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Renormalization Theory for Interacting Crumpled Manifolds

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Abstract

We consider a continuous model of D-dimensional elastic (polymerized) manifold fluctuating in d-dimensional Euclidean space, interacting with a single impurity via an attractive or repulsive $\boldsymbol{\delta}$-potential (but without self-avoidance interactions). Except for $D = 1$ (the polymer case), this model cannot be mapped onto a local field theory. We show that the use of intrinsic distance geometry allows for a rigorous construction of the high-temperature perturbative expansion and for analytic continuation in the manifold dimension D. We study the renormalization properties of the model for $0 < D < 2$, and show that for bulk space dimension d smaller that the upper critical dimension $d^\star = 2D/(2-D)$, the perturbative expansion is ultra-violet finite, while ultraviolet divergences occur as poles at $d = d^\star$. The standard proof of perturbative renormalizability for local field theories (the Bogoliubov Parasiuk Hepp theorem) does not apply to this model. We prove perturbative renormalizability to all orders by constructing a subtraction operator $\mathbf{R}$ based on a generalization of the Zimmermann forests formalism, and which makes the theory finite at $d = d^\star$. This subtraction operation corresponds to a renormalization of the coupling constant of the model (strength of the interaction with the impurity). The existence of a Wilson function, of an ϵ-expansion *à la* Wilson Fisher around the critical dimension, of scaling laws for $d < d^\star$ in the repulsive case, and of non-trivial critical exponents of the delocalization transition for $d > d^\star$ in the attractive case is thus established. To our knowledge, this study provides the first proof of renormalizability for a model of extended objects, and should be applicable to the study of self-avoidance interactions for random manifolds.

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1. Introduction

One general problem arising in statistical physics is the understanding of the effect of interactions on the thermodynamical properties of extended fluctuating geometrical objects. These objects may be (1-dimensional) lines, like long linear macromolecules or polymers, (2-dimensional) surfaces, like membranes or interfaces, or even (3-dimensional) volumes, like gels. The interactions involve in general two-body attractive or repulsive forces, and one may in general reduce such problems into two different classes: (i) either one deals with self-interactions between distinct points of the same fluctuating object, or mutual interactions between several fluctuating objects; (ii) or one deals with the interaction of a single freely fluctuating object with another non-fluctuating fixed object. Case (i) includes for instance self-avoiding polymers or membranes, polyelectrolytes and charged gels, as well as the description of intersections of random walks. Case (ii) includes the problems of binding/unbinding of a long molecule or a membrane on a wall, the wetting of an interface. One can also reduce to this class the problems of unbinding of two membranes or interfaces, and that of the steric repulsions between membranes in a lamellar phase.

Among the many different generic situations one can think of, one case is now well understood, namely that where the fluctuating objects are only *one dimensional objects*. Indeed, many problems in case (ii) can then be solved by simple analogy with quantum mechanics, *i.e.* by use of a diffusion equation. The situation is more complicated in case (i), a paradigm of which is the celebrated problem of self-avoiding polymers. Still in this case, the use of perturbative expansions and Renormalization Group techniques allows for explicit results on the thermodynamics of these objects. For instance, a self-avoiding polymer embedded in a d-dimensional external space can be described by the continuous Edwards Hamiltonian [1] [2]:

$$\mathcal{H} = \frac{1}{2}\int_0^S ds \left(\frac{d\vec{r}}{ds}\cdot\frac{d\vec{r}}{ds}\right) + \frac{b}{2}\int_0^S ds \int_0^S ds'\, \delta^d\big(\vec{r}(s)-\vec{r}(s')\big) \ . \tag{1.1}$$

This model can then be viewed as a 1-dimensional field theory, with position field $\vec{r}(s)$ at abscissa s along the chain of size S, and with a *non-local* interaction term. This field theory then has a formal perturbative expansion in b: this point of view dates back the work of Fixman [3] and has been developed by des Cloizeaux [2] [4]. The terms of this expansion are in general integrals over the *internal* coordinates s of the interaction points and may diverge when these interaction points come close to each other ($|s-s'| \to 0$).

The theory can then be regularized by analytic continuation in $d \geq 2$, and the natural expansion parameter is then $bS^{2-d/2}$, hence large in the thermodynamic limit $S \to \infty$ for $d < 4$. For dimensional reasons, the corresponding large distance divergences are twinned with the short-distance divergences, and appear as poles in d at $d = 4$. Within a double expansion in b and $\epsilon = 4 - d$, the structure of these poles is such that the theory is *renormalizable* for $\epsilon \geq 0$. This means that the poles at $\epsilon = 0$ can actually be absorbed into redefinitions of the parameters of the model, and that a scaling limit is obtained for the thermodynamical properties of the polymer when $\epsilon \geq 0$. Still, a rigorous proof of renormalizability requires the use of the famous equivalence of the Edwards model with the $\mathrm{O}(n)$ model for $n = 0$, that is a model with a n-component field $\vec{\Phi}(\vec{r})$ in the d-dimensional *external* space, as shown by de Gennes [5]. From this different point of view, which was the first to be developed in the 70's, the self-avoiding polymer problem is seen as a d-dimensional *local* field theory, that is a theory with local interactions, and amenable to the standard renormalization group treatments for critical phenomena [6] [7]. Again, this field theory can be studied via a perturbative expansion, the terms of which may diverge when two *external* interaction points $\vec{r}$ and $\vec{r}\,'$ come close to each other ($|\vec{r} - \vec{r}\,'| \to 0$). Now the general renormalization scheme for local field theories applies and ensures (perturbative) renormalizability, from which one deduces a posteriori the renormalizability of the direct approach "à la des Cloizeaux" [8] [9] [10]. This equivalence with a local field theory also holds for 1-dimensional problems in case (ii), and methods of perturbative field theory can also be applied in this case. Although they are in general more complicated than the simple diffusion equation, they give comparable results (see [11]).

Beside the perturbative framework, one should notice that rigorous non-perturbative results have been obtained for the Edwards model and related models: the mathematical construction of the measure on random paths associated with (1.1) [12]; the large distance behavior of intersection properties of independent random walks at $d = 4$ [13]; the large distance behavior of weakly self-avoiding polymers at $d = 4$ in constructive field theory [14]. These non-perturbative studies always corroborate the results of the perturbative renormalization group analysis.

The existence of an underlying local field theory in the external d-dimensional space, which is crucial to ensure renormalizability and allows for predictions from the perturbative expansion, is however directly related to the 1-dimensional nature of the object. When we now consider a *D-dimensional object* with $D \neq 1$, embedded in d dimensions, no such

equivalence with a d-dimensional local field theory exists. Still, the approach "à la des Cloizeaux" can be generalized, by considering a D-dimensional field theory. For instance, the Edwards Hamiltonian writes for a D-dimensional manifold with internal coordinate x [15] [16] [17] :

$$\mathcal{H} = \frac{1}{2}\int d^D x \left(\nabla_x \vec{r} \cdot \nabla_x \vec{r}\right) + \frac{b}{2}\int d^D x \int d^D x' \, \delta^d\big(\vec{r}(x) - \vec{r}(x')\big) \ . \tag{1.2}$$

This describes a polymerized or "tethered" manifold with a fixed internal metric (to be distinguished from the case of fluid membranes, with a fluctuating metric). The self-avoidance interaction term leads to a perturbative expansion in b, with poles in $\epsilon = 4D - d(2-D)$. This method has been used to first order in ϵ [16][17], and leads to first order estimates of critical exponents [16][17][18][19], *assuming that renormalizability holds* and that a Renormalization Group equation can thus be used.

Two crucial questions remain however open, which show that new mathematical developments are required:

(I) A perturbative approach cannot be performed *directly* at D larger or equal to 2. Indeed, for $D \geq 2$ (and $d \geq 0$), ϵ is never small ($\epsilon \geq 8$). The double expansion in b and ϵ requires to consider the case of *real* non-integer D (typically $1 \leq D < 2$). The term of order N in the perturbative expansion being an integral over $2N$ (resp. N) interaction points in case (i) (resp. case (ii)) in internal D-dimensional space, the meaning of these integrations for non integer D has to be defined.

(II) Since, as a D-dimensional field theory, the theory is either non local (case (i)) or local (case (ii)) but with a singular potential with explodes at the origin $\vec{r} = 0$ (typically $1/|\vec{r}|^\gamma$ or $\delta^d(\vec{r})$), standard methods of local field theory do not apply. Since furthermore, as mentioned above, we cannot rely (as for $D = 1$) on an equivalence with a d-dimensional local field theory, the question arises of the actual renormalizability of the theory, and in particular of the validity of the use of a (for instance first order) Renormalization Group equation to predict a scaling behavior.

Beyond the one-loop calculations of [16][17][18][19] for the model of self-avoiding random manifold, which assumes renormalizability, a next step in a general analysis of the problem of renormalization for interacting extended object with dimensionality $D \neq 1$ has been performed by one of the present authors in [20]. In [20] a model describing the simple avoidance interaction of a D-dimensional fluctuating manifold with a fixed Euclidean

element was considered. The leading UV divergences of the model were analyzed in perturbation theory and resummed, so that the consistency of a renormalization group equation at one loop was established for this model. A similar direct approach has been applied to the Edwards manifold model (1.2), and the one-loop renormalizability established [21].

The purpose of this paper is to present a general, mathematically rigorous, framework to study these questions, and to analyze the renormalizability of models of interacting objects *to all orders* in perturbation theory. In this paper, we shall discuss the simple model of reference [20], of a D-dimensional fluctuating manifold interacting with a single fixed point (or more generally a fixed Euclidean element), defined by the following Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int d^D x \left(\nabla_x \vec{r} \cdot \nabla_x \vec{r} \right) + \; b \int d^D x \, \delta^d \big(\vec{r}(x) \big) \; . \tag{1.3}$$

We prove perturbative renormalizability for this model, to all orders in perturbation theory, from the internal-space formulation of [20]. For that purpose we rely on methods devised in perturbative field theory, in particular by Bergère and Lam, for renormalizing the Feynman amplitudes in the so-called α-parameter or Schwinger representation. Indeed, our construction can be seen as a generalization of renormalization theory in Schwinger representation to the case of a D-dimensional α-parameter space.

This paper is organized as follows.

In section 2, we present the model of a D-dimensional manifold interacting with a single fixed point, discuss its physical relevance for the problem of entropic repulsion by an impurity (case of repulsive interaction) and of delocalization transition (case of attractive interaction), and discuss its formal perturbative expansion. To each order N of the expansion corresponds a unique diagram, which is an integral over the positions of N points in the D-dimensional Euclidean internal space.

In section 3, we give a mathematical prescription to define the analytic continuation of those integrals for non-integer D. The basic idea relies on the concept of "distance geometry": we use the Euclidean invariance in $\mathbb{R}^D$ to replace the integral over N D-dimensional vectors labeling the positions of the points, $x_i = \{x_i^\mu ; \mu = 1, \ldots, D\}$ $(i = 1, \ldots, N)$ by an integral over the $\frac{N(N-1)}{2}$ mutual squared distances $a_{ij} = (x_i - x_j)^2$, with possible constraints. The dimension D appears then only as a parameter for the measure term of the a_{ij}'s (which by analytic continuation in D has in general to be considered as a distribution).

In section 4, we analyze the short-distance (ultraviolet) divergences of these analytically continued integrals. We show that they lead to poles in $\epsilon \equiv D - d(2-D)/2$. They correspond to an upper critical dimension $d^\star = 2D/(2-D)$. We also analyze the large distance (infrared) divergences which occur when the internal dimensions of the D-dimensional manifold go to infinity. We show how to regularize these infrared divergences, simply by keeping a finite size for the manifold, in order to concentrate on the ultraviolet divergences.

The next four sections are devoted to the analysis and proof of renormalizability of the theory. Our analysis relies in fact heavily on concepts and mathematical tools developed in the 70's for the theory of perturbative renormalization of "ordinary" local field theories [22]. Since these concepts have to be strongly modified for our problem, and since they are not so well-known, they will be introduced from the beginning in this paper, which is therefore (hopefully) entirely self-contained.

In section 5, we show that a diagram of arbitrary order N is finite when $\epsilon > 0$. For that purpose, we introduce a "sector decomposition" of the domain of integration over distances in internal space, which is analogous to the Hepp sectors of renormalization theory.

The next three sections are devoted to the proof of renormalizability of the theory, that is the possibility of absorbing the poles at $\epsilon = 0$ into a redefinition of the coupling constant of the model, thus making the diagrams finite at $\epsilon = 0$ by appropriate counterterms. Renormalized amplitudes are defined in section 6, by subtraction of suitable counterterms. These counterterms are organized in families of divergent "subdiagrams", which correspond to the concepts of "forest" and of "nest".

To prove finiteness, we need to reorganize in each "sector" the counterterms. Such a reorganization is presented in section 7, and requires an elaborate "equivalence classes of nests" construction, inspired from [23].

Finally, in section 8, we show that the subtracted amplitudes are finite at $\epsilon = 0$, as long as the integration over the squared distances a_{ij} is given by a measure, while in section 9 we show that this remains true in the general case where the measure term is a distribution. This ends the proof of the renormalizability of the model. The rest of section 9 is devoted to some physical consequences of this renormalizability property, such as the existence of a Wilson-Fisher ϵ-expansion and of universal scaling behaviors.

In the concluding section 10, we summarize our work and discuss various prospects, in particular for the problem of self-avoiding random manifolds.

A lot of technical points are relegated into various appendices.

The reader not interested in the details of the proof of renormalizability may skip (at least in a first reading...) sections 5, 7 and 8.

2. The model

2.1. The action

We first define the model that we shall study and the formal structure of its perturbative expansion, without taking care of the possible infinities which may arise from short and/or large distance divergences. It is the purpose of next sections (in particular section 4) to define proper regularization schemes.

We start with the manifold Hamiltonian [20]:

$$\mathcal{H} = \int_{\mathcal{V}} d^D x \left[\frac{1}{2} \vec{\mathbf{r}}(x) (-\Delta)^{\frac{k}{2}} \vec{\mathbf{r}}(x) + b\ \delta^d(\vec{\mathbf{r}}(x)) \right] \ , \tag{2.1}$$

where x labels the internal position in the D-dimensional manifold with volume $\mathcal{V}$ and $\vec{\mathbf{r}}(x)$ is the corresponding position in the d-dimensional Euclidean space. For the physical case $k = 2$, the first term in (2.1)corresponds to the elastic energy of the Gaussian manifold (the internal tension is set to unity). For reasons of mathematical convenience, which will be clear in the following, we shall consider in full generality the more general class of elastic Hamiltonian with $k \geq 2$. This allows in particular to define in a proper way a consistent analytic continuation in the internal dimension D. The case $k = 4$ corresponds to a manifold with vanishing tension but with bending rigidity. The absence from Eq. (2.1) of a two-point self-avoidance interaction term (as compared to Eq. (1.2)) means that we are dealing with a "phantom" manifold which can intersect itself freely. The second term in (2.1) corresponds to the interaction of the manifold with a fixed impurity, that is a single point in the external d-dimensional space, here fixed at the origin $\vec{\mathbf{r}} = \vec{0}$. The coupling constant b may be either positive (repulsive interaction) or negative (attractive interaction).

As mentioned in the Introduction, this model is interesting as a toy model for the more complex problem of self-avoiding manifolds. In both cases the interaction term is a singular δ-function, and similar mathematical techniques can be used to write perturbative expansions and to study their properties. In the present case the interaction is much simpler, since it corresponds to a 1-body interaction, instead of a 2-body interaction in the case of self-avoidance. This model is also interesting in its own, since the Hamiltonian (2.1)

can also be used to describe the (attractive or repulsive) interaction of a fluctuating D-dimensional manifold with a fixed D'-dimensional Euclidean subspace in a d'-dimensional Euclidean space [20], with $d' = d + D'$. In this case $\vec{\mathbf{r}}$ describes the d coordinates of the fluctuating manifold *orthogonal* to the fixed Euclidean D'-subspace. The case $D = 1$, corresponding to a polymer interacting with some fixed object, has been already considered by several authors [11] [24] [25] . The case $D = 1$, $d = 2$ corresponds for instance to a polymer interacting with a rigid rod in 3-dimensional space. If $D = D'$ this model can also be used to describe a "directed manifold" (parallel to a flat Euclidean subspace). In this case the coordinates of the point with internal coordinate x in the external d'-dimensional space are $(x, \vec{\mathbf{r}}(x))$, and the first $D = D'$ longitudinal degrees of freedom are fixed. For instance the case $D = D' = 1$ describes a "directed polymer" interacting with a parallel rod in d'-dimensional space, the case $D = D' = 2$, $d' = 3$ ($d = 1$) describes a SOS-like fluctuating interface interacting with a parallel plane, ...

Fig. 1: *A D-dimensional fluctuating manifold (here $D = 2$) interacting: (a) with a point at the origin in $\mathbb{R}^d$ (here $d = 3$), (b) with a fixed D'-dimensional Euclidean subspace of $\mathbb{R}^{d'}$ (here $D' = 2$, $d = 1$, $d' = d + D' = 3$). (c) A "directed" manifold interacting with a "parallel" flat subspace of same dimension D in $\mathbb{R}^{d'}$.*

The "engineering" dimensions of the position field $\vec{\mathbf{r}}$ and of the coupling constant b are respectively

$$\begin{aligned} [\vec{\mathbf{r}}] &= [x^{\nu}] \qquad & \nu &= \frac{k-D}{2} \ , \\ [b] &= \left[x^{-\epsilon}\right] \qquad & \epsilon &= D - \nu d \ . \end{aligned} \tag{2.2}$$

Therefore the interaction is expected to be relevant (that is to change the large distance properties of the manifold) if $\epsilon > 0$, that is if $D > D^{\star}$, where $D^{\star}$ is the critical internal dimension, given by

$$D^{\star} = \ k \, \frac{d}{d+2} \ , \tag{2.3}$$

or equivalently if $d < d^\star$, where $d^\star$ is the critical embedding dimension

$$d^\star \;=\; \frac{2D}{k-D}\;, \tag{2.4}$$

simply equal to the fractal dimension of the manifold. In particular, this model possesses an upper critical dimension $0 < d^\star < \infty$ for a "membrane" dimension $0 < D < k$. For the standard interface model ($k = 2$), we recover the conditions $0 < D < 2$ [16] [17][20]. The exponent ν plays the role of the size exponent of the elastic manifold. For fluctuating interfaces, that is "directed manifolds", it is also called in the literature the wandering exponent, and denoted by ζ [26]. ν has its natural range between 0 (collapsed manifold) and 1 (stretched manifold). This corresponds exactly to

$$k-2 \leq D \leq k\;, \tag{2.5}$$

or equivalently to the "physical" conditions $D \leq d^\star \leq \infty$.

In [20], a *dimensionless effective coupling constant* g was introduced, which measures the effective strength of the interaction as a function of the length scale X measuring the linear internal extent of the manifold, defined by $\mathcal{V} = X^D$. In the vicinity of the critical dimension ($\epsilon \simeq 0$), and for the physical case $k = 2$, a one-loop calculation [20] shows that this effective coupling constant obeys a renormalization group (RG) flow equation, which writes

$$X\frac{\partial g}{\partial X} \;=\; W(g) \;=\; \epsilon\, g - \frac{1}{2} S_D\, g^2 \;+\; \mathcal{O}(g^3) \tag{2.6}$$

with $S_D \;=\; 2\,\pi^{\frac{D}{2}}/\Gamma(\frac{D}{2})$ the volume of the unit sphere in $\mathbb{R}^D$. Apart from the trivial $g = 0$ solution, this flow equation has a fixed point solution at the non-trivial zero of the Wilson function $W(g)$

$$g^\star = \frac{2\epsilon}{S_D} + \mathcal{O}(\epsilon^2) \tag{2.7}$$

At large negative g, $W(g)$ behaves like

$$W(g) \simeq D\, g \log(-g). \tag{2.8}$$

The physical consequences of these equations are the following:

(I) $\epsilon > 0$: This corresponds to $D > D^\star$ or $d < d^\star$. The RG flow has an infrared (IR) stable fixed point at $g^\star > 0$ and an IR unstable (ultraviolet (UV) stable) fixed point at $g = 0$, as depicted on fig. 2. For arbitrarily small negative b (attractive interaction), g is negative and flows to $(-\infty)$ at large length scale X; the manifold is localized

Fig. 2: *The Wilson W function and the Renormalization Group (IR) flow (for increasing manifold size X) for the dimensionless coupling constant g: (a) in the case $\epsilon > 0$, (b) in the case $\epsilon < 0$, (c) in the case $\epsilon = 0$.*

(or pinned) at the origin $\vec{\mathbf{r}} = \vec{0}$, and its average distance to the origin stays finite. For arbitrarily small positive b (repulsive interaction), g is positive and flows to $g^\star$ at large X; the manifold is delocalized, and furthermore repelled from the origin. The UV Gaussian fixed point at $g = 0$ thus describes a *delocalization transition*, whose critical properties are given by mean field theory [1]. The nontrivial IR fixed point at $g = g^\star$ describes the *universal large distance properties* of the delocalized state [20],[25], and of the long range repulsive force away from the origin generated by the fluctuations of the manifold.

(II) $\epsilon < 0$: This corresponds to $D < D^\star$ or $d > d^\star$. The RG flow has now an IR unstable (UV stable) fixed point at $g^\star < 0$ and an IR stable fixed point at $g = 0$. The delocalization transition now occurs for some $b = b^\star < 0$, *i.e.* for a non-zero, large enough attractive interaction. For $b < b^\star$, g is negative and flows to $(-\infty)$ at large X; the manifold is pinned at the origin. The UV non-trivial fixed point at $b = b^\star$ describes the delocalization transition. At this point $g(b^\star) = g^\star$ for any value of the size X. The critical properties of the transition are now anomalous, *i.e.* no longer given by mean field theory. For smaller attractive interaction ($b^\star < b < 0$), g is negative but now flows to 0 at large X. For repulsive interactions ($b > 0$), g is positive and flows to 0 at large X. In these latter two cases ($b > b^\star$), the manifold is delocalized, and

[1] This transition occurs at vanishing b, which corresponds to infinite temperature. Thus it cannot be induced by a simple change in the temperature but requires a qualitative change from attractive to repulsive interaction.

no longer feels at large distance the existence of the singular interaction at the origin, since the IR behavior is now governed by the trivial Gaussian fixed point at $b = 0$.

(III) $\epsilon = 0$: Finally, at the critical dimension, we are in the marginal situation where the localization transition occurs at $g = 0$ ($b = 0$), but where calculable logarithmic corrections to scaling occur [20].

As discussed previously, this picture is valid provided that the renormalization group calculations which lead to fig. 2 make sense. This point has been discussed at one loop by one of us in [20]. For the case of a one-dimensional manifold ($D = 1$), exact solutions corroborate this picture. Finally, let us mention the exact treatment of the renormalization group flow for small b ($b \simeq 0$) for the problem of interface pinning of [27]. This corresponds to the case $D = 2$, $d = 1$ and $k = 2$ ($\epsilon = 2$).

2.2. The partition function

The partition function $\mathcal{Z}$ for the model is defined by

$$\mathcal{Z} = \int \mathcal{D}[\vec{\mathbf{r}}] \exp(-\mathcal{H}) \ . \tag{2.9}$$

Its perturbative expansion in the coupling constant b is

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{(-b)^N}{N!} \mathcal{Z}_N \ , \tag{2.10}$$

where

$$\mathcal{Z}_N = \Big\langle \int_{\mathcal{V}} \prod_{i=1}^{N} d^D x_i \, \delta^d(\vec{\mathbf{r}}(x_i)) \Big\rangle_0 \tag{2.11}$$

and $\langle \ldots \rangle_0$ is the average with respect to the Gaussian measure $\exp\left[-\int_{\mathcal{V}} d^D x \frac{1}{2} \vec{\mathbf{r}} (-\Delta)^{\frac{k}{2}} \vec{\mathbf{r}}\right]$. The evaluation of $\mathcal{Z}_N$ is best performed in Fourier space by introducing the vertex function

$$V(x, \vec{\mathbf{k}}) = \exp(i \, \vec{\mathbf{k}} \cdot \vec{\mathbf{r}}(x)) \ , \tag{2.12}$$

with $\vec{\mathbf{k}}$ a d-dimensional vector, and by writing $\mathcal{Z}_N$ as

$$\mathcal{Z}_N = \Big\langle \prod_{i=1}^{N} \int_{\mathcal{V}} d^D x_i \int \frac{d^d \vec{\mathbf{k}}_i}{(2\pi)^d} V(x_i, \vec{\mathbf{k}}_i) \Big\rangle_0 \ . \tag{2.13}$$

We compute the above functional average by taking care of the overall displacement of the manifold (zero-mode):

$$\vec{\mathbf{r}}_G = \frac{1}{\mathcal{V}} \int_{\mathcal{V}} d^D x \ \vec{\mathbf{r}}(x) \ . \tag{2.14}$$

We have explicitly:

$$\langle \prod_{i=1}^{N} V(x_i, \vec{\mathbf{k}}_i) \rangle_0 = \int d^d\vec{\mathbf{r}}_0 \int \mathcal{D}[\vec{\mathbf{r}}(x)]\delta^d(\vec{\mathbf{r}}_G - \vec{\mathbf{r}}_0) \exp\left[-\int_{\mathcal{V}} d^D x \frac{1}{2}\vec{\mathbf{r}}(-\Delta)^{\frac{k}{2}}\vec{\mathbf{r}} + i\sum_{i=1}^{N} \vec{\mathbf{k}}_i \cdot \vec{\mathbf{r}}(x_i) \right] \tag{2.15}$$

Performing the shift $\vec{\mathbf{r}} = \vec{\mathbf{r}}_G + \vec{\tilde{\mathbf{r}}}$, we get:

$$\int d^d\vec{\mathbf{r}}_0 \int \mathcal{D}[\vec{\tilde{\mathbf{r}}}(x)]\delta^d(\vec{\tilde{\mathbf{r}}}_G) \exp\left[-\int_{\mathcal{V}} d^D x \frac{1}{2}\vec{\tilde{\mathbf{r}}}(-\Delta)^{\frac{k}{2}}\vec{\tilde{\mathbf{r}}} + i\sum_{i=1}^{N} \vec{\mathbf{k}}_i \cdot (\vec{\tilde{\mathbf{r}}}(x_i) + \vec{\mathbf{r}}_0) \right] \ . \tag{2.16}$$

Integrating over the displacement $\vec{\mathbf{r}}_0$, and performing the Gaussian average, with normalization

$$\int \mathcal{D}[\vec{\tilde{\mathbf{r}}}(x)]\delta^d(\vec{\tilde{\mathbf{r}}}_G) \exp\left[-\frac{1}{2}\int_{\mathcal{V}} d^D x\, \vec{\tilde{\mathbf{r}}}(-\Delta)^{\frac{k}{2}}\vec{\tilde{\mathbf{r}}} \right] \ = \ 1 \ , \tag{2.17}$$

we finally get

$$\mathcal{Z}_N = \int \prod_{i=1}^{N} \frac{d^D x_i\, d^d\vec{\mathbf{k}}_i}{(2\pi)^d}\, (2\pi)^d\, \delta^d(\sum_{i=1}^{N} \vec{\mathbf{k}}_i)\ \exp\left[-\frac{1}{2}\sum_{i,j=1}^{N} \vec{\mathbf{k}}_i \cdot \vec{\mathbf{k}}_j\, G(x_i, x_j) \right] \tag{2.18}$$

where $G(x, y)$ is the propagator, solution (in infinite flat D-dimensional space) of

$$(-\Delta_x)^{\frac{k}{2}} G(x, y) = \delta^D(x - y) \ , \tag{2.19}$$

namely:

$$G(x, y) \ = \ \frac{1}{2^k\, \pi^{\frac{D}{2}}}\, \frac{\Gamma(\frac{D-k}{2})}{\Gamma(\frac{k}{2})}\, |x - y|^{k-D} \ . \tag{2.20}$$

This propagator, which is a Coulomb-like potential, will play a fundamental role in what follows. In the range of parameters (2.5) , it vanishes at $|x - y| = 0$.

The first term of the expansion of $\mathcal{Z}$ ($N = 0$) is simply the (infinite) volume of external space

$$\mathcal{Z}_0 \ = \ (2\pi)^d \delta^d(\vec{\mathbf{k}} = \vec{0}) \ = \ \int d^d\vec{\mathbf{r}}_G \equiv V_{\mathbb{R}^d} \ . \tag{2.21}$$

But the next terms are finite. Indeed, for $N > 0$ we can deal with the δ^d constraint in (2.18) by setting $\vec{\mathbf{k}}_1 = -\sum_{i=2}^{N} \vec{\mathbf{k}}_i$. The integration over $\vec{\mathbf{k}}$ becomes Gaussian and leads for $N = 1$ to

$$\mathcal{Z}_1 \ = \ \int d^D x_1 \ = \ \mathcal{V} \tag{2.22}$$

and for $N > 1$ to the basic formula [20]:

$$\boxed{\mathcal{Z}_N \;=\; (2\pi)^{-\frac{d}{2}(N-1)} \int \prod_{i=1}^{N} d^D x_i \, \left(\det\left[\Pi_{ij}\right]_{2\leq i,j\leq N}\right)^{-\frac{d}{2}}} \;, \tag{2.23}$$

where Π_{ij} $(2 \leq i, j \leq N)$ is the $(N-1) \times (N-1)$ matrix

$$\Pi_{ij} \;=\; G(x_i, x_j) - G(x_1, x_j) - G(x_i, x_1) + G(x_1, x_1) \;. \tag{2.24}$$

Notice that Π_{ij} is function of the point x_1 which acts as a reference point, and that $G(x_1, x_1)$ is actually equal to zero.

2.3. Correlation functions

Similarly, all expectation values of observables can be obtained from the partition functions with inserted vertex operators (2.12)

$$\mathcal{Z}^{(M)}(X_a, \vec{\mathbf{k}}_a) \;=\; \mathcal{Z} \cdot \langle \prod_{a=1}^{M} V(X_a, \vec{\mathbf{k}}_a) \rangle \;=\; \int \mathcal{D}[\vec{\mathbf{r}}] \exp(-\mathcal{H} + \sum_{a=1}^{M} i\, \vec{\mathbf{k}}_a \cdot \vec{\mathbf{r}}(X_a)) \;. \tag{2.25}$$

Each term of their perturbative expansion

$$\mathcal{Z}^{(M)}(X_a, \vec{\mathbf{k}}_a) \;=\; \sum_{N=0}^{\infty} \frac{(-b)^N}{N!} \mathcal{Z}_N^{(M)}(X_a, \vec{\mathbf{k}}_a) \tag{2.26}$$

can be computed by the same techniques. The final result is for $N > 1$

$$\mathcal{Z}_N^{(M)}(X_a, \vec{\mathbf{k}}_a) = (2\pi)^{-\frac{d}{2}(N-1)} \int \prod_{i=1}^{N} d^D x_i \, \left(\det\left[\Pi_{ij}\right]_{2\leq i,j\leq N}\right)^{-\frac{d}{2}} \exp\left(-\frac{1}{2} \sum_{a,b=1}^{M} \vec{\mathbf{k}}_a \cdot \vec{\mathbf{k}}_b \, \Delta_{ab}\right) \tag{2.27}$$

where Δ_{ab} is a ratio of determinants:

$$\Delta_{ab} \;=\; \frac{\det_N \begin{pmatrix} \Pi_{ab} & \Pi_{aj} \\ \Pi_{ib} & \Pi_{ij} \end{pmatrix}}{\det_{N-1}(\Pi_{ij})} \;, \tag{2.28}$$

with an obvious extension of the definition of the Π matrix (2.24) to include external points (in particular $\Pi_{ab} = G(X_a, X_b) - G(x_1, X_b) - G(X_a, x_1) + G(x_1, x_1)$). The cases $N = 0$ and $N = 1$ require a specific analysis. For $N = 0$ we get simply

$$\mathcal{Z}_0^{(M)}(X_a, \vec{\mathbf{k}}_a) \;=\; (2\pi)^d \delta^d(\sum_{a=1}^{M} \vec{\mathbf{k}}_a) \, \exp(-\frac{1}{2} \sum_{a,b=1}^{M} \vec{\mathbf{k}}_a \cdot \vec{\mathbf{k}}_b \, G(X_a, X_b)) \;, \tag{2.29}$$

and for $N = 1$

$$\mathcal{Z}_1^{(M)}(X_a, \vec{\mathbf{k}}_a) \;=\; \int d^D x_1 \, \exp(-\frac{1}{2} \sum_{a,b=1}^{M} \vec{\mathbf{k}}_a \cdot \vec{\mathbf{k}}_b \, \Pi_{ab}) \tag{2.30}$$

(Notice in this last equation that Π_{ab} actually depends on x_1).

2.4. Mean squared distances

From (2.29) one can in particular derive the mean squared distance between any two points x and y for the free model ($b = 0$):

$$\frac{1}{2d}\,\langle(\vec{\mathbf{r}}(x)-\vec{\mathbf{r}}(y))^2\rangle_0 \;=\; -\,G(x,y) \;=\; \frac{1}{4^\nu(4\pi)^{D/2}}\frac{\Gamma(1-\nu)}{\nu\Gamma(\nu+\frac{D}{2})}|x-y|^{2\nu}\;, \tag{2.31}$$

which is IR- and UV-finite and positive for $0 < \nu < 1$ ($k-2 < D < k$).

3. Analytic continuation in the internal dimension D

3.1. Independent set of parameters: D, ν and ϵ

We now want to give a meaning to the above expressions for arbitrary real D, d and k, so as to have a continuous approach to the "physical" elastic membrane problem $D = 2$ and $k = 2$. As is clear from (2.27), the general observables of the form (2.25) depend on the external dimension d only through: *(i)* the external invariants $\vec{\mathbf{k}}_a \cdot \vec{\mathbf{k}}_b$, *(ii)* the exponent $-d/2$ in (2.27). We can therefore, as usual in field theory, consider d as a continuous parameter. The same is true for the exponent k associated with the internal Laplacian, which appears only as a parameter in the propagator (2.20) . Since we shall be interested in the range d close to $d^\star$, it is natural to substitute to the continuous parameters d and k the set of continuous parameters ϵ and ν. Their relevant range is $\epsilon \simeq 0$ (where we expect a non-trivial universal fixed point) and $0 < \nu < 1$ (where the manifold is crumpled, that is neither collapsed nor stretched).

The analytic continuation in the internal dimension D is a new feature of this model and requires a separate analysis, namely that of the signification of the measure $\prod_i d^D x_i$ for non-integer D. We now discuss equivalent geometric definitions of this measure, which have a natural extension to non-integer D.

3.2. Distance geometry in D dimensions

We are looking at generalized integrals of the type $\int d^D x_1 \ldots d^D x_N\, f(x_1,\ldots,x_N)$ where f is invariant by rotation in D-dimensional space and thus depends only on the invariant scalar products

$$u_{ij} = x_i \cdot x_j \tag{3.1}$$

Fig. 3: *Equivalent representations of the positions of a given set of N interaction points (here $N = 6$). The points are described (a) by their position x_i in $\mathbb{R}^D$ or $\mathbb{R}^{N-1}$ or (b) by the set of their mutual squared distances $a_{ij} = (x_i - x_j)^2$ or (c) by their relative vector $y_i = x_{i+1} - x_1$ in $\mathbb{R}^D$ or $\mathbb{R}^{N-1}$ (relative to the point x_1) or (d) by the line vectors (labeled by α) of an arbitrary spanning tree joining these points.*

which form a symmetric matrix $[u_{ij}]$. For $D \geq N$ we can reduce the integration over the x_i's to an integral over the u_{ij}'s of the form (see Appendix A)

$$\int \prod_{i=1}^{N} d^D x_i \, f(u_{ij}) \;=\; \int_{\mathcal{U}_N} \prod_{i \leq j} du_{ij} \, \sigma_N^{(D)}([u_{ij}]) \, f([u_{ij}]) \;, \tag{3.2}$$

where

$$\sigma_N^{(D)}([u_{ij}]) \;=\; \frac{S_D}{2} \frac{S_{D-1}}{2} \ldots \frac{S_{D-N+1}}{2} \; (\det{}_N [u_{ij}])^{\frac{D-N-1}{2}} \;. \tag{3.3}$$

S_D is the volume of the unit sphere in $\mathbb{R}^D$, $S_D = 2\,\pi^{\frac{D}{2}}/\Gamma(\frac{D}{2})$. The domain of integration $\mathcal{U}_N$ for u_{ij} is such that u_{ij} is the actual scalar product of vectors in Euclidean space, *i.e.* $[u_{ij}]$ is a positive matrix.

If moreover the integrand is translationally invariant in D-dimensional space, we can go to relative vectors $y_i = x_{i+1} - x_1$ $(1 \leq i \leq N-1)$ and reduce by one unit the number of points, *i.e* use $\sigma_{N-1}^{(D)}([y_i \cdot y_j])$.

$$\prod_{i=1}^{N} d^D x_i \;=\; d^D x_1 \prod_{1 \leq i \leq j \leq N-1} d(y_i \cdot y_j) \; \frac{S_D}{2} \frac{S_{D-1}}{2} \ldots \frac{S_{D-N+2}}{2} \; (\det{}_{N-1} [y_i \cdot y_j])^{\frac{D-N}{2}} \;. \tag{3.4}$$

This is equivalent to a measure expressed uniquely in terms of the complete set of $N(N-1)/2$ squared distances

$$a_{ij} = (x_i - x_j)^2 \tag{3.5}$$

by simply rewriting $y_i \cdot y_j$ as

$$\begin{aligned} y_{i-1} \cdot y_{j-1} &= D_{ij}(a) \\ D_{ij}(a) &\equiv \frac{1}{2}(a_{i1} + a_{j1} - a_{ij}) \qquad 2 \le i, j \le N \end{aligned} \tag{3.6}$$

Finally, after the simple change of variables (3.6) we arrive, for a translationally and rotationally invariant integrand, at an integral over distances

$$\int_{\mathbb{R}^D} \prod_{i=1}^{N} d^D x_i \, f(a_{ij}) \;=\; \mathcal{V} \int_{\mathcal{A}_N} \prod_{1 \le i < j \le N} da_{ij} \, \mu_N^{(D)}([a_{ij}]) \, f([a_{ij}]) \; . \tag{3.7}$$

where

$$\mu_N^{(D)}([a_{ij}]) \;=\; 2^{-\frac{(N-1)(N-2)}{2}} \, \frac{S_D}{2} \frac{S_{D-1}}{2} \ldots \frac{S_{D-N+2}}{2} \, \left(\det_{N-1}[D_{ij}(a)]_{2 \le i,j \le N}\right)^{\frac{D-N}{2}} \quad . \tag{3.8}$$

This last formula is valid for $D \ge N-1$. Indeed, $D = N-1$ is the smallest dimension for which N linearly independent points can be embedded in Euclidean space. The domain of integration $\mathcal{A}_N$ for a_{ij} is then simply the set for which $[D_{ij}(a)]$ is a positive matrix.

In (3.8) appears the important quantity

$$P_N(a) \equiv \det_{N-1}[D_{ij}(a)] = \det_{N-1}[y_i \cdot y_j] \tag{3.9}$$

which is a homogeneous polynomial of degree $N-1$ in the a_{ij}. $P_N(a)$ is actually fully symmetric under permutations of the indices i or j in $[a_{ij}]$, as can be seen from its expression as a Cayley-Menger determinant [2] well-known in distance geometry [28]

$$P_N(a) \;=\; \frac{(-1)^N}{2^{N-1}} \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & a_{12} & \ldots & a_{1N} \\ 1 & a_{21} & 0 & \ldots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{N1} & a_{N2} & \ldots & 0 \end{vmatrix} \; . \tag{3.10}$$

[2] This determinant appears, in a different disguise, in a letter by Descartes to the Princess Elisabeth of Bohemia (1643), as quoted by Coxeter in [29].

We have for instance for $N = 2$ and 3 points

$$P_2(a) \;=\; a_{12}\;, \quad P_3(a) \;=\; \frac{1}{4}(2a_{12}a_{23} + 2a_{23}a_{31} + 2a_{31}a_{12} - a_{12}^2 - a_{23}^2 - a_{31}^2) \tag{3.11}$$

The matrix $[D_{ij}]$ will be positive iff any bordered principal minor $P_K(a)$ is ≥ 0 for any $K \leq N$:

$$P_K(a) \;=\; \frac{(-1)^K}{2^{K-1}} \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_{12} & \dots & a_{1K} \\ 1 & a_{21} & 0 & \dots & a_{2K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{K1} & a_{K2} & \dots & 0 \end{vmatrix} \geq 0\;. \tag{3.12}$$

For $K = 2$, this is simply the positivity condition $a_{12} \geq 0$. For $K = 3$, one recovers the familiar triangular inequality

$$(a_{12} - a_{13} - a_{23})^2 \leq 4a_{13}a_{23} \quad \Leftrightarrow \quad |a_{13}^{\frac{1}{2}} - a_{23}^{\frac{1}{2}}| \leq a_{12}^{\frac{1}{2}} \leq a_{13}^{\frac{1}{2}} + a_{23}^{\frac{1}{2}}\;. \tag{3.13}$$

For $K > 3$ one gets more general inequalities which are the necessary and sufficient conditions for the a_{ij} to be realized as squared distances between N points of the Euclidean space $\mathbb{R}^{N-1}$. The volume $\mathcal{V}(x_1, \dots, x_K)$ of the (possibly degenerate) parallelotope [30] ($(K-1)$-dimensional parallelepiped) with vertices $x_1, x_2, \dots, x_K$ is given by

$$\mathcal{V}^2(x_1, \dots, x_K) \;=\; P_K(a)\;. \tag{3.14}$$

Thus $P_K(a) = 0$ indicates that the first K points are linearly dependent, *i.e.* can be embedded in $\mathbb{R}^{K-2}$.

For $D \leq N-2$, the expression (3.8) becomes singular due to the appearance of zeros in the sphere volumes S_{D-K+2} for $D+2 \leq K \leq N$ on the one hand, and due to divergences of the term $(P_N(a))^{\frac{D-N}{2}}$, which occur when $P_N(a)$ vanishes, that is on the boundary of the domain $\mathcal{A}_N$, on the other hand. Nevertheless $\mu_N^{(D)}(a)$ can now be considered as a distribution with a support in submanifolds of $\mathcal{A}_N$ of dimension $D(N - \frac{D+1}{2})$, which correspond to D-dimensional Euclidean subspaces of $\mathbb{R}^{N-1}$. One therefore still reproduces the natural Euclidean measure in $\mathbb{R}^D$, as can be shown by analytic continuation, which we now describe.

3.3. Analytic continuations in D

◇ *1- Distance geometry for non-integer D*

The first way to define integrals of the form $\int d^D x_1 \dots d^D x_N \, f(x_1, \dots, x_N)$ for non integer D is to start from (3.2) and (3.3) or equivalently from (3.7) and (3.8). The measures (3.3) and (3.8) now involve D as a simple parameter and therefore provide a natural basis for analytic continuation. For *real* $D > N-2$, $\mu_N^{(D)}(a)$ remains a positive measure density on $\mathcal{A}_N$. Therefore it can be considered as a distribution, over the space $\mathbb{R}^{\frac{N(N-1)}{2}}$ of all squared distances a_{ij}, with support $\mathcal{A}_N$ (i.e. by definition it vanishes outside $\mathcal{A}_N$). As a distribution it can be extended to $0 \leq D \leq N-2$ by analytic continuation. This amounts to treat by a finite part prescription all the divergences which occur at the boundaries of $\mathcal{A}_N$ (see below the spherical coordinate representation for more details). As a distribution, it is not singular for positive *integer* $D \leq N-2$, but becomes a measure density concentrated on the submanifold such as the principal minors $P_K(a)$ vanish for all K such that $D+1 < K \leq N$.

As an example let us consider the case of two points. For $N = 2$ we have the distribution (denoting $\chi(\mathcal{A})$ the characteristic function of support $\mathcal{A}$)

$$\mu_2^{(D)}(a) \; \chi(\mathcal{A}_2) \;=\; \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \, |a_{12}|^{\frac{D}{2}-1} \, \theta(a_{12}) \tag{3.15}$$

When $D \to 0$ the r.h.s. of (3.15) tends to

$$\frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \, |a_{12}|^{\frac{D}{2}-1} \, \theta(a_{12}) \stackrel{D\to 0}{\longrightarrow} \delta(a_{12}) \tag{3.16}$$

Thus the support of the distribution becomes restricted to the zero-dimensional subspace (where all points coincide).

Similarly for $N = 3$ we have

$$\begin{aligned} \mu_3^{(D)}(a) \; \chi(\mathcal{A}_3) \;&=\; \frac{1}{2} \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \frac{\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \, |\det_2 D|^{\frac{D-1}{2}-1} \, \theta(\det_2 D) \; \theta(a_{12})\theta(a_{13})\theta(a_{23}) \\ &\stackrel{D\to 1}{\longrightarrow} \frac{1}{2} \delta(\det_2 D) \; \theta(a_{12})\theta(a_{13})\theta(a_{23}) \end{aligned} \tag{3.17}$$

where $\det_2 D \equiv P_3(a)$ reads:

$$\det_2 D = \frac{1}{4}(a_{12}^{\frac{1}{2}} + a_{13}^{\frac{1}{2}} + a_{23}^{\frac{1}{2}})(a_{12}^{\frac{1}{2}} + a_{13}^{\frac{1}{2}} - a_{23}^{\frac{1}{2}})(a_{13}^{\frac{1}{2}} + a_{23}^{\frac{1}{2}} - a_{12}^{\frac{1}{2}})(a_{12}^{\frac{1}{2}} + a_{23}^{\frac{1}{2}} - a_{13}^{\frac{1}{2}}) \; . \tag{3.18}$$

Separating three different boundary sectors of $\mathcal{A}_3$, we get

$$\mu_3^{(1)}(a)\chi(\mathcal{A}_3)da_{12}da_{13}da_{23} \;=\; 2\delta(a_{12}^{\frac{1}{2}}+a_{23}^{\frac{1}{2}}-a_{13}^{\frac{1}{2}})\;\theta(a_{12})\theta(a_{23})\theta(a_{13})da_{12}^{\frac{1}{2}}\,da_{23}^{\frac{1}{2}}\,da_{13}^{\frac{1}{2}} + \text{perm.} \tag{3.19}$$

which represents indeed all possible relative positions of three points on an oriented line.

◇ *2- Cartesian coordinates in* $\mathbb{R}^{N-1}$

Realizing that $N-1$ is the minimal dimension of Euclidean space in which one can embed N points with given squared distances a_{ij} (in $\mathcal{A}_N$), we can use (3.4) back to reexpress the measure over the scalar products $d(y_i\cdot y_j)$ as a measure over N points in $\mathbb{R}^{N-1}$

$$\prod_{i=1}^{N-1} d^{N-1}y_i \;=\; \prod_{1\le i\le j\le N-1} d(y_i\cdot y_j)\;\frac{S_{N-1}}{2}\frac{S_{N-2}}{2}\dots\frac{S_1}{2}\;(\det{}_{N-1}[y_i\cdot y_j])^{-\frac{1}{2}}\;. \tag{3.20}$$

Thus we can implement the analytic continuation in D by modifying the Euclidean measure in $\mathbb{R}^{N-1}$ by a suitable analytic measure term:

$$\prod_{i=1}^{N-1} d^{D}y_i \;\equiv\; \prod_{i=1}^{N-1} d^{N-1}y_i\frac{S_DS_{D-1}\dots S_{D-N+2}}{S_{N-1}S_{N-2}\dots S_1}\Big[\det[y_i\cdot y_j]_{1\le i,j\le N-1}\Big]^{\frac{D-N+1}{2}}\;. \tag{3.21}$$

Analytic continuation can thus be summarized in the following compact formula, which is a formal rewriting of (3.21):

$$\boxed{\begin{aligned}\prod_{i=1}^{N-1} d^{D}y_i &= \prod_{i=1}^{N-1} d^{N-1}y_i\;\Omega(D,N)\;\left(\mathcal{V}\left(0,y_1,\dots,y_{N-1}\right)\right)^{D-N+1}\\ \Omega(D,N) &= \frac{\mathrm{Vol}(\mathrm{SO}(D))}{\mathrm{Vol}(\mathrm{SO}(D-N+1))\mathrm{Vol}(\mathrm{SO}(N-1))}\end{aligned}}\;, \tag{3.22}$$

where $\mathrm{Vol}(\mathrm{SO}(D))$ is the volume of the special orthogonal group in D dimensions:

$$\mathrm{Vol}(\mathrm{SO}(D)) = \frac{S_D}{2}\frac{S_{D-1}}{2}\dots\frac{S_1}{2}\;. \tag{3.23}$$

When M external points X_a are present (that is points over which we do not integrate), Eq. (3.22) has to be replaced by the more general formula:

$$\begin{aligned}\prod_{i=1}^{N} d^{D}x_i &= \prod_{i=1}^{N} d^{M+N-1}x_i\;\Omega(D,M,N)\;\left(\frac{\mathcal{V}\left(x_1,x_2,\dots,x_N,X_1,\dots,X_M\right)}{\mathcal{V}\left(X_1,\dots,X_M\right)}\right)^{D-M-N+1}\\ \Omega(D,M,N) &= \frac{\mathrm{Vol}(\mathrm{SO}(D-M+1))}{\mathrm{Vol}(\mathrm{SO}(D-M-N+1))\mathrm{Vol}(\mathrm{SO}(N))}\;.\end{aligned} \tag{3.24}$$

◇ *3- Spherical coordinates*

A third (equivalent) way to perform an analytic continuation in D is the use of spherical coordinates. We first consider again the case of N points in $\mathbb{R}^D$ with D integer and $D \geq N-1$. We take x_1 as the center of the spherical coordinates, and describe the $N-1$ other points by their relative coordinate, as before

$$y_i = x_{i+1} - x_1 \qquad i = 1, \ldots, N-1 \ . \tag{3.25}$$

Introducing generalized spherical coordinates for the y_i, we write

$$\begin{aligned} y_{i,1} &= |y_i| \cos\theta_{i,1} \\ y_{i,2} &= |y_i| \sin\theta_{i,1} \cos\theta_{i,2} \\ &\vdots \\ y_{i,D-1} &= |y_i| \sin\theta_{i,1} \sin\theta_{i,2} \ldots \sin\theta_{i,D-2} \cos\theta_{i,D-1} \\ y_{i,D} &= |y_i| \sin\theta_{i,1} \sin\theta_{i,2} \ldots \sin\theta_{i,D-2} \sin\theta_{i,D-1} \end{aligned} \tag{3.26}$$

where $\theta_{i,n} \in [0, \pi]$ for $1 \leq n \leq D-2$ and $\theta_{i,D-1} \in [0, 2\pi[$. The corresponding measure is given by

$$d^D y_i = |y_i|^{D-1} d|y_i| \prod_{n=1}^{D-1} (\sin\theta_{i,n})^{D-1-n} d\theta_{i,n} \ . \tag{3.27}$$

For rotationally invariant integrands, we can furthermore fix successively

$$\theta_{i,n} = 0 \qquad n \geq i \tag{3.28}$$

Taking care of the successive rotational symmetries, we arrive at

$$\boxed{\prod_{i=1}^{N-1} d^D y_i \ = \ S_D S_{D-1} \ldots S_{D-N+2} \prod_{i=1}^{N-1} |y_i|^{D-1} d|y_i| \prod_{i=2}^{N-1} \prod_{n=1}^{i-1} (\sin\theta_{i,n})^{D-1-n} d\theta_{i,n}} \tag{3.29}$$

with all the $\theta_{i,n}$ now integrated from 0 to π. In (3.29) , D again appears only as a parameter. This therefore provides another natural path to analytic continuation in D. Indeed, possible singularities at integer D arise from the negative powers of the $\sin\theta_{i,n}$'s, which diverge at $\theta_{i,n} = 0$ or π. It is clear from the spherical coordinates representation (3.26) that when some of the θ's are equal to 0 or π the vectors y_i are not linearly independent and the

N points x_i are on an Euclidean subspace with dimension smaller than $N-1$. Away from integer values of D (with $0 < D < N-1$), these divergences can be treated by the standard finite part prescription (independently for each $\theta_{i,n}$). To prove that for integer D, (3.29) remains a distribution and can be rewritten as a finite measure localized on some subspace (corresponding to spherical coordinates in some D dimensional submanifold) requires a more elaborate discussion, not presented here.

This analytic continuation (3.29) is totally equivalent to the analytic continuation (3.22), as can be seen by going back as before to coordinates in $\mathbb{R}^{N-1}$. Using (3.29), we have formally

$$\prod_{i=1}^{N-1} d^D y_i \;=\; \prod_{i=1}^{N-1} d^{N-1} y_i \frac{S_D S_{D-1} \dots S_{D-N+2}}{S_{N-1} S_{N-2} \dots S_1} \left[\prod_{i=1}^{N-1} |y_i| \times \prod_{i=2}^{N-1} \prod_{n=1}^{i-1} \sin\theta_{i,n} \right]^{D-N+1} \tag{3.30}$$

where the $\theta_{i,n}$ are spherical angles in $\mathbb{R}^{N-1}$. We read on this equation the angular representation of the squared parallelotope volume [28]

$$\begin{aligned} P_N(a) \;=\; \det[y_i \cdot y_j]_{1 \le i,j \le N-1} \;&=\; \prod_{i=1}^{N-1} |y_i|^2 \times \prod_{i=2}^{N-1} \prod_{n=1}^{i-1} \sin^2\theta_{i,n} \\ &= \mathcal{V}^2(0, y_1, \dots, y_{N-1}) \;. \end{aligned} \tag{3.31}$$

and (3.30) is therefore identical to (3.22).
Finally, when M external points are present, (3.29) has to be replaced by

$$\prod_{i=1}^{N} d^D x_i = S_{D-M+1} S_{D-M} \dots S_{D-M-N+2} \prod_{i=1}^{N} |x_i|^{D-1} d|x_i| \prod_{i=1}^{N} \prod_{n=1}^{M+i-2} (\sin\theta_{i,n})^{D-1-n} d\theta_{i,n} \tag{3.32}$$

where the $\theta_{i,n}$ are the $M+i-2$ successive relative spherical angles for x_i necessary to assign position to the vector $x_i - X_1$ with respect to the $M-1$ external vectors $X_2 - X_1, \dots, X_M - X_1$ and to the $i-1$ internal vectors $x_j - X_1$ for $j < i$, in a reference frame where X_1 is at the origin.

3.4. Factorization

Of course, for integer D, the measure $\prod_i d^D y_i$ is naturally factorized, when applied to a product of functions of independent variables:

$$\int \prod_{k=1}^{P+Q} d^D y_k \, f(\{y_{k;k=1,P}\}) \, g(\{y_{k;k=P+1,P+Q}\}) \;=\; \int \prod_{i=1}^{P} d^D y_i \, f(y_i) \cdot \int \prod_{j=P+1}^{P+Q} d^D y_j \, g(y_j) \;. \tag{3.33}$$

This important factorization property becomes non trivial when extended to arbitrary D, as can be seen from (3.21) . Still, if we consider the scalar product matrix $[u_{ij}]_{1\leq i,j\leq P+Q}$ and denote by $[u]_P$ (respectively $[u]_Q$) the submatrix $[u_{ij}]_{1\leq i,j\leq P}$ (respectively $[u_{ij}]_{P+1\leq i,j\leq P+Q}$), one has (see Appendix B)

$$\begin{aligned}\int_{\mathcal{U}_{P+Q}} d[u]\, \sigma^{(D)}_{P+Q}([u]) f([u]_P) g([u]_Q) \;&=\; \int_{\mathcal{U}_P} d[u]_P\, \sigma^{(D)}_P([u]_P)\, f([u]_P) \\ &\quad\times \int_{\mathcal{U}_Q} d[u]_Q\, \sigma^{(D)}_Q([u]_Q)\, g([u]_Q)\end{aligned} \tag{3.34}$$

which means that the integration over the mixed scalar products u_{ij}, $1 \leq i \leq P < j \leq P+Q$ can be performed and amounts to factorize $\sigma^{(D)}_{P+Q}$ into $\sigma^{(D)}_P \sigma^{(D)}_Q$. *The factorization property of the measure is thus preserved under analytic continuation in D.*

3.5. The interaction as a Cayley-Menger determinant

The N point interaction term $(\det_{N-1}[\Pi_{ij}])^{-\frac{d}{2}}$ depends explicitly on D through the occurrence of the Green function (2.31) and is readily analytically continued to non-integer D. Let us recall that we consider D, ν and ϵ as the three independent parameters of the model, so that d itself is a function of D given by $d = (D-\epsilon)/\nu$. From a distinct, geometrical point of view, it is particularly interesting to notice that the interaction term also involves a determinant of the Cayley-Menger type with a_{ij} replaced by its power $(a_{ij})^\nu$

$$P_N(a^\nu) \;\equiv\; \frac{(-1)^N}{2^{N-1}} \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a^\nu_{12} & \dots & a^\nu_{1N} \\ 1 & a^\nu_{21} & 0 & \dots & a^\nu_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a^\nu_{N1} & a^\nu_{N2} & \dots & 0 \end{vmatrix} . \tag{3.35}$$

Indeed, from definition (2.24) and from (2.31) , we have

$$\Pi_{ij} \;=\; A_D(\nu)\; D_{ij}(a^\nu)\;, \tag{3.36}$$

with

$$D_{ij}(a^\nu) \;=\; \frac{1}{2}(a^\nu_{i1} + a^\nu_{j1} - a^\nu_{ij}) \tag{3.37}$$

and the factor

$$A_D(\nu) \;=\; \frac{2}{4^\nu (4\pi)^{\frac{D}{2}}} \frac{\Gamma(1-\nu)}{\nu\Gamma(\nu+\frac{D}{2})}\;, \tag{3.38}$$

and therefore

$$\det_{N-1}[\Pi_{ij}] \;=\; [A_D(\nu)]^{N-1}\; P_N(a^\nu)\;. \tag{3.39}$$

Finally we have the compact formula, analytic in D, ϵ and ν, for the term of order N of the partition function (2.10)

$$\boxed{\mathcal{Z}_N = \mathcal{V}\,(2\pi A_D(\nu))^{-\frac{d}{2}(N-1)} \prod_{K=2}^{N} \left(\frac{S_{D-K+2}}{2^{K-1}}\right) \int_{\mathcal{A}_N} \prod_{1\le i<j\le N} da_{ij} [P_N(a)]^{\frac{D-N}{2}} [P_N(a^\nu)]^{-\frac{d}{2}}} \tag{3.40}$$

with again $d = (D-\epsilon)/\nu$.

3.6. Analytic expression of $\mathcal{Z}_N$ in Cartesian coordinates

An immediate corollary of the above formalism is the following alternative formula for $\mathcal{Z}_N$, now in Cartesian coordinates in $\mathbb{R}^{N-1}$, which provides an equivalent definition of the analytic continuation of $\mathcal{Z}_N$:

$$\begin{aligned}\mathcal{Z}_N \;&= (2\pi)^{-\frac{d}{2}(N-1)}\,\mathcal{V} \int \prod_{i=1}^{N-1} d^{N-1}y_i \frac{S_D\ldots S_{D-N+2}}{S_{N-1}\ldots S_1} \left(\det\left[y_i\cdot y_j\right]_{1\le i,j\le N-1}\right)^{\frac{D-N+1}{2}} \\ &\qquad\qquad \times \left(\det\left[\Pi_{ij}\right]_{2\le i,j\le N}\right)^{-\frac{d}{2}}\;.\end{aligned} \tag{3.41}$$

3.7. Determinant attached to trees

In the following, we shall find useful to express both the measure and the interaction contributions in terms of more general variables λ_α obtained from the positions x_i and attached to arbitrary oriented trees. A spanning tree is a connected graph whose vertices are the previous N points x_i, and without loops. This graph therefore has $N-1$ internal lines labeled by $\alpha = 1,\ldots,N-1$ for which one also specifies an orientation. An oriented tree is characterized by its incidence $N\times(N-1)$ matrix $[\epsilon_{i\alpha}]$ defined by $\epsilon_{i\alpha}=1$ if the line α is incident to i and points toward i, $\epsilon_{i\alpha}=-1$ if α is incident to i and points outward i, $\epsilon_{i\alpha}=0$ otherwise. One has

$$\sum_{i=1}^{N} \epsilon_{i\alpha} = 0\;. \tag{3.42}$$

For each line α of the tree we define the *line vector* (or *edge vector*) λ_α in $\mathbb{R}^{N-1}$ by

$$\lambda_\alpha = \sum_{i=1}^{N} \epsilon_{i\alpha}\ x_i \ = \ \sum_{i=1}^{N-1} \epsilon_{i+1\alpha}\ y_i \ , \tag{3.43}$$

where the y_i's have been defined in (3.25).

◇ *Expression for the measure*

Since the Jacobian of the linear transformation (3.43) from the y_i's to the λ_α's is $|\det'[\epsilon]| = |\det[\epsilon_{i\alpha}]_{\substack{2\le i\le N\\ 1\le\alpha\le N-1}}| = 1$ and $\det[\lambda_\alpha\cdot\lambda_\beta] = (\det'[\epsilon])^2\det[y_i\cdot y_j] = \det[y_i\cdot y_j]$. one has directly from (3.21)

$$\begin{aligned}\prod_{i=1}^{N-1} d^D y_i \ &= \prod_{\alpha=1}^{N-1} d^D\lambda_\alpha \\ &\equiv \prod_{\alpha=1}^{N-1} d^{N-1}\lambda_\alpha \frac{S_D S_{D-1}\dots S_{D-N+2}}{S_{N-1}S_{N-2}\dots S_1}\Big[\det[\lambda_\alpha\cdot\lambda_\beta]_{1\le\alpha,\beta\le N-1}\Big]^{\frac{D-N+1}{2}} \ .\end{aligned} \tag{3.44}$$

This also means that one can replace in (3.4) the integration over the matrix elements $u_{ij} = y_i\cdot y_j$ by an integration over matrix elements $u_{\alpha\beta} = \lambda_\alpha\cdot\lambda_\beta$ associated with an arbitrary tree.

◇ *Expression for the interaction*

We now derive the expression of the determinant $P_N(a^\nu)$ which enters the interaction factor in terms of the λ_α's. Equation (2.23) was actually a particular representation of the interaction, associated with a particular choice of a tree, namely the star centered at x_1 and lines pointing toward the other points. This can be seen in our choice $\vec{\mathbf{k}}_1 = -\sum_{i=2}^N \vec{\mathbf{k}}_i$ to account for the $\delta^d(\sum_i \vec{\mathbf{k}}_i)$ constraint in the momentum integral (2.18) . We can generalize this construction to an arbitrary oriented tree $\mathbf{T}$ by writing $\vec{\mathbf{k}}_i$ as

$$\vec{\mathbf{k}}_i = -\sum_{\alpha=1}^{N-1} \epsilon_{i\alpha}\ \vec{\mathbf{q}}_\alpha \ . \tag{3.45}$$

These vectors $\vec{\mathbf{q}}_\alpha$ can be seen as flowing along the lines of the tree while the vectors $\vec{\mathbf{k}}_i$ can be thought of as being injected at the nodes of the tree. Equation (3.45) expresses the momentum conservation at the nodes and moreover, together with (3.42), ensures $\sum_i \vec{\mathbf{k}}_i = \vec{0}$ for any set of $\vec{\mathbf{q}}_\alpha$'s. Using then

$$\prod_{i=1}^{N} d^d\vec{\mathbf{k}}_i\ \delta^d(\sum_{i=1}^{N}\vec{\mathbf{k}}_i) \ = \ \prod_{\alpha=1}^{N-1} d^d\vec{\mathbf{q}}_\alpha \ , \tag{3.46}$$

we get for the interaction term (2.18)

$$\begin{aligned}\mathcal{Z}_N &= \mathcal{V}\int\prod_{\alpha=1}^{N-1}\frac{d^D\lambda_\alpha d^d\vec{\mathbf{q}}_\alpha}{(2\pi)^d}\exp\left[-\frac{1}{2}\sum_{\alpha,\beta=1}^{N-1}\vec{\mathbf{q}}_\alpha\cdot\vec{\mathbf{q}}_\beta\,\Pi^{\mathbf{T}}_{\alpha\beta}\right]\\ &= (2\pi)^{-\frac{d}{2}(N-1)}\mathcal{V}\int\prod_{\alpha=1}^{N-1}d^D\lambda_\alpha\,\left(\det\left[\Pi^{\mathbf{T}}_{\alpha\beta}\right]_{1\leq\alpha,\beta\leq N-1}\right)^{-\frac{d}{2}}\ ,\end{aligned}\tag{3.47}$$

where we take advantage of (3.44) and define a new matrix $\Pi^{\mathbf{T}}$ attached to the tree $\mathbf{T}$:

$$\Pi^{\mathbf{T}}_{\alpha\beta} = \sum_{i,j=1}^{N}\epsilon_{i\alpha}G(x_i,x_j)\epsilon_{j\beta}\ .\tag{3.48}$$

Indeed $\det[\Pi^{\mathbf{T}}_{\alpha\beta}]$ is independent of the choice of the tree $\mathbf{T}$.

In terms of pairs of oriented lines α,β of the tree, with extremities (i_α, i'_α) and (i_β, i'_β) respectively, the matrix element $\Pi^{\mathbf{T}}_{\alpha\beta}$ is associated with the quadrilateral $(i_\alpha, i'_\alpha; i_\beta, i'_\beta)$

$$\Pi^{\mathbf{T}}_{\alpha\beta} = G(x_{i_\alpha},x_{i_\beta}) + G(x_{i'_\alpha},x_{i'_\beta}) - G(x_{i_\alpha},x_{i'_\beta}) - G(x_{i'_\alpha},x_{i_\beta})\ .\tag{3.49}$$

It can be viewed as an interaction potential between two dipoles λ_α and λ_β and has the following pictorial representation:

$$\Pi^{\mathbf{T}}_{\alpha\beta} \ = \tag{3.50}$$

◇ *Expression for correlation functions*

For correlation functions $\mathcal{Z}^{(M)}(X_a,\vec{\mathbf{k}}_a)$ (2.25) one can generalize the above construction simply *(i)* by considering the spanning star tree $\mathbf{T}_{\rm ex}$ with line vectors $\Lambda_a = X_a - X_1$ $(a>1)$ for the external points, *(ii)* by choosing an arbitrary tree $\mathbf{T}_{\rm in}$ with line vectors λ_α for the internal points, and *(iii)* by attaching these two trees by a line vector Λ_1 joining the external point X_1 to an arbitrary internal point. In this way, we obtain a larger tree $\mathbf{T}$ to which we can associate a generalized form of (2.27) :

$$\mathcal{Z}_N^{(M)}(X_a,\vec{\mathbf{k}}_a) \;=\; (2\pi)^{-\frac{d}{2}(N-1)} \int d^D\Lambda_1 \prod_{\alpha=1}^{N-1} d^D\lambda_\alpha \left(\det\left[\Pi^{\mathbf{T}}_{\alpha\beta}\right]_{1\le\alpha,\beta\le N-1}\right)^{-\frac{d}{2}} \times \exp\left(-\frac{1}{2}\sum_{a,b=1}^{M} \vec{\mathbf{k}}_a\cdot\vec{\mathbf{k}}_b\,\Delta_{ab}\right) \tag{3.51}$$

$$\Delta_{ab} \;=\; \frac{\det_N\begin{pmatrix}\Pi^{\mathbf{T}}_{ab} & \Pi^{\mathbf{T}}_{a\beta}\\ \Pi^{\mathbf{T}}_{\alpha b} & \Pi^{\mathbf{T}}_{\alpha\beta}\end{pmatrix}}{\det_{N-1}(\Pi^{\mathbf{T}}_{\alpha\beta})}\;. \tag{3.52}$$

As discussed above the determinants in (3.52) are independent of the tree $\mathbf{T}$ chosen. In (3.51), the integral over the λ_α's and Λ_1 has to be understood, for real D, as

$$\int d^D\Lambda_1 \prod_{\alpha=1}^{N-1} d^D\lambda_\alpha \;=\; \int d^{M+N-1}\Lambda_1 \prod_{\alpha=1}^{N-1} d^{M+N-1}\lambda_\alpha \;\frac{S_{D-M+1}\ldots S_{D-M-N+2}}{S_N\ldots S_1} \times\left\{\frac{\left[\det_{N+M-1}\begin{pmatrix}\Lambda_a\cdot\Lambda_b & \Lambda_a\cdot\lambda_\beta\\ \lambda_\alpha\cdot\Lambda_b & \lambda_\alpha\cdot\lambda_\beta\end{pmatrix}\right]}{\left[\det_{M-1}\left(\Lambda_a\cdot\Lambda_b\right)_{2\le a,b\le M}\right]}\right\}^{\frac{D-M-N+1}{2}} \tag{3.53}$$

and (3.51) is a function of the invariants $a_{ab}=(X_a-X_b)^2$, which are quadratic forms in terms of the line vectors Λ_a.

3.8. The limit D =1 and the Schwinger representation

As an example, for a manifold with internal dimension $D=1$, one can recover the standard Schwinger representation[3] of an interacting field theory with interaction term $(\mathbf{\Phi})^2(\vec{0})$ (see subsection 6.1 for further details), here in direct correspondence with the continuous Edwards-like model for a polymer interacting with a single fixed point at the origin. Choosing $D=1$ and $k=2$ in (2.1) corresponding to the Gaussian weight of a Brownian chain, one has $\nu=\frac{1}{2}$ and the propagator along the chain

$$G(x,y) = -\frac{1}{2}|x-y|\;. \tag{3.54}$$

Furthermore, for the perturbative order N, the measure term (3.8) reconstructs in the limit $D=1$ (like in (3.19)) the measure over all relative distances of N ordered points

[3] In the context of polymers, it is also known as the Fixman representation [3] .

along the chain, as well as all their permutations. For a given permutation $x_{i_1} \leq \ldots \leq x_{i_N}$, the measure term is simply

$$\prod_{\alpha=1}^{N-1} da_{i_\alpha i_{\alpha+1}}^{\frac{1}{2}} \ . \tag{3.55}$$

Choosing as a particular tree $\mathbf{T}$ the successive oriented links $(i_\alpha, i'_\alpha) = (i_\alpha, i_{\alpha+1})$ the matrix $\Pi^{\mathbf{T}}$ (3.49) is diagonal

$$\Pi^{\mathbf{T}}_{\alpha\beta} \ = \ s_\alpha \, \delta_{\alpha\beta} \quad \text{with} \quad s_\alpha \ = \ a_{i_\alpha i_{\alpha+1}}^{\frac{1}{2}} \ = \ x_{i_{\alpha+1}} - x_{i_\alpha} \ . \tag{3.56}$$

The s_α are nothing but the usual Schwinger parameters (proper time) for the propagator lines α, or in polymer theory the lengths of the successive polymer segments. The interaction gives for the partition function a term of the form

$$\mathcal{Z}_N \ = \ \int \prod_{\alpha=1}^{N-1} ds_\alpha \, (\ldots) \Big(\prod_{\alpha=1}^{N-1} s_\alpha \Big)^{-\frac{d}{2}} \tag{3.57}$$

which is nothing but the Schwinger representation for the "daisy" diagram in d dimensions.

Fig. 4: *The daisy diagram corresponding to the term (3.57)* .

4. Ultraviolet and infrared properties of the integrand

4.1. Existence and positiveness of the integrand

The rules that we have proposed above for defining the perturbative expansion of the model in non-integer dimension D remain formal. Indeed, we have not shown yet that the integrands do exist and that the integrals are convergent (for D large enough), and define an analytic function in D. Let us concentrate on the N-th term for the partition function, $\mathcal{Z}_N$, which is explicited by the integral (3.40) in terms of distance variables a_{ij}, by the integral (3.41) in terms of Cartesian coordinates in $\mathbb{R}^{N-1}$ or by the integral (3.47)

in terms of tree-variables λ_α. We shall furthermore assume in the following sections that $D \geq N-1$, that is D large enough for $\mu_N^{(D)}$ to be a measure density (similarly, for $\mathcal{Z}_N^{(M)}$, we shall assume $D \geq N+M-1$). We shall discuss in section 9 how our results can be extended to smaller D.

$\diamond$ *Schoenberg's theorem*

First, in view of the formula (3.40), the positiveness of the Cayley-Menger determinant $P_N(a^\nu)$ (3.35) has to be ensured inside the domain of integration $\mathcal{A}_N$ for the variables a_{ij}. For $0 < \nu \leq 1$, this actually is just a consequence of a remarkable theorem in distance geometry due to I.J. Schoenberg [31].

THEOREM I *If we change the metric of the Euclidean space $\mathbb{R}^m$ from the Euclidean distance $d(x,y) = |x-y|$ to the new distance*

$$\tilde{d}(x,y) = (d(x,y))^\nu \qquad\qquad 0 < \nu \leq 1 \tag{4.1}$$

the new metric space $\mathbb{R}^m_{(\nu)}$ thus arising may be embedded isometrically in the Hilbert space $\mathbb{R}^\infty$ with the L^2-norm.

A practical (equivalent) statement is that any set of N distinct points of $\mathbb{R}^m_{(\nu)}$ can be embedded in the Euclidean space $\mathbb{R}^{N-1}$. In our language, this means that, if the a_{ij} are actual squared distances of N points in $\mathbb{R}^{N-1}$, then a_{ij}^ν with $0 < \nu \leq 1$ can also be realized as actual squared distances between N transformed points in $\mathbb{R}^{N-1}$. An immediate consequence is that $P_N(a^\nu) \geq 0$, as well as all the lower rank polynomials $P_K(a^\nu) \geq 0$.

We moreover have the useful refined result for $0 < \nu < 1$ [31]:

THEOREM II *If $x_1, \ldots, x_N$ are N distinct points in $\mathbb{R}^m$, and $[a_{ij}]$ the corresponding squared distance matrix, the matrix $D_{ij}(a^\nu) = \frac{1}{2}(a_{i1}^\nu + a_{j1}^\nu - a_{ij}^\nu)$, $(0 < \nu < 1)$, is positive definite.*

The positiveness is a consequence of the previous theorem. The novelty here concerns the definiteness and states that the determinant $P_N(a^\nu)$ vanishes if and only if two points at least coincide, that is $a_{ij} = 0$ for some $i \neq j$. Notice that this property does not hold for the case $\nu = 1$ for which we already know that $P_N(a)$ vanishes as soon the a_{ij} can be realized as distances between N points in $\mathbb{R}^K$ for $K \leq N-2$, which can be obtained with none of the a_{ij} $(i \neq j)$ vanishing.

4.2. Short distance divergences

The above result ensures that for $0 < \nu < 1$ the only possible divergences in (3.40) occur when some distances a_{ij} go to 0 (UV divergences) or ∞ (IR divergences). Let us first discuss the UV behavior.

If one scales the distances by a global factor ρ:

$$a_{ij} \longrightarrow \rho^2 a_{ij} \ , \tag{4.2}$$

the measure term in (3.40) is scaled according to

$$\prod_{1\leq i<j\leq N} da_{ij}[P_N(a)]^{\frac{D-N}{2}} \longrightarrow \rho^{D(N-1)} \prod_{1\leq i<j\leq N} da_{ij}[P_N(a)]^{\frac{D-N}{2}} \tag{4.3}$$

while the interaction term scales as

$$[P_N(a^\nu)]^{-\frac{d}{2}} \longrightarrow \rho^{-(N-1)\nu d}[P_N(a^\nu)]^{-\frac{d}{2}} \ . \tag{4.4}$$

We therefore obtain a global scaling factor $\rho^{(N-1)(D-\nu d)} = \rho^{(N-1)\epsilon}$. This means that the contribution to $\mathcal{Z}_N$ of the region of $\mathcal{A}_N$ such that all distances $a_{ij} \leq \rho^2$ is of order $\rho^{(N-1)\epsilon}$, indicating that $\mathcal{Z}_N$ is superficially UV convergent for $\epsilon > 0$, but divergent for $\epsilon \leq 0$.

Similarly, we expect that when the distances between some subset of P points are $\leq \rho^2$, we get a contribution of order $\rho^{(P-1)\epsilon}$ to $\mathcal{Z}_N$. This is indeed what occurs, due to the following crucial factorization property of the interaction term.

THEOREM *Short distance factorization of the interaction term*

Consider the subset $\mathcal{P}$ of (for instance) the first P interacting points (considered as embedded in $\mathbb{R}^{N-1}$) $x_1, \ldots, x_P$ and let us contract it toward one of its points, which we choose to be x_1. We set

$$x_k(\rho) \ = \ \begin{cases} x_1 + \rho(x_k - x_1) & \text{if } 1 \leq k \leq P \\ x_k & \text{if } P < k \leq N \ . \end{cases} \tag{4.5}$$

This defines a mapping in distance variables

$$a_{ij}(\rho) \ = \ \begin{cases} \rho^2 a_{ij} & \text{if } 1 \leq i \leq j \leq P \\ a_{1j} - \rho(a_{1i} + a_{1j} - a_{ij}) + \rho^2(a_{1i}) & \text{if } 1 \leq i \leq P < j \leq N \\ a_{ij} & \text{if } P < i \leq j \leq N \quad . \end{cases} \tag{4.6}$$

Then, in the limit $\rho \to 0$, the determinant of the matrix $D_{ij}(a^\nu)$ (3.37) factorizes as

$$\begin{aligned} \det_{N-1}[D_{ij}\left(a^\nu(\rho)\right)] \ &= \ \rho^{2\nu(P-1)} \det_{P-1}[D_{ij}(a^\nu)]_{2\leq i,j\leq P} \\ &\quad \times \det_{N-P}[D_{ij}(a^\nu)]_{P+1\leq i,j\leq N} \left\{1 + \mathcal{O}(\rho^{2\delta})\right\} \end{aligned} \tag{4.7}$$

Fig. 5: *Schematic picture of the short-distance factorization of the interaction term relative to some set $\mathcal{G}$ of N interaction points (here $N = 10$). When the points of a subset $\mathcal{P}$ of $\mathcal{G}$ are contracted toward one of its point x_1, the interaction term factorizes into the product of the interaction term relative to $\mathcal{P}$ and the interaction term relative to $\overline{\mathcal{P}} = (\mathcal{G} \setminus \mathcal{P}) \cup \{x_1\}$.*

with

$$\delta = \min(\nu, 1-\nu) > 0 \ . \tag{4.8}$$

Proof:

The matrix D_{ij} transforms under a contraction according to

$$D_{ij}(a^\nu(\rho)) \ = \ \begin{cases} \rho^{2\nu} D_{ij}(a^\nu) & \text{if } 1 \le i \le j \le P \\ \\ \frac{1}{2}\Big\{\rho^{2\nu} a^\nu_{1i} + a^\nu_{1j} & \\ \quad -[a_{1j} - \rho(a_{1i} + a_{1j} - a_{ij}) + \rho^2(a_{1i})]^\nu\Big\} & \text{if } 1 \le i \le P < j \le N \\ \\ D_{ij}(a^\nu) & \text{if } P < i \le j \le N \end{cases} \ . \tag{4.9}$$

For small ρ, the mixed term D_{ij} $i \le P < j$ has the expansion

$$\begin{aligned} D_{ij}(a^\nu(\rho)) &= \rho^{2\nu} a^\nu_{1i} + \rho\, \nu a^{\nu-1}_{1j}(a_{1i} + a_{1j} - a_{ij}) + \mathcal{O}(\rho^2) \\ &= \rho^\nu \mathcal{O}(\rho^\delta) \end{aligned} \tag{4.10}$$

since the leading term is $\propto \rho^{2\nu}$ or $\propto \rho$, depending on whether ν is greater or less than $1/2$. Thus we can write the matrix $D_{ij}(a^\nu(\rho))$ in blocks associated respectively with the subset

$\mathcal{P}$ and $\overline{\mathcal{P}} = \{x_1\} \cup \{x_{P+1}, \ldots, x_N\}$

$$D(a^\nu(\rho)) \;=\; \begin{array}{c} \begin{array}{cc} {\scriptstyle 2,\ldots\ldots,P} & {\scriptstyle P+1,\ldots,N} \end{array} \\ \begin{pmatrix} \rho^{2\nu} D_{\mathcal{P}}(a^\nu) & \rho^\nu \mathcal{O}(\rho^\delta) \\ \rho^\nu \mathcal{O}(\rho^\delta) & D_{\overline{\mathcal{P}}}(a^\nu) \end{pmatrix} \end{array} . \tag{4.11}$$

Hence [4] $\det(D(a^\nu(\rho))) \;=\; \rho^{2\nu(P-1)} \left[\det(D_{\mathcal{P}}(a^\nu)) \det(D_{\overline{\mathcal{P}}}(a^\nu)) + \mathcal{O}(\rho^{2\delta})\right]$. Furthermore, from Schoenberg's theorem, if $\det(D_{\mathcal{P}}(a^\nu))$ or $\det(D_{\overline{\mathcal{P}}}(a^\nu))$ vanishes, some subset of points $x_k(\rho)$ coincide for any ρ and therefore $\det(D(a^\nu(\rho)))$ also vanishes. The equivalence in Eq. (4.7) and the theorem follow.

The consequences of this theorem are twofold. First, as expected, when a subset $\mathcal{P}$ of P points coalesce to a single point p, this gives a divergence in $\mathcal{Z}_N$, as well as in any correlation function $\mathcal{Z}_N^{(M)}$, since from (2.27) the same interaction determinant $(\det(\Pi))^{-\frac{d}{2}}$ is present. Second, this divergence is formally equal to the global divergence of the partition function amplitude $\mathcal{Z}_P$ for the P contracted points times the amplitude obtained by replacing those points by the single contraction point p, $\mathcal{Z}_{N-P+1}$. This is a key point for ensuring renormalizability, since this shows that short-distance divergences can be absorbed into an effective interaction term, thanks to a short-distance operator product expansion for "interaction operators"

$$\prod_{i\in\mathcal{P}} \delta^d(\vec{\mathbf{r}}(x_i)) \;\overset{\forall i,\; x_i \to x_p}{\sim}\; |\mathrm{size}(\mathcal{P})|^{-d\nu(P-1)}\, \delta^d(\vec{\mathbf{r}}(x_p)) \;, \tag{4.12}$$

where size($\mathcal{P}$) is a "typical distance" between the points x_i of $\mathcal{P}$ in D-dimensional space (which depends on the precise way the limit $x_i \to x_p$ is taken).

It is the purpose of the next sections to give a precise meaning to these assertions, to provide rigorous arguments, and to discuss their consequences for the physics of the model.

One can regularize those short-distance divergences and make the integrals (3.40), (3.51) UV-finite by changing the short-distance behavior of the propagator $G(x, y)$. However, it is both convenient and natural to use dimensional regularization, that is to consider the amplitudes as analytic functions of the parameters D (the dimension of internal space), ν (the scaling dimension of the field $\vec{\mathbf{r}}$), and ϵ (the scaling dimension of the interaction).

[4] This follows for instance from $\det \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \;=\; \det(A)\det(B)\det(1 - A^{-1}BC^{-1}B^t)$ for invertible matrices A and C.

As we shall argue below, for fixed D and $0 < \nu < 1$, the amplitudes are expected to be UV-finite, and therefore analytic functions of ϵ in the half-plane $\mathrm{Re}(\epsilon) > 0$. Because of the short-distance behavior of its integrand, $\mathcal{Z}_N$ will exhibit poles at $\epsilon = 0$. For instance, the singular contribution to the integral (3.40) arising from the integration over the global dilation parameter of the N-interaction point set gives a single pole $\propto 1/\epsilon$. More generally, we expect that multiple poles in $1/\epsilon^k$ $(1 \leq k \leq N-1)$ will occur at $\epsilon = 0$, corresponding to the dominant singularities appearing when k successive subsets of interaction points coalesce [20]. Apart from these poles at $\epsilon = 0$, subdominant divergences will be shown to give poles in the ϵ plane for $\mathrm{Re}(\epsilon) \leq -\delta/(N-1)$. In field theory, the factorization property of the integrand under partial contractions of subdiagrams determines the pole structure of the resulting Feynman amplitude and is the key point that ensures renormalizability. Here, although the interacting manifold model is not mapped onto a standard field theory, a similar pole structure of $\mathcal{Z}_N$ will be found, due to the factorization property of the interaction term that we just discussed.

4.3. IR regularization

By similar power counting arguments (*i.e.* dimensional analysis), it is expected that the integrals will diverge for large distances $a_{ij} \to \infty$ (when $\mathrm{Re}(\epsilon) \geq 0$). As usual in field theory, we shall deal with this problem by introducing an infra-red regulator, and by showing that such a regulator does not change the short-distance properties and the renormalization of the model.

The simplest kind of regulator is to work in a finite D-dimensional space, *i.e.* to consider a "membrane" of finite size. This is in fact what is usually done for the continuous polymer Edwards model. Indeed, the polymer is taken to have a finite total "length" S, which amounts to constrain the length variables s_α in (3.57) by a measure term $(S - \sum_\alpha s_\alpha)\, \theta(S - \sum_\alpha s_\alpha)$.

In our case, our formulation of the model in non-integer dimension relies on the invariance of the observables under Euclidean motions in $\mathbb{R}^D$. A simple way to keep a similar symmetry over a finite manifold is to start from the D-dimensional hypersphere $\mathcal{S}_D$ with radius R and volume $\mathcal{V}_{\mathcal{S}_D} = S_{D+1} R^D$, so that the group of invariance is now $\mathrm{SO}(D+1)$. One can easily generalize the concept of distance geometry on $\mathcal{S}_D$, and its analytic continuation for non-integer D. Indeed, we can embed the sphere into $\mathbb{R}^{D+1}$, and write the

integral of a SO($D+1$) invariant function of N variables as an integral over scalar products $u_{ij} = x_i \cdot x_j$:

$$\int \prod_{i=1}^{N} d^{D+1}x_i \, \delta(|x_i| - R) f(u_{ij}) \;=\; \int_{\mathcal{U}_N(R)} \prod_{i<j} du_{ij} \, \sigma_N^{(D)}([u_{ij}], R) \, f([u_{ij}]) \;, \tag{4.13}$$

with $u_{ij} = R^2$ if $i = j$, and the measure

$$\sigma_N^{(D)}([u_{ij}], R) \;=\; S_{D+1} \ldots S_{D-N+2} \, R^N \; (\det_N [u_{ij}])^{\frac{D-N}{2}} \;, \tag{4.14}$$

$\mathcal{U}_N(R)$ being the domain of u_{ij} $(i < j)$ where the matrix $[u_{ij}]$ is positive with all the u_{ii} set equal to R^2. Equivalently we can express the integral (4.13) in terms of squared distances $a_{ij} = 2(R^2 - u_{ij})$ *in* $D+1$*-dimensional space* (this defines the so-called cord distance on $\mathcal{S}_D$ which differs from the geodesic distance):

$$\int \prod_{i=1}^{N} d^{D+1}x_i \, \delta(|x_i| - R) f(a_{ij}) \;=\; \mathcal{V}_{\mathcal{S}_D} \int_{\mathcal{A}_N(R)} \prod_{i<j} da_{ij} \, \mu_N^{(D)}([a_{ij}], R) \, f([a_{ij}]) \;, \tag{4.15}$$

with the measure

$$\mu_N^{(D)}([a_{ij}], R) = 2^{-\frac{N(N-1)}{2}} \, S_D \ldots S_{D-N+2} \left(\frac{1}{R^2} \det_N [R^2 - \frac{1}{2} a_{ij}] \right)^{\frac{D-N}{2}} \tag{4.16}$$

and $\mathcal{A}_N(R)$ the domain of a_{ij} where the matrix $[R^2 - \frac{1}{2} a_{ij}]$ is positive. In particular, the positiveness of the 2×2 minors ensures for any two points the diameter inequality $a_{ij} \leq 4R^2$. Hence, $\mathcal{A}_N(R)$ is a bounded subset of $\mathbb{R}^{\frac{N(N-1)}{2}}$.

One can check the identity:

$$\det_N([R^2 - \frac{1}{2} a_{ij}]) = R^2 \det_{N-1}([D_{ij}(a)]) + \det_N([-\frac{1}{2} a_{ij}]) \tag{4.17}$$

where $D_{ij}(a)$ is defined in (3.6) (indeed the $N-2$ highest degree terms in the polynomial expansion in R^2 of the l.h.s of (4.17) vanish identically !). This implies that in the thermodynamic limit $R \to \infty$ one recovers the measure (3.8) in Euclidean (infinite flat) space. Conversely, for a finite R, formula (4.17) shows that, at short-distances, the measure is dominated by the first term of the r.h.s, *i.e.* the Euclidean one, while the second term, which is one degree higher in a_{ij}, becomes relevant for distances of order R only, hence providing an IR regulator.

It remains to write the expression for the interaction term. In fact, the latter is the same as in (2.23), with the matrix Π_{ij} (2.24), or more generally the tree matrix $\Pi^{\mathbf{T}}_{\alpha\beta}$ (3.48), involving the massless propagator $G(x,y) = \left[(-\Delta)^{\frac{k}{2}}\right]^{-1}(x,y)$ now on $\mathcal{S}_D$. There is however no general simple analytic expression for $G(x_i,x_j)$ as a function of the distance variable a_{ij} defined above for general D and k. For definiteness, another simple possibility then consists in keeping a propagator on the sphere of the form (2.31)

$$-G(x_i,x_j) \;=\; \frac{1}{4^\nu(4\pi)^{D/2}}\frac{\Gamma(1-\nu)}{\nu\Gamma(\nu+\frac{D}{2})}\,|a_{ij}|^\nu \;. \tag{4.18}$$

This amounts to modify the "elastic" term of the Hamiltonian (2.1) by finite volume corrections

$$\vec{\mathbf{r}}(x)(-\Delta)^{\frac{k}{2}}\vec{\mathbf{r}}(x) \;\to\; \vec{\mathbf{r}}(x)\left[(-\Delta)^{\frac{k}{2}} + \mathrm{cst}R^{-2}(-\Delta)^{\frac{k-2}{2}}) + \mathrm{cst}R^{-4}(-\Delta)^{\frac{k-4}{2}}) + \ldots\right]\vec{\mathbf{r}}(x) \tag{4.19}$$

which change its large distance behavior (IR regulator), but not its short-distance behavior. In particular, Schoenberg's theorem II, which is readily satisfied by the propagator G given by (4.18), is expected to remain valid for the exact massless propagator on the sphere. The corrections in (4.19) vanish in the limit $R\to\infty$. In the following, we will keep in mind that the model is defined with the measure (4.16) and the propagator (4.18) . However, since we shall be concerned with the UV renormalization of the model, we shall use formally the simpler Euclidean ($R\to\infty$) limit (3.8) of (4.16) . As discussed above, they actually share the same short-distance properties.

5. Absolute convergence for $\epsilon = D - \nu\, d > 0$

In this section, we want to prove that:

THEOREM I *For $\epsilon > 0$ (i.e. $d < d^\star$), the integrals $\mathcal{Z}_N$ and $\mathcal{Z}_N^{(M)}$ are absolutely (UV) convergent.*

As in field theory, this actually is a consequence of (i) the superficial convergence of $\mathcal{Z}_P$ for any $P \le N$ and (ii) the basic factorization property (4.7), and generalizations thereof. Since the formalism developed above can be thought of as a natural generalization of the Schwinger representation of Feynman integrals, the proof of absolute convergence will be inspired by the standard method based on decomposition into Hepp sectors [32]. As discussed just above, we shall always assume the (implicit) presence of an IR regulator.

Fig. 6: *(a) An example of construction of the ordered tree* $\mathbf{T} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *for a set of interaction points with* $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq |\lambda_4|$. *This tree defines the generalized Hepp sector* $\mathcal{H}^{\mathbf{T}}$ *to which this set of points belongs. (b) Moving the point* x_2 *toward the point* x_1 *results in a change of generalized Hepp sector.*

5.1. Generalized Hepp sectors

We start with formula (3.41) and partition the domain of integration for the y_i's into generalized Hepp sectors as follows. Let us consider the N points in $\mathbb{R}^{N-1}$ with Cartesian coordinates $0, y_1, \ldots, y_{N-1}$. We first singularize the pair of points having the minimum mutual distance, and define λ_1 as the vector in $\mathbb{R}^{N-1}$ joining these two points, with an arbitrary orientation. We define λ_2 in a similar way, as the vector associated with the minimal distance among all the remaining mutual distances. λ_2 can (i) either share one of its extremities with λ_1, or (ii) be disjoint. At the next step, we define λ_3 as the vector associated with the minimal distance among all the remaining ones *and* such that $(\lambda_1, \lambda_2, \lambda_3)$ do not form a closed loop (this may occur only in case (i)). We iterate this construction, by requiring at each step that no loop ever appears, up to the emergence

of the last vector λ_{N-1}. We thus have constructed an oriented *ordered* tree $\mathbf{T}$ with line vectors $(\lambda_1, \ldots, \lambda_{N-1})$, which spans the N points and is such that

$$|\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_{N-1}| \ . \tag{5.1}$$

We shall denote $\mathbf{T} = (\lambda_1, \ldots, \lambda_{N-1})$ although the tree $\mathbf{T}$ is not strictly speaking characterized by the line vectors λ_α but only by the incidence matrix $\epsilon_{i\alpha}$ of the linear transformation from the x_i (or y_i) to the λ_α. With any ordered tree $\mathbf{T}$, we can therefore associate the Hepp sector $\mathcal{H}^{\mathbf{T}}$ defined as the domain of the y_i's in $\mathbb{R}^{N-1}$ leading after this construction to this ordered tree $\mathbf{T}$, regardless of its orientation. It is clear that $\mathbb{R}^{N-1} = \bigcup_{\mathbf{T}} \mathcal{H}^{\mathbf{T}}$.

In a given sector $\mathcal{H}^{\mathbf{T}}$, we make a change of variables from the y_i's to the λ_α's associated with the ordered tree $\mathbf{T}$ (with an arbitrary choice of orientation) and, in particular, use $\Pi^{\mathbf{T}}_{\alpha\beta}$ to evaluate the interaction term. We parametrize the λ_α by their spherical coordinates in $\mathbb{R}^{N-1}$, namely by their modules $|\lambda_\alpha|$ and relative angles $\theta_{\alpha,1}, \ldots \theta_{\alpha,\alpha-1}$ as in (3.26) and (3.28) . The variables $|\lambda_\alpha|$ will play the role of the Schwinger parameters s_α in field theory. Since $|\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_{N-1}|$, it is natural to rewrite the $|\lambda|$'s as

$$\begin{aligned} |\lambda_1| &= \beta_1 \beta_2 \ldots \beta_{N-1} \\ |\lambda_2| &= \beta_2 \ldots \beta_{N-1} \\ &\vdots \\ |\lambda_{N-1}| &= \beta_{N-1} \end{aligned} \tag{5.2}$$

with $0 \leq \beta_\alpha \leq 1$ for $1 \leq \alpha \leq N-2$ and $0 \leq \beta_{N-1} < \infty$ (in the Euclidean version of the problem, thus without IR regulator). The domain of integration $\mathcal{D}^{\mathbf{T}}$ for the β and θ variables which reconstructs the domain $\mathcal{H}^{\mathbf{T}}$ for the y_i's in $\mathbb{R}^{N-1}$, depends on the topology of the ordered tree. For instance, the value $\beta_\alpha = 1$ can in general be reached inside the sector only for some domain of the angle θ between λ_α and $\lambda_{\alpha+1}$. Still, the domain $\mathcal{D}^{\mathbf{T}}$ has the following general structure:

$$\begin{aligned} & 0 \leq \theta_{\alpha,n} \leq \pi && 1 \leq n < \alpha \leq N-1 \\ & \beta_\alpha^{\min}(\mathbf{T}\ ; \beta_{\gamma:\gamma<\alpha}\, ; \theta\text{ 's}) \leq \beta_\alpha \leq \beta_\alpha^{\max}(\mathbf{T}\ ; \beta_{\gamma:\gamma<\alpha}\, ; \theta\text{ 's}) && 1 \leq \alpha \leq N-2 \\ & 0 \leq \beta_{N-1} \end{aligned} \tag{5.3}$$

where $\beta_\alpha^{\min}(\mathbf{T}\ ; \beta\text{ 's}, \theta\text{ 's})$ and $\beta_\alpha^{\max}(\mathbf{T}\ ; \beta\text{ 's}, \theta\text{ 's})$ are (positive and possibly vanishing) functions of the θ's and of the β_γ's for $\gamma < \alpha$. The inequality $\beta_\alpha^{\min} > \beta_\alpha^{\max}$ for some

θ's and $\beta_{\gamma:\gamma<\alpha}$ would indicate that such a partial configuration of θ's and β_γ always lies outside the given sector. The only important properties of $\mathcal{D}^{\mathbf{T}}$ that we shall use are:
(i) $\mathcal{D}^{\mathbf{T}}$ is by definition bounded, if one excepts the variable β_{N-1}, since $\beta_\alpha^{\max}(\mathbf{T};\beta'\text{s};\theta'\text{s}) \leq 1$ by construction. The variable β_{N-1} itself stays bounded due to the implicit presence of an IR regulator.
(ii) $\det([\Pi^{\mathbf{T}}_{\alpha,\beta}])$, when expressed in terms of the β's and the θ's, is a continuous function of these variables and vanishes in $\mathcal{D}^{\mathbf{T}}$ if and only if one at least of the β's vanishes. Indeed, from Schoenberg's theorem, $\det([\Pi^{\mathbf{T}}_{\alpha,\beta}]) = 0$ iff two points coincide, that is if their mutual distance is zero. Since this distance is by construction larger than or equal to $|\lambda_1|$ in the sector, this implies $|\lambda_1| = 0$, or equivalently $\beta_1\beta_2\ldots\beta_{N-1} = 0$.

5.2. Absolute convergence

It is enough to prove the absolute convergence in each Hepp sector $\mathcal{H}^{\mathbf{T}}$. Omitting global factors in (3.41) we consider the integral:

$$\int_{\mathcal{H}^{\mathbf{T}}} \prod_{i=1}^{N-1} d^{N-1}y_i \left(\det\left[y_i\cdot y_j\right]\right)^{\frac{D-N+1}{2}} \left(\det\left[\Pi_{ij}\right]\right)^{-\frac{d}{2}}$$
$$= \int_{\mathcal{D}^{\mathbf{T}}} \prod_{\alpha=1}^{N-1} (\beta_\alpha)^{\alpha D-1} d\beta_\alpha \prod_{\alpha=2}^{N-1}\prod_{n=1}^{\alpha-1} \left(\sin\left(\theta_{\alpha,n}\right)\right)^{D-1-n} d\theta_{\alpha,n} \left(\det\left[\Pi^{\mathbf{T}}_{\alpha\beta}(\beta\text{'s},\theta\text{'s})\right]\right)^{-\frac{d}{2}} . \tag{5.4}$$

As already mentioned, we shall limit ourselves to the case $D \geq N-1$. We shall discuss in section 9 how our results can then be extended to $D < N-1$. The product of sinuses in (5.4) is thus a bounded function on $\mathcal{D}^{\mathbf{T}}$. Possible ultraviolet divergences may only arise from the vanishing of $\det[\Pi_{\alpha\beta}]$, that is when some β's vanish. For $\epsilon > 0$ ($d < d^\star = D/\nu$), it is sufficient to show that, on $\mathcal{D}^{\mathbf{T}}$,

$$\prod_{\alpha=1}^{N-1} (\beta_\alpha)^{\alpha D} \left(\det\left[\Pi_{\alpha\beta}\right]\right)^{-\frac{d}{2}} = \mathcal{O}(\prod_{\alpha=1}^{N-1} (\beta_\alpha)^{\alpha\epsilon}) . \tag{5.5}$$

As is clear from its definition, $\Pi^{\mathbf{T}}_{\alpha\beta}$ vanishes when λ_α and/or λ_β vanish. The key point is that while $\Pi^{\mathbf{T}}_{\alpha\alpha} = A_D(\nu)|\lambda_\alpha|^{2\nu}$, $\Pi^{\mathbf{T}}_{\alpha\beta}$ vanishes more rapidly than $|\lambda_\alpha|^\nu|\lambda_\beta|^\nu$ if $\alpha \neq \beta$ (see Appendix C). This property is best expressed by introducing the "normalized" matrix

$$Y^{\mathbf{T}}_{\alpha\beta} \equiv \frac{1}{A_D(\nu)} \frac{\Pi^{\mathbf{T}}_{\alpha\beta}}{|\lambda_\alpha|^\nu|\lambda_\beta|^\nu} \tag{5.6}$$

(such that $Y^{\mathbf{T}}_{\alpha\alpha} = 1$).

In term of the β's, we can write:

$$\begin{aligned} \Pi^{\mathbf{T}}_{\alpha\alpha} &= A_D(\nu)\, \beta_\alpha^{2\nu} \beta_{\alpha+1}^{2\nu} \dots \beta_{N-1}^{2\nu} Y^{\mathbf{T}}_{\alpha\alpha} \\ \Pi^{\mathbf{T}}_{\alpha\beta} &= A_D(\nu)\, \beta_\alpha^{\nu} \dots \beta_{\beta-1}^{\nu} \beta_\beta^{2\nu} \dots \beta_{N-1}^{2\nu} Y^{\mathbf{T}}_{\alpha\beta}(\beta\text{ 's}, \theta\text{ 's}) \qquad (\alpha < \beta)\ , \end{aligned} \tag{5.7}$$

leading to the identity

$$\det_{N-1}\left(\left[\Pi^{\mathbf{T}}_{\alpha\beta}\right]\right) \;=\; (A_D(\nu))^{N-1}\, \beta_1^{2\nu} \dots \beta_{N-1}^{2\nu(N-1)}\, \det_{N-1}\left(\left[Y^{\mathbf{T}}_{\alpha\beta}\right]\right)\ . \tag{5.8}$$

This amounts to factorize out the maximal powers of β's. In particular, $\det(Y^{\mathbf{T}})$ is independent of β_{N-1}. In order to obtain (5.5), one has to show that on $\mathcal{D}^{\mathbf{T}}$ the positive quantity $\det(Y^{\mathbf{T}})$ in (5.8) *cannot vanish* and is actually bounded from below by a strictly positive number. This property is proven in Appendix C. Indeed, if $\det(Y^{\mathbf{T}})$ were to vanish, $\det(\Pi^{\mathbf{T}})$ would also vanish and, from Schoenberg's theorem, some subset of the β's must vanish. This corresponds to contract successively some subsets of points (by a contracting scale factor β) to single points. A generalization of the factorization property (4.7) (see Appendix C) shows that, in such a limit, the determinant $\det(\Pi^{\mathbf{T}})$ factorizes into a product of similar determinants associated with subtrees of $\mathbf{T}$. The normalized determinant $\det(Y^{\mathbf{T}})$ then becomes exactly equal to a product of normalized subdeterminants, each of them corresponding to a subtree of $\mathbf{T}$. In the sector, these subtrees have no coinciding (with vanishing distance) points and therefore their determinants do not vanish. Thus, $\det(Y^{\mathbf{T}})$ does not vanish even in this limit where some β's tend to zero.

From the above results, the quantity $\det(Y^{\mathbf{T}})$ in (5.8) , seen as a function of β_γ ($1 \leq \gamma \leq N-2$) and of the θ's, is a continuous positive non-vanishing function on the *compact* restriction of $\mathcal{D}^{\mathbf{T}}$ obtained by omitting the (here dummy) variable β_{N-1}. Therefore it admits a strictly positive lower bound on $\mathcal{D}^{\mathbf{T}}$ and thus (since $d > 0$)

$$(\det_{N-1}[\Pi^{\mathbf{T}}_{\alpha\beta}])^{-\frac{d}{2}} \;<\; \text{cst} \cdot \beta_1^{-d\nu} \dots \beta_{N-1}^{-(N-1)d\nu}\ , \tag{5.9}$$

which is equivalent to (5.5). The convergence of the integral (5.4) in the Hepp sector $\mathcal{H}^{\mathbf{T}}$ for $\epsilon = D - \nu d > 0$ follows.

We thus have proven the convergence of the generic perturbative term $\mathcal{Z}_N$ of the partition function $\mathcal{Z}$ (for $D \geq N-1$). Similarly, the perturbative terms $\mathcal{Z}_N^{(M)}$ (Eq. (2.27)) of the vertex operators $\mathcal{Z}^{(M)}$ (Eq. (2.25)) can be shown to be UV convergent for $\epsilon > 0$

and D large enough ($D \geq N + M - 1$). This follows from the same decomposition into Hepp sectors and the use of (3.51) . The proof is then exactly the same up to the following modifications:

(I) The measure term in (5.4) is replaced by a measure similar to (3.32) for tree variables. The difference between this measure and that of (5.4) concerns only angular terms, which are bounded functions on $\mathcal{D}^{\mathbf{T}}$ (provided now that $D \geq N + M - 1$).

(II) The exponential term, depending of the external momenta, has for argument a negative quadratic form $-\frac{1}{2}\sum_{a,b} \vec{\mathbf{k}}_a \cdot \vec{\mathbf{k}}_b \, \Delta_{ab}$, and is therefore bounded between 0 and 1.

The above proof therefore carries over to this generalized case.

6. The subtraction operation R

6.1. Renormalization: introductory remarks

The purpose of renormalization is to show that the short-distance divergences that occur at $\epsilon = 0$ can be absorbed into a redefinition of the coupling constants of the model. If true, this property allows us (i) to give a meaning to the theory at $\epsilon = 0$, and (ii) to write a Renormalization Group equation and deduce the scaling behavior of the model for $\epsilon \lg 0$. From the analysis of divergences performed in sections 4 and 5, we expect that the correlation functions can be made finite by a simple renormalization of the bare coupling constant b in the action (2.1)[20]

$$b \;=\; \mu^{\epsilon}\, \hat{b}_R \, Z(\hat{b}_R, \epsilon) \tag{6.1}$$

where μ is an (internal) momentum scale and $\hat{b}_R$ a finite dimensionless renormalized coupling constant. In the case of a finite manifold with volume $\mathcal{V}_{\mathcal{S}_D}$, a convenient and natural choice of momentum scale is $\mu = R^{-1} \propto (\mathcal{V}_{\mathcal{S}_D})^{-1/D}$. The renormalization factor $Z(\hat{b}_R, \epsilon)$ will be an implicit function of the parameters D (internal dimension of the manifold) and ν (scaling dimension of the $\vec{\mathbf{r}}$ field). It will be defined in perturbation theory as

$$Z(\hat{b}_R, \epsilon) \;=\; 1 \,+\, \hat{b}_R \, a_1(\epsilon) \,+\, \hat{b}_R^2 \, a_2(\epsilon) \,+\, \ldots \tag{6.2}$$

where the coefficients a_n diverge as ϵ^{-n} when $\epsilon \to 0$.

If it is possible to construct, at least in perturbation theory, a function Z such that the partition function $\mathcal{Z}(b)$ (2.9) and the correlation functions $\mathcal{Z}^{(M)}(X_a, \vec{\mathbf{k}}_a; b)$ (2.25) are UV-finite in the limit $\epsilon \to 0$, $\hat{b}_R$ and μ finite, then the model will be perturbatively

renormalizable. The validity of the approach initiated in [16],[17] and [20] will then be ensured, since the standard techniques of renormalization group theory can be applied to the model, and can (in principle) be extended to all orders in perturbation theory.

It is interesting to compare our construction with what is usually done for a "standard" local field theory, such as the O(n)-symmetric Φ^4_D theory, whose action in D-dimensions is written as:

$$\mathcal{H} = \int_{\mathcal{V}} d^D x \left[\frac{1}{4}(\partial \mathbf{\Phi})^2 + \frac{m^2}{2}\mathbf{\Phi}^2 + \frac{b}{2}(\mathbf{\Phi}^2)^2 \right] \tag{6.3}$$

where $\mathbf{\Phi} = \{\Phi^i,\, i = 1, \ldots n\}$ is an n-component field. There are basically two kinds of approaches to prove renormalizability of this theory at the critical dimension $D = 4$.

The first approach (à la Wilson) consists in introducing explicitly a short-distance cut-off, in integrating over the high momenta modes, and in showing that the UV divergent terms in the effective action which arise from this integration can be absorbed into a redefinition of the physical coupling constants of the theory, so that a finite continuum limit can be reached by letting the cut-off go to zero and the bare coupling constants flow along RG trajectories [33] [34]. This approach is physically transparent, appropriate for the applications of renormalization group to critical phenomena in statistical mechanics [6], and has in some cases gained a rigorous status at the non-perturbative level [35]. However, it requires a formulation of the theory through a lattice regularization, or a phase space formulation, which is possible for integer space dimension D only. It does not seem possible (up to now) to apply these methods in the framework of an analytic continuation in non integer space dimension so as for instance to justify the ϵ-expansion used in the description of critical phenomena by a $\Phi^4_{4-\epsilon}$ theory.

The second, perturbative approach à la Bogoliubov–Parasiuk–Hepp–Zimmerman (BPHZ) [22] consists in working in perturbation theory and in constructing, directly or by a recursive process, a subtraction operation on the Feynman amplitudes of the theory, which makes all the terms of perturbation theory finite and well defined through convergent integrals; then one shows that this operation corresponds, in the field theory language, to a renormalization of the action by local counterterms, and that it preserves the equation of motions of the theory and the Ward identities associated with its symmetries. From the statistical mechanics point of view, this amounts to a change of variables from microscopic to effective coupling constants. Renormalization group equations and scaling behaviors are then derived from the renormalized theory. This BPHZ formulation of renormalization has a simple and general perturbative formulation for theories in non integer dimensions

D, since there are now well defined recipes of "dimensional regularization" which allows to construct Feynman amplitudes for non-integer D, and to study their properties, either in the real space representation, or in the momentum space representation, or in the so-called Schwinger parametric α-representation. The BPHZ subtraction operation can then easily be extended to the case of non-integer space dimensions, at least in momentum space or in the α-representation.

For our model the action (2.1) can be seen also as a local field theory in D-dimensional space for a scalar d-component $\vec{\mathbf{r}}$ field

$$\mathcal{H} = \int_{\mathcal{V}} d^D x \left[\frac{1}{2} \vec{\mathbf{r}}(x) (-\Delta)^{\frac{k}{2}} \vec{\mathbf{r}}(x) + b \; \delta^d(\vec{\mathbf{r}}(x)) \right] \; , \tag{6.4}$$

but the interaction δ-term is singular and non-polynomial, which makes the perturbative expansion very different from that of the ordinary case, since it does not involve usual Feynman diagrams. Furthermore, the dimension of the interaction term depends explicitly on the number of components of the field, here d.

In principle, nothing prevents the application of a renormalization program à la Wilson in the physical case of objects described by (6.4) with integer dimension ($D = 1,\ 2$). Some preliminary rigorous results have indeed been obtained (for the case $D = 2$, $d = 1$, $k = 2$) in [27]. However, it is probably impossible to study by such methods the renormalizability of the model at (or near) its critical dimension $D^\star$ (Eq. (2.3)), since the latter is in general non-integer (even for integer d), and between 0 and 2 (for the elastic membrane $k = 2$ case). The so-called "functional renormalization", which is an approximate renormalization group scheme, has also been applied to the study of the specific case $d = 1$ in [36]. Such schemes are well defined by analytic continuations at non integer D but are only approximate and have no rigorous status.

On the other hand, in section 3, we constructed a perturbation theory for the model in non integer dimension D, via distance geometry, which correspond to a dimensional regularization scheme in (internal) *real–* or *position–space*. In sections 4 and 5, we have shown that the structure of the UV divergences of the amplitude (poles in ϵ) is quite similar to that of Feynman amplitudes of ordinary local field theories. It is the purpose of the rest of this article to show that it is possible to develop a BPHZ-like formalism to prove renormalizability of this model. In this Section we shall propose a subtraction operation, which will appear to be a generalization of the BPHZ subtraction operation for ordinary Feynman integrals, with a similar structure in term of the so-called "Zimmerman forests".

This subtraction operator, which in our case acts directly on the integrands of interaction terms like (3.40) and (3.47), involving positions x_i or squared distances a_{ij}, will be shown to make the integrals UV-finite (for $\epsilon = 0$), and to correspond to a renormalization of the coupling constant b. This will ensure (in perturbation theory) the renormalizability of the model, the validity of renormalization group equations, and of an ϵ-expansion about the critical dimension.

Another fundamental structure underlies our approach, since the position variables x_i, (or the a_{ij} in distance geometry) can be thought of as a D-dimensional generalization of the Feynman α-parameters in the Schwinger representation. In field theory, this representation consists in writing the propagators in terms of an auxiliary α parameter via a Laplace transform of the free field propagator (in momentum space)

$$\frac{1}{p^2 + m^2} \;=\; \int_0^\infty d\alpha \, \mathrm{e}^{-\alpha(p^2+m^2)} \tag{6.5}$$

and in writing all the Feynman amplitudes as multiple integrals over these α-variables. As we have seen for our model (6.4) in section 3, the integrals giving the perturbative terms (3.40) of the partition function have a form generalizing that of a Feynman amplitude in α-representation. Indeed, the subtraction operation and the mathematical techniques that we shall use to prove renormalizability are in fact extensions of techniques developed by Bergère and Lam in [23] to study renormalization of local field theory precisely in the α-representation.

This analogy of the internal position D-space representation of a statistical mechanics model with the α-representation of a local field theory is not surprising. Indeed, for $D = 1$, it is well-known that the Edwards model for self-avoiding polymer (1.1) embedded in d dimensions, can be formulated as a local Φ^4 theory in d-dimensional space, with Hamiltonian (6.3) (with D now formally replaced by d), in the limit where the number of components of the field $\mathbf{\Phi}$, n, goes to zero (this is the well-known de Gennes equivalence). The length S of the polymer is conjugate, via a Laplace transform, to the squared-mass m^2 of the corresponding $n \to 0$ field theory. Similarly, for our model (6.4) (and for $k = 2$), in the case $D = 1$ (polymer interacting with an impurity), the same mapping allows to write it as a $n \to 0$ field theory in the *external* d-dimensional space $\mathbb{R}^d$, with Hamiltonian

$$\mathcal{H}[\mathbf{\Phi}] \;=\; \int_{\mathbb{R}^d} d^d\vec{\mathbf{r}} \left(\frac{1}{4}(\partial\mathbf{\Phi})^2 \;+\; \frac{m^2}{2}\mathbf{\Phi}^2 \right) \;+\; b\,\mathbf{\Phi}^2(\vec{\mathbf{r}} = \vec{0}). \tag{6.6}$$

The interaction between the polymer and the impurity located at the origin is represented by the last term in the r.h.s. of (6.6), which is a singular mass term located at the origin. Here also, the length of the polymer S (which corresponds to the radius R of the manifold in the case $D = 1$) is conjugate to the squared mass m^2 of the field $\mathbf{\Phi}$ in (6.6). The diagrams associated with (6.6) are "daisy diagrams" identical to those of fig. 4, with the α-parameters for the propagators identified with the internal relative distances in (3.57) $s_\beta = |x_{i_{\beta+1}} - x_{i_\beta}|$ between successive vertices i_β and $i_{\beta+1}$ in the *internal* 1-dimensional manifold, i.e. the polymer itself (see Eq. (3.56)).

Thus, it will appear that our BPHZ renormalization scheme in position space for the theory (2.1) defined in $\mathbb{R}^D$ is a generalization to continuous values of D of the ordinary BPHZ renormalization in α-representation of the theory (6.6) defined in $\mathbb{R}^d$, with α viewed as a $D = 1$ relative position [5]. Finally let us stress that this remarkable mathematical analogy makes us hope that in a similar way, it will be possible to develop renormalization techniques in position space for the non-local theory (1.2) (which describes self-avoiding D-dimensional manifold), which would reduce for $D = 1$ to the ordinary renormalization theory for the Edwards model (formulated either as a direct renormalization à la des Cloizeaux for the Edwards model, or equivalently as a BPHZ renormalization for the $n = 0$ Φ^4_d field theory in the α-representation).

6.2. The subtraction operation

We first give a heuristic presentation of the recursive subtraction process that we shall use to prove renormalizability. As we have seen, the term of order N of a M-point correlation function, $\mathcal{Z}_N^{(M)}(X_a, \vec{\mathbf{k}}_a)$, is given by an integral over the positions of N internal points (2.27), that we write schematically (omitting the external momenta $\vec{\mathbf{k}}_a$, and the parameters D, ν and ϵ), and denoting by $\mathcal{G}$ the set of these N internal points

$$\mathcal{Z}_N^{(M)}(X_a) \;=\; \int \prod_{i\in\mathcal{G}} d^D x_i \, I_{\mathcal{G}}(x_i, X_a). \tag{6.7}$$

To subtract the "superficial UV divergences" which occur in the integral (6.7) when some subset $\mathcal{P}$ of points collapses toward a single point, we can use the factorization theorem of section 4.2 (and Appendix C), which implies that when the points of the

[5] For this theory, the renormalization is quite trivial, since the only divergent diagram is the tadpole, i.e. the petal of the daisy.

subset $\mathcal{P}$ tend altogether toward some (arbitrary) point p in $\mathcal{P}$, the integrand in (6.7) behaves as

$$I_{\mathcal{G}}(x_i, X_a) \ \sim \ I_{\mathcal{G}/_p\mathcal{P}}(x_i, X_a) \cdot I_{\mathcal{P}}(x_i) \tag{6.8}$$

where $I_{\mathcal{G}/_p\mathcal{P}}$ denotes the integrand of the "reduced" set $\mathcal{G}/_p\mathcal{P}$ with the M external points (with positions X_a), and the $N - \mathrm{Card}(\mathcal{P}) + 1$ internal points obtained from $\mathcal{G}$ by removing all the points of $\mathcal{P}$ but p, and $I_{\mathcal{P}}$ is the integrand for the subset $\mathcal{P}$ with no external points and $\mathrm{Card}(\mathcal{P})$ internal points. Therefore, we expect that by subtracting the divergences associated with all families of *mutually disjoint subsets* $\{\mathcal{P}_k\}$ in $\mathcal{G}$, we deal with all *superficial* short-distance divergences. This can be performed by changing in (6.7) the integrand into

$$I_{\mathcal{G}}(x_i, X_a) \ \to \ I^{\bullet}_{\mathcal{G}}(x_i, X_a) \equiv I_{\mathcal{G}}(x_i, X_a) \ + \ \sum_{\{\mathcal{P}_k\}} I_{\mathcal{G}/_{\{p_k\}}\{\mathcal{P}_k\}}(x_i, X_a) \prod_k \big(- I_{\mathcal{P}_k}(x_i)\big), \tag{6.9}$$

where the reduced set $\mathcal{G}/_{\{p_k\}}\{\mathcal{P}_k\}$ is obtained by replacing each subset $\mathcal{P}_k$ by one of its points p_k (chosen arbitrarily). To this subtracted amplitude, we associate the subtracted partition function term:

$$\mathcal{Z}^{\bullet(M)}_N(X_a) \equiv \int \prod_{i \in \mathcal{G}} d^D x_i I^{\bullet}_{\mathcal{G}}(x_i, X_a). \tag{6.10}$$

Thanks to the factorization property of the measure (section 3.4), we can integrate separately $I_{\mathcal{G}/_{\{p_k\}}\{\mathcal{P}_k\}}$ over the positions of the internal points of $\mathcal{G}/_{\{p_k\}}\{\mathcal{P}_k\}$, and each counterterm $I_{\mathcal{P}_k}$ over the positions of all the points of $\mathcal{P}_k$ but one, p_k, thus obtaining for each of these counterterms one term of the expansion of the partition function $\mathcal{Z}$. Evaluating all subset integrals in (6.9) leads to the explicit formula:

$$\mathcal{Z}^{\bullet(M)}_N(X_a) = \sum_{N'=1}^{N} \frac{1}{(N')!} \mathcal{Z}^{(M)}_{N'}(X_a) \sum_{\substack{\{N_k,\, k=1,\ldots,N'\} \\ N_k \geq 1,\, \sum_k N_k = N}} \frac{N!}{\prod_k (N_k)!} \prod_{\substack{k: \\ N_k > 1}} \left(-\frac{\mathcal{Z}_{N_k}}{\mathcal{V}_{\mathcal{S}_D}}\right) \tag{6.11}$$

where $\mathcal{V}_{\mathcal{S}_D}$ is the internal volume of the manifold. One can check that this subtraction operation on integrands corresponds to a perturbative expansion of the partition functions $\mathcal{Z}^{(M)}$ with respect to a "renormalized" coupling constant $b_\bullet$ such that:

$$\begin{aligned} \mathcal{Z}^{(M)} &= \mathcal{Z}^{(M)}(b) = \sum_N (-b)^N \frac{\mathcal{Z}^{(M)}_N}{N!} \\ &= \mathcal{Z}^{\bullet(M)}(b_\bullet) \equiv \sum_{N'} (-b_\bullet)^{N'} \frac{\mathcal{Z}^{\bullet(M)}_{N'}}{N'!} \end{aligned} \tag{6.12}$$

with $b_\bullet$ given implicitly by the equation:

$$b \;=\; b_\bullet \;+\; \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \sum_{N=2}^{\infty} \frac{(-b_\bullet)^N}{N!} \mathcal{Z}_N \tag{6.13}$$

However, this subtraction is not sufficient to make the $\mathcal{Z}^{(M)}$ finite, since it does not deal with sub-divergences inside the subsets $\mathcal{P}$. As in standard renormalization theory, one deals with that problem by repeating this subtraction operation inside these subsets, that is by subtracting to each $I_{\mathcal{P}_k}$ the divergent parts associated to families of mutually disjoint subsets in $\mathcal{P}_k$, and iterating the process. One thus obtains at a given order N a subtraction operation expressed in terms of the sets $\mathcal{F} = \{\mathcal{P}_k\}$ of *mutually disjoint or strictly included* subsets $\mathcal{P}_k$ of $\mathcal{G}$. In analogy with renormalization theory in field theory, such a set $\mathcal{F}$ will be called a *forest*[6] of $\mathcal{G}$.

In addition, for a given forest $\mathcal{F}$, at each subtraction step, that is for each subset $\mathcal{P}_k$ of $\mathcal{F}$, we have to specify a *root* p_k of $\mathcal{P}_k$, toward which we contract $\mathcal{P}_k$, in order to calculate the associated counterterm. It is quite clear that, after integration over the position variables, the result of the subtraction operation does not depend on the specific choice of roots. However, it is natural to choose for each forest a set of roots in a way which is consistent with the geometrical picture of the subtraction operation as successive contractions of subsets toward their root. This leads to the notion of a *compatibly rooted* forest, which will be discussed below.

After these somehow heuristic considerations, let us give the precise definition of the subtraction operation that we shall use.
Let us consider a set $\mathcal{G}$ of N abstract points, that we call vertices.

Definition 6.1

A *rooted subset* of $\mathcal{G}$ is a couple $(\mathcal{P}, p)$ of a subset $\mathcal{P}$ of $\mathcal{G}$ and of a vertex p which belongs to $\mathcal{P}$, that we call the *root* of $\mathcal{P}$.

Definition 6.2

A *forest* $\mathcal{F}$ of $\mathcal{G}$ is a set of subsets $\mathcal{P}_i$ of $\mathcal{G}$ such that:

- two elements of $\mathcal{F}$ are disjoint or strictly included into one another, *i.e.*

$$\begin{aligned} &\mathcal{P}_i \neq \mathcal{P}_j \qquad \text{if } i \neq j \\ \text{and} \qquad &\mathcal{P}_i \cap \mathcal{P}_j = \mathcal{P}_i \text{ , or } \mathcal{P}_j \text{ , or } \emptyset, \qquad \forall i, j. \end{aligned}$$

[6] In renormalization theory, a forest is a family of diagrams $\mathcal{P}_k$ such that for any $k \neq l$ one has either $\mathcal{P}_k \subset \mathcal{P}_l$, or $\mathcal{P}_l \subset \mathcal{P}_k$, or $\mathcal{P}_k \cap \mathcal{P}_l = \emptyset$.

- all elements of $\mathcal{F}$ have at least two elements, i.e.

$$\mathrm{Card}(\mathcal{P}_i) = |\mathcal{P}_i| > 1.$$

Let us note that, by convention, the empty set $\emptyset$ is a forest.

Definition 6.3

A *rooted forest* $\mathcal{F}_\oplus$ is a set of rooted subsets $(\mathcal{P}_i, p_i)$ of $\mathcal{G}$ such that $\{\mathcal{P}_i\}$ is a forest.

Definition 6.4

A *compatibly rooted forest* is a rooted forest such that, if, for some i j, $\mathcal{P}_i \supset \mathcal{P}_j$ and $p_i \in \mathcal{P}_j$, then $p_i = p_j$.

Definition 6.5

Finally to any rooted forest $\mathcal{F}_\oplus$ we associate its compatibly rooted forest ${}^c\mathcal{F}_\oplus$ by simply changing its roots according to the following recursion:

- First, replace the root p_i of each $\mathcal{P}_i$ of the forest by the root p_j of the *smallest* subset $\mathcal{P}_j$ of the forest such that $p_i \in \mathcal{P}_j$ (p_j may coincide with p_i). One thus obtain a new rooted forest.
- Then, repeat this process for the new forest. One can easily show that after a finite number of iterations ($\leq \mathrm{Card}(\mathcal{F})$), this process will leave the roots unchanged, so that one obtains a compatibly rooted forest ${}^c\mathcal{F}_\oplus$.

Of course, a forest $\mathcal{F}_\oplus$ is compatibly rooted iff ${}^c\mathcal{F}_\oplus = \mathcal{F}_\oplus$.

$\diamond$ *Dilation operation*

For a rooted subset $(\mathcal{P}, p)$, we define the dilation operation $\mathcal{D}^\rho_{(\mathcal{P},p)}$ as the transformation acting on the positions of the vertices according to (as in (4.5)):

$$\mathcal{D}^\rho_{(\mathcal{P},p)} \quad : \quad x_i \;\rightarrow\; x_i(\rho) \;=\; \begin{cases} x_p + \rho\,(x_i - x_p) & \text{if } i \in \mathcal{P} \\ x_i & \text{if } i \notin \mathcal{P} \end{cases} \tag{6.14}$$

or equivalently in distance space, according to:

$$\mathcal{D}^\rho_{(\mathcal{P},p)} \quad : \quad a_{ij} \;\rightarrow\; a_{ij}(\rho) \;=\; \begin{cases} \rho^2 a_{ij} & \text{if } i \in \mathcal{P},\ j \in \mathcal{P} \\ a_{pj} - \rho\,(a_{pi} + a_{pj} - a_{ij}) + \rho^2\, a_{pi} & \text{if } i \in \mathcal{P},\ j \notin \mathcal{P} \\ a_{ij} & \text{if } i \notin \mathcal{P},\ j \notin \mathcal{P}\ . \end{cases} \tag{6.15}$$

More generally, for a function I, expressed as a function of the positions x_i or the distances a_{ij}, we denote by $\mathcal{D}^\rho_{(\mathcal{P},p)} I$ the value of this function at the positions (or distances) modified according to (6.14) (or (6.15)).

◇ *Taylor operator*

We then define the "Taylor" operator $\mathfrak{T}_{(\mathcal{P},p)}$ acting on functions I by:

$$\mathfrak{T}_{(\mathcal{P},p)} I = \lim_{\rho\to 0} \rho^{d\nu(|\mathcal{P}|-1)} \mathcal{D}^{\rho}_{(\mathcal{P},p)} I. \tag{6.16}$$

The functions that we shall consider are the integrands in (2.27) and (2.23), which are of the form

$$\begin{aligned} I_{\mathcal{G}}(x_i, X_a) &= \left(\det\left[\Pi\left(x_{i,\, i\in\mathcal{G}}\right)\right]\right)^{-\frac{d}{2}} \ \exp\left(-\frac{1}{2}\sum_{a,b} \vec{\mathbf{k}}_a \cdot \vec{\mathbf{k}}_b\, \Delta_{ab}(x_{i,\, i\in\mathcal{G}}, X_a)\right) \\ I_{\mathcal{G}}(x_i) &= \left(\det\left[\Pi\left(x_{i,\, i\in\mathcal{G}}\right)\right]\right)^{-\frac{d}{2}} \end{aligned} \tag{6.17}$$

where the Π and Δ matrices, defined in (2.24), (3.49) and (2.28), (3.52), are functions of the positions of the internal vertices i in $\mathcal{G}$ and external vertices a. On such functions, the effect of $\mathfrak{T}_{(\mathcal{P},p)}$ is to keep the most singular term in ρ when performing the dilation $\mathcal{D}^{\rho}_{(\mathcal{P},p)}$. For instance one operator $\mathfrak{T}_{(\mathcal{P},p)}$ factorizes $I_{\mathcal{G}}$ into

$$\mathfrak{T}_{(\mathcal{P},p)} I_{\mathcal{G}}(x_i, X_a) = I_{\mathcal{P}}(x_i) I_{\mathcal{G}/\!\!_p\mathcal{P}}(x_i, X_a) \tag{6.18}$$

where

$$\mathcal{G}/\!\!_p\mathcal{P} \equiv \mathcal{G} \setminus (\mathcal{P} \setminus \{p\}) \tag{6.19}$$

is the reduced set obtained by contracting $\mathcal{P}$ into p ($\setminus$ is the usual subtraction of sets). This operation can be repeated for rooted subsets which form a *compatibly rooted* forest, and the result does not depend on the order of the $\mathfrak{T}$ operators in this case (commutativity). The result is a product of integrands $I(x_i)$ of reduced internal subsets, times the integrand $I(x_i, X_a)$ of the set $\mathcal{G}$ reduced by all elements of the forest.

◇ *The subtraction operator*

With those notations, we define the subtraction operation $\mathbf{R}$ as a sum of subtractions for all forests. For a given forest $\mathcal{F}$, subtractions associated with different roots give different results on the integrand. We shall sum over the subtractions for all compatibly rooted forests $\mathcal{F}_{\oplus^c}$, with some weight factor $W(\mathcal{F}_{\oplus^c})$ associated with the (compatible) rooting of $\mathcal{F}$. In order to ensure the finiteness of the subtracted integrals, the weights $W(\mathcal{F}_{\oplus^c})$ must be such that the sum of the W for all rooted forests which correspond to the same unrooted forest $\mathcal{F}$ gives 1. A convenient choice of weight factor $W(\mathcal{F}_{\oplus^c})$ for $\mathcal{F}_{\oplus^c}$ is to make it proportional to the number of different (not necessarily compatibly)-rooted

forests $\mathcal{F}_\oplus$ which gives $\mathcal{F}_{\oplus^c}$ by the compatibilization operation ^{c} (i.e. $^c\mathcal{F}_\oplus = \mathcal{F}_{\oplus^c}$). Our final definition for $\mathbf{R}$ is therefore expressed as a sum over all rooted forests, or equivalently as a sum over all compatibly rooted forests. It reads[7]:

$$\boxed{\begin{aligned}\mathbf{R} &= \sum_{\mathcal{F}_\oplus}\left[\prod_{(\mathcal{P},p)\in{}^c\mathcal{F}_\oplus}\frac{1}{|\mathcal{P}|}\big(-\mathfrak{T}_{(\mathcal{P},p)}\big)\right]\\ &= \sum_{\mathcal{F}_{\oplus^c}} W(\mathcal{F}_{\oplus^c})\left[\prod_{(\mathcal{P},p)\in\mathcal{F}_{\oplus^c}}\big(-\mathfrak{T}_{(\mathcal{P},p)}\big)\right]\end{aligned}} \tag{6.20}$$

The weight factors are given explicitly by a product over all *different roots* p of $\mathcal{F}_{\oplus^c}$

$$W(\mathcal{F}_{\oplus^c}) = \prod_{\substack{p \text{ root}\\ \text{of } \mathcal{F}_{\oplus^c}}}\frac{1}{|\mathcal{P}_p|}, \tag{6.21}$$

where $\mathcal{P}_p$ is the *largest* subset of the forest whose root is p.

◇ *Subtracted amplitudes and renormalization*

We now restrict ourselves to the case of amplitudes defined in a finite volume, by using the IR regulator introduced in section 4.3 (D-dimensional sphere), that is by defining the integration over the positions of the vertices by (4.15) and (4.16). The subtracted correlation functions at order N are simply defined by applying the subtraction operator $\mathbf{R}$ to the integrand of (6.7)

$$\mathcal{Z}^{\mathbf{R}}{}_N^{(M)}(X_a) \equiv \int\prod_{i\in\mathcal{G}} d^Dx_i\,\mathbf{R}\,[I_{\mathcal{G}}(x_i,X_a)]. \tag{6.22}$$

Let us note that, since the integrand for the partition function is homogeneous under global rescaling, one has

$$\mathbf{R}\,[I_{\mathcal{G}}\,(x_i)] = 0 \tag{6.23}$$

(as soon as $|\mathcal{G}|\geq 2$, of course). This means that with our choice of subtraction, for $N\geq 2$, in the absence of external correlation points,

$$\mathcal{Z}_N^{\mathbf{R}} = 0\ , \quad N\geq 2. \tag{6.24}$$

[7] In this equation, $\mathcal{F}_{\oplus^c}$ denotes an arbitrary compatibly rooted forest, while $^c\mathcal{F}_\oplus$ denotes the compatibly rooted forest obtained from the (non necessarily compatibly) rooted forest $\mathcal{F}_\oplus$ by the compatibilization procedure of Definition 6.5.

The purpose of the next sections is to prove that this subtraction operation makes all correlation functions finite, as summarized in the theorem:

THEOREM *For $0 < \nu < 1$, the renormalized integral (6.22) is convergent for $\epsilon = 0$ and defines a finite function $\mathcal{Z}^{\mathbf{R}}{}_N^{(M)}(X_a)$ for $D \geq N + M - 1$.*

The renormalized correlation functions are defined by their perturbative expansion in powers of a renormalized coupling constant b_R

$$\mathcal{Z}^{\mathbf{R}(M)}(X_a; b_R) \;=\; \sum_{N=0}^{\infty} \frac{(-b_R)^N}{N!} \mathcal{Z}^{\mathbf{R}}{}_N^{(M)}(X_a). \tag{6.25}$$

As discussed above, the forest structure of the subtraction operation $\mathbf{R}$ ensures that for $\epsilon > 0$, there exists a renormalized coupling constant $b_R(b)$ such that the renormalized correlation functions $\mathcal{Z}^{\mathbf{R}(M)}(X_a; b_R)$ are equal to the original "bare" correlation functions $\mathcal{Z}^{(M)}(X_a; b)$ for the model (2.1).

The relation between b and b_R can be obtained directly from the identity of the partition functions

$$\mathcal{Z}(b) \;=\; \mathcal{Z}^{\mathbf{R}}(b_R). \tag{6.26}$$

From (6.24) we have $\mathcal{Z}^{\mathbf{R}}(b_R) \;\equiv\; (2\pi)^d \delta^d(\vec{\mathbf{k}} = \vec{0}) - b_R\, \mathcal{V}_{\mathcal{S}_D} \equiv V_{\mathbb{R}^d} - b_R\, \mathcal{V}_{\mathcal{S}_D}$ and therefore, equating to $\mathcal{Z}(b)$, we get

$$b_R = -\frac{1}{\mathcal{V}_{\mathcal{S}_D}} \left(\mathcal{Z} - V_{\mathbb{R}^d} \right), \tag{6.27}$$

or the explicit series expansion in b:

$$b_R \;=\; b \;-\; \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \sum_{N \geq 2} \frac{(-b)^N}{N!} \mathcal{Z}_N. \tag{6.28}$$

Notice that the fully renormalized coupling constant b_R satisfies the indentity $b \;=\; b_R + \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \sum_{N \geq 2} \frac{(-b)^N}{N!} \mathcal{Z}_N$, while the former partially renormalized coupling constant $b_\bullet$ (built so as to absorb the superficial divergences) satisfies the *truncated* equation (6.13), $b \;=\; b_\bullet + \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \sum_{N \geq 2} \frac{(-b_\bullet)^N}{N!} \mathcal{Z}_N$, obtained from the equation for b_R mentioned just above by replacing $(-b)$ by $(-b_\bullet)$ in the r.h.s.

Equation (6.27) shows that, in this scheme, renormalization simply amounts to a change of variable from the microscopic b to an effective coupling constant b_R, directly proportional to the *connected* partition function of the manifold interacting with a point. This scheme is precisely that used in [20], and generalizes that of the "direct renormalization method" [4] for the polymer Edwards model.

Let us stress that b_R as defined above is not dimensionless. The corresponding dimensionless coupling constant can be conveniently chosen as

$$g = \Big(2\pi\, A_D(\nu)\Big)^{-d/2} b_R \mathcal{V}_{\mathcal{S}_D}^{1-\nu d/D} \tag{6.29}$$

for which the Wilson function (2.6) has been calculated explicitly at one loop [20]. In this subtraction scheme, the subtraction scale μ of the general equation (6.1) is fixed by the D-dimensional volume (which fixes the IR cut-off) $\mu \sim (\mathcal{V}_{\mathcal{S}_D})^{-1/D}$. In these notations, this precisely corresponds to $\hat{b}_R = b_R\,(\mathcal{V}_{\mathcal{S}_D})^{\epsilon/D}$ and $Z^{-1}(\hat{b}_R, \epsilon) \equiv b_R/b = 1 + \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \sum_{N\geq 2} (-b)^{N-1} \frac{\mathcal{Z}_N}{N!}$, where b is an implicit function of b_R, thus $\hat{b}_R$. Of course, other subtraction schemes can be chosen where the subtraction scale μ is not related to the volume of internal D-dimensional space. They are needed in order to define the theory (*e.g.* the normalized correlation functions) in the infinite volume limit.

7. Reorganization of the counterterms

7.1. Formulation of the subtraction operation in terms of nests

As we shall see later, it will be more convenient in the proof of the finiteness of the renormalized amplitudes to express the subtraction operation $\mathbf{R}$ in term of *nested* subdiagrams. In the formalism of BPHZ renormalization in the Schwinger representation in field theory, a subdiagram is a set of lines (propagators) of a Feynman graph (and has in general many connected components). A *nest* is then a family of subdiagrams P_k which are nested, that is included into one another (for any $k \neq l$, $P_k \subset P_l$ or $P_l \subset P_k$).

In our case we shall introduce a different notion of diagram, now in terms of vertices, rather than lines. Indeed, we have seen that the natural generalization of Schwinger parameters s_α is given by the larger set of all mutual distances a_{ij} between points on the manifold. In terms of links, we thus would have to deal with the large number of interdependent mutual distances, which are constrained by triangular inequalities. Therefore, we prefer to define diagrams in terms of vertices. Denoting again by $\mathcal{G}$ a set of N vertices, a diagram of $\mathcal{G}$ will now be a collection of disjoint vertex-subsets of $\mathcal{G}$. Each of these subsets of vertices can be thought of as a connected set (which stands for the the complete set of its pairwise mutual distances in the link representation). These ideas will be embodied in the following definitions.

Fig. 7: *A subdiagram.*

Fig. 8: *A complete diagram, with connected components* $\mathcal{P}_i$.

We recall that a partition P of a set $\mathcal{S}$ is a set of mutually disjoint non empty subsets $\mathcal{S}_i$ of $\mathcal{S}$, whose union is $\mathcal{S}$ itself.

Definition 7.1 (see fig. 7, 8)

We shall call a *subdiagram* (respectively *complete diagram*) of $\mathcal{G}$ any partition P of some subset $\mathcal{S}$ of $\mathcal{G}$ (respectively of $\mathcal{G}$ itself). The generic word *diagram* will be used in both cases.

The elements of this partition P are called the *connected components* of the diagram P.

Definition 7.2 (see fig. 9)

A diagram P is *contained* in a diagram Q if any connected component of P is included in one of the connected components of Q. This will be denoted $P \prec Q$. [8] This defines a partial ordering among the diagrams of $\mathcal{G}$.

[8] Let us stress that $P \prec Q$ does not mean that P, considered as a set (whose elements are subsets of $\mathcal{G}$), is included in Q. Still if $P \subset Q$, then $P \prec Q$.

Fig. 9: *The complete diagram with connected components $\mathcal{P}_i$ (dashed line) is contained in the complete diagram with connected components $\mathcal{Q}_j$ (full line).*

Fig. 10: *The intersection diagram (dark-grey diagram) of two diagrams (grey and white diagrams).*

Definition 7.3 (see fig. 10)

We define the *intersection* of two diagrams P and Q as the maximal diagram which is contained in both P and Q (it is unique), and denote it by $P \wedge Q$. Its connected components are nothing but the (non-empty) intersections of a connected component of P and one of Q.

Definition 7.4 (see fig. 11)

We define the *union* of two diagrams P and Q as the minimal diagram which contains both P and Q (it is also unique), and denote it by $P \vee Q$. Let us note that the connected components of $P \vee Q$ are unions of connected components of P and Q, but in general not simply the union of one connected component of P and of one of Q.

Notice that the union and the intersection of complete diagrams of $\mathcal{G}$ are complete. The maximal complete diagram of $\mathcal{G}$ is $G = \{\mathcal{G}\}$. We shall denote by $G_\odot$ the (unique) minimal complete diagram of $\mathcal{G}$. Its connected components are the N single vertex subsets of $\mathcal{G}$ (see fig. 12). For any complete diagram P, we have $G_\odot \prec P \prec G$.

Fig. 11: *The union diagram (dark-grey diagram) of two diagrams (grey and white diagrams).*

Fig. 12: *The minimal complete diagram* $G_{\odot}$.

Fig. 13: *The subtraction diagram (dark-grey diagram) of a diagram (grey diagram) from another diagram (white diagram).*

Definition 7.5 (see fig. 13)

We define the *subtraction* of a diagram P from a diagram Q as the (unique) maximal diagram contained in Q and whose intersection with P is empty, and denote it by $Q \setminus P$.

The usual properties of commutativity and associativity are satisfied by $\wedge$ and $\vee$.

However these operations are not distributive with respect to one another. They still satisfy the weaker relations

$$\begin{aligned} P \wedge (Q \vee R) &\succ (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) &\prec (P \vee Q) \wedge (P \vee R) \end{aligned} \tag{7.1}$$

Fig. 14: *A rooted subdiagram. The roots are specified by squares.*

Definition 7.6 (see fig. 14)

A *rooted diagram* $P_\oplus$ is a family $\{(\mathcal{P}_1, p_1), \ldots, (\mathcal{P}_k, p_k)\}$ of rooted subsets $(\mathcal{P}_i, p_i)$ of $\mathcal{G}$ such that $P = \{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ is a diagram of $\mathcal{G}$.
We call

$$P = \mathrm{comp}(P_\oplus) = \{\mathcal{P}_1, \ldots, \mathcal{P}_k\} \tag{7.2}$$

the component diagram of $P_\oplus$, and

$$\wp = \mathrm{root}(P_\oplus) = \{\{p_1\}, \ldots, \{p_k\}\} \tag{7.3}$$

the root diagram of $P_\oplus$. We shall use for a rooted diagram the equivalent notations:

$$P_\oplus \equiv \big(\mathrm{comp}(P_\oplus), \mathrm{root}(P_\oplus)\big) \equiv \big(P, \wp\big). \tag{7.4}$$

Definition 7.7 (see fig. 15)

A complete rooted diagram is a rooted diagram such that its component diagram is complete.

Fig. 15: *A complete rooted diagram. Its elements are rooted subsets* $(\mathcal{P}_i, p_i)$.

Definition 7.8

A *nest* $\mathcal{N}$ is a set of $\tau + 1$ complete diagrams $\{T_0, T_1, \ldots, T_\tau\}$ such that

$$\begin{aligned} &T_0 = G_\odot \\ &T_0 \prec T_1 \prec T_2 \prec \ldots \prec T_\tau. \end{aligned} \tag{7.5}$$

Fig. 16: *Two successive complete rooted diagrams* $T_{\jmath\oplus}$, *with connected components* $\mathcal{T}_{\jmath,j}$ *(dashed lines) and* $T_{\jmath+1\oplus}$ *with connected components* $\mathcal{T}_{\jmath+1,k}$ *(full lines) of a rooted nest. The roots of these two diagrams are not compatible.*

Definition 7.9 (see fig. 16)

A *rooted nest* $\mathcal{N}_\oplus$ is a set of complete rooted diagrams $\{T_{0\oplus}, T_{1\oplus}, \ldots, T_{\tau\oplus}\}$ such that the associated component diagrams form a nest

$$\mathrm{comp}(T_{0\oplus}) \prec \mathrm{comp}(T_{1\oplus}) \prec \ldots \prec \mathrm{comp}(T_{\tau\oplus}). \tag{7.6}$$

Fig. 17: *The two successive diagrams of fig. 16, with compatible roots. The roots $w_{J+1,k}$ have been obtained from the roots of fig. 16 by the construction of Definition 7.11.*

Definition 7.10 (see fig. 17)

A rooted nest is said to be *compatibly rooted* if we have moreover

$$\mathrm{root}(T_{0\oplus}) \succ \mathrm{root}(T_{1\oplus}) \succ \ldots \succ \mathrm{root}(T_{T\oplus}). \tag{7.7}$$

(Notice that $\mathrm{root}(T_{0\oplus}) = G_{\odot}$).

At level J, the generic element of the rooted nest $\mathcal{N}_{\oplus}$ reads explicitly:

$$T_{J\oplus} = \big\{(\mathcal{T}_{J,j}, t_{J,j}),\, j = 1, \ldots, \mathrm{Card}(T_J)\big\}. \tag{7.8}$$

Eq. (7.7) means that when we consider two successive rooted complete diagrams of the rooted nest, $T_{J\oplus}$ and $T_{J+1\oplus}$, if we consider a connected component $\mathcal{T}_{J+1,k}$ of T_{J+1} and its root $t_{J+1,k}$, this root must coincide with the root $t_{J,j}$ of the connected component $\mathcal{T}_{J,j}$ of T_J to which $t_{J+1,k}$ belongs (since T_J is complete, $t_{J+1,k}$ belongs necessarily to some connected component of T_J). This property then implies by recursion that, at each level $L \leq J$, $t_{J+1,k}$ coincides with the root $t_{L,l}$ of the connected component $\mathcal{T}_{L,l}$ of T_L to which it belongs.

Definition 7.11

To any rooted nest $\mathcal{N}_{\oplus}$ with elements given by (7.8), we associate the compatibly rooted nest

$$\begin{aligned} {}^c\mathcal{N}_{\oplus} &= \big\{{}^cT_{J\oplus}\big\} \\ {}^cT_{J\oplus} &= \big\{(\mathcal{T}_{J,j}, w_{J,j}),\, j = 1, \ldots, \mathrm{Card}(T_J)\big\}, \end{aligned} \tag{7.9}$$

with the same connected components $\mathcal{T}_{J,j}$ at each level J, and whose roots $w_{J,j}$ are obtained from the roots $t_{J,j}$ by the following recursion:

- at level 0, the roots of T_0 are fixed since $\mathrm{root}(T_{0\oplus}) = G_{\odot}$;

- at level 1, we identify $w_{1,j}$ with the original root $t_{1,j}$, that is set $w_{1,j} = t_{1,j}$ for all $j = 1, \ldots, \mathrm{Card}(T_1)$;
- at level $\jmath + 1$ and for each connected component $\mathcal{T}_{\jmath+1,k}$, we look for the component $\mathcal{T}_{\jmath,j(k)}$ of the complete diagram $T_\jmath$ at the preceding level $\jmath$ to which the original root $t_{\jmath+1,k}$ belongs. The root $w_{\jmath,j(k)}$ has already been constructed at level $\jmath$ and we make the roots compatible between level $\jmath$ and $\jmath + 1$ by substituting to the original root $t_{\jmath+1,k}$ the root $w_{\jmath+1,k} = w_{\jmath,j(k)}$ (Notice that, since $T_\jmath \prec T_{\jmath+1}$, $\mathcal{T}_{\jmath,j(k)} \subset \mathcal{T}_{\jmath+1,k}$ and therefore $w_{\jmath,j(k)} \in \mathcal{T}_{\jmath+1,k}$) .

By construction, the rooted nest ${}^c\mathcal{N}_\oplus$ is compatibly rooted. Of course, a rooted nest $\mathcal{N}_\oplus$ is compatibly rooted if and only if ${}^c\mathcal{N}_\oplus = \mathcal{N}_\oplus$, and in this case, $w_{\jmath,j} \equiv t_{\jmath,j}$ for all $\jmath$ and j.

To a rooted diagram $T_\oplus$, we associate the Taylor operator $\mathfrak{T}_{T_\oplus}$ defined simply as the product of the Taylor operators $\mathfrak{T}_{(\mathcal{T},t)}$ of its rooted connected components:

$$\mathfrak{T}_{T_\oplus} = \prod_{(\mathcal{T}_j,t_j)\in T_\oplus} \mathfrak{T}_{(\mathcal{T}_j,t_j)} \tag{7.10}$$

with the convention $\mathfrak{T}_{(\mathcal{T},t)} = \mathbb{1}$ if $|\mathcal{T}| = 1$ (*i.e.* $\mathcal{T} = \{t\}$), which in particular implies that $\mathfrak{T}_{T_{0\oplus}} = \mathbb{1}$ for $T_0 = G_\odot$. We denote by $\|T_\oplus\|$ the product of the cardinals of the connected components $\mathcal{T}_j$ of the diagram $\mathrm{comp}(T_\oplus)$

$$\|T_\oplus\| \;=\; \prod_{\mathcal{T}_j\in\mathrm{comp}(T_\oplus)} |\mathcal{T}_j| \tag{7.11}$$

PROPOSITION:

The subtraction operator $\mathbf{R}$ (6.20) can be rewritten as a sum over rooted nests:

$$\boxed{\begin{aligned}\mathbf{R} \;&= -\sum_{\mathcal{N}_\oplus}\Big[\prod_{{}^cT_{\jmath\oplus}\in{}^c\mathcal{N}_\oplus}\big(-\frac{1}{\|{}^cT_{\jmath\oplus}\|}\mathfrak{T}_{{}^cT_{\jmath\oplus}}\big)\Big]\\ &= -\sum_{\mathcal{N}_{\oplus^c}} W(\mathcal{N}_{\oplus^c})\Big[\prod_{T_{\jmath\oplus}\in\mathcal{N}_{\oplus^c}}\big(-\mathfrak{T}_{T_{\jmath\oplus}}\big)\Big]\end{aligned}} \tag{7.12}$$

where the second formula is a sum over compatibly rooted nests with the appropriate weight factor:

$$W(\mathcal{N}_{\oplus^c}) \;=\; \prod_w \frac{1}{|\mathcal{T}_w|} \tag{7.13}$$

with $\mathcal{T}_w$ being as before the largest connected component (among all connected components of all diagrams of $\mathcal{N}$) whose root is w. In (7.13) the products is over all vertices of $\mathcal{G}$ since any point w of $\mathcal{G}$ is the root of at least one connected component in the nest, namely the connected component $\{w\}$ of T_0.

Proof:

The global (-1) factor in (7.12) is introduced to reverse the global (-1) sign coming from the contribution $\big(-\frac{1}{\|T_{0\oplus}\|}\mathfrak{T}_{T_{0\oplus}}\big) = -\mathbb{1}$ which is present for each nest (compatible or not).

To prove that (7.12) coincides with (6.20) one can proceed in two steps, that we indicate below. The details are left to the reader.

We start from (7.12) as a sum over compatibly rooted nests $\mathcal{N}_{\oplus^c}$,

First, we notice that the family of all distinct rooted components, excluding single vertex components, of the rooted diagrams of some compatibly rooted nest $\mathcal{N}_{\oplus^c}$ form a compatibly rooted forest. Moreover, if two different compatibly rooted nests yield the same compatibly rooted forest $\mathcal{F}_{\oplus^c}$, the products of $\mathfrak{T}$ for these two different nests give the same result, which is nothing but the product of $\mathfrak{T}$ associated with the compatibly rooted forest $\mathcal{F}_{\oplus^c}$. This allows us to regroup all compatibly rooted nests which yield the same compatibly rooted forest.

Second, we have to check that the (-1) factors and weights associated with each diagram of this group of nests sum up in order to give the correct factor $W(\mathcal{F}_{\oplus^c})$ (6.21) for this forest. This can be seen in two steps: First, the weights $W(\mathcal{N}_{\oplus^c})$ (7.13) are in fact equal to $W(\mathcal{F}_{\oplus^c})$ (6.21) , for each $\mathcal{N}_{\oplus^c}$ yielding $\mathcal{F}_{\oplus^c}$. Therefore, at that stage, we can forget about the roots and the weights W and concentrate on the (-1) factors associated to the diagrams of the nests. It remains to show that, when summing over all nests $\mathcal{N}$ which yield a given forest $\mathcal{F}$, one has $(-1) \times \sum_{\mathcal{N}\to\mathcal{F}} (-1)^{\mathrm{Card}(\mathcal{N})} = (-1)^{\mathrm{Card}(\mathcal{F})}$. This relation can be easily checked for forests made out of two subsets, which are either disjoint or included into one another[9] , and then extended by a recursion on the number of elements of the forest.

[9] If $\mathcal{F} = \{S_1, S_2\}$, either $S_1 \cap S_2 = \emptyset$ and there are three nests $\{G_\odot, G_\odot \vee \{S_1, S_2\}\}$, $\{G_\odot, G_\odot \vee \{S_1\}, G_\odot \vee \{S_1, S_2\}\}$ and $\{G_\odot, G_\odot \vee \{S_2\}, G_\odot \vee \{S_1, S_2\}\}$, with respectively 2, 3 and 3 diagrams, or $S_1 \subset S_2$ and there is only one nest $\{G_\odot, G_\odot \vee \{S_1\}, G_\odot \vee \{S_2\}\}$ with 3 diagrams.

7.2. Sectors

Definition 7.12: Saturated nest

A *saturated nest* $\mathcal{S}$ of $\mathcal{G}$ is a nest with $N = \mathrm{Card}(\mathcal{G})$ (distinct) elements [10], which we call $R^0, \ldots, R^{N-1}$.

The cardinal of a saturated nest is therefore maximal. A saturated nest is actually constructed from $G_\odot$ (the complete diagram made of N single point connected components) by fusing recursively at each level $R^\imath$ exactly two connected components of the preceding level $R^{\imath-1}$ until $G = \{\mathcal{G}\}$ is obtained. A saturated nest is therefore characterized as follows:

- its minimal diagram is $R^0 = G_\odot$,
- its maximal diagram is $R^{N-1} = G = \{\mathcal{G}\}$,
- $\mathrm{Card}(R^{\imath+1}) = \mathrm{Card}(R^\imath) - 1$ for all $\imath = 0, \ldots, N-1$

$\diamond$ *Saturated nest associated with ordered trees*

The notion of saturated nest occurs naturally when spanning integration points by *trees*, as was done formally in section 3.7 . Indeed, let us consider a tree $\mathbf{T} = (\lambda_\alpha;\, \alpha = 1, \ldots N-1)$, considered as *ordered* by increasing values of α (this order will actually correspond to increasing mutual distances, in a generalized sense to be made precise below). Such an ordered tree $\mathbf{T}$ generates naturally a saturated nest $\mathcal{S}(\mathbf{T})$ as follows:

- $R^0 = G_\odot$
- at level $\imath$ $(1 \leq \imath \leq N-1)$, we consider the line $\alpha = \imath$ with end points i_α, i'_α and set $R^\imath = R^{\imath-1} \vee \{\{i_\alpha, i'_\alpha\}\}$, which corresponds to the fusion of the connected component of $R^{\imath-1}$ containing i_α with that containing i'_α.

Of course, different trees $\mathbf{T}$ can yield the same $\mathcal{S}(\mathbf{T})$. This allows us to classify trees into equivalence classes, by regrouping all the trees $\mathbf{T}$ such that $\mathcal{S}(\mathbf{T}) = \mathcal{S}$ for any given saturated nest $\mathcal{S}$. If two ordered trees $\mathbf{T} = (\lambda_\alpha;\, \alpha = 1, \ldots N-1)$ and $\mathbf{T}' = (\lambda'_\alpha;\, \alpha = 1, \ldots N-1)$ are equivalent, then the transformation from λ to λ' is such that:

$$\lambda_\alpha = \pm\lambda'_\alpha + \sum_{\gamma<\alpha} c_\alpha^\gamma \lambda'_\gamma \qquad\qquad c_\alpha^\gamma = 0, \pm 1 \tag{7.14}$$

where c_α^γ are coefficients equal to 0 or ± 1 (which are in general further constrained so that $\mathbf{T}$ and $\mathbf{T}'$ actually span the same set of integration points).

[10] We use superscripts here in $R^\imath$ rather than subscripts as before in $T_\jmath$ for future convenience.

◇ *Oriented ordered tree associated with a compatibly rooted saturated nest*

Conversely, if the saturated nest $\mathcal{S}$ is *compatibly rooted*, there is a natural way to associate with $\mathcal{S}_\oplus$ an oriented ordered tree $\mathbf{T}(\mathcal{S}_\oplus)$. Indeed, by definition, a saturated nest $\mathcal{S} = \{R^I\}$ is constructed by fusing recursively at each level R^I exactly two connected components $\mathcal{R}^{I-1,k}$ and $\mathcal{R}^{I-1,k'}$ of the preceding level R^{I-1}. Denoting by i_I and i'_I their respective roots in $R^{I-1}_\oplus$, one of these roots, say i_I, is the root of $\mathcal{R}^{I-1,k} \cup \mathcal{R}^{I-1,k'}$ in $R^I_\oplus$, since the rooting is compatible. In this case the other root i'_I can no longer be the root of any connected component of the diagrams $R^{I'}$ for $I' \geq I$. Therefore, if we define by $\lambda_I = x_{i'_I} - x_{i_I}$ the oriented line vector joining the positions of the roots i_I and i'_I, the set of λ_I for $I = 1, \ldots, N-1$ defines an *oriented ordered* (by I) tree, which we denote by $\mathbf{T}(\mathcal{S}_\oplus)$. Of course, we have by construction $\mathcal{S}(\mathbf{T}(\mathcal{S}_\oplus)) = \mathcal{S}$. Moreover, one can easily check that the tree $\mathbf{T}(\mathcal{S}_\oplus)$ has the following property: for any I and I', the path on the tree joining the two origins x_I and $x_{I'}$ of the vectors λ_I and $\lambda_{I'}$ passes only through vectors λ_K for $K > \min(I, I')$.

Although this construction does not play any role in the present section 7, it will turn out to be useful in section 8.

Fig. 18: *(a) Saturated nest associated with an ordered tree* $\mathbf{T} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. *The nest is made of four diagrams. Each diagram is represented by the contour of its connected components with at least two vertices (the diagrams 1, 3 and 4 have only one such connected component, the diagram 2 has two such connected components). (b) Oriented ordered tree associated with a compatibly rooted saturated nest. We have first assigned compatible roots to the saturated nest of (a) (here the diagrams 3 and 4, and the connected component on the right of the diagram 2 have the same root) and then constructed the oriented ordered tree from these roots.*

Definition 7.13: Extended Hepp Sectors

Now we want to associate with an *unrooted* saturated nest $\mathcal{S}$ an *extended* Hepp sector, defined from the Hepp sectors attached to ordered trees constructed in section 5.1 .

If we consider as in section 5.1 the N points as being embedded in $\mathbb{R}^{N-1}$ with Cartesian coordinates $0, y_1, \ldots, y_{N-1}$, and denote as before $\mathcal{H}^{\mathbf{T}}$ the domain of the y_i's defining the Hepp sector attached to the ordered tree[11] $\mathbf{T}$, we define the Hepp sector $\mathcal{H}^{\mathcal{S}}$ as the union of all Hepp sectors attached to all ordered trees $\mathbf{T}$ such that $\mathcal{S}(\mathbf{T}) = \mathcal{S}$, that is the domain of the y_i's given by:

$$\mathcal{H}^{\mathcal{S}} = \bigcup_{\mathbf{T}:\, \mathcal{S}(\mathbf{T})=\mathcal{S}} \mathcal{H}^{\mathbf{T}}. \tag{7.15}$$

This extended Hepp sector is best described by the vectors λ_α associated with a given (arbitrary) tree $\mathbf{T}$ such that $\mathcal{S}(\mathbf{T}) = \mathcal{S}$. Let us stress that now the λ_α's are no longer successive minimal distances when the y_i's move everywhere inside $\mathcal{H}^{\mathcal{S}}$, but are so only for y_i's inside the subset $\mathcal{H}^{\mathbf{T}}$ of $\mathcal{H}^{\mathcal{S}}$. In particular, the inequalities $|\lambda_\alpha| \leq |\lambda_{\alpha+1}|$ of (5.1) are not necessarily satisfied inside $\mathcal{H}^{\mathcal{S}}$. Still, for y_i's inside $\mathcal{H}^{\mathcal{S}}$, one can find a tree $\mathbf{T^0}$ such that $\mathcal{S}(\mathbf{T^0}) = \mathcal{S}(\mathbf{T})$ and $\{y_i\} \in \mathcal{H}^{\mathbf{T^0}}$. The λ^0_α associated with $\mathbf{T^0}$ satisfy for this set of y_i's the inequalities $|\lambda^0_1| \leq \ldots \leq |\lambda^0_{N-1}|$. By construction, one has inside $\mathcal{H}^{\mathbf{T^0}}$ at each level α: $|\lambda^0_\alpha| \leq |\lambda_\alpha|$ and, as in (7.14), a relation between the λ_α's and the λ^0_α's of the form $\lambda_\alpha = \pm\lambda^0_\alpha + \sum_{\gamma<\alpha} c^\gamma_\alpha \lambda^0_\gamma$ with some coefficients c^γ_α equal to 0 or ± 1. We can thus write:

$$\begin{aligned} |\lambda_\alpha| &= |\pm\lambda^0_\alpha + \sum_{\gamma<\alpha} c^\gamma_\alpha \lambda^0_\gamma| \\ &\leq |\lambda^0_\alpha| + \sum_{\gamma<\alpha} |c^\gamma_\alpha||\lambda^0_\gamma| \\ &\leq (1 + \sum_{\gamma<\alpha} |c^\gamma_\alpha|)|\lambda^0_\alpha| \\ &\leq \alpha|\lambda^0_\alpha|. \end{aligned} \tag{7.16}$$

We thus have the set of inequalities:

$$|\lambda^0_\alpha| \leq |\lambda_\alpha| \leq \alpha|\lambda^0_\alpha| \tag{7.17}$$

[11] We recall that the domain $\mathcal{H}^{\mathbf{T}}$ corresponds to the domain where the λ_α's obtained from the y_i's by Eq. (3.43) are actual successive minimal distances, and in particular satisfy $|\lambda_1| \leq \ldots \leq |\lambda_{N-1}|$.

which, together with $|\lambda^0_\alpha| \leq |\lambda^0_{\alpha+1}|$ implies

$$\frac{|\lambda_\alpha|}{|\lambda_{\alpha+1}|} \leq \alpha. \tag{7.18}$$

This is an example of constraints satisfied by all tree variables compatible with the nest $\mathcal{S}$ in the extended sector $\mathcal{H}^{\mathcal{S}}$, which is a relaxed extension of (5.1) . Another consequence of (7.17) is that if $\mathbf{T}$ and $\mathbf{T}'$ are two trees such that $\mathcal{S}(\mathbf{T}) = \mathcal{S}(\mathbf{T}') = \mathcal{S}$, then inside $\mathcal{H}^{\mathcal{S}}$, the corresponding line vectors satisfy:

$$\begin{aligned} \frac{1}{\alpha} \leq \frac{|\lambda_\alpha|}{|\lambda'_\alpha|} \leq \alpha \\ \frac{|\lambda_\alpha|}{|\lambda'_{\alpha'}|} \leq \alpha \qquad \text{for} \quad \alpha' > \alpha. \end{aligned} \tag{7.19}$$

These bounds will be useful in section 8.

The corresponding extended Hepp sector $\mathcal{A}^{\mathcal{S}}_N$ in the space $\mathcal{A}_N$ of mutual squared distances a_{ij} between vertices (see section 3.2) can be described simply, without reference to ordered trees. Given a saturated nest $\mathcal{S} = \{R^0, \ldots, R^{N-1}\}$, let us consider, for a given diagram R^I, the smallest squared distance between vertices which belong to two *different* connected components of the diagram R^I (minimal squared distance between connected components):

$$a_{\min}(R^I) \;=\; \min_{\mathcal{R}^{I,k} \neq \mathcal{R}^{I,l} \in R^I} \left(\min_{i \in \mathcal{R}^{I,k}, j \in \mathcal{R}^{I,l}} (a_{ij}) \right).$$

For the minimal diagram $R^0 = G_\odot$ one has obviously $a_{\min}(G_\odot) = \min\limits_{i \neq j}(a_{ij})$, and by convention for the maximal diagram $G = \{\mathcal{G}\}$ (which has only one connected component) we set $a_{\min}(G) = \infty$. One can check that one has always, for any saturated nest, $a_{\min}(R^0) \leq a_{\min}(R^1) \leq \ldots \leq a_{\min}(R^{N-2}) < a_{\min}(R^{N-1})$.

The extended Hepp sector $\mathcal{A}^{\mathcal{S}}_N$ associated with the saturated nest $\mathcal{S}$ is the subset of $\mathcal{A}_N$ such that

$$a_{\min}(R^0) < a_{\min}(R^1) < \ldots < a_{\min}(R^{N-2}) < a_{\min}(R^{N-1}). \tag{7.20}$$

One can check that the sectors associated with two different saturated nests are disjoints $\mathcal{A}^{\mathcal{S}}_N \cap \mathcal{A}^{\mathcal{S}'}_N = \emptyset$, and that $\mathcal{A}_N$ is the union of the closure of sectors over all saturated nests $\mathcal{A}_N = \bigcup\limits_{\mathcal{S} \text{ saturated}} \overline{\mathcal{A}^{\mathcal{S}}_N}$.

7.3. Equivalence classes of nests: an example

In order to prove the finiteness of subtracted correlation functions $\mathcal{Z}^{\mathbf{R}}{}_N^{(M)}$ in (6.22) when $\epsilon = 0$, we shall proceed in a way similar to what was done in section 5, by decomposing the domain of integration over positions into *extended* Hepp sectors and prove that the integration of $\mathbf{R}[I_{\mathcal{G}}(x_i, X_a)]$ inside each extended Hepp sector yields a finite result.

We have seen that UV divergences arise generally when successive subsets of points coalesce. Inside the Hepp sector $\mathcal{H}^{\mathcal{S}}$, these successions must be compatible with the nested structure of $\mathcal{S}$. From (7.12) the subtracted integrand is a sum of contributions associated with (rooted) nests $\mathcal{N}_{\oplus}$, and many contributions (for different nests) give the same divergences inside $\mathcal{H}^{\mathcal{S}}$. The general strategy to prove that the subtracted integrand $\mathbf{R}[I_{\mathcal{G}}(x_i, X_a)]$ is convergent inside the sector $\mathcal{H}^{\mathcal{S}}$ is to regroup the nests giving the same UV divergences into equivalence classes, and to show that all divergences cancel within each equivalence class.

Let us first consider the simple example of a sector associated with a saturated nest $\mathcal{S}$ such that, at some level $\imath_0$, the diagram $R \equiv R^{\imath_0}$ has one and only one connected component $\mathcal{R}$ with $|\mathcal{R}| > 1$ and let us focus on the behavior of the subtracted integrand when the points of $\mathcal{R}$ coalesce. More precisely, let us consider the contribution in $\mathbf{R}$ of a rooted nest $\mathcal{N}_{\oplus}$ with one single rooted diagram $T_{\oplus}$ where $T_{\oplus}$ also has one and only one element $(\mathcal{T}, w)$ with $|\mathcal{T}| > 1$ (notice that the nest $\mathcal{N}_{\oplus}$ is automatically compatible). The corresponding contribution is (up to a factor $\frac{-1}{|\mathcal{T}|}$):

$$\mathfrak{T}_{(\mathcal{T},w)} I_{\mathcal{G}}(x_i, X_a) = I_{\mathcal{T}}(x_i) I_{\mathcal{G}/_w\mathcal{T}}(x_i, X_a) \tag{7.21}$$

where we used as before in (6.19) the short-hand notation $\mathcal{G}/_w\mathcal{T} \equiv \mathcal{G} \setminus (\mathcal{T} \setminus \{w\})$ which simply corresponds to replacing $\mathcal{T}$ in $\mathcal{G}$ by its single vertex w. We now ask which are the nests whose contribution leads to the same UV behavior when the points of $\mathcal{R}$ coalesce, that is when the positions x_i for $i \in \mathcal{R}$ tend altogether to an arbitrary position x_0: we shall denote this limit by $\mathcal{R} \to 0$. In this limit, the first term $I_{\mathcal{T}}(x_i)$ in the r.h.s. of Eq. (7.21) factorizes into $I_{\mathcal{R}\cap\mathcal{T}}(x_i)\, I_{\mathcal{T}/_o(\mathcal{R}\cap\mathcal{T})}(x_i)$, where the notation "$/_o$" means that the vertices of $\mathcal{R} \cap \mathcal{T}$ have been replaced by a single contraction vertex 0 with position x_0. The factorization of the second term $I_{\mathcal{G}/_w\mathcal{T}}(x_i, X_a)$ depends on whether or not the point w belongs to $\mathcal{R}$.

- case (a): $w \in \mathcal{R}$

If $w \in \mathcal{R}$, then we get $I_{\mathcal{G}/_w\mathcal{T}}(x_i, X_a) \to I_{\mathcal{R}/_o(\mathcal{R}\cap\mathcal{T})}(x_i)\, I_{\mathcal{G}/_o(\mathcal{R}\cup\mathcal{T})}(x_i, X_a)$. The contribution of $T_\oplus$ (7.21) thus behaves as:

$$\mathfrak{T}_{(\mathcal{T},w)} I_{\mathcal{G}}(x_i, X_a) \overset{\mathcal{R}\to 0}{\sim} I_{\mathcal{R}\cap\mathcal{T}}(x_i)\, I_{\mathcal{T}/_o(\mathcal{R}\cap\mathcal{T})}(x_i)\, I_{(\mathcal{R}\cup\mathcal{T})/_o\mathcal{T}}(x_i)\, I_{\mathcal{G}/_o(\mathcal{R}\cup\mathcal{T})}(x_i, X_a) \tag{7.22}$$

, where we used the fact that $\mathcal{R}/_o(\mathcal{R}\cap\mathcal{T}) = (\mathcal{R}\cup\mathcal{T})/_o\mathcal{T}$. In view of (7.22), let us now consider the product of Taylor operators associated with the larger rooted nest $\tilde{\mathcal{N}}_\oplus$ defined as:

$$\tilde{\mathcal{N}}_\oplus = \Big\{\big\{(\mathcal{R}\cap\mathcal{T}, w)\big\}, \big\{(\mathcal{T}, w)\big\}, \big\{(\mathcal{R}\cup\mathcal{T}, \bullet)\big\}\Big\} \tag{7.23}$$

with " $\bullet$ " standing for an arbitrary compatible root[12] [13]. This new nest can be seen as resulting from the superposition of the two nests $\mathcal{N}_\oplus$ and $\mathcal{S}$ at level $\imath_0$. Applying the corresponding three $\mathfrak{T}$ on $I_{\mathcal{G}}$ one obtains

$$\prod_{\tilde{T}_\oplus\in\tilde{\mathcal{N}}_\oplus} \mathfrak{T}_{\tilde{T}_\oplus}\big[I_{\mathcal{G}}(x_i, X_a)\big] \propto I_{\mathcal{R}\cap\mathcal{T}}(x_i)\, I_{\mathcal{T}/_w(\mathcal{R}\cap\mathcal{T})}(x_i)\, I_{(\mathcal{R}\cup\mathcal{T})/_w\mathcal{T}}(x_i)\, I_{\mathcal{G}/_\bullet(\mathcal{R}\cup\mathcal{T})}(x_i, X_a). \tag{7.24}$$

In the same limit when all points in $\mathcal{R}$ coalesce to the single point 0, w and the compatible root $\bullet$ are replaced by 0 since they both belong to $\mathcal{R}$, and (7.24) is equal to the r.h.s. of (7.22).

- case (b): $w \notin \mathcal{R}$

If $w \notin \mathcal{R}$, then we get $I_{\mathcal{G}/_w\mathcal{T}}(x_i, X_a) \to I_{(\mathcal{R}\setminus\mathcal{T})}(x_i)\, I_{(\mathcal{G}/_w\mathcal{T})/_o(\mathcal{R}\setminus\mathcal{T})}(x_i, X_a)$ and the contribution of $T_\oplus$ (7.21) behaves as:

$$\mathfrak{T}_{(\mathcal{T},w)} I_{\mathcal{G}}(x_i, X_a) \overset{\mathcal{R}\to 0}{\sim} I_{\mathcal{R}\cap\mathcal{T}}(x_i)\, I_{\mathcal{T}/_o(\mathcal{R}\cap\mathcal{T})}(x_i)\, I_{\mathcal{R}\setminus\mathcal{T}}(x_i)\, I_{(\mathcal{G}/_w\mathcal{T})/_o(\mathcal{R}\setminus\mathcal{T})}(x_i, X_a). \tag{7.25}$$

The larger rooted nest $\tilde{\mathcal{N}}_\oplus$ which gives a similar contribution when $\mathcal{R}\to 0$ is now defined as:

$$\tilde{\mathcal{N}}_\oplus = \Big\{\big\{(\mathcal{R}\cap\mathcal{T}, \bullet)\big\}, \big\{(\mathcal{T}, w)\big\}, \big\{(\mathcal{R}\setminus\mathcal{T}, \bullet), (\mathcal{T}, w)\big\}\Big\}. \tag{7.26}$$

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[12] This root is either w or some vertex of $\mathcal{R}\setminus\mathcal{T}$.

[13] We use here the convention that a diagram is explicited by keeping only each of its connected components having more that one element. For instance, $\big\{(\mathcal{T}, w)\big\}$ is a short-hand notation for $(G_\odot \vee \{\mathcal{T}\}, \omega)$ which means that the diagram must be completed by the set of all remaining isolated points not already in $\mathcal{T}$, while ω consists of the root w plus these isolated points. Similarly, Eq. (7.23) is a short-hand notation for $\tilde{\mathcal{N}}_\oplus = \Big\{(G_\odot, G_\odot), (G_\odot \vee \big\{\mathcal{R}\cap\mathcal{T}\big\}, \omega), (G_\odot \vee \big\{\mathcal{T}\big\}, \omega), (G_\odot \vee \big\{\mathcal{R}\cup\mathcal{T}\big\}, \bullet)\Big\}$.

Fig. 19: *Schematic picture of the rooted nest $\tilde{\mathcal{N}}_{\oplus}$ when the root w of $\mathcal{T}$ (a) belongs to $\mathcal{R}$, or (b) does not belong to $\mathcal{R}$.*

Notice that the largest element of $\tilde{\mathcal{N}}_{\oplus}$ is now a diagram with *two* connected components $\mathcal{R}\backslash\mathcal{T}$ and $\mathcal{T}$.

The two cases (a) and (b) can be unified in a single formula. If we denote $T_{\oplus}$ by (T,ω) where $\omega = \mathrm{root}(T_{\oplus}) = \big\{\{w\}\big\}$, the nest $\tilde{\mathcal{N}}_{\oplus}$ can be written in both cases as:

$$\tilde{\mathcal{N}}_{\oplus} = \Big\{(R \wedge T, \bullet), (T,\omega), (R \vee_{\omega} T, \bullet)\Big\} \tag{7.27}$$

where we introduce the union operation $\vee_{\omega}$ of an unrooted diagram R and a rooted diagram (T,ω)

$$R \vee_{\omega} T \equiv \Big[R \setminus (T \setminus \omega)\Big] \vee T = \Big[R \setminus \big\{\mathrm{comp}(T_{\oplus}) \setminus \mathrm{root}(T_{\oplus})\big\}\Big] \vee \mathrm{comp}(T_{\oplus}) \tag{7.28}$$

where "$\setminus$" is the subtraction operation acting on diagrams as defined in definition 7.5 in section 7 . The result of this operation is an unrooted diagram equal to $\{\mathcal{R} \cup \mathcal{T}\}$ if the

root w of T belongs to the connected component $\mathcal{R}$ of R, and equal to $\{(\mathcal{R}\backslash\mathcal{T},\mathcal{T})\}$ if w does not belong to $\mathcal{R}$:

$$R \vee_\omega T = \begin{cases} \{\mathcal{R}\cup\mathcal{T}\} & \text{if} \quad w\in\mathcal{R} \\ \{\mathcal{R}\setminus\mathcal{T},\mathcal{T}\} & \text{if} \quad w\notin\mathcal{R}. \end{cases} \tag{7.29}$$

The operation $R \vee_\omega T$ thus consists in a *fusion* operation of $\mathcal{R}$ and $\mathcal{T}$ into $\mathcal{R}\cup\mathcal{T}$, followed by a *cutting out* of $\mathcal{T}$ from $\mathcal{R}\cup\mathcal{T}$ if the root w is not shared by $\mathcal{R}$.

Fig. 20: *The unrooted complete diagram $R \vee_\omega T$ (thick full lines) obtained from the unrooted complete diagram R (dashed lines) and the complete rooted diagram T (thin full lines). The diagram $R \vee_\omega T$ is obtained by fusing each connected component of T to the connected component of R to which its root belongs, and cutting it out from all the other connected components of R.*

The above expression for $R \vee_\omega T$ can be applied to the more general case when $R = \{\mathcal{R}^i\}$ and $T_\oplus = (T,\omega) = (\{\mathcal{T}_j\},\{w_j\})$ have more that one connected component, with the result that each connected component $\mathcal{T}_j$ of T is fused to the connected component $\mathcal{R}^i$ of R which contains its root w_j, and cut out from all the other connected components of R which it intersects (see fig. 20). Note that the operation $\vee_\omega$ crucially depends on the position of the roots of the diagram T on the right with respect to the connected components of the diagram R on the left, but that these roots are not retained as roots of the resulting diagram $R \vee_\omega T$ which *by definition* is unrooted. The product of Taylor operators associated with the nest $\tilde{\mathcal{N}}_\oplus$ as given by (7.27) still corresponds in this case to the combined result of the Taylor operation $\mathfrak{T}_{T_\oplus}$ followed by the coalescence of the Card(R) connected components of R toward arbitrary points.

Finally, we return to the original question of finding the nests $\mathcal{N}'_\oplus$ which give the same UV behavior as $T_\oplus$ when components of R coalesce. These are the rooted nests

which build the same factorized integrand (7.22) or (7.25) (possibly generalized to several connected components). They are characterized by $\mathcal{N}_\oplus \subset \mathcal{N}'_\oplus \subset \tilde{\mathcal{N}}_\oplus$. We therefore get the four nests:

$$\begin{aligned}
\mathcal{N}_\oplus &= \Big\{(T,\omega)\Big\} \\
\mathcal{N}_{\oplus_2} &= \Big\{(R\wedge T,\bullet),(T,\omega)\Big\} \\
\mathcal{N}_{\oplus_3} &= \Big\{(T,\omega),(R\vee_\omega T,\bullet)\Big\} \\
\tilde{\mathcal{N}}_\oplus &= \Big\{(R\wedge T,\bullet),(T,\omega),(R\vee_\omega T,\bullet)\Big\}.
\end{aligned} \tag{7.30}$$

One can check (see Appendix D) that the (-1) and symmetry factors associated with these four nests sum up to give zero exactly (this includes a sum over the unspecified compatible roots $\bullet$). As a consequence, the divergences induced in the contributions of the four nests above by the coalescence of the points in the subset $\mathcal{R}$ cancel exactly. This property can be generalized to nests $\mathcal{N}_\oplus$ with an arbitrary number of diagrams as well as to successive coalescences associated with a saturated nest $\mathcal{S}$. Indeed, from the nest $\mathcal{N}_\oplus$, we can build a family of nests $\mathcal{N}'_\oplus$ giving the same divergences when points coalesce successively according to the nested structure of $\mathcal{S}$; we then can check that these divergences cancel exactly within the obtained family. The details of this construction will be discussed in the next section.

7.4. Equivalence classes of nests: general construction

In this section, we present a general procedure for classifying nests according to the diverging behavior of the associated counterterm in a given sector. Our construction is inspired by a construction by Bergère and Lam in [23] in the context of local field theories in the Schwinger representation. Extensive modifications are however necessary in order to make this construction applicable in our context.

We denote by $\mathcal{S} = \{R^0, R^1 \ldots, R^{N-1}\}$ a saturated nest of $\mathcal{G}$, which will be kept fixed throughout this section. We are going to regroup all rooted nests into equivalence classes, associated with $\mathcal{S}$.

$\diamond$ *Tableau construction*

From now on and until the end of the article, the only rooted nests which we shall consider will be *compatibly rooted nests*.

Let us thus consider an arbitrary compatibly rooted nest $\mathcal{N}_\oplus = \{T_{J\oplus}; J = 0, \ldots, T\}$ where $T_{J\oplus} = (T_J, \omega_J)$. For this compatibly rooted nest, we define the (unrooted) complete diagram

$$R^I_J \equiv R^I \vee_{\omega_J} T_J \equiv \big(R^I \setminus (T_J \setminus \omega_J)\big) \vee T_J \tag{7.31}$$

and build the tableau

$$\boxed{\begin{array}{cccccccc}
T_0 & R^1_0 \wedge T_1 & R^2_0 \wedge T_1 & \ldots & R^I_0 \wedge T_1 & \ldots & R^{N-2}_0 \wedge T_1 & R^{N-1}_0 \wedge T_1 \\
T_1 & R^1_1 \wedge T_2 & R^2_1 \wedge T_2 & \ldots & R^I_1 \wedge T_2 & \ldots & R^{N-2}_1 \wedge T_2 & R^{N-1}_1 \wedge T_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
T_J & R^1_J \wedge T_{J+1} & R^2_J \wedge T_{J+1} & \ldots & R^I_J \wedge T_{J+1} & \ldots & R^{N-2}_J \wedge T_{J+1} & R^{N-1}_J \wedge T_{J+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
T_T & R^1_T \wedge T_{T+1} & R^2_T \wedge T_{T+1} & \ldots & R^I_T \wedge T_{T+1} & \ldots & R^{N-2}_T \wedge T_{T+1} & R^{N-1}_T \wedge T_{T+1}
\end{array}} \tag{7.32}$$

where by convention $T_{T+1} \equiv G = \big\{\mathcal{G}\big\}$. Notice that for $R^0 = G_\odot$, we have $R^0_J = R^0_J \wedge T_{J+1} = T_J$. Hence the first column $T_J = R^0_J \wedge T_{J+1}$ of the tableau can be seen as being build from R^0, with the same structure as the other columns. Notice also that since $R^{N-1} = G$, $R^{N-1}_J = G$ for any J, hence $R^{N-1}_J \wedge T_{J+1} = T_{J+1}$. Therefore the last element of a given line of the tableau is identical to the first element of the following line. Finally, since $R^I \prec R^{I+1}$, then $R^I_J \prec R^{I+1}_J$ and

$$R^I_J \wedge T_{J+1} \prec R^{I+1}_J \wedge T_{J+1}. \tag{7.33}$$

Therefore, reading the tableau in the natural order, *i.e.* reading successive lines from the left to the right, we get a totally nested structure, which defines an unrooted nest $\tilde{\mathcal{N}}$. This nest $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus)$ depends on both the sector nest $\mathcal{S}$ and the subtraction nest $\mathcal{N}_\oplus$. By construction, $\tilde{\mathcal{N}}$ contains all the diagrams of $\mathcal{N}$. Of course, it may happen that two successive elements of the tableau are identical (this is for instance the case for the last element of a line and the first element of the next line), hence the tableau contains redundant information.

The nest $\tilde{\mathcal{N}}$ is a generalization of the one constructed in the previous section (Eq. (7.27)). Indeed, if we consider the nest $\mathcal{N}_\oplus = \big\{(G_\odot, G_\odot), (T, \omega)\big\}$ and set $R^{I_0} = R$ at level I_0 of

the nest $\mathcal{S}$, we obtain in this case the simple tableau

$$\begin{array}{llll} T_0 = G_\odot & \dots & R_0^{\imath_0} \wedge T_1 = (R \vee_{G_\odot} G_\odot) \wedge T = R \wedge T & \dots \\ T_1 = T & \dots & R_1^{\imath_0} \wedge T_2 = (R \vee_\omega T) \wedge G = R \vee_\omega T & \dots \end{array} \tag{7.34}$$

where only columns 1 and $\imath_0$ are specified. The general construction (7.32) therefore reproduces in this simple case exactly the largest nest $\tilde{\mathcal{N}}$ (here unrooted) of (7.30) .

◇ *Reduction of the tableau*

Going back to the general case, we are now interested in finding the *smallest* rooted nest $\mathcal{N}_\oplus^0$ which, under a construction similar to (7.32), gives the same nest $\tilde{\mathcal{N}}$ (that is $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus^0) = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_\oplus)$). More precisely, we must remove from $\mathcal{N}_\oplus$ the diagrams $T_\jmath$ which are not necessary to build $\tilde{\mathcal{N}}$. Since $T_\jmath$ is involved is the construction of the two lines $\jmath - 1$ and $\jmath$, removing $T_\jmath$ from the nest $\mathcal{N}_\oplus$ amounts to replace these two lines by a single line, which will be built directly from $T_{\jmath-1}$ and $T_{\jmath+1}$. In this process, N diagrams will be lost. Therefore, removing $T_\jmath$ will be possible if the tableau contains N redundant diagrams, which happens when at least $N+1$ successive diagrams of the two lines $\jmath - 1$ and $\jmath$ are identical. This implies that there exists an $\imath_0$ such that the two vertically adjacent elements of the column $\imath_0$ *coincide* at levels $\jmath - 1$ and $\jmath$:

$$R_{\jmath-1}^{\imath_0} \wedge T_\jmath = R_\jmath^{\imath_0} \wedge T_{\jmath+1} \tag{7.35}$$

that is, on the tableau:

$$\begin{array}{cccccc} T_0 & R_0^1 \wedge T_1 & \dots & R_0^{\imath_0} \wedge T_1 & \dots & R_0^{N-1} \wedge T_1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ T_{\jmath-1} & R_{\jmath-1}^1 \wedge T_\jmath & \dots & R_{\jmath-1}^{\imath_0} \wedge T_\jmath & \dots & R_{\jmath-1}^{N-1} \wedge T_\jmath \\ & & & \| & & \\ T_\jmath & R_\jmath^1 \wedge T_{\jmath+1} & \dots & R_\jmath^{\imath_0} \wedge T_{\jmath+1} & \dots & R_\jmath^{N-1} \wedge T_{\jmath+1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ T_\imath & R_\imath^1 \wedge T_{\imath+1} & \dots & R_\imath^{\imath_0} \wedge T_{\imath+1} & \dots & R_\imath^{N-1} \wedge T_{\imath+1} \end{array} \tag{7.36}$$

Then, by the inclusion property (7.33), all the diagrams of $\tilde{\mathcal{N}}$ between $R_{\jmath-1}^{\imath_0} \wedge T_\jmath$ and $R_\jmath^{\imath_0} \wedge T_{\jmath+1}$ are identical, hence equal to $T_\jmath$ itself. We thus don't loose any information by replacing the two lines $\jmath - 1$ and $\jmath$ by the single line:

$$\begin{aligned} T_{\jmath-1}, \quad R_{\jmath-1}^1 \wedge T_\jmath, \quad \dots, \quad & R_{\jmath-1}^{\imath_0} \wedge T_\jmath \\ & = T_\jmath \\ & = R_\jmath^{\imath_0} \wedge T_{\jmath+1}, \quad \dots, \quad R_\jmath^{N-1} \wedge T_{\jmath+1}. \end{aligned} \tag{7.37}$$

The important point is that this new line is precisely the one which would have been constructed directly by (7.32), when applied to the nest

$$\mathcal{N}'_{\oplus} = (T_{0\oplus}, T_{1\oplus}, \ldots, T_{J-1\oplus}, T_{J+1\oplus}, \ldots, T_{T\oplus})$$

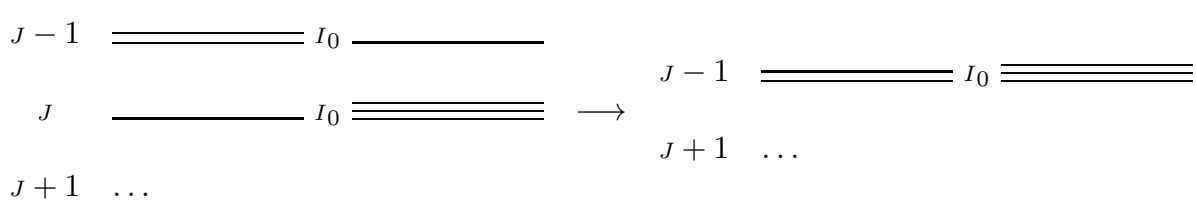
obtained from $\mathcal{N}_{\oplus}$ by removing $T_{J\oplus}$ (notice that the induced rooting of this nest remains compatible). Indeed, the construction (7.32) for $\mathcal{N}'_{\oplus}$ simply corresponds to suppressing the J–line and to substituting to the $(J-1)$–line the new line, constructed from $T_{J-1\oplus}$ and $T_{J+1\oplus}$:

$$T_{J-1}, \quad R^1_{J-1} \wedge T_{J+1}, \quad \ldots, \quad R^{I_0}_{J-1} \wedge T_{J+1}, \quad \ldots, \quad R^{N-1}_{J-1} \wedge T_{J+1}, \tag{7.38}$$

the other lines staying unchanged. It is the purpose of Appendix E to establish in detail the statement, on which all our construction will rely, that the lines (7.37) and (7.38) are actually *identical* when (7.35) is satisfied. As a consequence, the nests $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}_{\oplus})$ and $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}'_{\oplus})$ are equal. In particular, we note that T_J, while absent from $\mathcal{N}'_{\oplus}$, is still present in $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}'_{\oplus})$ since

$$T_J = R^{I_0}_{J-1} \wedge T_{J+1}. \tag{7.39}$$

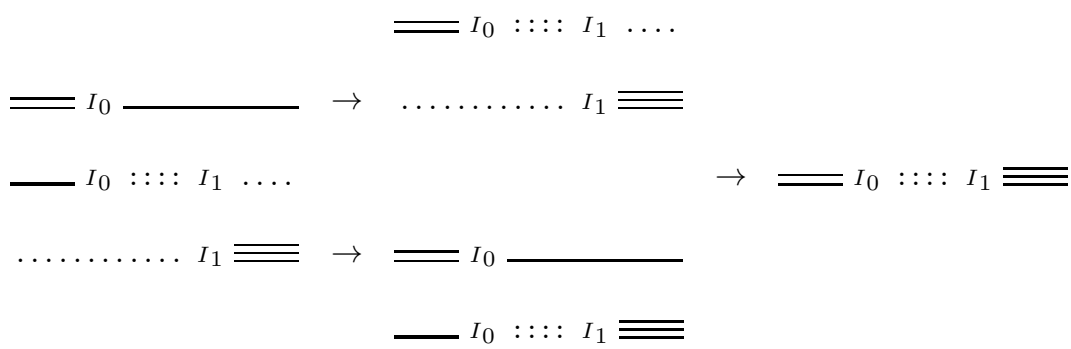
The "suppression" of line J from (7.36) when (7.35) is satisfied, consistent with the construction of $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}'_{\oplus})$, can be visualized as follows:



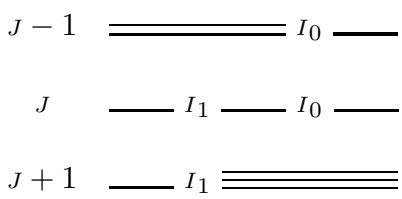
where the double and triple lines represent successively nested (in general distinct) diagrams, while the single line represents a series of identical diagrams.

We therefore have at our disposal a reduction procedure, which allows for the substitution to the nest $\mathcal{N}_{\oplus}$ of the reduced nest $\mathcal{N}'_{\oplus}$, with one diagram less, which still generates the same nest $\tilde{\mathcal{N}}$. This process can be iterated to suppress all the diagrams T_J of the original nest $\mathcal{N}$ which are such that they satisfy the coincidence property (7.35) for at least one I_0 $(1 \leq I_0 \leq N-1)$. When two successive lines possess this coincidence property, for

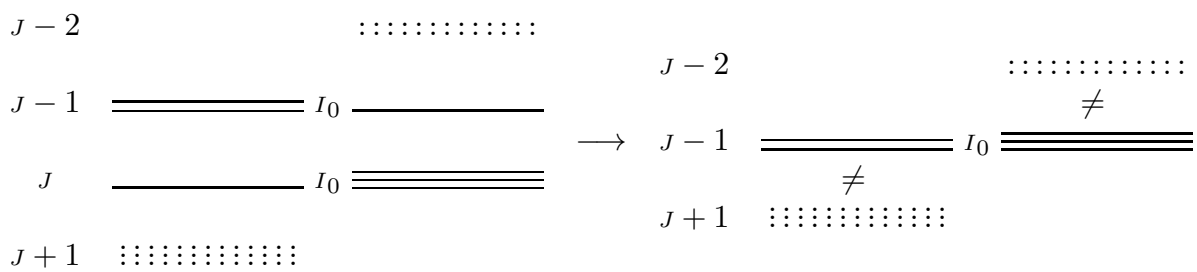
some $\imath_0$ and $\imath_1$, the reduction is *associative*, that is its result is independent of the order of the operations, as represented on the following picture:



Notice furthermore that a configuration like



which would cause obstruction to associativity, is actually forbidden since it would imply $T_\jmath = T_{\jmath+1}$, which is ruled out by definition. Notice finally that the "suppression" of a line $\jmath$ does not create new coincidences (that is coincidences which did not exist before suppression). Indeed, the only pairs of vertical neighbors which are modified by the suppression are those of the lines $\jmath-2$ and $\jmath-1$ for $\imath > \imath_0$ one the one hand, and those of the lines $\jmath$ and $\jmath+1$ for $\imath < \imath_0$ on the other hand, as can be seen on the following picture:



A new coincidence would imply $T_{\jmath-1} = T_\jmath$ in the first (upper right) case, and $T_\jmath = T_{\jmath+1}$ in the second (lower left) case, and is thus impossible. Therefore, after "suppression" of all the lines of the original nest which present a vertical coincidence with the preceding line, we end up with a tableau which no longer contains any pair of coinciding vertical neighbors. We denote by $\mathcal{N}_\oplus^0$ the nest resulting from this reduction procedure, that is the subset of $\mathcal{N}_\oplus$ made of the diagrams $T_{\jmath\oplus}$ for values of $\jmath$ corresponding to lines which *remain* after reduction.

◇ *Equivalence classes of nests*

The above reduction allows to assign to any compatibly rooted nest $\mathcal{N}_\oplus$ a unique minimal nest $\mathcal{N}^0_\oplus$, which is a subset of the original nest $\mathcal{N}_\oplus$ (and in particular whose compatible rooting is the restriction of the original rooting of $\mathcal{N}_\oplus$ to $\mathcal{N}^0$), such that $\tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}^0_\oplus) = \tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}_\oplus)$, and whose tableau (7.32) is "minimal", *i.e.* has no vertically adjacent coinciding elements[14].

We define the equivalence class $\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$ of a minimal (with respect to $\mathcal{S}$) nest $\mathcal{N}^0_\oplus$ as the set of all compatibly rooted nests $\mathcal{N}_\oplus$ which lead by reduction of their $\mathcal{S}$-tableau to that minimal nest $\mathcal{N}^0_\oplus$.

$$\mathcal{N}_\oplus \in \mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus) \quad \Longleftrightarrow \quad \mathcal{N}_\oplus \stackrel{\text{tableau}}{\longrightarrow} \tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}_\oplus) \stackrel{\text{reduction}}{\longrightarrow} \mathcal{N}^0_\oplus \ .$$

Of course, if $\mathcal{N}^0_\oplus$ is minimal with respect to $\mathcal{S}$, one has $\mathcal{N}^0_\oplus \in \mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$. For any $\mathcal{N}_\oplus \in \mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$, one has $\tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}_\oplus) = \tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}^0_\oplus)$.

We have the following characterization, for any compatibly rooted nest $\mathcal{N}_\oplus$ (with $\mathcal{N}$ the corresponding unrooted nest):

THEOREM *Characterization of* $\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$

$$\mathcal{N}_\oplus \in \mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus) \qquad \Longleftrightarrow \qquad \text{(a) } \mathcal{N}^0_\oplus \subset \mathcal{N}_\oplus \quad \text{and} \quad \text{(b) } \mathcal{N} \subset \tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}^0_\oplus). \tag{7.40}$$

A nest of the equivalence class $\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$ is thus constituted of *all* the diagrams of $\mathcal{N}^0$ plus *some* of the diagrams of $\tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}^0_\oplus)$ not in $\mathcal{N}^0$. Its rooting is constrained to be both compatible and such that its restriction to $\mathcal{N}^0$ is the rooting of $\mathcal{N}^0_\oplus$. Conversely, one builds all the elements of $\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$ by completing $\mathcal{N}^0_\oplus$ by an arbitrary number of diagrams of $\tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}^0_\oplus) \setminus \mathcal{N}^0$ (that is diagrams of $\tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}^0_\oplus)$ not in $\mathcal{N}^0$), and assigning to these extra elements any roots compatible with the roots of $\mathcal{N}^0_\oplus$.

The direct implication ($\Longrightarrow$) is immediate since

- the reduced rooted nest is always a subset of the original rooted nest, hence (a);
- any diagram of T_J of $\mathcal{N}_\oplus$ belong to $\tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}_\oplus)$ and the reduction process is defined so as to leave $\tilde{\mathcal{N}}$ invariant. Thus $T_J \in \tilde{\mathcal{N}}(\mathcal{S},\mathcal{N}^0_\oplus)$, hence (b).

The reverse implication ($\Longleftarrow$) is not immediate and is proven in Appendix F.

[14] In general, this tableau still contains series of identical successive elements, but not more than that N successive elements can be identical.

Notice finally that the diagram $G = \{\mathcal{G}\}$ is always a diagram of $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}^0_\oplus)$ since the last element ($\imath = N-1$) of the last line ($\jmath = \tau$) of the tableau of any nest is always equal to G. As a consequence, G is never a diagram of $\mathcal{N}^0_\oplus$ since it can be rebuilt from $\mathcal{N}^0_\oplus$ by the tableau construction. Actually, if a nest contains the diagram G, the line of its tableau built from G has all its elements equal to G, while the preceding line has its last element equal to G; this leads to the coincidence property for this two lines for $\imath_0 = N-1$, indicating that G is to be suppressed in the construction of $\mathcal{N}^0_\oplus$. Therefore, for any minimal nest $\mathcal{N}^0_\oplus$, one has $G \in \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}^0_\oplus) \setminus \mathcal{N}^0_\oplus$.

7.5. Factorization of the R operator inside an equivalence class

As we have seen before, the reason for classifying nests into equivalence classes was to regroup nests whose diverging contributions in a given sector $\mathcal{S}$ in the $\mathbf{R}$ operator (7.12) cancel exactly. Given a sector nest $\mathcal{S}$, it is therefore natural to rewrite the $\mathbf{R}$ operator, which is a sum over all compatibly rooted nests, as a sum of reduced operators $\mathbf{R}_{\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)}$, each of them involving all the nests in the equivalence class $\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$ of a minimal (w.r.t. $\mathcal{S}$) nest $\mathcal{N}^0_\oplus$. This reads:

$$\mathbf{R} = \sum_{\substack{\mathcal{N}^0_\oplus \text{ minimal} \\ \text{w.r.t.}\mathcal{S}}} \mathbf{R}_{\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)} \ , \qquad \mathbf{R}_{\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)} = - \sum_{\mathcal{N}_\oplus \in \mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)} W(\mathcal{N}_\oplus) \prod_{T_\oplus \in \mathcal{N}_\oplus} \Big(- \mathfrak{T}_{T_\oplus} \Big). \quad (7.41)$$

Each operator $R_{\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)}$ can then be rewritten as a sum of *factorized* contributions associated with different rootings of the elements of the equivalence class, as explained now.

We will need a lemma about partial sums over compatible rootings of nests. Let us consider a nest $\mathcal{M} = \{T_\jmath; \jmath = 1, \ldots, \tau\}$. We denote by $\oplus_\mathcal{M}$ a *compatible rooting* of $\mathcal{M}$, that is simply the specification for each diagram $T_\jmath$ of $\mathcal{M}$ of a root diagram $\omega_\jmath$ such that $\mathcal{M}_{\oplus_\mathcal{M}} \equiv \{(T_\jmath, \omega_\jmath); \jmath = 1, \ldots \tau\}$ is a compatibly rooted nest.

Lemma:

Given a compatibly rooted nest $\mathcal{N}_\oplus$ and an unrooted nest $\mathcal{M}$ such that $\mathcal{N} \subset \mathcal{M}$ (that is all the diagrams of $\mathcal{N}$ are diagrams of $\mathcal{M}$), we can consider all the compatible rootings $\oplus_\mathcal{M}$ of $\mathcal{M}$ such that $\mathcal{N}_\oplus \subset \mathcal{M}_{\oplus_\mathcal{M}}$, that is the compatible rootings of $\mathcal{M}$ whose restriction to $\mathcal{N}$ is the rooting in $\mathcal{N}_\oplus$; we then have the useful sum rule for the weights (7.13):

$$\boxed{\sum_{\oplus_\mathcal{M}:\ \mathcal{N}_\oplus \subset \mathcal{M}_{\oplus_\mathcal{M}}} W(\mathcal{M}_{\oplus_\mathcal{M}}) = W(\mathcal{N}_\oplus)} \quad (7.42)$$

This lemma is proven in Appendix G.

We can now use this property in the case of an arbitrary nest $\mathcal{N}_\oplus \in \mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$ if we choose:

$$\mathcal{M} = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}^0_\oplus) \tag{7.43}$$

since, from (7.40), we have $\mathcal{N} \subset \mathcal{M}$. Inserting (7.42) in the formula (7.41) for $\mathbf{R}_{\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)}$, we get:

$$\begin{aligned}
\mathbf{R}_{\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)} &= - \sum_{\mathcal{N}_\oplus \in \mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)} \sum_{\substack{\oplus_\mathcal{M}: \\ \mathcal{N}_\oplus \subset \mathcal{M}_\oplus}} W(\mathcal{M}_\oplus) \prod_{T_\oplus \in \mathcal{N}_\oplus} \big(- \mathfrak{T}_{T_\oplus} \big) \\
&= - \sum_{\substack{\oplus_\mathcal{M}: \\ \mathcal{N}^0_\oplus \subset \mathcal{M}_\oplus}} W(\mathcal{M}_\oplus) \sum_{\substack{\mathcal{N}_\oplus: \\ \mathcal{N}^0_\oplus \subset \mathcal{N}_\oplus \subset \mathcal{M}_\oplus}} \prod_{T_\oplus \in \mathcal{N}_\oplus} \big(- \mathfrak{T}_{T_\oplus} \big) \\
&= - \sum_{\substack{\oplus_\mathcal{M}: \\ \mathcal{N}^0_\oplus \subset \mathcal{M}_\oplus}} W(\mathcal{M}_\oplus) \prod_{T^0_\oplus \in \mathcal{N}^0_\oplus} \big(- \mathfrak{T}_{T^0_\oplus} \big) \prod_{T_\oplus \in (\mathcal{M}_\oplus \setminus \mathcal{N}^0_\oplus)} \big(1 - \mathfrak{T}_{T_\oplus} \big)
\end{aligned} \tag{7.44}$$

where $\mathcal{M}_\oplus$ stands here for $\mathcal{M}_{\oplus_\mathcal{M}}$. In the second equation, we used the characterization (7.40) of $\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$. The sum rule (7.42) allows us to reconstruct all possible rootings of the nests $\mathcal{N}$ in $\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$ with the appropriate weight, by first fixing the roots of $\mathcal{M}_\oplus$ by a compatible extension of the roots of $\mathcal{N}^0_\oplus$, and then restricting these roots of $\mathcal{M}_\oplus$ to all intermediate subnests $\mathcal{N}$ between $\mathcal{N}^0$ and $\mathcal{M}$ (notice that a given rooting of such a nest $\mathcal{N}$ can come from different rootings of $\mathcal{M}$). In the last equation, we used the fact that the set of rooted nests $\mathcal{N}_\oplus$ such that $\mathcal{N}^0_\oplus \subset \mathcal{N}_\oplus \subset \mathcal{M}_\oplus$ is built by taking necessarily, on the one hand all the diagrams $T^0_\oplus$ of the minimal nest $\mathcal{N}^0_\oplus$ and, for each diagram $T_\oplus$ of $\mathcal{M}_\oplus \setminus \mathcal{N}^0_\oplus$ on the other hand, deciding whether to take it or not, hence choosing 1 or $-\mathfrak{T}_{T_\oplus}$ in the expansion of the product of Taylor operators.

Notice finally that the compatibly rooted nests $\mathcal{M}_\oplus$ involved in (7.44) can actually be characterized independently of the minimal nest $\mathcal{N}^0_\oplus$ from which they are built, by the property:

$$\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{M}_\oplus) = \mathcal{M}. \tag{7.45}$$

A compatibly rooted nest satisfying (7.45) will be called *maximal* with respect to $\mathcal{S}$. With this definition, the equations (7.41) and (7.44) can be replaced by the single equation:

$$\boxed{\mathbf{R} = \sum_{\substack{\mathcal{M}_\oplus \\ \text{maximal} \\ \text{w.r.t.}\mathcal{S}}} W(\mathcal{M}_\oplus)\, \mathbf{R}_{\mathcal{M}_\oplus}} \tag{7.46}$$

with

$$\boxed{\mathbf{R}_{\mathcal{M}_\oplus} = -\prod_{T^0_\oplus \in \mathcal{N}^0_\oplus} \Big(-\mathfrak{T}_{T^0_\oplus} \Big) \prod_{T_\oplus \in (\mathcal{M}_\oplus \setminus \mathcal{N}^0_\oplus)} \Big(1 - \mathfrak{T}_{T_\oplus} \Big)} \tag{7.47}$$

where $\mathcal{N}^0_\oplus$ is now the minimal nest obtained by reducing the tableau of the maximal nest $\mathcal{M}_\oplus$.

8. Proof of UV convergence

We are now in a position to prove the finiteness of subtracted correlation functions $\mathcal{Z}^{\mathbf{R}^{(M)}_N}$ in (6.22) when $\epsilon = 0$. Our strategy is the following:

(I) First we partition the domain of integration over positions into *extended* Hepp sectors (as defined in section 7.2), each of them being characterized by a saturated nest $\mathcal{S}$.

(II) In each sector $\mathcal{S}$, we reorganize the $\mathbf{R}$ operator by use of (7.46) as a sum of operators $\mathbf{R}_{\mathcal{M}_\oplus}$ associated with the different nests $\mathcal{M}_\oplus$ maximal with respect to $\mathcal{S}$.

(III) At the end, one can write $\mathcal{Z}^{\mathbf{R}^{(M)}_N}$ as

$$\mathcal{Z}^{\mathbf{R}^{(M)}_N}(X_a) = \sum_{\mathcal{S}} \sum_{\substack{\mathcal{M}_\oplus \\ \text{maximal} \\ \text{w.r.t.}\mathcal{S}}} W(\mathcal{M}_\oplus) \int_{\mathcal{H}^{\mathcal{S}}} \prod_{i \in \mathcal{G}} d^D x_i\, \mathbf{R}_{\mathcal{M}_\oplus}\, [I_{\mathcal{G}}(x_i, X_a)]. \tag{8.1}$$

It is therefore sufficient to prove the finiteness of the integral:

$$\int_{\mathcal{H}^{\mathcal{S}}} \prod_{i \in \mathcal{G}} d^D x_i\, \mathbf{R}_{\mathcal{M}_\oplus}\, [I_{\mathcal{G}}(x_i, X_a)] \tag{8.2}$$

where we integrate over the domain $\mathcal{H}^{\mathcal{S}}$ defined by (7.15) [15] with the measure (3.24), and where $\mathcal{M}_\oplus$ is any nest maximal with respect to $\mathcal{S}$.

[15] More precisely, we integrate over the $x_i's$ such that the y_i's defined by $y_i = x_{i+1} - x_1$ are in $\mathcal{H}^{\mathcal{S}}$, since a Hepp sector is actually defined in terms of relative positions.

(IV) Using the factorized form (7.47) for $\mathbf{R}_{\mathcal{M}_\oplus}$, we first apply the Taylor operators $\mathfrak{T}_{T^0_J}$ associated with diagrams of the minimal nest $\mathcal{N}^0_\oplus$. This results in factorizing $I_{\mathcal{G}}(x_i, X_a)$ into a product of amplitudes $I_{\tilde{T}} = \prod_{\tilde{\mathcal{T}} \in \tilde{T}} I_{\tilde{\mathcal{T}}}$ for suitable reduced diagrams $\tilde{T}$ made of subsets $\tilde{\mathcal{T}}$ of $\mathcal{G}$.

(V) We show that the product of the remaining $(1 - \mathfrak{T}_{T_\oplus})$ operators acts independently on each subdiagram $\tilde{T}$, leading to a subtracted integrand for $\tilde{T}$.

(VI) We show that this subtracted integrand, once integrated over points in the Hepp sector $\mathcal{H}^{\mathcal{S}}$, yields a finite result when $\epsilon = 0$.

Points (I), (II) and (III) have been already discussed in section 7. We now show points (IV), (V) and (VI) precisely.

8.1. Factorization of $I_{\mathcal{G}}(x_i, X_a)$

In order to precise the action of $\mathbf{R}_{\mathcal{M}_\oplus}$ on $I_{\mathcal{G}}(x_i, X_a)$, let us first have a closer look at the tableau $\mathcal{M} = \tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}^0_\oplus)$. We denote by T^0_J, $J = 0, \ldots, T$ the diagrams of $\mathcal{N}^0$, and by T^I_J

$$T^I_J = (R^I \vee_{\omega^0_J} T^0_J) \wedge T^0_{J+1} \qquad 0 \leq I \leq N-1; \quad J = 0, \ldots, T \tag{8.3}$$

the diagrams of $\mathcal{M}$. By convention, we have set $T^0_{T+1} = G$. Starting from the factorized form (7.47) for $\mathbf{R}_{\mathcal{M}_\oplus}$ (and using the fact that the $\mathfrak{T}$'s commute), we first apply the Taylor operators $\mathfrak{T}_{T^0_{J\oplus}}$ associated with diagrams of the minimal nest $\mathcal{N}^0_\oplus$. This results in factorizing $I_{\mathcal{G}}(x_i, X_a)$ into:

$$\prod_{T^0_{J\oplus} \in \mathcal{N}^0_\oplus} \left(\mathfrak{T}_{T^0_{J\oplus}}\right) I_{\mathcal{G}}(x_i, X_a) = I_{\tilde{T}_{T+1}}(x_i, X_a) \prod_{J=1}^{T} I_{\tilde{T}_J}(x_i) \tag{8.4}$$

where

$$\tilde{T}_J \equiv T^0_J /_{\omega^0_{J-1}} T^0_{J-1} \tag{8.5}$$

is the (uncomplete) diagram obtained from T^0_J by replacing by its root each component of the preceding diagram T^0_{J-1} in $\mathcal{N}^0_\oplus$. Each $\tilde{T}_J$ is made of $\mathrm{Card}(T^0_J)$ connected components $\tilde{\mathcal{T}}_{J,j}$ and in (8.4) the amplitude for $\tilde{T}_J$ is by definition equal to

$$I_{\tilde{T}_J} \equiv \prod_{j=1}^{\mathrm{Card}(T^0_J)} I_{\tilde{\mathcal{T}}_{J,j}}. \tag{8.6}$$

By convention, if some connected component is reduced to one single vertex, the corresponding amplitude is 1. Eq. (8.4) establishes point (IV).

Similarly to (8.5), it is convenient to define $\tilde{T}^{\imath}_{\jmath-1}$ as the (uncomplete) diagram obtained by reducing in some diagram of the tableau $T^{\imath}_{\jmath-1}$ the preceding minimal diagram $T^{0}_{\jmath-1}$ to its root diagram $\omega^{0}_{\jmath-1}$:

$$\tilde{T}^{\imath}_{\jmath-1} \equiv T^{\imath}_{\jmath-1} /_{\omega^{0}_{\jmath-1}} T^{0}_{\jmath-1}. \tag{8.7}$$

Notice that $\tilde{T}^{0}_{\jmath-1} = \omega^{0}_{\jmath-1}$ and that $\tilde{T}^{N-1}_{\jmath-1} = \tilde{T}_{\jmath}$.

From (8.3), the connected components of the diagram $\tilde{T}^{\imath}_{\jmath-1}$ are made out of the intersection of the connected components $\tilde{\mathcal{T}}_{\jmath,j}$ of $\tilde{T}_{\jmath}$ and of the connected components $\mathcal{R}^{\imath,i}$ of $R^{\imath}$

$$\tilde{\mathcal{T}}^{\imath,i}_{\jmath-1,j} \;\equiv\; \mathcal{R}^{\imath,i} \cap \tilde{\mathcal{T}}_{\jmath,j} \tag{8.8}$$

Furthermore, from the compatibility requirement for $\mathcal{M}_{\oplus}$, the root in $\mathcal{M}_{\oplus}$ of any connected component of the diagram $T^{\imath}_{\jmath-1}$ automatically belongs to the corresponding reduced connected component of the reduced diagram $\tilde{T}^{\imath}_{\jmath-1}$. Therefore, the rooting of $\mathcal{M}_{\oplus}$ naturally induces a rooting for the diagrams $\tilde{T}^{\imath}_{\jmath-1}$. We denote by $w^{\imath,i}_{\jmath-1,j}$ the root of $\tilde{\mathcal{T}}^{\imath,i}_{\jmath-1,j}$ and by $\tilde{T}^{\imath}_{\jmath-1\oplus}$ the set of all $(\tilde{\mathcal{T}}^{\imath,i}_{\jmath-1,j}, w^{\imath,i}_{\jmath-1,j})$ for varying i and j.

Let us for a while concentrate on what happens inside some given subset $\tilde{\mathcal{T}}_{\jmath,j}$ which we shall assume to have at least two vertices ($\mathrm{Card}(\tilde{\mathcal{T}}_{\jmath,j}) > 1$). We can consider the family of different (and non empty) rooted subsets $(\tilde{\mathcal{T}}^{\imath,i}_{\jmath-1,j}, w^{\imath,i}_{\jmath-1,j})$ for all $i = 1, \ldots, \mathrm{Card}(R^{\imath})$ (with $\jmath$ and j fixed) as a complete rooted diagram $\tilde{T}^{\imath}_{\jmath-1,j\oplus}$ of the subset $\tilde{\mathcal{T}}_{\jmath,j}$ in which we are now working. From (8.8), this is nothing but the restriction of the diagram $R^{\imath}$ to this subset $\tilde{\mathcal{T}}_{\jmath,j}$, together with a set of roots. The family of distinct $\tilde{T}^{\imath}_{\jmath-1,j\oplus}$ for varying $\imath$ forms a *compatibly rooted and saturated nest*, $\mathcal{S}_{\jmath,j\oplus}$, of $\tilde{\mathcal{T}}_{\jmath,j}$, which is nothing but the restriction of the saturated nest $\mathcal{S}$ to $\tilde{\mathcal{T}}_{\jmath,j}$, with a given rooting. We define:

$$\overline{\mathrm{Ind}}(\jmath, j) = \left\{ \imath \geq 1 :\; \tilde{T}^{\imath}_{\jmath-1,j} \neq \tilde{T}^{\imath-1}_{\jmath-1,j} \right\} \tag{8.9}$$

as the set of indices $\imath$ (of the sector $\mathcal{S}$) such that inside $\tilde{\mathcal{T}}_{\jmath,j}$, a new element $\tilde{T}$ appears at level $\imath$ in the saturated nest $\mathcal{S}_{\jmath,j}$.

We now again consider the whole diagram $\tilde{T}_{\jmath}$ and define, in a way similar to (8.9):

$$\overline{\mathrm{Ind}}(\jmath) = \left\{ \imath \geq 1 :\; \tilde{T}^{\imath}_{\jmath-1} \neq \tilde{T}^{\imath-1}_{\jmath-1} \right\} = \left\{ \imath \geq 1 :\; T^{\imath}_{\jmath-1} \neq T^{\imath-1}_{\jmath-1} \right\} \tag{8.10}$$

as the set of $\imath$ such that a new diagram appears in the tableau at level $\imath$ between $T^0_{\jmath-1}$ and $T^0_\jmath$. Of course, if $\tilde{T}^\imath_{\jmath-1} \neq \tilde{T}^{\imath-1}_{\jmath-1}$, there exists at least one j such that $\tilde{T}^\imath_{\jmath-1,j} \neq \tilde{T}^{\imath-1}_{\jmath-1,j}$ and thus:

$$\overline{\mathrm{Ind}}(\jmath) = \bigcup_{j=1}^{\mathrm{Card}(T^0_\jmath)} \overline{\mathrm{Ind}}(\jmath, j) \tag{8.11}$$

We moreover denote by

$$\imath^{\min}(\jmath) = \min\left(\overline{\mathrm{Ind}}(\jmath)\right), \qquad \imath^{\max}(\jmath) = \max\left(\overline{\mathrm{Ind}}(\jmath)\right) \tag{8.12}$$

with the property that

$$\imath^{\max}(\jmath) = \min\left\{\imath \,:\, \tilde{T}^\imath_{\jmath-1} = \tilde{T}_\jmath\right\} = \min\left\{\imath \,:\, T^\imath_{\jmath-1} = T^0_\jmath\right\} \tag{8.13}$$

is the index $\imath$ such that $T^0_\jmath$ appears at first in the tableau We set:

$$\mathrm{Ind}(\jmath) = \overline{\mathrm{Ind}}(\jmath) \setminus \{\imath^{\max}(\jmath)\} \tag{8.14}$$

(which may be empty).

Finally, we define

$$\overline{\mathrm{Ind}} = \bigcup_{\jmath=1}^{\tau+1} \overline{\mathrm{Ind}}(\jmath). \tag{8.15}$$

With these notations, the $(1-\mathfrak{T})$ operators in (7.44) act independently on each amplitude $I_{\tilde{T}_\jmath}$. The operator $(1-\mathfrak{T}_{T^\imath_{\kappa\oplus}})$ acts on $I_{\tilde{T}_\jmath}$ only if $\kappa = \jmath - 1$, and results in this case in $(1-\mathfrak{T}_{\tilde{T}^\imath_{\jmath-1\oplus}})[I_{\tilde{T}_\jmath}]$. We thus can express $\mathbf{R}_{\mathcal{M}_\oplus}[I_\mathcal{G}]$ as a product of subtracted amplitudes for each reduced diagram $\tilde{T}_\jmath$. The subtracted amplitude for $\tilde{T}_\jmath$ is obtained by the successive action on $I_{\tilde{T}_\jmath}$ of a $(1-\mathfrak{T}_{\tilde{T}^\imath_{\jmath-1\oplus}})$ operator for each $\imath \in \mathrm{Ind}(\jmath)$. The case $\jmath = \tau+1$ is special since, since in addition to the $(1-\mathfrak{T}_{\tilde{T}^\imath_{\tau\oplus}})$ operator for each $\imath \in \mathrm{Ind}(\tau+1)$, a $(1-\mathfrak{T})$ operator is also associated with $\tilde{T}^{\imath^{\max}(\tau+1)}_{\tau\oplus}$ [16]. The factorization of $\mathbf{R}_{\mathcal{M}_\oplus}[I_\mathcal{G}]$ is then expressed in the following equation:

$$\begin{aligned}\mathbf{R}_{\mathcal{M}_\oplus}[I_\mathcal{G}(x_i, X_a)] = &\prod_{\imath\in\overline{\mathrm{Ind}}(\tau+1)} (1-\mathfrak{T}_{\tilde{T}^\imath_{\tau\oplus}}) I_{\tilde{T}_{\tau+1}}(x_i, X_a) \\ &\times \prod_{\jmath=1}^{\tau} \Big[\prod_{\imath\in\mathrm{Ind}(\jmath)} (1-\mathfrak{T}_{\tilde{T}^\imath_{\jmath-1\oplus}})[I_{\tilde{T}_\jmath}(x_i)]\Big].\end{aligned} \tag{8.16}$$

We recall that

$$\mathfrak{T}_{\tilde{T}^\imath_{\jmath-1\oplus}} = \prod_{j=1}^{\mathrm{Card}(T^0_\jmath)} \mathfrak{T}_{\tilde{T}^\imath_{\jmath-1,j\oplus}} = \prod_{j=1}^{\mathrm{Card}(T^0_\jmath)} \prod_{(\tilde{\mathcal{T}}^{\imath,i}_{\jmath-1,j}, w^{\imath,i}_{\jmath-1,j}) \in \tilde{T}^\imath_{\jmath-1,j\oplus}} \mathfrak{T}_{(\tilde{\mathcal{T}}^{\imath,i}_{\jmath-1,j}, w^{\imath,i}_{\jmath-1,j})}. \tag{8.17}$$

We have thus achieved point (V). It remains to show that the subtractions associated with the $(1-\mathfrak{T})$'s are sufficient to make (8.16) integrable in the sector $\mathcal{S}$.

[16] Notice that $\tilde{T}_{\tau+1}$ has only one connected component.

8.2. Appropriate tree variables

In section 5, in order to prove the convergence of the original (unsubtracted) integral in some Hepp sector (for $\epsilon > 0$), we found useful to express the measure in terms of tree variables for some specific tree (which defined the sector). Those tree variables are no longer adapted to deal with the subtracted amplitude $\mathbf{R}_{\mathcal{M}_\oplus}[I_{\mathcal{G}}(x_i, X_a)]$ since they do not take into account the factorization (8.16) of $\mathbf{R}_{\mathcal{M}_\oplus}[I_{\mathcal{G}}(x_i, X_a)]$. Instead, we shall look for tree variables associated with a tree that, inside each subset $\tilde{\mathcal{T}}_{\jmath,j}$, forms a subtree compatible with the sector.

The basic idea is that, since the nest $\mathcal{S}_{\jmath,j\oplus}$, which has been defined above as the restriction of the sector nest $\mathcal{S}$ to $\tilde{\mathcal{T}}_{\jmath,j}$, is both saturated in $\tilde{\mathcal{T}}_{\jmath,j}$ and rooted, it naturally defines a unique oriented ordered tree $\mathbf{T}_{\jmath,j}$ spanning the vertices of $\tilde{\mathcal{T}}_{\jmath,j}$, as discussed in section 7.2. The corresponding line vectors are naturally ordered by increasing values of $\imath$ in $\overline{\mathrm{Ind}}(\jmath, j)$ and denoted by

$$\lambda^{\imath}_{\jmath,j}; \quad \imath \in \overline{\mathrm{Ind}}(\jmath, j). \tag{8.18}$$

From the nested structure of $\mathcal{N}^0_\oplus$, we deduce that the union of the trees $\mathbf{T}_{\jmath,j}$ for varying $\jmath$ and j (including $\jmath = \tau + 1$) forms a tree of the set $\mathcal{G}$:

$$\mathbf{T} = \bigcup \mathbf{T}_{\jmath,j} = \Big(\lambda^{\imath}_{\jmath,j}; \quad \jmath = 1, \ldots, \tau + 1;\ j = 1, \ldots, \mathrm{Card}(T^0_\jmath);\ \imath \in \overline{\mathrm{Ind}}(\jmath, j)\Big) \tag{8.19}$$

(see fig. 21). In particular, this tree has $N - 1$ line vectors. We can therefore use the corresponding tree variables $\lambda^{\imath}_{\jmath,j}$ as integration variables, instead of the $N - 1$ relative positions $y_i = x_{i+1} - x_1$ in $\mathbb{R}^{N-1}$ (or $\mathbb{R}^{N+M-1}$ when M external points are present).

Notice that the tree $\mathbf{T}$ is not in general compatible with the sector $\mathcal{S}$, that is in general, $\mathcal{S}(\mathbf{T}) \neq \mathcal{S}$. Still, since $\mathcal{S}_{\jmath,j}$ is the restriction of $\mathcal{S}$ to the subset $\tilde{\mathcal{T}}_{\jmath,j}$, the subtree $\mathbf{T}_{\jmath,j}$ of $\mathbf{T}$ remains compatible with $\mathcal{S}$. By this we mean that one can find ordered trees of $\mathcal{G}$ compatible with $\mathcal{S}$, and which contain $\mathbf{T}_{\jmath,j}$ as an ordered subtree. We can therefore take advantage of the inequalities (7.19) and get the following bounds for ratios of length of $\lambda^{\imath}_{\jmath,j}$ inside the domain $\mathcal{H}^{\mathcal{S}}$:

$$\begin{aligned} &\frac{1}{\imath} \leq \frac{|\lambda^{\imath}_{\jmath,j}|}{|\lambda^{\imath}_{\jmath',j'}|} \leq \imath \\ &\frac{|\lambda^{\imath}_{\jmath,j}|}{|\lambda^{\imath'}_{\jmath',j'}|} \leq \imath \qquad \text{for} \quad \imath < \imath'. \end{aligned} \tag{8.20}$$

Fig. 21: *Appropriate tree variables. At level $\jmath$, inside a connected component $\tilde{\mathcal{T}}_{\jmath,j}$ of $\tilde{T}_\jmath$ (dashed circles), we build an oriented ordered tree with line vectors $\lambda^{\imath}_{\jmath,j}$. As shown in the framed box, this tree is built in a way similar to what was done in fig. 18 (b), now from the rooted saturated nest $\mathcal{S}_{\jmath,j\oplus}$. This nest is here made of the three diagrams $\tilde{T}^{\imath_1}_{\jmath-1,j}$, $\tilde{T}^{\imath_2}_{\jmath-1,j}$ and $\tilde{T}^{\imath_3}_{\jmath-1,j}$ (i.e. $\overline{\mathrm{Ind}}(\jmath,j) = \{\imath_1,\imath_2,\imath_3\}$) whose roots are represented by the dashed squares. At level $\jmath+1$, the connected components of $\tilde{T}_\jmath$ are fully contracted toward their roots (big black dots), which are the vertices of $\tilde{\mathcal{T}}_{\jmath+1,k}$. An oriented ordered tree with line vectors $\lambda^{\imath'}_{\jmath+1,k}$ is then built inside $\tilde{\mathcal{T}}_{\jmath+1,k}$. The trees at level $\jmath$ and $\jmath+1$ can be fused into a single oriented (but only partially ordered) larger tree contributing to (8.19) .*

This means that two λ's with the same index $\imath$ are of the same order, while the λ's with higher index $\imath' > \imath$ cannot vanish more rapidly than those with index $\imath$.

Finally, since the vectors $\lambda^{\imath}_{\jmath,j}$ defining the subtree $\mathbf{T}_{\jmath,j}$ are built from the *rooted* nest $\mathcal{S}_{\jmath,j\oplus}$, whose roots are precisely the roots $w^{\imath,i}_{\jmath-1,j}$ of the subsets $\tilde{\mathcal{T}}^{\imath,i}_{\jmath-1,j}$, the action of dilation operations (6.14)

$$\prod_i \mathcal{D}^{\rho}_{(\tilde{\mathcal{T}}^{\imath,i}_{\jmath-1,j},w^{\imath,i}_{\jmath-1,j})} \tag{8.21}$$

(for some fixed $\jmath$ and j) on the positions of the vertices of $\tilde{\mathcal{T}}_{\jmath,j}$ is exactly performed by the transformation

$$|\lambda^{\imath'}_{\jmath,j}| \to \rho|\lambda^{\imath'}_{\jmath,j}| \qquad \text{for} \qquad \imath' \leq \imath \tag{8.22}$$

on the *modules* of the λ variables.

In a way similar to what we did in section 5.1, it is natural to rewrite the vectors λ's in terms of real variables $\beta^\imath$ which measure ratios of successive modules $|\lambda|$'s, together with angular variables θ.

For definiteness, we write the elements of $\overline{\text{Ind}}$ ((8.15)) as

$$\overline{\text{Ind}} = \{\imath_1 < \imath_2 < \dots \imath_E\}. \tag{8.23}$$

For each $\imath$ in $\overline{\text{Ind}}$, we choose one of the $\lambda^{\imath}_{\jmath,j}$ of the tree $\mathbf{T}$ as a representative of all the lines which appear at level $\imath$, and denote it by $\lambda^{\imath}$. We then define the β variables as the ratios of these representatives $\lambda^{\imath}$ for successive $\imath$ in $\overline{\text{Ind}}$:

$$\begin{aligned} \beta^{\imath_k} &= \frac{|\lambda^{\imath_k}|}{|\lambda^{\imath_{k+1}}|} \qquad k = 1, \dots, E-1 \\ \beta^{\imath_E} &= |\lambda^{\imath_E}|. \end{aligned} \tag{8.24}$$

From (8.20), they satisfy

$$\beta^{\imath_k} \leq \imath_k \qquad k = 1, \dots, E-1. \tag{8.25}$$

Eq. (8.24) defines $\beta^{\imath}$ for $\imath \in \overline{\text{Ind}}$. We shall also use the convention

$$\beta^{\imath} = 1 \quad \text{if} \quad \imath \notin \overline{\text{Ind}}. \tag{8.26}$$

In order to compare $\lambda^{\imath}_{\jmath,j}$ to its representative $\lambda^{\imath}$, we define

$$\chi^{\imath}_{\jmath,j} = \frac{|\lambda^{\imath}_{\jmath,j}|}{|\lambda^{\imath}|}, \tag{8.27}$$

with of course $\chi^{\imath}_{\jmath,j} = 1$ if $\lambda^{\imath}_{\jmath,j}$ has been chosen as a representative. We thus have

$$|\lambda^{\imath}_{\jmath,j}| = \chi^{\imath}_{\jmath,j} \beta^{\imath} \beta^{\imath+1} \dots \beta^{N-1} \tag{8.28}$$

and the bounds

$$\frac{1}{\imath} \leq \chi^{\imath}_{\jmath,j} \leq \imath. \tag{8.29}$$

Finally, in addition to their moduli, the line vectors $\lambda^{\imath}_{\jmath,j}$ are characterized by a set of relative angles $\theta^{\imath,n}_{\jmath,j}$ labeled by some extra index n. These angles can be constructed in different ways, corresponding in particular to different orderings of the tree $\mathbf{T}$. In any case, as in section 5.2, these angle variables do not actually play any role in the proof of the finiteness of the integral (8.2). Therefore, we shall not make their construction more explicit.

In terms of the $|\lambda|$ and θ variables, the measure term can be written, up to a global numerical factor, as in *e.g.* (3.32), as

$$\prod_{\jmath=1}^{\mathcal{T}+1} \prod_{j=1}^{\text{Card}(T^0_{\jmath})} \prod_{\imath \in \overline{\text{Ind}}(\jmath,j)} \Big[d|\lambda^{\imath}_{\jmath,j}| \, |\lambda^{\imath}_{\jmath,j}|^{D-1} \Big(\prod_n (\sin \theta^{\imath,n}_{\jmath,j})^{p(D,n)} d\theta^{\imath,n}_{\jmath,j} \Big) \Big] \tag{8.30}$$

where $p(D,n)$ is some positive number (when $D \geq N + M - 1$).

8.3. Subtracted integrand

The tree variables of the preceding section, together with the factorization (8.16) allow us to work separately inside each reduced diagram $\tilde{T}_{\jmath}$. Indeed, the amplitude $I_{\tilde{T}_{\jmath}}$ for $\jmath \leq \tau$ is a function of the variables $\lambda^{\imath}_{\jmath,j}$ for the same $\jmath$ only, with $\imath \in \overline{\mathrm{Ind}}(\jmath, j)$ (the case $\jmath = \tau + 1$ which is special since it also involves the set of external points, will be discussed separately). Going back to the definition of the amplitude I, we can write (8.2) in a form where the measure and the integrand are factorized simultaneously. For each $\tilde{T}_{\jmath}$ (with $\jmath \leq \tau$), we get

$$\begin{aligned} &\prod_{j=1}^{\mathrm{Card}(T^0_{\jmath})} \prod_{\imath \in \overline{\mathrm{Ind}}(\jmath,j)} d|\lambda^{\imath}_{\jmath,j}||\lambda^{\imath}_{\jmath,j}|^{D-1} \Big(\prod_n \sin(\theta^{\imath,n}_{\jmath,j})^{p(D,n)} \Big) \\ &\qquad \times \prod_{\imath \in \mathrm{Ind}(\jmath)} \big(1 - \mathfrak{T}_{\tilde{T}^{\imath}_{\jmath-1\oplus}}\big) \Big[\prod_{j=1}^{\mathrm{Card}(T^0_{\jmath})} \det(\Pi^{\mathbf{T}_{\jmath,j}}) \Big]^{-\frac{d}{2}} \end{aligned} \tag{8.31}$$

where $\Pi^{\mathbf{T}_{\jmath,j}}$ is the matrix defined by (3.49),(3.50) for the subtree $\mathbf{T}_{\jmath,j}$. Its elements $\Pi^{\mathbf{T}_{\jmath,j}}_{\imath,\imath'}$ are labeled by elements $\imath, \imath'$ of $\overline{\mathrm{Ind}}(\jmath, j)$. As in the convergence proof of section 5, we introduce the normalized matrix:

$$Y^{\mathbf{T}_{\jmath,j}}_{\imath,\imath'} \equiv \frac{1}{A_D(\nu)} \frac{\Pi^{\mathbf{T}_{\jmath,j}}_{\imath,\imath'}}{|\lambda^{\imath}_{\jmath,j}|^{\nu} |\lambda^{\imath'}_{\jmath,j}|^{\nu}} \tag{8.32}$$

which, inserted in (8.31), gives

$$\begin{aligned} &\prod_{j=1}^{\mathrm{Card}(T^0_{\jmath})} \prod_{\imath \in \overline{\mathrm{Ind}}(\jmath,j)} \frac{d|\lambda^{\imath}_{\jmath,j}|}{|\lambda^{\imath}_{\jmath,j}|^{1-\epsilon}} \Big(\prod_n \sin(\theta^{\imath,n}_{\jmath,j})^{p(D,n)} \Big) \\ &\qquad \times \prod_{\imath \in \mathrm{Ind}(\jmath)} \big(1 - \mathfrak{T}^{\,0}_{\tilde{T}^{\imath}_{\jmath-1\oplus}}\big) \Big[\prod_{j=1}^{\mathrm{Card}(T^0_{\jmath})} \det(Y^{\mathbf{T}_{\jmath,j}}) \Big]^{-\frac{d}{2}}. \end{aligned} \tag{8.33}$$

Since through (8.32) we have extracted the most singular factor of the Π matrices under rescalings $\mathcal{D}^{\rho}$ (see (6.14)), the Taylor operators $\mathfrak{T}^{\,0}$ appearing in (8.33) are now defined as

$$\mathfrak{T}^{\,0} \;=\; \lim_{\rho \to 0} \mathcal{D}^{\rho}. \tag{8.34}$$

The properties of $\det(Y^{\mathbf{T}_{\jmath,j}})$ are the same as those mentioned in section 5.2. In particular, $\det(Y^{\mathbf{T}_{\jmath,j}})$ is a function of the ratios of $\lambda^{\imath}_{\jmath,j}$ for successive $\imath$ in $\overline{\mathrm{Ind}}(\jmath, j)$, which play the

role of the β_α variables of section 5, and are now products of the β and χ variables defined above.

Then $\det(Y^{\mathbf{T}_{J,j}})$ is a bounded function of the β^I and $\chi^I_{J,j}$ variables on the domain $\mathcal{H}^{\mathcal{S}}$, and is equal to 1 when all the β^I are set to zero.

Due to our choice for the $\lambda^I_{J,j}$ variables, the action of $\mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}}$ on $\prod_{j=1}^{\mathrm{Card}(T^0_J)} \Big(\det(Y^{\mathbf{T}_{J,j}})\Big)^{-\frac{d}{2}}$ simply corresponds to set $\beta^I = 0$ in all the $Y^{\mathbf{T}_{J,j}}$ for different j (see (8.21) and (8.22)). Therefore $(1-\mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}})\Big[\prod_{j=1}^{\mathrm{Card}(T^0_J)} \Big(\det(Y^{\mathbf{T}_{J,j}})\Big)^{-\frac{d}{2}}\Big]$ vanishes when $\beta^I \to 0$. This is the key property which will ensure the finiteness of the subtracted integrals. First we have to generalize this fact to all the β variables. This is contained in the following stronger property, as shown in Appendix H:

PROPOSITION:

$$\boxed{\prod_{I\in\mathrm{Ind}(J)} (1-\mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}})\Big[\prod_{j=1}^{\mathrm{Card}(T^0_J)} \Big(\det(Y^{\mathbf{T}_{J,j}})\Big)^{-\frac{d}{2}}\Big] = \mathcal{O}\Big(\prod_{I^{\min}(J)\le I<I^{\max}(J)} (\beta^I)^\delta\Big)} \tag{8.35}$$

with $\delta = \min(\nu, 1-\nu)$ as in (4.8).

The above discussion holds for $J \le T$ only. The case $J = T+1$ (and $j=1$) requires a separate analysis. We now have

$$I_{\tilde{T}_{T+1\oplus}} = \Big(\det(Y^{\mathbf{T}_{T+1,1}})\Big)^{-\frac{d}{2}} \exp\Big(-\frac{1}{2}\sum_{a,b} \vec{\mathbf{k}}_a\cdot\vec{\mathbf{k}}_b\Delta_{ab}\Big) \tag{8.36}$$

and a property similar to (8.35):

$$\boxed{\prod_{I\in\overline{\mathrm{Ind}}(T+1)} (1-\mathfrak{T}^0_{\tilde{T}^I_{T\oplus}})\Big[\Big(\det(Y^{\mathbf{T}_{T+1,1}})\Big)^{-\frac{d}{2}} \exp\Big(-\frac{1}{2}\sum_{a,b} \vec{\mathbf{k}}_a\cdot\vec{\mathbf{k}}_b\Delta_{ab}\Big)\Big] = \mathcal{O}\Big(\prod_{I\ge I^{\min}(T+1)} (\beta^I)^\delta\Big)} \tag{8.37}$$

8.4. Proof of finiteness

From the above discussion, we arrive at the following form for (8.2) at $\epsilon = 0$:

$$\int_{\mathcal{D}^{\mathcal{S}}} \prod_{\jmath=1}^{\tau+1} \prod_{j=1}^{\mathrm{Card}(T^0_\jmath)} \Bigg[\prod_{\imath\in\overline{\mathrm{Ind}}(\jmath,j)} \Big(\prod_n (\sin\theta^{\imath,n}_{\jmath,j})^{p(D,n)} d\theta^{\imath,n}_{\jmath,j} \Big) \prod_{\imath\in\overline{\mathrm{Ind}}(\jmath,j)}{}' \frac{d\chi^{\imath}_{\jmath,j}}{\chi^{\imath}_{\jmath,j}} \Bigg] \times \prod_{\imath\in\overline{\mathrm{Ind}}} \frac{d\beta^{\imath}}{\beta^{\imath}} \; \mathcal{O}\Big(\prod_{\imath\geq\imath^{\min}(\tau+1)} (\beta^{\imath})^{\delta} \Big) \prod_{\jmath=1}^{\tau} \mathcal{O}\Big(\prod_{\imath^{\min}(\jmath)\leq\imath<\imath^{\max}(\jmath)} (\beta^{\imath})^{\delta} \Big) \tag{8.38}$$

where $\prod'$ means that we omit the values of $\imath$ such that $\lambda^{\imath}_{\jmath,j}$ is a representative, and where the domain of integration $\mathcal{D}^{\mathcal{S}}$ reproduces the domain of integration $\mathcal{H}^{\mathcal{S}}$ for the relative positions of internal points. Inside $\mathcal{D}^{\mathcal{S}}$, the variables $\chi^{\imath}_{\jmath,j}$ are bounded from below according to (8.29). Therefore, the integration over these $\chi^{\imath}_{\jmath,j}$ variables and the integration over the $\theta^{\imath,n}_{\jmath,j}$ variables do not produce any divergence. For the integral to be convergent, it is actually sufficient that, for each $\imath \in \overline{\mathrm{Ind}}$, at least one $(\beta^{\imath})^{\delta}$ is present in the product of $\mathcal{O}$'s appearing in (8.38), thus making the integration over $\beta^{\imath}$ UV convergent. This will be true if

$$\overline{\mathrm{Ind}} \subset \Big\{ \bigcup_{\jmath=1}^{\tau} \Big[\imath^{\min}(\jmath), \imath^{\max}(\jmath) \Big) \Big\} \cup \Big[\imath^{\min}(\tau+1), N-1 \Big]. \tag{8.39}$$

Now, from their definition (8.12), all the diagrams $T^{\imath}_{\jmath-1}$ for $\imath \geq \imath^{\max}(\jmath)$ and the diagrams $T^{\imath}_{\jmath}$ for $\imath < \imath^{\min}(\jmath+1)$ are equal and identical to $T^0_\jmath$. Since, by hypothesis, the nest $\mathcal{N}^0_{\oplus}$ is minimal and therefore its tableau has no equal vertically adjacent diagrams, we deduce that

$$\imath^{\max}(\jmath) \geq \imath^{\min}(\jmath+1). \tag{8.40}$$

Using this inequality for each $\jmath$, it is easy to check that the r.h.s of (8.39) is actually equal to

$$\Big[\min_{\jmath=1}^{\tau+1} \imath^{\min}(\jmath) \;, N-1 \Big] = \Big[\min(\overline{\mathrm{Ind}}) \;, N-1 \Big] \tag{8.41}$$

and the required property (8.39) follows. This proves the convergence of (8.38), Q.E.D.

9. Discussion

9.1. Analytic continuation and convergence at small D

Up to now, the finiteness of $\mathcal{Z}_N$ (resp. $\mathcal{Z}^{(M)}_N$) at $\epsilon > 0$ and that of $\mathcal{Z}^{\mathbf{R}}{}_N$ (resp. $\mathcal{Z}^{\mathbf{R}}{}^{(M)}_N$) at $\epsilon \geq 0$ were proven for large enough dimension D only, that is $D \geq N-1$ (resp.

$D \geq N+M-1$), ν being fixed. If we now want to recover the physical models with a fixed value of k (typically $k=2$) and of ν, this requires a fixed value of $D=k-2\nu$ (satisfying (2.5) so that $0<\nu<1$). All the diagram contributions to $\mathcal{Z}_N$ (resp. $\mathcal{Z}^{\mathbf{R}_N^{(M)}}$) with $N \geq D+1$ (resp. $D-M+1$) then have to be defined by the analytic continuation procedure discussed in section 3, in a regime where the products of the measure (in the distance or the tree variables) by the integrands (resp. the subtracted integrands) that we have considered become *distributions*. This is the case for all the diagrams but a finite number of these.

To end this study, we have to make sure that, in this regime, these integrals (resp. subtracted integrals) are still finite in the sense of distributions for $\epsilon>0$ (resp. $\epsilon \geq 0$). We shall not give a rigorous and complete proof of this fact, but we shall rather outline the main steps of the argument.

First we have to check that the absolute convergence of the unsubtracted amplitude $\mathcal{Z}_N$ for $\epsilon>0$ given in section 5 extends to $D<N-1$. Considering the integral representation (5.4) for the contribution to $\mathcal{Z}_N$ of a given generalized Hepp sector $\mathcal{H}^{\mathbf{T}}$, expressed in spherical coordinates, and using (5.8), we get for this integral:

$$\int_{\mathcal{D}^{\mathbf{T}}} \prod_{\alpha=1}^{N-1} (\beta_\alpha)^{\alpha\epsilon-1} d\beta_\alpha \prod_{\alpha=2}^{N-1} \prod_{n=1}^{\alpha-1} (\sin(\theta_{\alpha,n}))^{D-1-n} d\theta_{\alpha,n} \left(\det\left[Y^{\mathbf{T}}_{\alpha\beta}(\beta\text{'s},\theta\text{'s})\right]\right)^{-\frac{d}{2}} \quad ; \qquad (9.1)$$

one sees that the problem of UV convergence (which comes from the small β_α behavior) is completely decoupled from the problem of analytic continuation of the measure in D (which comes from the behavior of the integral when $\theta_{\alpha,n} \to 0$ or π for $n>D$). As already discussed in subsection 3.3, an explicit representation of the analytically continued amplitude can be written, for non integer D, by subtracting the divergent powers of θ and $\pi-\theta$ (this is the standard finite part prescription). The resulting integration over the θ's are convergent, for fixed non-zero β's. From the explicit form of the matrix $Y^{\mathbf{T}}_{\alpha\beta}$, one can check that the subtractions in θ do not introduce dangerous negative powers of the β's (at least in the sector $\mathcal{H}^{\mathbf{T}}$ *i.e.* $\mathcal{D}^{\mathbf{T}}$), so that the power counting argument in the β's stays valid. Finally one can check that (as already done in subsection 3.3), the poles that occur at integer D are cancelled by the corresponding zeros of the global factor $S_D S_{D-1} \ldots S_{D-N+2}$ in the measure (3.29), so that the unsubtracted amplitude $\mathcal{Z}_N$ is finite for any $D>0$ and $\epsilon>0$.

The same argument can be applied to the subtracted amplitude at $\epsilon = 0$. Starting from the expression (8.38) for the part associated with the maximal nest $\mathcal{M}_\oplus$ of the subtracted amplitude in an extended Hepp sector, some of the $p(D,n)$ exponents become negative for $D < N+M-1$, and the integration over the corresponding angular variables $\theta^{I,n}_{J,j}$ requires a finite part subtraction prescription. Again, one can argue that these subtractions do not interfere with the power counting in β's and χ's, and that the small β estimates (8.35) and (8.37) remain valid for the θ-subtracted integrands.

Finally, one can extend this analysis to small negative ϵ, and show that for a subtracted amplitude of order N, no UV divergences occur as long as $\mathrm{Re}(\epsilon) > -\delta/(N-1)$, with $\delta = \min(\nu, 1-\nu)$, as in (4.8). Indeed, for $\epsilon \neq 0$, we must modify (8.38) by inserting in the integrand

$$\prod_{\jmath=1}^{\tau+1} \prod_{j=1}^{\mathrm{Card}(T^0_\jmath)} \prod_{\imath\in\overline{\mathrm{Ind}}(\jmath,j)}{}' \left(\chi^{\imath}_{\jmath,j}\right)^{\epsilon} \times \prod_{\imath\in\overline{\mathrm{Ind}}} \left(\beta^{\imath}\right)^{n(\imath)\epsilon} , \tag{9.2}$$

where $n(\imath)$ is the number of line vectors $\lambda^{\imath'}_{\jmath,j}$ with an index $\imath' \leq \imath$. One has clearly $n(\imath) \leq \imath \leq N-1$. Since the subtracted interaction term is (from (8.38) and (8.39)) a $\mathcal{O}\left(\prod_{\imath\in\overline{\mathrm{Ind}}} \left(\beta^{\imath}\right)^{\delta}\right)$, the convergence at small β's is guaranteed for $\mathrm{Re}(\epsilon) > -\delta/(N-1)$.

Finally, we have not discussed the problem of the convergence or summability of the perturbative series for our model. Since the model is expected to make sense for both $b > 0$ and $b < 0$ (with a finite free energy proportional to the internal volume in the latter case), we expect that the radius of convergence of these series will be non-zero, and in fact infinite for the unrenormalized series (which exists for $\epsilon > 0$, thus defining entire functions of b).

9.2. Universal scaling properties of the manifold

In this subsection, we shall derive some physical implications of the existence of a renormalized theory, well defined at $\epsilon = 0$. We shall consider here explicitly the case of elastic membranes with $k = 2$ in (2.1).

The main result of the preceding sections is that the subtracted amplitudes (6.22) for the correlation functions remain finite at $\epsilon = 0$. In terms of these, the full correlation functions

$$\mathcal{Z}^{(M)}(X_a, \vec{\mathbf{k}}_a; b) = \mathcal{Z}^{\mathbf{R}(M)}(X_a, \vec{\mathbf{k}}_a; b_R) \;=\; \sum_{N=0}^{\infty} \frac{(-b_R)^N}{N!}\, \mathcal{Z}^{\mathbf{R}(M)}_N(X_a, \vec{\mathbf{k}}_a) \tag{9.3}$$

have a series expansion in terms of the effective excluded volume parameter:

$$b_R = \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \left(V_{\mathbb{R}^d} - \mathcal{Z} \right) \ , \tag{9.4}$$

which represents the resummed one-point interaction of the manifold with the impurity. As functions of b_R and ϵ, these correlation functions thus stay finite at $\epsilon = 0$.

◇ *Existence of a Wilson function*

Our renormalization operation involves a peculiar renormalized coupling constant b_R (9.4), which is a function:

$$b_R \equiv b_R(b, X; \epsilon) \tag{9.5}$$

where X is the internal linear size of the manifold, defined by

$$\mathcal{V}_{\mathcal{S}_D} \equiv X^D. \tag{9.6}$$

As usual, since the renormalization operator $\mathbf{R}$ deals only with local counterterms, other choices of the renormalized coupling constant are possible, keeping the correlation functions finite as in (9.3). In particular, the theory describing the manifold of a given size X remains finite when expressed in terms of the parameter

$$b_R(\lambda) \equiv b_R(b, \lambda X; \epsilon) \ , \tag{9.7}$$

which corresponds to the renormalized coupling constant of a (reference) manifold with different size λX. In particular, the original $b_R(b, X; \epsilon)$ itself can be expressed in terms of $b_R(\lambda)$ (and λ):

$$b_R(b, X; \epsilon) = B_R(b_R(\lambda), \lambda, X; \epsilon) \tag{9.8}$$

where B_R stays finite at $\epsilon = 0$. This information is best expressed by writing

$$0 = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} b_R(b, X; \epsilon) = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} b_R(\lambda) \ \left. \frac{\partial}{\partial b_R} \right|_{\lambda, X} B_R + \lambda \left. \frac{\partial}{\partial \lambda} \right|_{b_R(\lambda), X} B_R \ , \tag{9.9}$$

from which we deduce that the quantity: $\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} b_R(\lambda)$ remains finite at $\epsilon = 0$ when expressed in terms of $b_R(\lambda)$, X and λ. This ensures in particular the finiteness at $\epsilon = 0$ of the Wilson function:

$$X \left. \frac{\partial}{\partial X} \right|_b b_R \equiv \lambda \left. \frac{\mathrm{d}}{\mathrm{d}\lambda} b_R(\lambda) \right|_{\lambda=1} \ . \tag{9.10}$$

As in (6.29), it is convenient to introduce the dimensionless coupling constants

$$\begin{aligned} g &\equiv \Big(2\pi\, A_D(\nu)\Big)^{-d/2} b_R X^\epsilon \ , \\ z &\equiv \Big(2\pi\, A_D(\nu)\Big)^{-d/2} b X^\epsilon \ , \end{aligned} \tag{9.11}$$

with $A_D(\nu) = (S_D(2-D)/2)^{-1}$ for $k = 2$. The associated Wilson function then does not depend on X explicitly and reads:

$$W(g,\epsilon) \equiv X \frac{\partial}{\partial X}\Big|_b g = \epsilon z \frac{\mathrm{d}g}{\mathrm{d}z} \ . \tag{9.12}$$

It is finite at $\epsilon = 0$, to all orders in g, and has the first order expansion (2.6):

$$W(g) \ = \ \epsilon\, g - \frac{1}{2} S_D\, g^2 \ + \ \mathcal{O}(g^3, g^2\epsilon) \ , \tag{9.13}$$

with a fixed point at

$$g^\star = \frac{2\epsilon}{S_D} + \mathcal{O}(\epsilon^2) \ . \tag{9.14}$$

◇ *Universality for the excluded volume and the osmotic pressure*

Let us consider the quantity

$$\mathbb{A} = V_{\mathbb{R}^d} - \mathcal{Z} = b_R \mathcal{V}_{\mathcal{S}_D} \ , \tag{9.15}$$

which has the dimension of a d-volume. For $b > 0$ (repulsive interaction) it is positive and represents an effective hard-sphere like excluded volume for the manifold around the impurity.

According to the definition (9.11) of g, we have explicitly

$$\mathbb{A} = g\ \big(2\pi\, A_D(\nu)\big)^{d/2} \mathcal{V}_{\mathcal{S}_D}^{d\nu/D} \ . \tag{9.16}$$

The internal volume of the manifold, $\mathcal{V}_{\mathcal{S}_D}$, is not directly observable, but, according to (2.31) and (3.38), it is related to the geometrical extension of the membrane in bulk d-dimensional space, when no impurity is present ($b = 0$). This extension can be measured, for instance, by the radius of gyration $\mathbb{R}_G$ of the noninteracting manifold, defined as

$$\begin{aligned} \mathbb{R}_G^2 \ &\equiv \ \frac{1}{2\mathcal{V}_{\mathcal{S}_D}^2} \Big\langle \int_{\mathcal{S}_D} d^D x \int_{\mathcal{S}_D} d^D y\, [\vec{\mathbf{r}}(x) - \vec{\mathbf{r}}(y)]^2 \Big\rangle_0 \\ &= \ \mathrm{Tr}' \left(\frac{1}{-\Delta} \right) \ , \end{aligned} \tag{9.17}$$

where Tr' means the sum over the non-zero eigenvalues of the Laplacian Δ on the closed manifold. Consequently we have

$$\mathbb{R}_G^2 \;=\; \mathbf{c}\,\mathcal{V}_{\mathcal{S}_D}{}^{2\nu/D} \tag{9.18}$$

where the dimensionless constant $\mathbf{c}$ depends on the geometrical shape of the manifold (it will be different for a sphere, an ellipsoid, a torus, etc...), and requires the knowledge of the true massless propagator $G(x,y)$ on the manifold $\mathcal{V}$, solution of

$$-\Delta_x G(x,y) \;=\; \delta^D(x,y) - \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \tag{9.19}$$

We consider explicitly the case where the external space dimension is lower than $d^\star$, so that a repulsive interaction ($b > 0$) is relevant. When the size of the membrane becomes large, g then reaches its (IR stable) fixed point value $g^\star$ in (9.16), and we get the universal scaling law:

$$\mathbb{A} \;=\; \mathbf{a}^\star\,\mathbb{R}_G^d\ , \tag{9.20}$$

where the dimensionless constant $\mathbf{a}^\star \;=\; g^\star(2\pi A_D(\nu))^{d/2}\mathbf{c}^{-d/2}$ depends on the intrinsic geometrical shape of the manifold, but neither on its size, nor on the details and the amplitude of the repulsive interaction, and is therefore, in this restricted sense, universal.

An ideal solution of $\mathbb{N}$ identical membranes interacting with one impurity, with concentration $\mathbb{C} = \mathbb{N}/V_{\mathbb{R}^d}$ in a box of volume $V_{\mathbb{R}^d}$, presents a shift of the osmotic pressure $\mathbb{P}$ from its ideal gas value. Owing to its relation (9.15) to the one-manifold partition function, the excluded volume $\mathbb{A}$ directly yields, by standard rules of thermodynamics,

$$\mathbb{P}/k_BT = \frac{\mathbb{C}}{1-\mathbb{A}/V_{\mathbb{R}^d}} = \mathbb{C}\,(1+\mathbb{A}/V_{\mathbb{R}^d}+\ldots)\ . \tag{9.21}$$

This law expresses the increase of the pressure due to the presence of the impurity in the solution with finite volume, and can be thought of as a finite size effect. The thermodynamic limit can be reached for a finite concentration $\mathbb{C}_I$ of impurities. One then gets the virial expansion of the osmotic pressure:

$$\mathbb{P}/k_BT \;=\; \mathbb{C} + \mathbb{C}\,\mathbb{C}_I\mathbb{A} + \ldots \;=\; \mathbb{C} + \mathbf{a}^\star\mathbb{C}\,\mathbb{C}_I\,\mathbb{R}_G^d \;+\; \ldots \tag{9.22}$$

Let us stress that the dimensionless quantity $\mathbf{a}^\star$, which is independent of the microscopic parameters and appears in the expression for the osmotic pressure, is directly related

to the fixed point value $g^\star$ with the choice (9.11) for the renormalized constant g. This is entirely similar to the case of a polymer solution with excluded volume [2][37].

$\diamond$*Pinned manifold*

Let us introduce the partition function of a manifold pinned at the origin at one of its points X_1:

$$\mathcal{Z}^\diamond \equiv \int \mathcal{D}[\vec{\mathbf{r}}] \exp(-\mathcal{H})\ \delta^d(\vec{\mathbf{r}}(X_1))\ . \tag{9.23}$$

Owing to the internal spherical symmetry of the manifold, $\mathcal{Z}^\diamond$ is independent of X_1 and actually equals:

$$\mathcal{Z}^\diamond = \int \mathcal{D}[\vec{\mathbf{r}}] \exp(-\mathcal{H})\ \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \int_{\mathcal{S}_D} d^D x\, \delta^d(\vec{\mathbf{r}}(x))\ . \tag{9.24}$$

From (2.9), one has clearly:

$$\mathcal{Z}^\diamond = -\frac{1}{\mathcal{V}_{\mathcal{S}_D}} \frac{\partial}{\partial b}\Big|_X \mathcal{Z}(b, X) = \frac{\partial b_R}{\partial b}\Big|_X\ . \tag{9.25}$$

Notice that, while the unrestricted partition function $\mathcal{Z}$ has the dimension of a d-volume, the pinned-manifold partition function $\mathcal{Z}^\diamond$ is dimensionless and is thus a function $\mathcal{Z}^\diamond(z;\epsilon)$ of z (and ϵ) only. According to (9.11) and (9.12), we have

$$\mathcal{Z}^\diamond = \frac{\partial}{\partial b}\Big|_X b_R = \frac{\mathrm{d}g}{\mathrm{d}z} = \frac{1}{\epsilon z} W(g(z);\epsilon)\ . \tag{9.26}$$

Notice that $\mathcal{Z}^\diamond$ itself is not renormalized, *i.e.* not finite at $\epsilon = 0$ as a function of g, but that $\epsilon z \mathcal{Z}^\diamond = W(g,\epsilon)$ is renormalized. When the size X becomes large (for ϵ and b positive) z becomes large and $g(z)$ tends to its limit $g^\star$, the Wilson function vanishing as:

$$W(g(z);\epsilon) = (g(z) - g^\star) W'(g^\star) + \dots\ , \tag{9.27}$$

with

$$g(z) - g^\star \sim \text{const}\ z^{W'(g^\star)/\epsilon}\ ; \tag{9.28}$$

(Notice that $W'(g^\star) < 0$; see fig. 2). This finally leads to the scaling law for $\mathcal{Z}^\diamond$:

$$\mathcal{Z}^\diamond \sim \text{const}\ z^{-1+W'(g^\star)/\epsilon} \sim \text{const}\ (b^{1/\epsilon} X)^{W'(g^\star)-\epsilon}\ . \tag{9.29}$$

At first order in ϵ, $W'(g^\star) = -\epsilon + \mathcal{O}(\epsilon^2)$, whence

$$\mathcal{Z}^\diamond \sim \text{const}\ (b^{1/\epsilon} X)^{-2\epsilon}\ . \tag{9.30}$$

◇ *Universal* $1/r$ *repulsion law*

The pinned-manifold partition function $\mathcal{Z}^\diamond$ is a particular case of a more general restricted partition function to which we now turn. We introduce:

$$\mathcal{Z}^\diamond(X_1, \vec{\mathbf{r}}; X, b\,; \epsilon) = \int \mathcal{D}[\vec{\mathbf{r}}] \exp(-\mathcal{H})\ \delta^d(\vec{\mathbf{r}}(X_1) - \vec{\mathbf{r}}) \tag{9.31}$$

which describes the partition function of a manifold held by one of its points at the position $\vec{\mathbf{r}}$ relative to the origin. It is the Fourier transform of the one-point correlation function (2.25) for $M = 1$, that is:

$$\mathcal{Z}^\diamond(X_1, \vec{\mathbf{r}}; X, b\,; \epsilon) = \int d^d\vec{\mathbf{k}}_1\ \exp(-i\ \vec{\mathbf{k}}_1 \cdot \vec{\mathbf{r}})\ \mathcal{Z}^{(1)}(X_1, \vec{\mathbf{k}}_1; b, X, \epsilon)\ . \tag{9.32}$$

As above, for a closed manifold, $\mathcal{Z}^\diamond(X_1, \vec{\mathbf{r}}; X, b\,; \epsilon)$ is actually independent of X_1 and equal to

$$\mathcal{Z}^\diamond(\vec{\mathbf{r}}; X, b\,; \epsilon) = \int \mathcal{D}[\vec{\mathbf{r}}] \exp(-\mathcal{H})\ \frac{1}{\mathcal{V}_{\mathcal{S}_D}} \int_{\mathcal{S}_D} d^D x\, \delta^d(\vec{\mathbf{r}}(x) - \vec{\mathbf{r}})\ . \tag{9.33}$$

The relations of this partition function to the former ones are

$$\begin{aligned} \mathcal{Z}^\diamond(\vec{0}) &= \mathcal{Z}^\diamond\ , \\ \int_{\mathbb{R}^d} d^d\vec{\mathbf{r}}\ \mathcal{Z}^\diamond(\vec{\mathbf{r}}) &= \mathcal{Z}\ . \end{aligned} \tag{9.34}$$

By rotational symmetry, the quantity $\mathcal{Z}^\diamond$ depends only on $r \equiv |\vec{\mathbf{r}}|$. It is furthermore dimensionless, and thus can be written as a function of z and r/X^ν (and ϵ):

$$\mathcal{Z}^\diamond(\vec{\mathbf{r}}; X, b\,; \epsilon) \equiv \mathcal{Z}^\diamond[r/X^\nu, z\,; \epsilon]\ . \tag{9.35}$$

As we have seen for $\mathcal{Z}^\diamond$ (9.26), $\mathcal{Z}^\diamond[r/X^\nu, z\,; \epsilon]$ is not exactly renormalized, when expressed in terms of g, but $\epsilon z \mathcal{Z}^\diamond[r/X^\nu, z\,; \epsilon]$ is. It is interesting to consider the limit when the interaction parameter b goes to infinity, while keeping the size X of the manifold finite. We expect $\mathcal{Z}^\diamond[r/X^\nu, z\,; \epsilon]$ to reach a finite limit

$$\mathcal{Z}^\diamond_\infty[r/X^\nu\,; \epsilon] \equiv \lim_{z\to\infty} \mathcal{Z}^\diamond[r/X^\nu, z\,; \epsilon]\ . \tag{9.36}$$

According to (9.34) and (9.4), we have

$$\int_{\mathbb{R}^d} d^d\vec{\mathbf{r}}\ (\mathcal{Z}^\diamond[r/X^\nu, z\,; \epsilon] - 1) = -b_R \mathcal{V}_{\mathcal{S}_D} = -g\Big(2\pi\, A_D(\nu)\Big)^{d/2} X^{\nu d}\ . \tag{9.37}$$

In the limit $z \to \infty$, g tends to $g^\star$, and we therefore have

$$\int_{\mathbb{R}^d} d^d\vec{u}\ (\mathcal{Z}^\diamond_\infty[u\,;\epsilon] - 1) = -g^\star \Big(2\pi\, A_D(\nu)\Big)^{d/2}\ , \tag{9.38}$$

which is consistent with the assumption that the limit in (9.36) actually exists.

In the scaling regime $r/X^\nu \ll 1$, we expect the marked point to be strongly repelled from the origin, and thus $\mathcal{Z}^\diamond_\infty$ to vanish as a power law:

$$\mathcal{Z}^\diamond_\infty[r/X^\nu\,;\epsilon] \sim \text{const}\Big(\frac{r}{X^\nu}\Big)^\theta\ . \tag{9.39}$$

This vanishing of $\mathcal{Z}^\diamond[r/X^\nu, z\,;\epsilon]$ in the successive limits $z \to \infty$ and $r \to 0$ is consistent with that obtained in the reversed double limit $r = 0$, and $z \to \infty$, which corresponds to the vanishing of $\mathcal{Z}^\diamond$ at infinite z according to (9.29).

The contact exponent θ can be obtained as follows. For finite b and large X, we expect a universal X-dependence of $\mathcal{Z}^\diamond[r/X^\nu, z\,;\epsilon]$, irrespective of the particular value given to r. This dependence is in particular known exactly when $r = 0$, according to (9.29). It must also be the same for $r \neq 0$ fixed and $b \to \infty$, that is a behavior which is given by (9.39). This leads to identifying the contact exponent with:

$$\theta = \frac{\epsilon - W'(g^\star)}{\nu}\ . \tag{9.40}$$

Notice that the argument above, intuitively clear on physical grounds, is usually mathematically justified in field theory from the existence of a short-distance operator product expansion. A rigorous proof of the existence of such a short-distance expansion in our case is beyond the scope of this paper. The repeated appearance of $W'(g^\star)$ in (9.29) and (9.40) suggests that all scaling behaviors in this theory are controlled by a single scaling anomalous dimension, *i.e.* the universal slope of the Wilson function at the fixed point.

Equation (9.39) allows us to derive a universal expression for the repulsive force exerted by the impurity on the membrane,

$$\vec{\mathbf{f}}(\vec{\mathbf{r}})/k_BT = \nabla_{\vec{\mathbf{r}}} \log \mathcal{Z}^\diamond(\vec{\mathbf{r}}) = \theta\frac{\vec{\mathbf{r}}}{r^2}\ . \tag{9.41}$$

According to the discussion above, this force law is valid in the scaling regime $b^{-\nu/\epsilon} \ll r \ll X^\nu$, where $b^{-1/\epsilon}$ plays the same physical role as an ultraviolet cut-off for internal distances.

◇ *Scaling laws for the delocalization transition*

Finally, we have seen in subsection 2.1 that for $d > d^\star$ (that is $\epsilon < 0$), the non-trivial fixed point $g^\star$ is now negative and IR repulsive, and corresponds to a delocalization transition with non-trivial critical exponents, for a particular negative critical value $b^\star$ of the bare coupling constant b. In the localized phase ($b < b^\star$), the correlation functions such as $\langle \vec{\mathbf{r}}(x)\vec{\mathbf{r}}(y)\rangle$ and the associated correlation length $\xi_\parallel$ (in the internal D-dimensional space) should be finite, as well as the average distance $r = \langle|\vec{\mathbf{r}}|\rangle$ of the manifold to the attractive impurity. At the transition these quantities should diverge as

$$\xi_\parallel \propto (b^\star - b)^{-\nu_\parallel} \qquad ; \qquad r \propto (b^\star - b)^{-\nu_\perp} \ . \tag{9.42}$$

Standard arguments lead to

$$\nu_\parallel = \frac{1}{W'(g^\star)} = -\frac{1}{\epsilon} + \dots \tag{9.43}$$

and

$$\nu_\perp = \nu_\parallel \, \nu \qquad . \tag{9.44}$$

Indeed, $\vec{\mathbf{r}}$ has no anomalous dimension and therefore, r scales as ${\xi_\parallel}^\nu$ with $\nu = (2-D)/2$ from (2.2).

10. Conclusion

10.1. Summary

In this last section, we would like to summarize the main steps of our construction and outline the main ingredients which ensure the renormalizability of the theory. We then discuss some possible extensions of our results.

◇ *Existence of a perturbative expansion analytically continued in D*

(I) The first ingredient is the existence, for integer dimension D of the manifold, of a formal perturbative expansion for the model. The diagrams present an invariance under global Euclidean motions in $\mathbb{R}^D$ of the interaction points (or under the group $\mathrm{SO}(D+1)$ for finite volume manifolds with the internal geometry of the sphere $\mathcal{S}_D$). The interaction terms, which are determinants involving the internal Green functions between interaction points, can then be expressed in terms of mutual squared distances

only. On the other hand, the external dimension d appears only in the power $(-d/2)$ of the interaction determinant.

(II) The second step is the construction of a measure term, analytic in D, in terms of the above set of internal mutual squared distances. One can then use for convenience any equivalent measure, for instance in terms of Cartesian or spherical coordinates in a space with a given integer dimension (typically $\mathbb{R}^{N-1}$ for a diagram of order N), D itself appearing as an analytic variable. This measure has in general to be understood as a distribution.

Points (I) and (II) allow us to define a perturbative expansion for the model, analytically continued in D. Its main features are the following:

- It can be viewed as a generalization of the Schwinger parametric representation of Feynman amplitudes for local field theory, with the one-dimensional α-parameters replaced by D-dimensional parameters.
- It appears as a string-like theory, in the sense that it presents only one diagram to each order in perturbation.
- It reduces to the expansion of a local field theory when $D = 1$, expressed in the Schwinger α-representation. The field theoretic diagrammatic contributions are recovered in the limit $D \to 1$ through the analytic continuation of the measure term.

◇ *Renormalizability*

The essential properties which are key to renormalizability are the following:

(III) Schoenberg's theorem: this property of the interaction determinants ensures that divergences in the integrals of the diagrammatic expansion occur only at short-distances (UV), as in ordinary local field theories. Infrared (IR) divergences also can occur if the internal space is infinite, a problem which is dealt with by considering a finite membrane, *e.g.* the sphere $\mathcal{S}_D$ with finite volume $\mathcal{V}_{\mathcal{S}_D}$.

(IV) Factorization of the interaction term: this property states that, when a subset of interaction points contracts toward a vertex, the interaction determinant factorizes into the product of the interaction term of the contracting subset by that simply obtained by replacing the whole subset by its contraction vertex. The possibility of replacing a set of coalescing points by a single contraction vertex, and of factorizing out the corresponding divergence is the key for renormalizability. Mathematically, it allow us to make the theory finite by letting a subtraction operator act on the integrand. This

operator essentially subtracts factorized equivalents so as to remove the UV divergences. It is constructed from elementary Taylor operators associated with subsets of points, then organized in forests or nests, corresponding to the hierarchical structure of the divergences.

(V) Factorization of the measure: this property, obviously satisfied for integer D, is preserved by the analytic continuation of the measure to non integer D. It allows us to integrate separately the factorized determinants which are to be subtracted from the original amplitude, and thus to interpret them as counterterms: the subtraction operation is then a simple reexpression of the partition function (or correlation functions) in terms of an effective (renormalized) coupling constant.

Points (III) and (IV) are properties of the interaction determinant themselves, while point (V) is a general property of the measure.

10.2. Prospects

Let us finally discuss possible outcomes of our results. As already discussed, the model (2.1) of a manifold interacting with a single point serves indeed as a laboratory for studying the renormalizability of more general models of interacting crumpled manifolds. A prominent model of this class is of course the Edwards model (1.2) of a self-avoiding manifold interacting via a short range two-body pseudopotential. Its perturbative expansion is similar in structure to the one studied here. We indeed believe that the mathematical techniques developed in this article can be applied and generalized to the Edwards model, and provide both conceptually and practically a framework for a similar proof of its renormalizability.

When reviewing the general scheme above, we note that point (I) is already known for the self-avoiding model [21] . Points (II) and (V) are actually valid for any manifold Hamiltonian. The specificity of a given model is actually encoded in its interaction determinants, for which properties similar to those of (III) and (IV) have to be analyzed in each case, and established in order to eventually build a subtraction procedure and prove renormalizability [38].

This scheme should be directly applicable to a series of manifolds theories with interactions, such as many-body or long-range interactions ... These models generalize to arbitrary internal dimension D models of interacting polymers ($D = 1$). All the latter models are known to be equivalent to some n-component field theories in the limit $n = 0$, with standard Feynman diagram expansions. When extended to manifolds of arbitrary

internal dimension, these models become theories with a single diagram to each order in perturbation (a property which is shared with string theories, although in our case the manifold has a fixed internal metric). Interestingly enough, the topological complexity of the usual Feynman diagrams is encoded in the D-measure on the manifold, and arises in the limit $D = 1$ from the ordering constraints along the one-dimensional (polymer) line. More generally, it would be interesting to try and express field theories with an arbitrary number n of components as $D = 1$ limits of "manifolds" string-like models, yet to be invented.

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Appendix A. From vectors to scalar products

In this appendix we derive (3.2) (3.3). First we insert the relation $u_{ij} = x_i \cdot x_j$ in the l.h.s. of (3.2)

$$\int \prod_{i=1}^{N} d^D x_i \, f(x_i \cdot x_j) \;=\; \int \prod_{i \le j} du_{ij} \int \prod_i d^D x_i \prod_{i \le j} \delta(u_{ij} - x_i \cdot x_j) \; f([u_{ij}]) \; . \tag{A.1}$$

Second we use the fact that the function

$$\sigma_N^{(D)}(u_{ij}) \;=\; \int \prod_i d^D x_i \prod_{i \le j} \delta(u_{ij} - x_i \cdot x_j) \tag{A.2}$$

is invariant under SO(N) rotations R ($u \to R^t u R$) to diagonalize u_{ij} and express (A.2) in terms of the N eigenvalues λ_i $i = 1, \ldots, N$ of u_{ij}

$$\sigma_N^{(D)}(u_{ij}) \;=\; \int \prod_i d^D x_i \prod_{i \le j} \delta(\lambda_i \delta_{ij} - x_i \cdot x_j) \; . \tag{A.3}$$

Third we perform the change of variables $x_i \to \sqrt{\lambda_i} x_i$ and get

$$\sigma_N^{(D)}(u_{ij}) \;=\; \prod_{i=1}^{N} \lambda_i^{\frac{D-N-1}{2}} \int \prod_i d^D x_i \prod_{i \le j} \delta(\delta_{ij} - x_i \cdot x_j) \; . \tag{A.4}$$

The remaining integral over the x_i's gives the volume of SO(D)/SO($D-N$) and we obtain finally (3.3)

$$\begin{aligned} \sigma_N^{(D)}(u_{ij}) \;&=\; \left(\prod_i \lambda_i \right)^{\frac{D-N-1}{2}} \cdot \frac{\mathrm{Vol}(\mathrm{SO}(D))}{\mathrm{Vol}(\mathrm{SO}(D-N))} \\ &=\; (\det[u_{ij}])^{\frac{D-N-1}{2}} \; \frac{S_D}{2} \cdots \frac{S_{D-N+1}}{2} \; . \end{aligned} \tag{A.5}$$

Appendix B. Factorization of the measure

To prove (3.34) let us decompose the $N \times N$ symmetric positive definite scalar product matrix $[u]_N$ into blocks of size P and Q ($P + Q = N$):

$$[u]_N \;=\; \begin{pmatrix} [u]_P & [v] \\ [v]^t & [u]_Q \end{pmatrix} \; . \tag{B.1}$$

Equation (3.34) is equivalent to the fact that, given the positive definite matrices $[u]_P$ and $[u]_Q$, when integrating over all $P \times Q$ matrices $[v]$ such that $[u]_N$ (defined by (B.1)) is positive definite, we have for arbitrary non-integer D

$$\int d[v]\, \sigma_N^{(D)}([u]_N) \;=\; \sigma_P^{(D)}([u]_P)\, \sigma_Q^{(D)}([u]_Q)\ . \tag{B.2}$$

Since $[u]_P$ and $[u]_Q$ are positive definite we can take their square root $[u]_P^{\frac{1}{2}}$ and $[u]_Q^{\frac{1}{2}}$ and write $\det([u]_N)$ in the expression (3.3) for $\sigma_N^{(D)}$ as

$$\det[u]_N \;=\; \det([u]_P)\, \det([u]_Q)\, \det(\mathbf{1} - [u]_P^{-\frac{1}{2}}[v][u]_Q^{-1}[v]^t[u]_P^{-\frac{1}{2}})\ . \tag{B.3}$$

Now, one can perform the change of variable $[v] \to [u]_P^{\frac{1}{2}}[v][u]_Q^{\frac{1}{2}}$ which induces a Jacobian $J = \det([u]_P)^{\frac{Q}{2}} \det([u]_Q)^{\frac{P}{2}}$ in (B.2). We thus obtain finally that the l.h.s. of (B.2) is equal to the r.h.s. of (B.2), up to a constant C which depends on D, P and Q, but not on $[u]_P$ and $[u]_Q$, and which is given by

$$C \;=\; \frac{\mathrm{Vol}(\mathrm{SO}(D))}{\mathrm{Vol}(\mathrm{SO}(D-N))}\frac{\mathrm{Vol}(\mathrm{SO}(D-P))}{\mathrm{Vol}(\mathrm{SO}(P))}\frac{\mathrm{Vol}(\mathrm{SO}(D-Q))}{\mathrm{Vol}(\mathrm{SO}(Q))}\int d[v]\,\left(\det(\mathbf{1} - [v][v]^t)\right)^{\frac{D-N-1}{2}}\ . \tag{B.4}$$

(The domain of integration for $[v]$ is now such that $\begin{pmatrix} \mathbf{1}_P & [v] \\ [v]^t & \mathbf{1}_Q \end{pmatrix}$ is positive definite)

It remains to prove that $C = 1$. This can be done in a simple way by proving that the factorization identity (3.34) holds for some particular function $f([u])$. As an example we can take the exponential

$$f([u]_N) \;=\; \exp(-\mathrm{tr}[u]_N)\ , \tag{B.5}$$

since we can easily calculate explicitly (see below)

$$I_N \;=\; \int_{\mathcal{U}_N} d[u]_N\, \sigma_N^{(D)}([u]_N)\, \exp(-\mathrm{tr}([u]_N)) \;=\; (\pi)^{N\frac{D}{2}}\ , \tag{B.6}$$

and therefore factorization holds in this case since:

$$f([u]_N) \;=\; f([u]_P)f([u]_Q) \qquad \text{and} \qquad I_N \;=\; I_P I_Q\ . \tag{B.7}$$

The direct computation of I_N ((B.6)) for any D proceeds as follows. The set $\mathcal{U}_N$ is the set of symmetric positive matrices. By $\mathrm{SO}(N)$ orthogonal transformations, it can be reduced

to the set of diagonal matrices with positive eigenvalues λ_i $(i = 1, \ldots, N)$, with the new measure:

$$d[u]_N = \ \mathrm{Vol}(\mathrm{SO}(N)) \frac{1}{N!} \prod_{i=1}^{N} d\lambda_i \ \Delta(\lambda) \ , \tag{B.8}$$

where the λ's are integrated from 0 to ∞ and $\Delta(\lambda)$ is the Jacobian [39]

$$\Delta(\lambda) = \prod_{1 \leq j < l \leq N} |\lambda_j - \lambda_l| \ . \tag{B.9}$$

In terms of these variables, I_N reads explicitly:

$$I_N = \frac{\mathrm{Vol}(\mathrm{SO}(D))\mathrm{Vol}(\mathrm{SO}(N))}{N! \, \mathrm{Vol}(\mathrm{SO}(D-N))} \int_0^\infty \prod_{j=1}^{N} d\lambda_j \ \Delta(\lambda) \ \exp(-\sum_{j=1}^{N} \lambda_j) \left(\prod_{j=1}^{N} \lambda_j \right)^{\frac{D-N-1}{2}} . \tag{B.10}$$

The calculation is completed by using the Selberg integral formula [39] , [40]:

$$\int_0^\infty (\Delta(\lambda))^{2\gamma} \prod_{j=1}^{N} \left[\lambda_j^{\alpha-1} \ \exp(-\lambda_j) \ d\lambda_j \right] = \prod_{j=0}^{N-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)}{\Gamma(1+\gamma)} \tag{B.11}$$

for $\gamma = \frac{1}{2}$ and $\alpha = \frac{D-N+1}{2}$, which leads finally to (B.6) .

Appendix C. Factorization of $\det\left(\left[\boldsymbol{\Pi}^{\mathbf{T}}_{\alpha\beta}\right]\right)$

Let us consider an ordered tree $\mathbf{T}$ and the corresponding vectors $\lambda_1, \ldots, \lambda_{N-1}$ with $|\lambda_1| \leq \ldots \leq |\lambda_{N-1}|$. We have by definition

$$\Pi^{\mathbf{T}}_{\alpha\beta} = -\frac{A_D(\nu)}{2} \Big\{ |R_{\alpha\beta} + \lambda_\beta - \lambda_\alpha|^{2\nu} - |R_{\alpha\beta} + \lambda_\beta|^{2\nu} - |R_{\alpha\beta} - \lambda_\alpha|^{2\nu} + |R_{\alpha\beta}|^{2\nu} \Big\} \ , \tag{C.1}$$

where $R_{\alpha\beta}$ is one "basis" of the quadrilateral

$$R_{\alpha\beta} = x_{i_\beta} - x_{i_\alpha} \ . \tag{C.2}$$

Fig. 22: *The quadrilateral picturing the matrix element* $\Pi^{\mathbf{T}}_{\alpha\beta}$ *and its "basis" vector* $R_{\alpha\beta}$.

The vector $R_{\alpha\beta}$ is a linear combination of the λ's

$$R_{\alpha\beta} = \sum_{\gamma=1}^{N-1} c^{\gamma}_{\alpha\beta}\, \lambda_\gamma \tag{C.3}$$

where $c^{\gamma}_{\alpha\beta} = 0, \pm 1$. Suppose we make the following rescaling

$$\lambda_\alpha \to \lambda_\alpha(\rho) = \begin{cases} \rho\lambda_\alpha & \text{if} \quad \alpha \leq P-1 \\ \\ \lambda_\alpha & \text{if} \quad \alpha \geq P \end{cases} \tag{C.4}$$

for some P, $2 \leq P \leq N$ and with a contraction factor ρ, $0 \leq \rho \leq 1$. Under this rescaling, $R_{\alpha\beta}$ becomes

$$\begin{aligned} R_{\alpha\beta}(\rho) &= \sum_{\gamma=P}^{N-1} c^{\gamma}_{\alpha\beta}\lambda_\gamma + \rho \sum_{\gamma=1}^{P-1} c^{\gamma}_{\alpha\beta}\lambda_\gamma \\ &= R^0_{\alpha\beta} + \rho R^1_{\alpha\beta}\ . \end{aligned} \tag{C.5}$$

We therefore have two possibilities:

(a) $R^0_{\alpha\beta} = 0$. This means that $R_{\alpha\beta}$ is formed only of vectors λ_γ with $\gamma \leq P-1$, which are all contracted, hence $R_{\alpha\beta}$ itself is contracted. By definition, this is also the case when $R_{\alpha\beta}$ is 0, that is when $x_{i_\alpha} = x_{i_\beta}$.

(b) $R^0_{\alpha\beta} \neq 0$. This occurs when $R_{\alpha\beta}$ is spanned by at least one λ_γ which is not contracted, that is with $\gamma \geq P$.

This allows us to classify the λ's into subtrees as follows (see fig. 23):

- We regroup the λ_α's with $\alpha \leq P-1$ (*i.e.* corresponding to contracted lines) into equivalence classes by deciding that λ_α and λ_β are equivalent if $R^0_{\alpha\beta} = 0$. The equivalence classes $\mathbf{T}_1, \ldots, \mathbf{T}_{m-1}$ (with $2 \leq m \leq P$ depending on $\mathbf{T}$) correspond to the $m-1$ distinct connected subtrees which build the subset of the contracted lines. Case (a) above thus corresponds to λ_α and λ_β in the same equivalence class, that is in the same connected subtree of contracted lines. Case (b) corresponds to λ_α and λ_β in two distinct equivalence classes, that is in two distinct connected subtrees of contracted lines.
- We regroup the λ_α with $\alpha \geq P$ into a single connected tree $\mathbf{T}_m$ obtained by setting $\lambda_\beta = 0$ for $\beta \leq P-1$ in the original tree $\mathbf{T}$.

We will now show that, for $\rho \to 0$:

$$\det\left(\left[\Pi^{\mathbf{T}}(\rho)\right]\right) = \rho^{2\nu(P-1)} \prod_{i=1}^{m} \det\left(\left[\Pi^{\mathbf{T}_i}\right]\right)\left\{1 + \mathcal{O}(\rho^{2\delta})\right\}\ . \tag{C.6}$$

Fig. 23: *Classification of the line vectors of the tree* $\mathbf{T}$ *into subtrees* $\mathbf{T}_i$*. The dashed lines in (a) correspond to contracting branches of the tree* $\mathbf{T}$*, and are organized into two connected subtrees* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *in (b) . The full lines in (a) correspond to non-contracting branches and are organized into a single connected subtree* $\mathbf{T}_3$ *in (b), by fully contracting the dashed lines in (a).*

Let us consider two lines λ_α and λ_β.

Case 1: $\alpha \leq P-1$, $\beta \leq P-1$

Case 1(a): $R^0_{\alpha\beta} = 0$

This case corresponds to two λ's in the same contracting connected subtree $\mathbf{T}_i$ for some $i \leq m-1$. In (C.1) , λ_α, λ_β and $R_{\alpha\beta}$ all get a factor ρ, hence

$$\Pi^{\mathbf{T}}_{\alpha\beta}(\rho) = \rho^{2\nu}\Pi^{\mathbf{T}}_{\alpha\beta} \ . \tag{C.7}$$

It is furthermore clear that $R_{\alpha\beta}$ is spanned only by λ's in $\mathbf{T}_i$, hence

$$\Pi^{\mathbf{T}}_{\alpha\beta}(\rho) = \rho^{2\nu}\Pi^{\mathbf{T}_i}_{\alpha\beta} \ . \tag{C.8}$$

Case 1(b): $R^0_{\alpha\beta} \neq 0$

This case corresponds to two λ's in two distinct contracting connected subtrees $\mathbf{T}_{i_1}$ and $\mathbf{T}_{i_2}$. Since $R_{\alpha\beta}$ does not contract to zero, we can formally expand (C.1) in power of λ_α and λ_β. The matrix element $\Pi^{\mathbf{T}}_{\alpha\beta}$ is by definition the interaction between two dipoles λ_α, λ_β separated by $R_{\alpha\beta}$. It is therefore clear that the first term in the multipolar expansion is of order

$$\Pi^{\mathbf{T}}_{\alpha\beta} \propto |R_{\alpha\beta}|^{2\nu-2}\lambda_\alpha \cdot \lambda_\beta + \dots \ . \tag{C.9}$$

Therefore, expanding in ρ yields immediately

$$\begin{aligned}\Pi^{\mathbf{T}}_{\alpha\beta}(\rho) &\propto \rho^2\,|R^0_{\alpha\beta}|^{2\nu-2}\lambda_\alpha\cdot\lambda_\beta+\dots\\ &= \mathcal{O}(\rho^2)=\rho^{2\nu}\mathcal{O}(\rho^{2\delta})\ ;\end{aligned}\tag{C.10}$$

(See (4.8)). As we shall see below, this element, which mixes several subtrees $\mathbf{T}_i$, is vanishing sufficiently fast as to disappear in the limit $\rho\to 0$.

Case 2: $\alpha\le P-1$, $\beta\ge P$

In this case, we have

$$\Pi^{\mathbf{T}}_{\alpha\beta}(\rho)\propto \overset{(1)}{|R_{\alpha\beta}(\rho)+\lambda_\beta-\rho\lambda_\alpha|^{2\nu}}\quad \overset{(2)}{-|R_{\alpha\beta}(\rho)+\lambda_\beta|^{2\nu}}\quad \overset{(3)}{-|R_{\alpha\beta}(\rho)-\rho\lambda_\alpha|^{2\nu}}\quad \overset{(4)}{+|R_{\alpha\beta}(\rho)|^{2\nu}}\tag{C.11}$$

Case 2(a): $R^0_{\alpha\beta}=0$

Substituting $R_{\alpha\beta}(\rho)=\rho R^1_{\alpha\beta}$ in (C.11) , the last two terms (3) and (4) are homogeneous to $\rho^{2\nu}$, while the expansion of $(1)-(2)$ in power of ρ gives a leading term linear in ρ. On the whole, we can write

$$\Pi^{\mathbf{T}}_{\alpha\beta}=\rho^\nu\,\mathcal{O}(\rho^\delta)\ .\tag{C.12}$$

Case 2(b): $R^0_{\alpha\beta}\neq 0$

This time, the expansion of $(1)-(2)$ on the one hand, and $-(3)+(4)$ on the other hand, in formal powers of $\rho\lambda_\alpha$ leads immediately to a matrix element of order ρ, hence

$$\Pi^{\mathbf{T}}_{\alpha\beta}=\mathcal{O}(\rho)=\rho^\nu\,\mathcal{O}(\rho^\delta)\ .\tag{C.13}$$

Case 3: $\alpha\ge P$, $\beta\ge P$

In this case, λ_α and λ_β are not contracted and belong to $\mathbf{T}_m$. In the limit $\rho\to 0$, $R_{\alpha\beta}$ is simply replaced by $R^0_{\alpha\beta}$. Whatever the value of $R^0_{\alpha\beta}$, this corresponds precisely to

$$\begin{aligned}\Pi^{\mathbf{T}}_{\alpha\beta}(\rho) &= \Pi^{\mathbf{T}_m}_{\alpha\beta}+\rho^\nu\,\mathcal{O}(\rho^\delta)\\ &= \Pi^{\mathbf{T}_m}_{\alpha\beta}+\mathcal{O}(\rho^{2\delta})\ .\end{aligned}\tag{C.14}$$

We can summarize all these cases by writing the synoptic table

$$\det\left(\Pi^{\mathbf{T}}(\rho)\right) = \det\begin{pmatrix} \boxed{\begin{matrix} \boxed{\rho^{2\nu}\Pi^{\mathbf{T}_1}} & & \rho^{2\nu}\mathcal{O}(\rho^{2\delta}) \\ & \boxed{\rho^{2\nu}\Pi^{\mathbf{T}_2}} & \\ & & \ddots \\ \rho^{2\nu}\mathcal{O}(\rho^{2\delta}) & & \boxed{\rho^{2\nu}\Pi^{\mathbf{T}_{m-1}}} \end{matrix}} & \rho^{\nu}\mathcal{O}(\rho^{\delta}) \\ \rho^{\nu}\mathcal{O}(\rho^{\delta}) & \boxed{\Pi^{\mathbf{T}_m}+\mathcal{O}(\rho^{2\delta})} \end{pmatrix} \tag{C.15}$$

where we have permuted the $P-1$ first lines and columns so as to regroup the λ's according to their equivalence classes. Therefore, each of the first $m-1$ blocks corresponds to a connected fully contracting subtree, while the last block corresponds to $\mathbf{T}_m$. This rearrangement leaves the determinant invariant. The factorization property (C.6) can now be read from the block structure of the matrix in (C.15) .

Considering the reduced matrix $Y^{\mathbf{T}}$ defined in (5.6) , we have a similar block structure

$$\det\left(Y^{\mathbf{T}}(\rho)\right) = \det\begin{pmatrix} \boxed{\begin{matrix} \boxed{Y^{\mathbf{T}_1}} & & \mathcal{O}(\rho^{2\delta}) \\ & \boxed{Y^{\mathbf{T}_2}} & \\ & & \ddots \\ \mathcal{O}(\rho^{2\delta}) & & \boxed{Y^{\mathbf{T}_{m-1}}} \end{matrix}} & \mathcal{O}(\rho^{\delta}) \\ \mathcal{O}(\rho^{\delta}) & \boxed{Y^{\mathbf{T}_m}+\mathcal{O}(\rho^{2\delta})} \end{pmatrix} \tag{C.16}$$

and we can now let $\rho \to 0$ and get

$$\det\left(\left[Y^{\mathbf{T}}(\rho \to 0)\right]\right) = \prod_{i=1}^{m} \det\left(\left[Y^{\mathbf{T}_i}\right]\right) \ , \tag{C.17}$$

which means that, in this limit, the tree has been disconnected into several components on which its determinant is exactly factorized.

Let us know turn to the variables β's defined in (5.2) . Notice that due to the rescaling (5.6) , $\det(Y^{\mathbf{T}})$ is actually independent of the global scale factor β_{N-1}[17]. Each variable β_γ can be associated with a contracting factor $\rho = \beta_\gamma$. Therefore, once expressed in term of the β's , $\det(Y^{\mathbf{T}})$ is such that, if we let one β tend to zero (say β_γ), keeping the others non zero, we have

$$\det\left(Y^{\mathbf{T}}\left(\beta_1,\ldots,\beta_{\gamma-1},\beta_\gamma\to 0,\beta_{\gamma+1},\ldots,\beta_{N-2};\theta^{\mathbf{T}}\right)\right) = \prod_{i=1}^{m-1}\det\left(Y^{\mathbf{T}_i}\left(\beta_1,\ldots,\beta_{\gamma-1};\theta^{\mathbf{T}_i}\right)\right) \times \det\left(Y^{\mathbf{T}_m}\left(\beta_{\gamma+1},\ldots,\beta_{N-2};\theta^{\mathbf{T}_m}\right)\right) \tag{C.18}$$

where the $m-1$ first determinants in the r.h.s. involve β_α with $\alpha<\gamma$ only, while the last determinant involves β_α with $\alpha>\gamma$ only. The angular parameter set $\theta^{\mathbf{T}}$ associated with $\mathbf{T}$ is left untouched by the rescaling, but simply decomposed into subsets $\theta^{\mathbf{T}_i}$ associated with the line vectors of the distinct subtrees $\mathbf{T}_i$ (see fig. 23). We are now interested in values of β and θ varying inside the domain $\mathcal{D}^{\mathbf{T}}$ and look at the possible zeros of $\det(Y^{\mathbf{T}})$ inside $\mathcal{D}^{\mathbf{T}}$. We already know that such zeros can be reached only when one β at least goes to zero. We thus fix all the variables θ, and all the variables β non zero except for one of them, β_γ. The quantity $\beta_\gamma^{\min}$ in (5.3) is therefore fixed, either strictly positive or zero. If it is strictly positive, this means that β_γ cannot reach 0 within the domain $\mathcal{D}^{\mathbf{T}}$ for this particular configuration of the other variables. This happens when the tree $\mathbf{T}_m$, obtained by fully contracting the lines $\lambda_1,\ldots,\lambda_\gamma$ of $\mathbf{T}$, is not compatible with the definition of the sector $\mathcal{D}^{\mathbf{T}}$. The only relevant case is therefore $\beta_\gamma^{\min}=0$. When $\beta_\gamma\to 0$, we can use equation (C.18) . The trees $\mathbf{T}_i$, $1\le i\le m-1$, were already subtrees of $\mathbf{T}$, hence the associated determinants $\det(Y^{\mathbf{T}_i})$, which involve only non vanishing β's, do not vanish. The new tree $\mathbf{T}_m$, which appears in the contraction process, is now *compatible* with the sector, which again implies that no fortuitous coincidence of its vertices can occur, and $\det(Y^{\mathbf{T}_m})$ itself cannot vanish. Thus $\det(Y^{\mathbf{T}})$ cannot vanish in this limit $\beta_\gamma\to 0$. This process can be iterated on the remaining determinants in (C.18) for successive β's going to zero. This shows that $\det(Y^{\mathbf{T}})$ does not vanish for any number of β's going to zero.

[17] This homogeneity property holds only for the choice (4.18) for the propagator, even on the sphere. Otherwise, both β_{N-1} and the IR regulator R would appear and lead to a slightly more complicated discussion.

Hence we reach the important result that $\det(Y^{\mathbf{T}})$ cannot vanish inside the whole sector $\mathcal{D}^{\mathbf{T}}$. Since $\mathcal{D}^{\mathbf{T}}$ is bounded (excluding the variable β_{N-1} which does not enter in $\det(Y^{\mathbf{T}})$), $\det(Y^{\mathbf{T}})$ is moreover bounded from below by a strictly positive number.

Appendix D. Example of cancellation of symmetry factors

Let us consider as in (7.30) the four compatible nests:

$$\begin{aligned}
\mathcal{N}_{\oplus} &= \Big\{(T,\omega)\Big\} \\
\mathcal{N}_{\oplus_2} &= \Big\{(R\wedge T,\bullet),(T,\omega)\Big\} \\
\mathcal{N}_{\oplus_3} &= \Big\{(T,\omega),(R\vee_\omega T,\bullet)\Big\} \\
\tilde{\mathcal{N}}_{\oplus} &= \Big\{(R\wedge T,\bullet),(T,\omega),(R\vee_\omega T,\bullet)\Big\}
\end{aligned} \tag{D.1}$$

where $R=\{\mathcal{R}\}$, $T=\{\mathcal{T}\}$ and $\omega=\big\{\{w\}\big\}$ with $w\in\mathcal{T}$. We want to show that the sum of the (-1) and symmetry factors associated with these nests (taking into account the degeneracy coming from the unspecified compatible roots $\bullet$) is equal to 0. We recall that to a compatible nest ${\mathcal{N}'}_{\oplus}$ is associated the factor in front of the associated Taylor operators (here we forget about the first diagram $T_0=(G_\odot,G_\odot)$ implicit in all the nests of (D.1), and the corresponding global (-1) factor):

$$(-1)^{\mathrm{Card}({\mathcal{N}'}_{\oplus})}W({\mathcal{N}'}_{\oplus}) \;=\; (-1)^{\mathrm{Card}({\mathcal{N}'}_{\oplus})}\prod_{\substack{w' \text{ root}\\ \text{of } {\mathcal{N}'}_{\oplus}}}\frac{1}{|\mathcal{T}_{w'}|} \tag{D.2}$$

with $\mathcal{T}_{w'}$ being the largest connected component (among all connected components of all diagrams of ${\mathcal{N}'}_{\oplus}$) whose root is w'.

The factor associated with $\mathcal{N}_{\oplus}$ in (D.1) is thus $(-)\frac{1}{|\mathcal{T}|}$. Let us now discuss the three remaining nests in (D.1).

Case (a): $w\in\mathcal{R}$ (see fig. 19)

The root of the connected component $\mathcal{R}\cap\mathcal{T}$ of $R\wedge T$ must be equal to w. The factor associated with $\mathcal{N}_{\oplus_2}$ is then $\frac{1}{|\mathcal{T}|}$. In $\mathcal{N}_{\oplus_3}$, the root of the connected component $\mathcal{R}\cup\mathcal{T}$ of $R\vee_\omega T$ is either equal to w, or belongs to $\mathcal{R}\setminus\mathcal{T}$. The factor associated with $\mathcal{N}_{\oplus_3}$ is therefore $\frac{1}{|\mathcal{R}\cup\mathcal{T}|}$ in the first case, and $\frac{1}{|\mathcal{T}|}\frac{1}{|\mathcal{R}\cup\mathcal{T}|}$ in the second case, with degeneracy $|\mathcal{R}\setminus\mathcal{T}|$. Hence, the global factor associated with $\mathcal{N}_{\oplus_3}$ and its possible rootings is $\frac{1}{|\mathcal{R}\cup\mathcal{T}|}+\frac{1}{|\mathcal{T}|}\frac{|\mathcal{R}\setminus\mathcal{T}|}{|\mathcal{R}\cup\mathcal{T}|}$ which, using $|\mathcal{R}\setminus\mathcal{T}|+|\mathcal{R}\cap\mathcal{T}|=|\mathcal{R}\cup\mathcal{T}|$, is nothing but $\frac{1}{|\mathcal{T}|}$. The factor associated with

$\tilde{\mathcal{N}}_{\oplus}$ is similarly equal to $(-)\Big[\frac{1}{|\mathcal{R}\cup\mathcal{T}|}+\frac{1}{|\mathcal{T}|}\frac{|\mathcal{R}\setminus\mathcal{T}|}{|\mathcal{R}\cup\mathcal{T}|}\Big]=-\frac{1}{|\mathcal{T}|}$. By summing up all these factors for all elements of (D.1) , we get zero as expected.

Case (b): $w\notin\mathcal{R}$ (see fig. 19)

The root of the connected component $\mathcal{R}\cap\mathcal{T}$ of $R\wedge T$ can now be any vertex of $\mathcal{R}\cap\mathcal{T}$. The factor associated with $\mathcal{N}_{\oplus_2}$ is in this case $\frac{|\mathcal{R}\cap\mathcal{T}|}{|\mathcal{R}\cap\mathcal{T}|}\frac{1}{|\mathcal{T}|}=\frac{1}{|\mathcal{T}|}$, since $\mathcal{N}_{\oplus_2}$ has now two distinct roots. In $\mathcal{N}_{\oplus_3}$, the two roots of the two connected components $\mathcal{T}$ and $\mathcal{R}\setminus\mathcal{T}$ of $R\vee_\omega T$ are respectively w and any vertex in $\mathcal{R}\setminus\mathcal{T}$. The factor associated with $\mathcal{N}_{\oplus_3}$ is then $\frac{1}{|\mathcal{T}|}\frac{|\mathcal{R}\setminus\mathcal{T}|}{|\mathcal{R}\setminus\mathcal{T}|}=\frac{1}{|\mathcal{T}|}$, while the factor associated with $\tilde{\mathcal{N}}_{\oplus}$ is $(-)\frac{|\mathcal{R}\cap\mathcal{T}|}{|\mathcal{R}\cap\mathcal{T}|}\frac{1}{|\mathcal{T}|}\frac{|\mathcal{R}\setminus\mathcal{T}|}{|\mathcal{R}\setminus\mathcal{T}|}=(-)\frac{1}{|\mathcal{T}|}$. Here too the sum of these factors gives zero as expected.

Appendix E. "Suppression" of a reducible line from the tableau nest

We assume here that the coincidence (7.35) holds in the tableau (7.36). We therefore have as a starting point the set of identities:

$$\begin{aligned} R^{\imath}_{\jmath-1}\wedge T_{\jmath}&=T_{\jmath} \qquad \imath\geq\imath_0\\ T_{\jmath}&=R^{\imath}_{\jmath}\wedge T_{\jmath+1} \qquad \imath\leq\imath_0. \end{aligned} \tag{E.1}$$

We want to prove that the lines (7.37) and (7.38) are then identical, *i.e.* that $T_{\jmath}$ can be skipped in the construction of the tableau. We thus have to prove the two following sets of identities:

(1) For $\imath\geq\imath_0$:

$$R^{\imath}_{\jmath}\wedge T_{\jmath+1}=R^{\imath}_{\jmath-1}\wedge T_{\jmath+1}. \tag{E.2}$$

(2) For $\imath\leq\imath_0$:

$$R^{\imath}_{\jmath-1}\wedge T_{\jmath+1}=R^{\imath}_{\jmath-1}\wedge T_{\jmath}. \tag{E.3}$$

These two sets of indentities are consequences of the stronger equality:

$$R^{\imath}_{\jmath}=R^{\imath}_{\jmath-1}\ , \quad \forall\imath\geq\imath_0. \tag{E.4}$$

Indeed (E.4) clearly implies (E.2) for case **(1)**. Furthermore, for case **(2)**, we make the following argument:

We use (E.1) to write $T_{\jmath}$ as:

$$T_{\jmath}=R^{\imath_0}_{\jmath}\wedge T_{\jmath+1}. \tag{E.5}$$

Fig. 24: *An application of the operation of fig. 20 : the figure describes a connected component $\mathcal{R}^{I,i}$ of R^I and the corresponding connected component $\mathcal{R}^{I,i}_{J-1}$ of $R^I \vee_{\omega_{J-1}} T_{J-1}$, obtained by fusing to $\mathcal{R}^{I,i}$ those connected components of $T_{J-1\oplus}$ which have their root inside $\mathcal{R}^{I,i}$ and cutting out those which do not have their root inside $\mathcal{R}^{I,i}$, but still intersect $\mathcal{R}^{I,i}$.*

Hence, we have:

$$\begin{aligned} R^I_{J-1} \wedge T_J &= (R^I_{J-1} \wedge R^{I_0}_J) \wedge T_{J+1} \\ &= R^I_{J-1} \wedge T_{J+1} \ , \ \text{QED} \end{aligned} \tag{E.6}$$

where we have made use of $R^I_{J-1} \prec R^{I_0}_{J-1}$ since $I \leq I_0$, together with $R^{I_0}_{J-1} = R^{I_0}_J$ as a particular case of (E.4). We are thus left with proving (E.4).

Proof of (E.4):

It is first useful to characterize the connected components of R^I_{J-1} (or of R^I_J). Denoting by $\mathcal{R}^{I,i}$ the connected components of R^I, a generic component $\mathcal{R}^{I,i}_{J-1}$ of R^I_{J-1} is of the form:

$$\mathcal{R}^{I,i}_{J-1} = \left[\mathcal{R}^{I,i} \bigcup_{w_{J-1,k} \in \mathcal{R}^{I,i}} \mathcal{T}_{J-1,k} \right] \setminus \left(\bigcup_{w'_{J-1,k'} \notin \mathcal{R}^{I,i}} \mathcal{T}'_{J-1,k'} \right) , \tag{E.7}$$

which simply states that a connected component $\mathcal{R}^{I,i}_{J-1}$ of $R^I_{J-1} = R^I \vee_{\omega_{J-1}} T_{J-1}$ is obtained from a connected component $\mathcal{R}^{I,i}$ of R^I by (see fig. 24):

- considering all the connected components of T_{J-1} ;
- making the union with $\mathcal{R}^{I,i}$ of those $\mathcal{T}_{J-1,k}$ which share their root with $\mathcal{R}^{I,i}$;
- cutting out from $\mathcal{R}^{I,i}$ those $\mathcal{T}'_{J-1,k'}$ which do not.

Since the connected components of T_{J-1} are all disjoint, the order of the union and cutting operations in (E.7) is indifferent. Notice also that the connected components of T_{J-1} which do not intersect $\mathcal{R}^{I,i}$ do not affect $\mathcal{R}^{I,i}_{J-1}$ in the operation $\vee_{\omega_{J-1}}$. Of course, it may happen that $\mathcal{R}^{I,i}_{J-1}$ is empty and R^I_{J-1} has in general less connected components than R^I.

For convenience, we introduce the notations:

$$A^{I,i}_{J-1} = \bigcup_{w_{J-1,k} \in \mathcal{R}^{I,i}} \mathcal{T}_{J-1,k} \tag{E.8}$$

$$B^{I,i}_{J-1} = \bigcup_{w'_{J-1,k'} \notin \mathcal{R}^{I,i}} \mathcal{T}'_{J-1,k'} \tag{E.9}$$

which are *complementary sets* in $\mathcal{G}$ since T_{J-1} is a complete diagram. With these notations, we have:

$$\mathcal{R}^{I,i}_{J-1} = \left(\mathcal{R}^{I,i} \cup A^{I,i}_{J-1}\right) \setminus \left(\mathcal{R}^{I,i} \cap B^{I,i}_{J-1}\right) \tag{E.10}$$

and a similar equation for the connected components $\mathcal{R}^{I,i}_{J}$ of R^I_J. Therefore, to prove $R^I_{J-1} = R^I_J$, it is enough to prove:

$$\boxed{\begin{aligned} A^{I,i}_{J} &= A^{I,i}_{J-1} \\ B^{I,i}_{J} &= B^{I,i}_{J-1} \end{aligned}} \tag{E.11}$$

The main ingredient comes from the property in (E.1):

$$T_J = R^I_{J-1} \wedge T_J, \quad \forall I \geq I_0 \ ,$$

which implies

$$\boxed{T_J \prec R^I_{J-1}} \tag{E.12}$$

This means that *any connected component $\mathcal{T}_{J,j}$ of T_J which intersects a connected component $\mathcal{R}^{I,i}_{J-1}$ is actually entirely included it the latter.*

$\diamond$ We first prove $A^{I,i}_{J-1} \subset A^{I,i}_{J}$:

Let us consider a connected component $\mathcal{T}_{J-1,k}$ of T_{J-1}, such that $w_{J-1,k} \in \mathcal{R}^{I,i}$. From the nest property, this connected component is included in a connected component $\mathcal{T}_{J,j}$ of T_J. By definition, $\mathcal{T}_{J-1,k} \subset \mathcal{R}^{I,i}_{J-1}$ and therefore $\mathcal{T}_{J,j}$ intersects $\mathcal{R}^{I,i}_{J-1}$. From (E.12), $\mathcal{T}_{J,j}$ is necessarily included in $\mathcal{R}^{I,i}_{J-1}$ and in particular its root $w_{J,j}$ belongs to $\mathcal{R}^{I,i}_{J-1}$, thus to $\mathcal{R}^{I,i} \cup A^{I,i}_{J-1}$. One has either $w_{J,j} \in \mathcal{R}^{I,i}$, or $w_{J,j} \in \mathcal{T}_{J-1,l}$ for some connected component $\mathcal{T}_{J-1,l}$ (with $l \neq k$ in general) of T_{J-1} such that $w_{J-1,l} \in \mathcal{R}^{I,i}$. In the latter case, from

the compatibility condition for the roots, we have $w_{J,j} = w_{J-1,l} \in \mathcal{R}^{I,i}$. Therefore, in any case, $w_{J,j} \in \mathcal{R}^{I,i}$ and $\mathcal{T}_{J,j} \subset A_J^{I,i}$. This implies $\mathcal{T}_{J-1,k} \subset A_J^{I,i}$, which leads to:

$$A_{J-1}^{I,i} \subset A_J^{I,i}. \tag{E.13}$$

$\diamond$ We now prove $B_{J-1}^{I,i} \subset B_J^{I,i}$:
We can use the fact that R^I is a complete diagram, thus each root $w_{J-1,k}$ of a connected component $\mathcal{T}_{J-1,k}$ belongs to one and only one connected component $\mathcal{R}^{I,i}$ of R^I. The set $B_{J-1}^{I,i}$ can therefore be expressed as:

$$B_{J-1}^{I,i} = \bigcup_{i' \neq i} A_{J-1}^{I,i'}. \tag{E.14}$$

A similar equation holds for $B_J^{I,i}$. Making use of (E.13) for each i' in the r.h.s. of (E.14), we directly arrive at:

$$B_{J-1}^{I,i} \subset B_J^{I,i}. \tag{E.15}$$

The inclusion properties (E.13) and (E.15), together with the fact that $A_{J-1}^{I,i}$ and $B_{J-1}^{I,i}$ on the one hand, and $A_J^{I,i}$ and $B_J^{I,i}$ on the other hand, are pairs of complementary sets of $\mathcal{G}$, imply (E.11), hence (E.4).

Appendix F. Addition of reducible lines in the tableau nest

$\diamond$ We want to prove first that, if we consider a compatibly rooted nest $\mathcal{N}_\oplus = \{T_{0\oplus}, \ldots, T_{T\oplus}\}$ and build the larger nest $\mathcal{N}'_\oplus = \{T_{0\oplus}, \ldots, T_{J-1\oplus}, T_{J-1\oplus}^{I_0}, T_{J\oplus}, \ldots, T_{T\oplus}\}$ by inserting between the levels $J-1$ and J of $\mathcal{N}_\oplus$ an extra rooted diagram $T_{J-1\oplus}^{I_0} = (T_{J-1}^{I_0}, \omega_{J-1}^{I_0})$ with:

$$T_{J-1}^{I_0} = R_{J-1}^{I_0} \wedge T_J \equiv T' \tag{F.1}$$

and $\omega_{J-1}^{I_0} \equiv \omega'$ an arbitrary set of roots compatible with the rooting of $\mathcal{N}_\oplus$ (making $\mathcal{N}'_\oplus$ compatibly rooted), then the tableau of $\mathcal{N}'_\oplus$ can be reduced to that of $\mathcal{N}_\oplus$. For convenience, we denote $T_{J-1\oplus}^{I_0}$ by $T_\oplus{}' = (T', \omega')$.
More precisely, the tableau built from $\mathcal{N}'_\oplus$ is:

$$\begin{array}{cccccc}
\vdots & & & & & \\
T_{J-1} & R_{J-1}^1 \wedge T' & \ldots & \boxed{R_{J-1}^{I_0} \wedge T'} & \ldots & R_{J-1}^{N-1} \wedge T' \\
T' & R'^1 \wedge T_J & \ldots & \boxed{R'^{I_0} \wedge T_J} & \ldots & R'^{N-1} \wedge T_J \\
T_J & & & & & \\
\vdots & & & & &
\end{array} \tag{F.2}$$

where

$$R'^{I} \equiv R^{I} \vee_{\omega'} T'. \tag{F.3}$$

We want to prove that this tableau has the coincidence property for $I = I_0$:

$$R^{I_0}_{J-1} \wedge T' = R'^{I_0} \wedge T_J \ , \tag{F.4}$$

and therefore can be reduced to the tableau of $\mathcal{N}_{\oplus}$. From the definition of T', the term on the l.h.s. of (F.4) is nothing but $R^{I_0}_{J-1} \wedge T_J$ and the coincidence property is equivalent to:

$$R^{I_0}_{J-1} \wedge T_J = R'^{I_0} \wedge T_J. \tag{F.5}$$

This last equation is actually a consequence of the stronger indentity

$$R^{I_0}_{J-1} = R'^{I_0} \tag{F.6}$$

which we prove now.

Proof of (F.6):

As in Appendix F, we consider a typical connected component $\mathcal{R}^{I_0,i_0}_{J-1}$ of $R^{I_0}_{J-1}$, defined by:

$$\mathcal{R}^{I_0,i_0}_{J-1} = \left[\mathcal{R}^{I_0,i_0} \bigcup_{w_{J-1,k} \in \mathcal{R}^{I_0,i_0}} \mathcal{T}_{J-1,k}\right] \setminus \left(\bigcup_{w_{J-1,l} \notin \mathcal{R}^{I_0,i_0}} \mathcal{T}_{J-1,l}\right) , \tag{F.7}$$

or by the equivalent equation:

$$\mathcal{R}^{I_0,i_0}_{J-1} = \left(\mathcal{R}^{I_0,i_0} \cup A^{I_0,i_0}_{J-1}\right) \setminus \left(\mathcal{R}^{I_0,i_0} \cap B^{I_0,i_0}_{J-1}\right) \tag{F.8}$$

where

$$A^{I_0,i_0}_{J-1} = \bigcup_{w_{J-1,k} \in \mathcal{R}^{I_0,i_0}} \mathcal{T}_{J-1,k} \ , \tag{F.9}$$

$$B^{I_0,i_0}_{J-1} = \bigcup_{w_{J-1,l} \notin \mathcal{R}^{I_0,i_0}} \mathcal{T}_{J-1,l}. \tag{F.10}$$

The sets A^{I_0,i_0}_{J-1} and B^{I_0,i_0}_{J-1} are complementary subsets of $\mathcal{G}$ and, as in Appendix E:

$$B^{I_0,i_0}_{J-1} = \bigcup_{i \neq i_0} A^{I,i}_{J-1}. \tag{F.11}$$

We then can write for R'^{I_0} an equation similar to (F.8) with A^{I_0,i_0}_{J-1} and B^{I_0,i_0}_{J-1} replaced by:

$$A'^{I_0,i_0} = \bigcup_{w'^i_k \in \mathcal{R}^{I_0,i_0}} \mathcal{T}'^i_k \tag{F.12}$$

$$B'^{I_0,i_0} = \bigcup_{w'^i_l \notin \mathcal{R}^{I_0,i_0}} \mathcal{T}'^i_l \tag{F.13}$$

which are complementary subsets of $\mathcal{G}$ and satisfy an equation similar to (F.11). In (F.12) and (F.13), $\mathcal{T}'^i_k$ is the generic connected component of T' given by:

$$\mathcal{T}'^i_k = \mathcal{R}^{I_0,i}_{J-1} \cap \mathcal{T}_{J,k} \tag{F.14}$$

and w'^i_k is its root.

In order to prove (F.6), it is sufficient to prove that $A^{I_0,i_0}_{J-1} \subset A'^{I_0,i_0}$. Indeed, from (F.11) and the similar equation for B'^{I_0,i_0}, this inclusion will imply $B^{I_0,i_0}_{J-1} \subset B'^{I_0,i_0}$. From the complementarity property of A^{I_0,i_0}_{J-1} and B^{I_0,i_0}_{J-1} on the one hand, and that of A'^{I_0,i_0} and B'^{I_0,i_0} on the other hand, the two equalities:

$$A^{I_0,i_0}_{J-1} = A'^{I_0,i_0} \tag{F.15}$$

$$B^{I_0,i_0}_{J-1} = B'^{I_0,i_0} \tag{F.16}$$

follow, leading to (F.6).

$\diamond$ We are thus left with proving $A^{I_0,i_0}_{J-1} \subset A'^{I_0,i_0}$:
Let us consider a connected component $\mathcal{T}_{J-1,k}$ of T_{J-1}, such that $w_{J-1,k} \in \mathcal{R}^{I_0,i_0}$. From the nest property, this connected component is included in a connected component $\mathcal{T}_{J,j}$ of T_J. By definition, $\mathcal{T}_{J-1,k} \subset \mathcal{R}^{I_0,i_0}_{J-1}$ and therefore $\mathcal{T}_{J-1,k} \subset \mathcal{R}^{I_0,i_0}_{J-1} \cap \mathcal{T}_{J,j} \equiv \mathcal{T}'^{i_0}_j$. The root $w'^{i_0}_j$ of $\mathcal{T}'^{i_0}_j$ belongs to $\mathcal{R}^{I_0,i_0}_{J-1}$, thus to $\mathcal{R}^{I_0,i_0} \cup A^{I_0,i_0}_{J-1}$. One has either $w'^{i_0}_j \in \mathcal{R}^{I_0,i_0}$ or $w'^{i_0}_j \in \mathcal{T}_{J-1,l}$ for some connected component $\mathcal{T}_{J-1,l}$ (with $l \neq k$ in general) of T_{J-1} such that $w_{J-1,l} \in \mathcal{R}^{I_0,i_0}$. In the latter case, from the compatibility condition (in the nest $\mathcal{N}'_\oplus$) between the root $w'^{i_0}_j$ and the roots of T_{J-1}, one has $w'^{i_0}_j = w_{J-1,l} \in \mathcal{R}^{I_0,i_0}$. Therefore, in any case, $w'^{i_0}_j \in \mathcal{R}^{I_0,i_0}$ and $\mathcal{T}'^{i_0}_j \subset A'^{I_0,i_0}$. This implies $\mathcal{T}_{J-1,k} \subset A'^{I_0,i_0}$, which leads to:

$$A^{I_0,i_0}_{J-1} \subset A'^{I_0,i_0} \ , \tag{F.17}$$

which completes the proof.

$\diamond$ The above property generalizes to a nest $\mathcal{N}'_\oplus$ obtained from $\mathcal{N}_\oplus$ by inserting between the levels $\jmath-1$ and $\jmath$ an arbitrary number of diagrams $T^{\imath_0}_{\jmath-1\oplus}, T^{\imath_1}_{\jmath-1\oplus}, \ldots, T^{\imath_K}_{\jmath-1\oplus}$ with $1 \leq \imath_0 \leq \imath_1 \leq \ldots \leq \imath_K \leq N-2$, where as before:

$$T^{\imath}_{\jmath-1} = R^{\imath}_{\jmath-1} \wedge T_\jmath \ , \tag{F.18}$$

and where the roots of these extra diagrams are such that $\mathcal{N}'_\oplus$ is compatibly rooted. Indeed, one can proceed by recursion by adding first $T^{\imath_K}_{\jmath-1} = R^{\imath_K}_{\jmath-1} \wedge T_\jmath$ between $T_{\jmath-1}$ and $T_\jmath$. Then one can add $R^{\imath_{K-1}}_{\jmath-1} \wedge T^{\imath_K}_{\jmath-1}$ between $T_{\jmath-1}$ and $T^{\imath_K}_{\jmath-1}$. From the nest property of the sector nest $\mathcal{S}$, we have $R^{\imath_{K-1}}_{\jmath-1} \prec R^{\imath_K}_{\jmath-1}$ and this second added diagram is nothing but $R^{\imath_{K-1}}_{\jmath-1} \wedge R^{\imath_K}_{\jmath-1} \wedge T_\jmath = R^{\imath_{K-1}}_{\jmath-1} \wedge T_\jmath = T^{\imath_{K-1}}_{\jmath-1}$ as wanted. This process can be repeated until the first diagram $T^{\imath_0}_{\jmath-1}$ is inserted.

$\diamond$ Finally, the above property also generalizes to arbitrary insertions between several pairs $(\jmath-1, \jmath)$, each pair being actually decoupled from the other pairs.

When applied to a minimal nest $\mathcal{N}^0_\oplus$, this property means that all the nests $\mathcal{N}'_\oplus$ obtained from $\mathcal{N}^0_\oplus$ by inserting an arbitrary number of diagrams of $\tilde{\mathcal{N}}(\mathcal{S}, \mathcal{N}^0_\oplus) \setminus \mathcal{N}^0$ (rooted with compatible roots) lead by reduction to $\mathcal{N}^0_\oplus$, and therefore belong to $\mathcal{C}_\mathcal{S}(\mathcal{N}^0_\oplus)$.

Appendix G. Sum rule for the weights W

In this appendix, we prove (7.42). Given a nest $\mathcal{N}$, we first give an alternative procedure to construct all compatible rootings $\oplus_\mathcal{N}$ of $\mathcal{N}$, with their weight factor $W(\mathcal{N}_{\oplus_\mathcal{N}})$ (7.13).

Let σ be a bijection from $\{1, 2, \ldots, N\}$ into $\mathcal{G}$ (it is nothing but an ordering of the N vertices of $\mathcal{G}$). There are $N!$ such orderings. To any subset $\mathcal{P}$ of $\mathcal{G}$, we assign a root p through σ by the following definition:

$$p = \sigma(k) \quad \text{where} \quad k = \min(n \in \{1, \ldots, N\} \ : \ \sigma(n) \in \mathcal{P}). \tag{G.1}$$

We denote this assignment procedure by:

$$\mathcal{P} \xrightarrow{\sigma} p. \tag{G.2}$$

It is easy to check that, when applied to all connected components of all diagrams of $\mathcal{N}$, this rooting procedure builds a compatible rooting of $\mathcal{N}$. Moreover, all compatible rootings

of $\mathcal{N}$ can be built in that way. Given such a rooting $\oplus_{\mathcal{N}}$, the number of distinct orderings σ which build $\oplus_{\mathcal{N}}$ is:

$$K(\mathcal{N}_{\oplus_{\mathcal{N}}}) \equiv \mathrm{Card}(\{\sigma : \ \forall(\mathcal{T}, w) \text{ rooted connected comp. of } \mathcal{N}_{\oplus_{\mathcal{N}}} \ , \mathcal{T} \stackrel{\sigma}{\longrightarrow} w\}). \tag{G.3}$$

It is simply related to the weight $W(\mathcal{N}_{\oplus_{\mathcal{N}}})$ by

$$\frac{K(\mathcal{N}_{\oplus_{\mathcal{N}}})}{N!} = W(\mathcal{N}_{\oplus_{\mathcal{N}}}) = \prod_{w} \frac{1}{|\mathcal{T}_w|} \ . \tag{G.4}$$

Indeed, given a subset $\mathcal{P}$ of $\mathcal{G}$ and a vertex p in $\mathcal{P}$, the number of σ which assign p to $\mathcal{P}$ is $N!/|\mathcal{P}|$ (the probability for p to be the first vertex of $\mathcal{P}$ to appear in the sequence $\sigma(1), \ldots, \sigma(N)$ is $1/|\mathcal{P}|$). A compatible rooting $\oplus_{\mathcal{N}}$ of $\mathcal{N}$ is entirely known once one specifies for each vertex w the largest connected component of $\mathcal{N}$, $\mathcal{T}_w$, which has w as its root. The above argument can then be extended to all these largest connected components of $\mathcal{N}$ containing the roots of $\oplus_{\mathcal{N}}$, and leads to (G.4).

The proof of (7.42)is then straightforward. Indeed, the r.h.s. of (7.42) is simply $1/(N!)$ times

$$\mathrm{Card}(\{\sigma : \ \forall(\mathcal{T}, w) \text{ rooted connected comp. of } \mathcal{N}_{\oplus} \ , \mathcal{T} \stackrel{\sigma}{\longrightarrow} w\}), \tag{G.5}$$

while each term of the sum in the l.h.s of (7.42) is $1/(N!)$ times

$$\begin{aligned} \mathrm{Card}(\{\sigma : \ & \bullet \, \forall(\mathcal{T}, w) \text{ rooted connected comp. of } \mathcal{N}_{\oplus} \ , \mathcal{T} \stackrel{\sigma}{\longrightarrow} w \ , \\ & \bullet \, \forall(\mathcal{T}, w) \text{ rooted connected comp. of } \mathcal{M}_{\oplus_{\mathcal{M}}} \text{ not in } \mathcal{N}_{\oplus} \ , \mathcal{T} \stackrel{\sigma}{\longrightarrow} w\}). \end{aligned} \tag{G.6}$$

The sum over $\oplus_{\mathcal{M}}$ in (7.42) relaxes the second constraint on σ in (G.6), and reproduces (G.5). Hence (7.42) follows.

Appendix H. Estimates of subtracted integrands in a Hepp sector

In this appendix, we prove (8.35) and (8.37). We shall proceed in three steps:

(I) We first analyze the properties of the elements of the matrix $Y^{\mathbf{T}_{J,j}}$ in terms of the β^I variables.

(II) We then write an integral representation of the $(1 - \mathfrak{T} \ )$ operators appearing in the l.h.s. of (8.35) or (8.37).

(III) We finally show (8.35) and (8.37).

$\diamond$ (I) *Properties of* $Y^{\mathbf{T}_{J,j}}$

In this subsection, we shall work separately inside each connected component $\tilde{\mathcal{T}}_{J,j}$ of $\tilde{T}_J$. As explained in section 8.2, the line vectors $\lambda^I_{J,j}$ of the oriented ordered tree $\mathbf{T}_{J,j}$ spanning $\tilde{\mathcal{T}}_{J,j}$ are uniquely labeled by $I \in \overline{\mathrm{Ind}}(J,j)$. From now on, we shall suppress the indices (J,j) and thus denote $\lambda^I_{J,j}$ by λ^I. A typical element of the matrix $Y^{\mathbf{T}_{J,j}}$ writes:

$$Y^{\mathbf{T}_{J,j}}_{KL} = \frac{-1}{2|\lambda^K|^\nu\ |\lambda^L|^\nu}\Big\{|R^{KL}+\lambda^L-\lambda^K|^{2\nu}-|R^{KL}+\lambda^L|^{2\nu}-|R^{KL}-\lambda^K|^{2\nu}+|R^{KL}|^{2\nu}\Big\} \quad \text{(H.1)}$$

where R^{KL} is the "basis" of the quadrilateral

$$R^{KL} = x_{i_L} - x_{i_K}\ , \quad \text{(H.2)}$$

with i_K and i_L being the origins of λ^K and λ^L. The vector R^{KL} is a linear combination of the λ^I's joining x_{i_K} and x_{i_L}, and since the tree $\mathbf{T}_{J,j}$ has been built from the rooted sector $\mathcal{S}_{J,j\oplus}$, this linear combination involves only λ^I's for $I > \min(K,L)$ (see section 7.2):

$$R^{KL} = \sum_{M>\min(K,L)} c^{KL}_M\,\lambda^M \quad \text{(H.3)}$$

with $c^{KL}_M = 0, \pm 1$.

PROPOSITIONS:

• *Prop. 1:* $\det(Y^{\mathbf{T}_{J,j}})$ is a positive, non vanishing continuous function on the compact domain $\mathcal{H}^{\mathcal{S}}$, and is therefore bounded from below on $\mathcal{H}^{\mathcal{S}}$ by a strictly positive number. In particular, the matrix $Y^{\mathbf{T}_{J,j}}$ is invertible.

• *Prop. 2:* $Y^{\mathbf{T}_{J,j}}_{KL}$, as a function of the β, χ and θ variables, depends on the β^I's for I in some subset $\mathcal{J}_{J,j}(K,L)$ only, defined as

$$\mathcal{J}_{J,j}(K,L) = \Big\{I : \min(K,L) \le I < \max\big(K,L,\max(M : c^{KL}_M \neq 0)\big)\Big\} \quad \text{(H.4)}$$

with the convention that: $\max\big(K,L,\max(M : c^{KL}_M \neq 0)\big) = \max(K,L)$, if all the c^{KL} are zero (that is if $R^{KL}=0$).

• *Prop. 3:* Inside the sector $\mathcal{H}^{\mathcal{S}}$,

$$Y^{\mathbf{T}_{J,j}}_{KL} = \mathcal{O}\Big(\prod_{I\in\mathcal{J}_{J,j}(K,L)} (\beta^I)^\delta\Big). \quad \text{(H.5)}$$

• *Prop. 4:* The matrix $Y^{\mathbf{T}_{J,j}}$ is positive, and bounded from below by a strictly positive constant. By this we mean that there exist a strictly positive number C such that $(Y^{\mathbf{T}_{J,j}} - C\mathbb{1})$ is a positive matrix on $\mathcal{H}^{\mathcal{S}}$.

Proposition 1 has already been proven in Appendix C, in the restricted case of a generalized Hepp sector $\mathcal{H}^{\mathbf{T}}$ attached to some tree $\mathbf{T}$. The proof can be carried over to the whole extended Hepp sector $\mathcal{H}^{\mathcal{S}}$ attached to the nest $\mathcal{S} = \mathcal{S}(\mathbf{T})$. Indeed, the spirit of the proof is that $Y^{\mathbf{T}}$ depends only on ratios of successive λ's (β variables); from the bounds on those ratios inside $\mathcal{H}^{\mathbf{T}}$, we deduce that if some points coincide then one of these ratios at least must vanish, and $\det(Y^{\mathbf{T}})$ factorizes and remains strictly positive. Since, from Schoenberg's theorem, this is the only case when $\det(Y^{\mathbf{T}})$ might have vanished, we deduce that it actually never vanishes, and remains positive inside $\mathcal{H}^{\mathbf{T}}$. Inside $\mathcal{H}^{\mathcal{S}}$, we have weaker bounds on the ratios of λ's but one can check that this does not alter the proof.

To prove Propositions 2 and 3, we first consider the trivial case $K = L$. In this case $\mathcal{J}_{J,j}(K, K) = \emptyset$ but then $Y^{\mathbf{T}_{J,j}(K,K)} = 1$, which satisfies these propositions.
We can therefore assume that $K < L$. Four distinct situations may occur:

(a) $R^{KL} = 0$: then $\mathcal{J}_{J,j}(K, L) = \{I : K \leq I < L\}$;

If $R^{K,L} \neq 0$, we denote by:

$$P = \max(M : c_M^{KL} \neq 0) \ . \tag{H.6}$$

(b) If $P > L$, then $\mathcal{J}_{J,j}(K, L) = \{I : K \leq I < P\}$;

(c) If $K < P < L$, then $\mathcal{J}_{J,j}(K, L) = \{I : K \leq I < L\}$;

(d) If $P = L$, then $\mathcal{J}_{J,j}(K, L) = \{I : K \leq I < L\}$.

We shall use the property that, if $A > B$ and $A > C$, then the quantity:

$$\frac{|\lambda^A + (\sum_B \pm\lambda^B) \pm \lambda^C|^{2\nu} - |\lambda^A + (\sum_B \pm\lambda^B)|^{2\nu}}{|\lambda^A|^{2\nu-1}|\lambda^C|} \tag{H.7}$$

is bounded (in module) from above inside $\mathcal{H}^{\mathcal{S}}$. This follows from the fact that, inside the sector $\mathcal{H}^{\mathcal{S}}$, the ratios $|\lambda^B|/|\lambda^A|$, $|\lambda^C|/|\lambda^A|$ and $|\lambda^A|/|\lambda^A + \sum_B \pm\lambda^B|$ are bounded. Eq. (H.7) can then easily be obtained by use of the mean value theorem.

By a simple generalization of this property, one can show that, if $A > B$, $A > C$ and $A > D$, then the quantity:

$$\frac{|\lambda^A + (\sum_B \pm\lambda^B) \pm \lambda^C \pm \lambda^D|^{2\nu} - |\lambda^A + (\sum_B \pm\lambda^B) \pm \lambda^C|^{2\nu} - |\lambda^A + (\sum_B \pm\lambda^B) \pm \lambda^D|^{2\nu} + |\lambda^A + (\sum_B \pm\lambda^B)|^{2\nu}}{|\lambda^A|^{2\nu-2}|\lambda^C||\lambda^D|} \tag{H.8}$$

is also bounded (in module) from above inside $\mathcal{H}^{\mathcal{S}}$.

Let us now consider cases (a)–(d) above.

Case (a): We can write:

$$\begin{aligned} Y_{KL}^{\mathbf{T}_{J,j}} &= -\frac{1}{2}\Big\{ \Big(\frac{|\lambda^K|}{|\lambda^L|}\Big)^{1-\nu} \frac{|\lambda^L - \lambda^K|^{2\nu} - |\lambda^L|^{2\nu}}{|\lambda^L|^{2\nu-1}|\lambda^K|} - \Big(\frac{|\lambda^K|}{|\lambda^L|}\Big)^{\nu} \Big\} \\ &= \mathcal{O}\Big\{ \Big(\frac{|\lambda^K|}{|\lambda^L|}\Big)^{1-\nu} \Big\} + \mathcal{O}\Big\{ \Big(\frac{|\lambda^K|}{|\lambda^L|}\Big)^{\nu} \Big\} \\ &= \mathcal{O}\Big(\prod_{K \le I < L} (\beta^I)^{\delta} \Big) , \end{aligned} \tag{H.9}$$

which proves Proposition 3 in this case. In (H.9), we used (H.7) and the fact that $|\lambda^K|/|\lambda^L|$ is of the same order that $\prod_{K \le I < L} (\beta^I)$ since:

$$\frac{|\lambda^K|}{|\lambda^L|} = \frac{\chi^K}{\chi^L} \prod_{K \le I < L} (\beta^I) , \tag{H.10}$$

and since the χ variables are bounded from above and from below. From (H.9), we also deduce that $Y_{KL}^{\mathbf{T}_{J,j}}$ depends only on $|\lambda^K|/|\lambda^L|$, that is, from (H.10), depends only on β^I for $K \le I < L$, which precisely defines $\mathcal{J}_{J,j}(K, L)$ in this case, whence Proposition 2.

Case (b): We can now write:

$$\begin{aligned} Y_{KL}^{\mathbf{T}_{J,j}} &= -\frac{1}{2}\Big(\frac{|\lambda^K|}{|\lambda^P|}\Big)^{1-\nu}\Big(\frac{|\lambda^L|}{|\lambda^P|}\Big)^{1-\nu} \times \\ &\frac{|\lambda^P + \sum c_M^{KL}\lambda^M + \lambda^L - \lambda^K|^{2\nu} - |\lambda^P + \sum c_M^{KL}\lambda^M + \lambda^L|^{2\nu} - |\lambda^P + \sum c_M^{KL}\lambda^M - \lambda^K|^{2\nu} + |\lambda^P + \sum c_M^{KL}\lambda^M|^{2\nu}}{|\lambda^P|^{2\nu-2}|\lambda^K||\lambda^L|} \\ &= \mathcal{O}\Big\{ \Big(\frac{|\lambda^K|}{|\lambda^P|}\Big)^{1-\nu}\Big(\frac{|\lambda^L|}{|\lambda^P|}\Big)^{1-\nu} \Big\} \\ &= \mathcal{O}\Big(\prod_{K \le I < L} (\beta^I)^{1-\nu} \prod_{L \le I < P} (\beta^I)^{2-2\nu} \Big) \\ &= \mathcal{O}\Big(\prod_{K \le I < P} (\beta^I)^{\delta} \Big) , \end{aligned} \tag{H.11}$$

by use of (H.8). This proves Proposition 3 in this case. Moreover, from (H.11), $Y_{KL}^{\mathbf{T}_{J,j}}$ can be written as a function of the ratios $|\lambda^K|/|\lambda^P|$, $|\lambda^L|/|\lambda^P|$ and $|\lambda^M|/|\lambda^P|$. Since $K < M, L < P$, these ratios involve β^I for $K \leq I < P$ only. This again proves Proposition 2.

Case (c): We now write:

$$\begin{aligned}
Y_{KL}^{\mathbf{T}_{J,j}} &= \frac{1}{2}\Big(\frac{|\lambda^K|}{|\lambda^P|}\Big)^{1-\nu}\Big(\frac{|\lambda^P|}{|\lambda^L|}\Big)^{\nu}\Big\{\frac{|\lambda^P + \sum c_M^{KL}\lambda^M - \lambda^K|^{2\nu} - |\lambda^P + \sum c_M^{KL}\lambda^M|^{2\nu}}{|\lambda^P|^{2\nu-1}|\lambda^K|}\Big\} \\
&- \frac{1}{2}\Big(\frac{|\lambda^K|}{|\lambda^L|}\Big)^{1-\nu}\Big\{\frac{|\lambda^L + \lambda^P + \sum c_M^{KL}\lambda^M - \lambda^K|^{2\nu} - |\lambda^L + \lambda^P + \sum c_M^{KL}\lambda^M|^{2\nu}}{|\lambda^L|^{2\nu-1}|\lambda^K|}\Big\} \\
&= \mathcal{O}\Big\{\Big(\frac{|\lambda^K|}{|\lambda^P|}\Big)^{1-\nu}\Big(\frac{|\lambda^P|}{|\lambda^L|}\Big)^{\nu}\Big\} + \mathcal{O}\Big\{\Big(\frac{|\lambda^K|}{|\lambda^L|}\Big)^{1-\nu}\Big\} \\
&= \mathcal{O}\Big\{\prod_{K\leq I<P}(\beta^I)^{1-\nu}\prod_{P\leq I<L}(\beta^I)^{\nu}\Big\} + \mathcal{O}\Big\{\prod_{K\leq I<L}(\beta^I)^{1-\nu}\Big\} \\
&= \mathcal{O}\Big(\prod_{K\leq I<L}(\beta^I)^{\delta}\Big)\,,
\end{aligned} \tag{H.12}$$

which proves Proposition 2. Here again, we can write $Y_{KL}^{\mathbf{T}_{J,j}}$ as a function of the ratios $|\lambda^K|/|\lambda^L|$, $|\lambda^P|/|\lambda^L|$ and $|\lambda^M|/|\lambda^L|$. Since $K < M < P < L$, we deduce Proposition 3.

Case (d): In this case $R^{KL} = -\lambda^L + \sum c_M^{KL}\lambda^M$ and the propositions can be obtained from case (c) by simply interchanging R^{KL} and $R^{KL}+\lambda^L$. This achieves the proof of Propositions 2 and 3.

Finally, Proposition 4 is a consequence of Propositions 1 and 3. Indeed, from Proposition 3 and the fact that the β^I are bounded from above inside $\mathcal{H}^{\mathcal{S}}$, we obtain a uniform upper bound for $|Y_{KL}^{\mathbf{T}_{J,j}}|$ inside $\mathcal{H}^{\mathcal{S}}$. This upper bound, together with the lower bound of Proposition 1 on $\det(Y^{\mathbf{T}_{J,j}})$ gives a uniform upper bound for the modules $|(Y^{\mathbf{T}_{J,j}})^{-1}_{KL}|$ of the elements of the inverse matrix. This then implies that $(Y^{\mathbf{T}_{J,j}})^{-1}$ is bounded from above by a positive number C^{-1} (that is $(Y^{\mathbf{T}_{J,j}})^{-1} - C^{-1}\mathbb{1}$ is a negative matrix), and, since $(Y^{\mathbf{T}_{J,j}})^{-1}$ is a positive matrix, that $Y^{\mathbf{T}_{J,j}}$ is bounded from below by the strictly positive number C.

◇ (II) *Integral representation of* $(1 - \mathfrak{T})$

From now on, we shall work inside the whole diagram $\tilde{T}_J$ for fixed J and treat in parallel its distinct connected components $\tilde{\mathcal{T}}_{J,j}$ for varying j. This is achieved by introducing the

block diagonal matrix:

$$Y^J = \begin{bmatrix} \boxed{Y^{\mathbf{T}_{J,1}}} & 0 & \dots & 0 \\ 0 & \boxed{Y^{\mathbf{T}_{J,2}}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boxed{Y^{\mathbf{T}_{J,j^{\max}}}} \end{bmatrix} \tag{H.13}$$

with $j^{\max} = \mathrm{Card}(T^0_J)$. This matrix is such that (for $J \leq T$):

$$I_{\tilde{T}_J} = \Big(\det(Y^J)\Big)^{-\frac{d}{2}}. \tag{H.14}$$

Now we must consider the action of $(1 - \mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}})$ on $I_{\tilde{T}_J}$. For our particular choice of tree variables, the action of $\mathfrak{T}\ 0_{\tilde{T}^I_{J-1\oplus}}$ simply corresponds to set $\beta^I = 0$ in the matrix Y^J. From the Propositions 2 and 3 of the preceding subsection, we know that an element $Y^{\mathbf{T}_{J,j}}_{KL}$ of the matrix Y^J either is independent of β^I (if $I \notin \mathcal{J}_{J,j}(K,L)$ for this value of j), or vanishes with β^I at least as $(\beta^I)^\delta$ (if $I \in \mathcal{J}_{J,j}(K,L)$). Therefore the action of $\mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}}$ simply corresponds to set to zero those elements of Y^J which depend on β^I, leaving the other elements unchanged. In particular, this action is non trivial (*i.e.* non reduced to the identity) when

$$I \in \mathcal{J}_J \equiv \bigcup_{j,K,L} \mathcal{J}_{J,j}(K,L). \tag{H.15}$$

Conversely, if $I \notin \mathcal{J}_J$, then $(1 - \mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}})[I_{\tilde{T}_J}] = 0$. To perform the action of $\mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}}$, it is convenient to introduce an extra variable t^I which multiplies the elements $Y^{\mathbf{T}_{J,j}}_{KL}$ such that $I \in \mathcal{J}_{J,j}(K,L)$. We thus define

$$Y^{\mathbf{T}_{J,j}}_{KL}(\{t\}) \equiv \Big(\prod_{I \in \mathcal{J}_{J,j}(K,L)} t^I\Big)\ Y^{\mathbf{T}_{J,j}}_{KL}\ , \tag{H.16}$$

and obtain a matrix $Y^J(\{t\})$ which is a function of the t^I's for $I \in \mathcal{J}_J$. The action of $\mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}}$ then corresponds to set $t^I = 0$ (and set the other t^I's equal to 1). We then have the following integral representation of a $(1 - \mathfrak{T}_{\tilde{T}^I_{J-1\oplus}})$ operator with $I \in \mathcal{J}_J$:

$$(1 - \mathfrak{T}^0_{\tilde{T}^I_{J-1\oplus}})\Big[\prod_{j=1}^{\mathrm{Card}(T^0_J)} \Big(\det(Y^{\mathbf{T}_{J,j}})\Big)^{-\frac{d}{2}}\Big] = \int_0^1 dt^I\ \frac{\partial}{\partial t^I}\Big[\det\big(Y^J(\{t\})\big)\Big]^{-\frac{d}{2}}\ . \tag{H.17}$$

Now we must apply a product of such Taylor operators for all the $\imath \in \mathrm{Ind}(\jmath)$. We can use the fact that $(1-\mathfrak{T}\ )$ is a projector, and can thus be applied several times to the same diagram. Since all the reduced diagrams $\tilde{T}^{\imath}_{\jmath-1}$ for

$$\imath \in \overline{\mathcal{J}_\jmath} \equiv \{\imath :\ \imath^{\min}(\jmath) \leq \imath < \imath^{\max}(\jmath)\} \tag{H.18}$$

are equal to some $\tilde{T}^{\imath}_{\jmath-1}$ for $\imath \in \mathrm{Ind}(\jmath)$, we have

$$\prod_{\imath \in \mathrm{Ind}(\jmath)} (1-\mathfrak{T}^{0}_{\tilde{T}^{\imath}_{\jmath-1\oplus}}) \ = \ \prod_{\imath \in \overline{\mathcal{J}_\jmath}} (1-\mathfrak{T}^{0}_{\tilde{T}^{\imath}_{\jmath-1\oplus}})\ . \tag{H.19}$$

From their definitions and (8.11) and (8.12), it is clear that $\mathcal{J}_\jmath \subset \overline{\mathcal{J}_\jmath}$. If $\mathcal{J}_\jmath \subsetneq \overline{\mathcal{J}_\jmath}$, then the above product of $(1-\mathfrak{T}\ )$, when acting on $I_{\tilde{T}_\jmath}$, gives 0, as a consequence of the discussion above. (8.35) is then obviously satisfied. We can therefore assume that $\mathcal{J}_\jmath = \overline{\mathcal{J}_\jmath}$. We then write the l.h.s of (8.35) as:

$$\prod_{\imath \in \mathrm{Ind}(\jmath)} (1-\mathfrak{T}^{0}_{\tilde{T}^{\imath}_{\jmath-1\oplus}}) \Big[\prod_{j=1}^{\mathrm{Card}(T^0_\jmath)} \Big(\det(Y^{\mathbf{T}_{\jmath,j}}) \Big)^{-\frac{d}{2}} \Big] = \int_0^1 \prod_{\imath \in \mathcal{J}_\jmath} dt^\imath \, \frac{\partial}{\partial t^\imath} \Big[\det\big(Y^\jmath(\{t\})\big) \Big]^{-\frac{d}{2}}\ , \tag{H.20}$$

with $\mathcal{J}_\jmath = \overline{\mathcal{J}_\jmath} = \{\imath : \imath^{\min}(\jmath) \leq \imath < \imath^{\max}(\jmath)\}$.

◇ (III) *Proof of estimates (8.35) and (8.37).*

First we use the fact that the property 4 of matrix $Y^{\mathbf{T}_{\jmath,j}}$ extends to the matrix $Y^\jmath(\{t\})$. Specifically we have:

• *Prop. 5:* The matrix $Y^\jmath(\{t\})$ is positive and bounded from below (*i.e.* $Y^\jmath(\{t\}) - C\mathbb{1} > 0$ for some positive C) for all $0 \leq t^\imath \leq 1$, $\imath \in \mathcal{J}_\jmath$.

Indeed, this property holds when each $t^\imath$ equals 0 or 1. In this case, each block $Y^{\mathbf{T}_{\jmath,j}}$ of the matrix is "factorized" into a product of sub-blocks $Y^{\mathbf{T}}$ for subtrees $\mathbf{T}$'s (see Appendix C) compatible with the sector. Each of these sub-matrices $Y^{\mathbf{T}}$ then satisfies Proposition 4, as well as the matrix $Y^\jmath$. To complete the proof of Proposition 5, we use the fact that the matrix $Y^\jmath(\{t\})$ is a linear function of each $t^\imath$, and that it is thus sufficient to have a lower bound at each corner of the hypercube $0 \leq t^\imath \leq 1$ ($\imath \in \mathcal{J}_\jmath$) to have this bound inside the whole hypercube.

A direct consequence of Proposition 5 is that $Y^\jmath(\{t\})$ is invertible, and that $\big(Y^\jmath(\{t\})\big)^{-1}$ is positive and bounded from above uniformly in the sector. In particular, the module of all the elements $\big(Y^\jmath(\{t\})\big)^{-1}_{\kappa\lambda}$ is also bounded from above.

Another consequence of Proposition 5 is that $\det\big(Y^{J}(\{t\})\big)$ is uniformly bounded from below by a strictly positive number.

Finally, if $\mathcal{E}$ is some subset of $\mathcal{J}_J$, it is clear from Proposition 3 and the definition (H.16) of $Y^{J}(\{t\})$ that, in the considered Hepp sector, the partial set-derivative

$$\partial_{\mathcal{E}} Y^{J}_{KL} \equiv \Big(\prod_{I\in\mathcal{E}} \frac{\partial}{\partial t^{I}}\Big) Y^{J}_{KL}(\{t\}) = \mathcal{O}\Big(\prod_{I\in\mathcal{E}} (\beta^{I})^{\delta}\Big) \tag{H.21}$$

for the t^{I}'s between 0 and 1.

To prove (8.35), we now perform explicitly the derivatives with respect to the t^{I}'s in the r.h.s. of (H.20). This leads to an integral over the t^{I} of a finite sum of terms of the form

$$\mathrm{Tr}\Big(\partial_{\mathcal{E}^1_1} Y\cdot Y^{-1}\dots\partial_{\mathcal{E}^1_{k_1}} Y\cdot Y^{-1}\Big)\dots\mathrm{Tr}\Big(\partial_{\mathcal{E}^n_1} Y\cdot Y^{-1}\dots\partial_{\mathcal{E}^n_{k_n}} Y\cdot Y^{-1}\Big)\Big(\det(Y)\Big)^{-\frac{d}{2}}\ , \tag{H.22}$$

made of a product of an arbitrary number n of traces (the ith trace involving a product of k_i set-derivatives) and where the set of all $\mathcal{E}^l_m$'s form a partition of $\mathcal{J}_J$ (here, Y stands for $Y^{J}(\{t\})$). From the estimates (H.21), from the upper bound on $\big(Y^{J}(\{t\})\big)^{-1}$ and from the lower bound on $\det\big(Y^{J}(\{t\})\big)$, we deduce that in the Hepp sector:

$$\prod_{I\in\mathrm{Ind}(J)} (1-\mathfrak{T}^{0}_{\tilde{T}^{I}_{J-1\oplus}})\Big[\prod_{j=1}^{\mathrm{Card}(T^{0}_{J})}\Big(\det(Y^{\mathbf{T}_{J,j}})\Big)^{-\frac{d}{2}}\Big] = \mathcal{O}\Big(\prod_{I\in\mathcal{J}_J}(\beta^{I})^{\delta}\Big)\ , \tag{H.23}$$

which is just the announced estimate (8.35).

It is not very difficult to extend the above analysis to the case of the largest diagram $\tilde{T}_{T+1}$ which contains the external points. Indeed, the action of the $\mathfrak{T}^{0}$'s on the extra term $\exp(-\frac{1}{2}\sum_{a,b}\vec{\mathbf{k}}_a\cdot\vec{\mathbf{k}}_b\Delta_{ab})$ can also be implemented through the t variables, and one can check that the quadratic form $\Delta_{ab}(\{t\})$ is still definite positive. This ensures that the estimate (8.37) is valid, Q.E.D.

References

[1] S. F. Edwards, Proc. Phys. Soc. Lond. **85** (1965) 613.

[2] J. des Cloizeaux and G. Jannink, *Polymers in Solution, their Modelling and Structure*, Clarendon Press, Oxford (1990).

[3] M. Fixman, J. Chem. Phys. **23** (1955) 1656.

[4] J. des Cloizeaux, J. Phys. France **42** (1981) 635.

[5] P.-G. de Gennes, Phys. Lett. **38A** (1972) 339.

[6] *Phase Transitions and Critical Phenomena*, Vol. **6**, eds. C. Domb and M. S. Green, Academic Press, New York (1976).

[7] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press, Oxford (1989).

[8] M. Benhamou and G. Mahoux, J. Phys. France **47** (1986) 559.

[9] B. Duplantier, J. Phys. France **47** (1986) 569.

[10] L. Schäfer and T. A. Witten, J. Phys. France **41** (1980) 459.

[11] J. F. Douglas, S.-Q. Wang and K. F. Freed, Macromolecules **19** (1986) 2207; see also M. K. Kosmas, Makromol. Chem., Rapid Commun. **2** (1981) 563.

[12] S. R. S. Varadhan, Appendix to *Euclidean Quantum Field Theory* by K. Symanzik, in l*Local Quantum Theory*, R. Jost Ed. (Varenna, 1968), Academic Press, New York (1969) p. 285;
J. Westwater, Commun. Math. Phys. **72** (1980) 131; **79** (1981) 53;
J. F. Le Gall, Commun. Math. Phys. **104** (1986) 471; *ibid.* 505.

[13] G. F. Lawler, Commun. Math. Phys. **86** (1982) 539,
M. Aizenmann, Commun. Math. Phys. **97** (1985) 91,
G. Felder and J. Fröhlich, Commun. Math. Phys. **97** (1985) 111.

[14] D. Arnaudon, D. Iagolnitzer, and J. Magnen, Phys. Lett. **B 273** (1991) 268.

[15] R. C. Ball, (1981) unpublished.

[16] M. Kardar and D. R. Nelson, Phys. Rev. Lett. **58** (1987) 12.

[17] J. A. Aronowitz and T. C. Lubensky, Europhys. Lett. **4** (1987) 395.

[18] B. Duplantier, Phys. Rev. Lett. **58** (1987) 2733; and in *Statistical Mechanics of Membranes and Surfaces*, Proceedings of the Fifth Jerusalem Winter School for Theoretical Physics (1987), D. R. Nelson, T. Piran and S. Weinberg Eds., World Scientific, Singapore (1989).

[19] M. Kardar and D. R. Nelson, Phys. Rev. Lett. **58** (1987) 2280(E); Phys. Rev. **A 38** (1988) 966.

[20] B. Duplantier, Phys. Rev. Lett. **62** (1989) 2337.

[21] B. Duplantier, T. Hwa and M. Kardar, Phys. Rev. Lett. **64** (1990) 2022.

[22] N. N. Bogoliubov and O. S. Parasiuk, Acta Math. **97** (1957) 227;
K. Hepp, Commun. Math. Phys. **2** (1966) 301;
W. Zimmermann, Commun. Math. Phys. **15** (1969) 208.

[23] M. C. Bergère and Y.-M. P. Lam, J. Math. Phys. **17** (1976) 1546.

[24] R. J. Rubin, J. Math. Phys. **8** (1967) 576, and references therein.

[25] J.-F. Joanny, J. Phys. France **49** (1988) 1981.

[26] M. E. Fisher, in *Statistical Mechanics of Membranes and Surfaces*, Proceedings of the Fifth Jerusalem Winter School for Theoretical Physics (1987), D. R. Nelson, T. Piran and S. Weinberg Eds., World Scientific, Singapore (1989).

[27] F. Dunlop, J. Magnen, V. Rivasseau and P. Roche, J. Stat. Phys. **66** (1992) 71.

[28] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford (1953).

[29] H. M. S. Coxeter, Aeq. Math. **1** (1968) 104.

[30] *Encyclopedic Dictionary of Mathematics*, Second Edition, edited by Kiyosi Itô, The MIT Press, Cambridge, London (1987).

[31] I. J. Schoenberg, Ann. Math. **38** (1937) 787.

[32] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York (1985).

[33] K. Wilson and J. Kogut, Phys. Rep. **12C** (1974) 75.

[34] J. Polchinski, Nucl. Phys. **B231** (1984) 269.

[35] V. Rivasseau, *From Perturbative to Constructive Renormalization*, Princeton University Press, Princeton (1991); and references therein.

[36] S. Grotehans and R. Lipowsky, Phys. Rev. **A 45** (1992) 8644.

[37] J. des Cloizeaux and I. Noda, Macromolecules **15** (1982) 1505.

[38] F. David, B. Duplantier and E. Guitter, work in progress.

[39] M. L. Mehta, *Random matrices*, second edition, Academic Press, New York (1991).

[40] A. Selberg, Norsk Matematisk Tidsskrift **26** (1944) 71-78.

Figure Captions

Fig. 1. A D-dimensional fluctuating manifold (here $D = 2$) interacting: (a) with a point at the origin in $\mathbb{R}^d$ (here $d = 3$), (b) with a fixed D'-dimensional Euclidean subspace of $\mathbb{R}^{d'}$ (here $D' = 2$, $d = 1$, $d' = d + D' = 3$). (c) A "directed" manifold interacting with a "parallel" flat subspace of same dimension D in $\mathbb{R}^{d'}$.

Fig. 2. The Wilson W function and the Renormalization Group (IR) flow (for increasing manifold size X) for the dimensionless coupling constant g: (a) in the case $\epsilon > 0$, (b) in the case $\epsilon < 0$, (c) in the case $\epsilon = 0$.

Fig. 3. Equivalent representations of the positions of a given set of N interaction points (here $N = 6$). The points are described (a) by their position x_i in $\mathbb{R}^D$ or $\mathbb{R}^{N-1}$ or (b) by the set of their mutual squared distances $a_{ij} = (x_i - x_j)^2$ or (c) by their relative vector $y_i = x_{i+1} - x_1$ in $\mathbb{R}^D$ or $\mathbb{R}^{N-1}$ (relative to the point x_1) or (d) by the line vectors (labeled by α) of an arbitrary spanning tree joining these points.

Fig. 4. The daisy diagram corresponding to the term (3.57) .

Fig. 5. Schematic picture of the short-distance factorization of the interaction term relative to some set $\mathcal{G}$ of N interaction points (here $N = 10$). When the points of a subset $\mathcal{P}$ of $\mathcal{G}$ are contracted toward one of its point x_1, the interaction term factorizes into the product of the interaction term relative to $\mathcal{P}$ and the interaction term relative to $\overline{\mathcal{P}} = (\mathcal{G} \setminus \mathcal{P}) \cup \{x_1\}$.

Fig. 6. (a) An example of construction of the ordered tree $\mathbf{T} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for a set of interaction points with $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq |\lambda_4|$. This tree defines the generalized Hepp sector $\mathcal{H}^{\mathbf{T}}$ to which this set of points belongs. (b) Moving the point x_2 toward the point x_1 results in a change of generalized Hepp sector.

Fig. 7. A subdiagram.

Fig. 8. A complete diagram, with connected components $\mathcal{P}_i$.

Fig. 9. The complete diagram with connected components $\mathcal{P}_i$ (dashed line) is contained in the complete diagram with connected components $\mathcal{Q}_j$ (full line).

Fig. 10. The intersection diagram (dark-grey diagram) of two diagrams (grey and white diagrams).

Fig. 11. The union diagram (dark-grey diagram) of two diagrams (grey and white diagrams).

Fig. 12. The minimal complete diagram $G_\odot$.

Fig. 13. The subtraction diagram (dark-grey diagram) of a diagram (grey diagram) from another diagram (white diagram).

Fig. 14. A rooted subdiagram. The roots are specified by squares.

Fig. 15. A complete rooted diagram. Its elements are rooted subsets $(\mathcal{P}_i, p_i)$.

Fig. 16. Two successive complete rooted diagrams $T_{\jmath\oplus}$, with connected components $\mathcal{T}_{\jmath,j}$ (dashed lines) and $T_{\jmath+1\oplus}$ with connected components $\mathcal{T}_{\jmath+1,k}$ (full lines) of a rooted nest. The roots of these two diagrams are *not compatible*.

Fig. 17. The two successive diagrams of fig. 16, with *compatible* roots. The roots $w_{\jmath+1,k}$ have been obtained from the roots of fig. 16 by the construction of *Definition 7.11*.

Fig. 18. (a) Saturated nest associated with an ordered tree $\mathbf{T} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The nest is made of four diagrams. Each diagram is represented by the contour of its connected components with at least two vertices (the diagrams 1, 3 and 4 have only one such connected component, the diagram 2 has two such connected components). (b) Oriented ordered tree associated with a compatibly rooted saturated nest. We have first assigned compatible roots to the saturated nest of (a) (here the diagrams 3 and 4, and the connected component on the right of the diagram 2 have the same root) and then constructed the oriented ordered tree from these roots.

Fig. 19. Schematic picture of the rooted nest $\tilde{\mathcal{N}}_\oplus$ when the root w of $\mathcal{T}$ (a) belongs to $\mathcal{R}$, or (b) does not belong to $\mathcal{R}$.

Fig. 20. The unrooted complete diagram $R \vee_\omega T$ (thick full lines) obtained from the unrooted complete diagram R (dashed lines) and the complete rooted diagram T (thin full lines). The diagram $R \vee_\omega T$ is obtained by fusing each connected component of T to the connected component of R to which its root belongs, and cutting it out from all the other connected components of R.

Fig. 21. Appropriate tree variables. At level $\jmath$, inside a connected component $\tilde{\mathcal{T}}_{\jmath,j}$ of $\tilde{T}_\jmath$ (dashed circles), we build an oriented ordered tree with line vectors $\lambda^{\imath}_{\jmath,j}$. As shown in the framed box, this tree is built in a way similar to what was done in fig. 18 (b), now from the rooted saturated nest $\mathcal{S}_{\jmath,j\oplus}$. This nest is here made of the three diagrams $\tilde{T}^{\imath_1}_{\jmath-1,j}$, $\tilde{T}^{\imath_2}_{\jmath-1,j}$ and $\tilde{T}^{\imath_3}_{\jmath-1,j}$ (*i.e.* $\overline{\mathrm{Ind}}(\jmath, j) = \{\imath_1, \imath_2, \imath_3\}$) whose roots are represented by the dashed squares. At level $\jmath + 1$, the connected components of $\tilde{T}_\jmath$ are fully contracted toward their roots (big black dots), which are the vertices of $\tilde{\mathcal{T}}_{\jmath+1,k}$. An oriented ordered tree with line vectors $\lambda^{\imath'}_{\jmath+1,k}$ is

then built inside $\tilde{\mathcal{T}}_{J+1,k}$. The trees at level J and $J+1$ can be fused into a single oriented (but only partially ordered) larger tree contributing to (8.19) .

Fig. 22. The quadrilateral picturing the matrix element $\Pi^{\mathbf{T}}_{\alpha\beta}$ and its "basis" vector $R_{\alpha\beta}$.

Fig. 23. Classification of the line vectors of the tree $\mathbf{T}$ into subtrees $\mathbf{T}_i$. The dashed lines in (a) correspond to contracting branches of the tree $\mathbf{T}$, and are organized into two connected subtrees $\mathbf{T}_1$ and $\mathbf{T}_2$ in (b) . The full lines in (a) correspond to non-contracting branches and are organized into a single connected subtree $\mathbf{T}_3$ in (b), by fully contracting the dashed lines in (a).

Fig. 24. An application of the operation of fig. 20 : the figure describes a connected component $\mathcal{R}^{I,i}$ of R^I and the corresponding connected component $\mathcal{R}^{I,i}_{J-1}$ of $R^I \vee_{\omega_{J-1}} T_{J-1}$, obtained by fusing to $\mathcal{R}^{I,i}$ those connected components of $T_{J-1\oplus}$ which have their root inside $\mathcal{R}^{I,i}$ and cutting out those which do not have their root inside $\mathcal{R}^{I,i}$, but still intersect $\mathcal{R}^{I,i}$.