mathcal{Z}_N$ and $\mathcal{Z}_N^{(0)}$ for $\epsilon > 0$. Indeed, the superficial degree of divergence of $\mathcal{Z}_N$ (in distance units) is $(N-1)\epsilon$, as can be read from (8), already ensuring the superficial convergence when $\epsilon > 0$. The above factorizations ensure that the superficial degree of divergence in $\mathcal{Z}_N$ or $\mathcal{Z}_N^{(0)}$ of any subgraph $\mathcal{P}$ of $\mathcal{G}$ is exactly that of $\mathcal{Z}_{|\mathcal{P}|}$ itself, *i.e.* $(|\mathcal{P}|-1)\epsilon > 0$. By recursion, this ensures the absolute convergence of the manifold Feynman integrals. A complete discussion has recourse to a generalized notion of Hepp sectors and is given elsewhere [15]. In the proof, it is convenient to first consider D large enough where $d\mu_{\mathcal{G}}^{(D)}$ is a non singular measure, with a fixed ν considered as an independent variable $0 < \nu < 1$, and to then continue to $D = 2 - 2\nu$, $0 < D < 2$, corresponding to the physical case.

When $\epsilon = 0$, the integrals giving $\mathcal{Z}_N$ and $\mathcal{Z}_N^{(0)}$ are (logarithmically) divergent. Another main consequence of Eqs. (9) and (10) is then the possibility to devise a renormalization operation $\mathbf{R}$, as follows. To each contracting rooted subgraph $(\mathcal{P}, p)$ of $\mathcal{G}$, we associate a Taylor operator $\mathfrak{T}_{(\mathcal{P},p)}$, performing on interaction integrands the exact factorization corresponding to (9):

$$\mathfrak{T}_{(\mathcal{P},p)} I_{\mathcal{G}}^{(0)} = I_{\mathcal{P}}\, I_{\mathcal{G}/\!\!/_p\mathcal{P}}^{(0)} \ , \tag{11}$$

and similarly $\mathfrak{T}_{(\mathcal{P},p)} I_{\mathcal{G}} = I_{\mathcal{P}}\, I_{\mathcal{G}/\!\!/_p\mathcal{P}}$. As in standard field theory [12], the subtraction renormalization operator $\mathbf{R}$ is then organized in terms of forests à la Zimmermann. In manifold theory, we define a *rooted forest* as a set of rooted subgraphs $(\mathcal{P}, p)$ such that any two subgraphs are either disjoint or nested, *i.e.* never partially overlap. Each of these subgraphs in the forest will be contracted toward its root under the action (11) of the corresponding

Taylor operator. When two subgraphs $\mathcal{P} \subset \mathcal{P}'$ are nested, the smallest one is contracted first toward its root p, the root p' of $\mathcal{P}'$ being itself attracted toward p if p' happened to be in $\mathcal{P}$. This hierarchical structure is anticipated by choosing the roots of the forest as *compatible*: in the case described above, if $p' \in \mathcal{P}$, then $p' \equiv p$. Finally, the renormalization operator is written as a sum over all such compatibly rooted forests of $\mathcal{G}$, denoted by $\mathcal{F}_\oplus$:

$$\mathbf{R} \;=\; \sum_{\mathcal{F}_\oplus} W(\mathcal{F}_\oplus) \left[\prod_{(\mathcal{P},p)\in\mathcal{F}_\oplus} \big(-\mathfrak{T}_{(\mathcal{P},p)} \big) \right] . \tag{12}$$

Here W is a necessary combinatorial weight associated with the degeneracy of compatible rootings, $W(\mathcal{F}_\oplus) \;=\; \prod_{\substack{p \text{ root} \\ \text{of } \mathcal{F}_\oplus}} 1/|\mathcal{P}(p)|$ with $\mathcal{P}(p)$ being the largest subgraph of the forest $\mathcal{F}_\oplus$ whose root is p. An important property is that, with compatible roots, the Taylor operators of a given forest now commute [15]. The renormalized amplitudes are defined as

$$\mathcal{Z}^{\mathbf{R}\,(0)}_{N} \;\equiv\; \int_{\mathcal{A}_{\mathcal{G}\cup\{0\}}} d\mu^{(D)}_{\mathcal{G}\cup\{0\}}\, \mathbf{R}\,[I^{(0)}_{\mathcal{G}}] \; . \tag{13}$$

The same operation $\mathbf{R}$ acting on $I_{\mathcal{G}}$ leads automatically by homogeneity to $\mathbf{R}\,[I_{\mathcal{G}}] = 0$ for $|\mathcal{G}| \geq 2$. We state the essential result that now *the renormalized Feynman integral* (13) *is convergent*: $\mathcal{Z}^{\mathbf{R}\,(0)}_{N} < \infty$ for $\epsilon = 0$. A complete proof of this renormalizability property goes well beyond the scope of this Letter and is given elsewhere [15]. the analysis being inspired from the direct proof by Bergère and Lam of the renormalizability in field theory of Feynman amplitudes in the α-representation [17].

The physical interpretation of the renormalized amplitude (13) and of (12) is now fairly simple. Eqs.(10) and (11) show that the substitution to the bare amplitudes (8) of the renormalized ones (13) amounts to a reorganization to all orders of the original perturbation series in b, leading to the remarkable identity:

$$\mathcal{Z}^{(0)} \;=\; \sum_{N=1}^{\infty} \frac{(\mathcal{Z})^N}{N!}\, \mathcal{Z}^{\mathbf{R}\,(0)}_{N} \; . \tag{14}$$

This actually extends to any vertex function, showing that the theory is made perturbatively finite (at $\epsilon = 0$) by a simple renormalization of the coupling constant b into $\mathcal{Z}$ itself. From this result, one establishes the existence of a Wilson function $\nu \frac{\partial \mathcal{Z}}{\partial \nu}\big|_b$, describing the scaling properties of the interacting manifold for ϵ close to zero [15]. For $\epsilon > 0$, an IR fixed point at $b > 0$ yields universal excluded volume exponents; for $\epsilon < 0$, the associated UV fixed point at $b < 0$ describes a localization transition.

In summary, we have shown how to define an interacting manifold model with continuous internal dimension, by use of distance geometry, a natural extension of Schwinger representation of field theories. Furthermore, in the case of a pointwise interaction, we have shown that the manifold model is indeed renormalizable to all orders. The main ingredients are the Schoenberg's theorem of distance geometry, insuring that divergences occur only at short distances for (finite) manifolds, and the short-distance factorization of the generalized Feynman amplitudes. The renormalization operator is a combination of Taylor operators associated with rooted diagrams, a specific feature of manifold models. This is probably the first example of a perturbative renormalization established for extended geometrical objects. This opens the way to a similar study of self-avoiding manifolds, as well as to other generalizations of field theories.

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