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# **A general dead-time correction method based on live-time stamping. Application to the measurement of short-lived radionuclides**

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## **Abstract**

Dead-time correction formulae are established in the general case of superimposed non-homogeneous Poisson processes. Based on the same principles as conventional live-timed counting, this method exploits the additional information made available using digital signal processing systems, and especially the possibility to store the time stamps of live-time intervals. No approximation needs to be made to obtain those formulae. Estimates of the variances of corrected rates are also presented. This method is applied to the activity measurement of short-lived radionuclides.

## **Keywords**

Dead-time correction ; Live-timed counting ; Digital processing ; Time stamping ; Short-lived radionuclides

## **1. Introduction**

The problem of dead-time correction in nuclear measurements has been addressed for a long time. Formulae have been established to get corrected count rates from measured ones, according to the different types of dead times (non-extending, extending or generalized). The most recent review article on this topic is by Pommé et al. (2015), where many references can be found. A prerequisite of the valid application of these formulae is that dead times be well defined and that the theoretical ones accurately reproduce the experimental ones. To comply with this requirement, a dead time generator is introduced in the electronic circuit of measuring systems. Triggered by input events due to the detection of radiation, this generator superposes an artificial dead time of fixed, larger duration  $\tau$  to the experimental dead times generated by the detection. To apply correction formulae, this length has to be known accurately. In the case of non-extending dead times, this artificial dead time is triggered only by counted events, i.e. those occurring during live time. In the case of extending dead times, it is also prolonged by the events occurring during dead-time intervals. In that case, the only events that can be counted are those preceded by no event during the time  $\tau$ . The main advantages over non-extending dead times are then to avoid creating hidden dead times due to the events occurring just before the end of apparent dead times, and to avoid counting a series of correlated events several times by setting dead times of length adapted to their time distribution (Bouchard, 2000). For example, this can be applied to get rid of spurious afterpulses in gas proportional counters, of delayed-fluorescence pulses in liquid scintillation counting, and of delayed radiation emission from radionuclides with a metastable state in their decay scheme.

A significant progress was achieved by introducing the live-timed counting technique (Baerg et al., 1976, Bouchard, 2000). The total duration of live-time intervals is measured by counting the number of pulses delivered by a clock during those time intervals. The dead-time correction factor is then simply equal to the ratio of the total measurement time to the live-time duration.

All these methods work well for the measurement of events that follow a homogeneous (stationary) Poisson process, e.g. when measuring radionuclides whose decay is negligible during the time of measurement. For short-lived radionuclides, approximated correction factors have been obtained for dead times using the live-timed counting method (Fitzgerald, 2016).

In the present paper, a general method for correcting count rates for dead times is introduced based on the recent developments of digital processing systems, and the possibility to store the time stamps of input events. As an extension of the live-timed counting method, it starts from the dead times as they are implemented in a measuring system, assuming they are correctly defined. The handling of specific

problems, like those due to hidden dead times, pile-up (Pommé, 2008), sub-threshold or correlated pulses, is related to the implementation as such of dead times, and is here out of scope. This method is shown to be particularly adapted to the measurement of short-lived radionuclides.

## 2. Properties of the live-timed counting method

### 2.1. Corrected rate and associated variance

The live-timed counting method is based on the measurement of periods of “calm” during the measurement time  $T_{tot}$ . These periods of total duration  $T_L$  can be characterized by the absence of any event and they end at the arrival of a new event.

The method relies on the stationary- and independent-increment properties of Poisson processes. Basically a Poisson process of rate  $\rho$  is defined by the following characteristics. At any time  $t$ , the probability of occurrence of one event during the infinitesimal time interval  $dt$  after  $t$  is independent of the preceding events and constant over time. This probability is proportional to  $dt$ , and equal to  $\rho dt$ . In addition, the probability of occurrence of more than one event during  $dt$  is negligible. Then the probability of non-occurrence during  $dt$  is given by  $(1 - \rho dt)$ .

As a consequence, the expected number of events that occur during any time interval of duration  $T$  is obtained by summing the probabilities of occurrence of one event during  $dt$ ,  $\rho dt$ , over this time interval. The result is equal to  $\rho T$ . An estimated value of that number is given by the number  $N$  of events that actually occur during the time  $T$ . Likewise, the number  $N$  is an estimate of the variance of the number of events arriving during  $T$ , equal to  $\rho T$ .

On the other hand, in the presence of dead times, the expected number of events that occur over all the live-time intervals of total duration  $T_L$  is the same as the expected number of events that occur during the time  $T_L$  in the absence of dead times, namely  $\rho T_L$ . An estimate of that number can be given by the number  $N$  of events counted during  $T_{tot}$ . Indeed because of the stationary- and independent-increment properties, a series of counted events of a Poisson process subjected to dead times and measured during  $T_{tot}$  behave like a series of events of the same process occurring during  $T_L$  in the absence of dead times. In the latter case, the time intervals between successive events are reduced to the live-time intervals between the forementioned counted events, dead-time intervals being ignored. It follows that an estimate of the rate  $\rho$  can be obtained from the number of counted events and the measured total live-time duration,

$$\hat{\rho} = \frac{N}{T_L} \quad (1)$$

In the same way, the variance of  $\hat{\rho}$  for a given value of  $T_L$  can be derived from the variance of the  $N$  events of the Poisson process occurring during  $T_L$ , whose estimated value is given by  $N$ ,

$$\frac{Var(\hat{\rho})}{\hat{\rho}^2} = \frac{Var(N)}{N^2} \approx \frac{1}{N} \quad (2)$$

It can be noted that in the foregoing considerations, there is no special requirement on the way in which live-time intervals are stopped. The dead times can be triggered not only by an event of the measured process of rate  $\rho$ , but also any event of another process, Poisson or not. In those cases, the number  $N_L$  of live-time intervals is larger than  $N$ . Then an additional condition is required for making relation (1) applicable; this is the possibility of selectively counting the events due to the Poisson process of rate  $\rho$ , which involves that they can be discriminated between all the events that can trigger a dead time.

### 2.2. Application of the maximum-likelihood method

Another way to get an estimate of  $\rho$  is based on the maximum-likelihood method. It consists in calculating the probability to get the measured sequence of  $N$  counted events, given the starting times of all the live-time intervals, and then calculating the value of the rate  $\rho$  which maximizes this probability. This is done by calculating the derivative of this probability with respect to the variable  $\rho$  and by calculating the value of  $\rho$  that makes this derivative equal to zero.

The  $k^{\text{th}}$  live-time interval of duration  $i_k$  starts at time  $t_k$  and ends at time  $t_k + i_k$  at the arrival of the  $k^{\text{th}}$  event. Denoting  $dp(t_k, i_k)$  the probability for the  $k^{\text{th}}$  counted event to occur within the infinitesimal time interval  $di$  following the live time of duration  $i_k$ , one can write

$$dp(t_k, i_k) = \rho \exp(-\rho i_k) di \quad (3)$$

This expression is the product of the probability of occurrence of one event during  $di$ ,  $\rho di$ , by the probability of non-occurrence during  $i_k$ ,  $\exp(-\rho i_k)$ .

If needed, to account for the possibility for dead times to be triggered by other events than those measured, the  $k^{\text{th}}$  live-time interval can be understood as the addition of all the live-time intervals between the  $(k-1)^{\text{th}}$  and the  $k^{\text{th}}$  counted events of the process of rate  $\rho$ ,  $i_k$  denoting then the sum of their durations.

Due to the independent-increment property, the  $N$  probabilities  $dp(t_k, i_k)$  are independent from each other. The probability of occurrence, given the starting times of the live-time intervals, of the  $N$  counted events according to the measured sequence is then equal to the product of those probabilities,

$$\prod_{k=1}^N dp(t_k, i_k) = \rho^N \exp(-\rho \sum_{k=1}^N i_k) (di)^N.$$

This expression can finally be written

$$\prod_{k=1}^N dp(t_k, i_k) = \rho^N \exp(-\rho T_L) (di)^N$$

It can be noted that this probability depends only on the number of live-time intervals and on their total duration, and not on their individual starting times and durations. Since the process is stationary, this expression remains unchanged for any sequence of  $N$  events of rate  $\rho$  counted during a total live time of duration  $T_L$ .

The derivative of this expression with respect to  $\rho$  is equal to

$$\rho^{N-1} (N - \rho T_L) \exp(-\rho T_L) (di)^N$$

This quantity is equal to zero when the value of the rate  $\rho$  is taken equal to  $N/T_L$ . It is positive for smaller values of  $\rho$  and negative for higher ones. This shows that the probability is maximal for  $\rho$  equal to  $N/T_L$ .

### 2.3. Comments

It is worth noting that these relations do not depend on what happens during dead-time intervals and particularly on the type and characteristics of the imposed dead times (extending, non-extending or mixed). The only condition of their validity is that the process be Poisson, and especially that all the counted events be uncorrelated to any preceding event. This means that the live-timed counting method is applicable to any type of dead time, provided that the measured events follow a Poisson process. Another conclusion is that the dead-time correction factor can be obtained without exact knowledge of the type of dead time introduced and of its setting parameters.

On the other hand, the principle of extending dead times is based on the introduction of a dead time of given duration after each detected event. This ensures that the distance between a counted event and the preceding event is always longer than this duration. If this minimal dead time is set long enough to cover all the perturbances that can be caused by the triggering event (in the detector, in the counting electronics,...), then all the counted events are “clean” and can give reliable information. As a result, the combination of live-timed counting with extending dead times seems so far the best option for accurate radionuclide activity measurements, as far as conventional analog counting systems are concerned (Bouchard, 2000).

## 3. Case of short-lived radionuclides

The preceding method holds for radionuclide sources undergoing a negligible decay during the time of measurement. Fitzgerald (2016) has recently reviewed and reported the corrections to apply for the combined effects of decay and dead times in the case of the live-timed counting of short-lived radionuclides. Formulae are obtained for the counting of one radionuclide in the absence of background in the cases of extending dead times and non-extending dead times of fixed duration  $\tau$ .

Exact forms of the correction factors are given in function of two parameters, the product of the decay constant  $\lambda$  by the total measurement time  $T_{\text{tot}}$ , and the product of the dead time  $\mathcal{T}$  by the count rate at time 0,  $\rho(0)$ , whose

exact value is the unknown. As the formulae are complex (including exponential integrals for extending dead times), series expansions including both parameters are calculated to the fourth power of the first parameter aforementioned ( $\lambda T_{\text{tot}}$ ) for practical applications.

From this work, it can be seen that for short-lived radionuclides, the precise determination of the dead-time correction factor in live-timed counting is hampered by the lack of available information, which is limited to the total number of counted events and the total live time.

The next sections are devoted to the possibilities offered by digital treatment systems which are of increasing use in nuclear electronics, to solve this problem. In particular, these systems enable the acquisition and storage of the time stamps of detected events and of the starts and stops of live-time intervals. They also enable superposing artificial dead times of chosen type and length to the real sequence of events, in off-line mode, to study the influence of the type of dead times and of their characteristics on the counting of events.

## 4. Non-homogeneous Poisson processes

### 4.1. Introduction to the measurement of non-homogeneous Poisson processes

The disintegrations of short-lived radionuclides do not follow a homogeneous Poisson process, since the occurrence rate can no longer be considered as constant during the measurement but undergoes an exponential variation with time according to their decay constant. This process belongs to the category of non-homogeneous Poisson processes. The main objective of the study presented here is to determine a dead-time correction factor for the counting of such radionuclides. However, in this section the choice is made to enlarge the study to the general case of non-homogeneous Poisson processes for which the exponential is replaced by any integrable function. Far from complicating the calculations, this generalization leads to formulae that remain simple, and offers a larger number of possible applications.

Unlike in previously published studies on dead times, it is assumed here that the starting time and the duration of each live-time interval of the measured sequence of events are accessible data, namely that live-time stamping is possible. No assumption is made concerning dead times, their type and characteristics. Indeed, it will be seen in what follows that dead times play no explicit role in this calculation. In this approach, dead times can be considered just as periods of time of arbitrary duration triggered by the first event that occurs during a live time.

### 4.2. Properties of non-homogeneous Poisson processes

The non-homogeneous Poisson process followed by the measured events can be described by a rate of occurrence  $\rho(t)$  expressed as the product of a constant rate,  $\rho$ , by a function of time  $f(t)$  supposed to be integrable within the measurement time.

The probability of occurrence of one event during the infinitesimal time interval  $dt$  after  $t$ , is equal to  $\rho f(t) dt$ . This is a time-dependent probability; therefore, compared to homogeneous processes, the stationary-increment property is lost, while the independent-increment one is maintained. The probability of occurrence of more than one event during  $dt$  is negligible, and consequently the non-occurrence probability during the time interval  $dt$  after  $t$  is equal to  $(1 - \rho f(t) dt)$ .

During the infinitesimal time interval  $dt$  following any time  $t$ , the process is assumed to be stationary because of the negligible variation of  $f$  during  $dt$ . As a consequence it is considered in this interval as a homogeneous Poisson process of rate  $\rho f(t)$  with an expected number of events equal to  $\rho f(t) dt$ .

Therefore the expected number of events occurring between times  $t$  and  $t + i$ ,  $E(N(t, t+i))$ , can be given by the integral

$$E(N(t, t+i)) = \int_t^{t+i} \rho f(u) du$$

Denoting  $F$  the integral of the function  $f$  such as  $F(t) = \int_0^t f(u) du$ , one can write

$$E(N(t, t+i)) = \rho (F(t+i) - F(t)).$$

The probability of non-occurrence between  $t$  and  $t+i$ , denoted  $P_0(t, t+i)$ , can be calculated as well assuming as above that the process is homogeneous on infinitesimal time intervals. To this end, this probability can be considered as the product of the probabilities of non-occurrence in all the infinitesimal intervals  $du$  from  $t$  to  $t+i$ . Then, one can write

$$P_0(t, t+i) = \prod_{u=t}^{t+i} P_0(u, u+du) ;$$

$$P_0(t, t+i) = \prod_{u=t}^{t+i} \exp(-\rho f(u) du) = \exp(\sum_{u=t}^{t+i} (-\rho f(u) du)) = \exp\left(-\int_t^{t+i} \rho f(u) du\right) ;$$

$$P_0(t, t+i) = \exp(-\rho(F(t+i) - F(t))).$$

Then the probability, given a time  $t$ , for the next event to occur during the infinitesimal time interval  $di$  after the time interval  $i$ , can be expressed by

$$dp(t, i) = \rho \cdot f(t+i) \cdot \exp(-\rho(F(t+i) - F(t))) \cdot di \quad (4)$$

Going further in the properties of non-homogeneous Poisson processes, it is established that such a process of rate  $\rho f(t)$  can be converted into a homogeneous one of rate  $\rho$  by a monotone transformation of time using the function  $F$  defined before (see Kingman, 1993, Sections 2.3 and 4.5). More specifically, a series of events of the non-homogeneous process of rate  $\rho f(t)$  that occur at successive times  $t_k, t_{k+1}, \dots$ , can be transformed into a series of events arriving at successive times  $F(t_k), F(t_{k+1}), \dots$ , which then belong to a homogeneous Poisson process of rate  $\rho$ .

For example, the probability of occurrence of one event during the infinitesimal time interval  $dt$  after  $t$ , equal to  $\rho f(t) dt$ , can be rewritten  $\rho dF(t)$ . After a transformation of time where  $t$  is replaced with  $F(t)$ , this can be interpreted as the probability of occurrence of one event during an infinitesimal time interval for a homogeneous Poisson process of rate  $\rho$  in the transformed time.

Similarly, the expected number of events that occur between times  $t$  and  $t+i$ , equal to  $\rho (F(t+i) - F(t))$  can be viewed as the expected number of events of the homogeneous process of rate  $\rho$  between  $F(t)$  and  $F(t+i)$  in this transformed time. This holds also for the probability  $dp(t, i)$  defined above, noting that relation (4) can be rewritten

$$dp(t, i) = \rho \cdot \exp(-\rho(F(t+i) - F(t))) \cdot d(F(t+i) - F(t)).$$

Properties of the measured non-homogeneous Poisson process can then be deduced from the associated homogeneous process by inversion of  $F$  (Kingman, 1993, section 4.5).

## 5. Relations obtained for non-homogeneous Poisson processes

### 5.1. Corrected rate and associated variance

It can be noted that, in the presence of dead times, the transformation of time described in the previous section applies also to the starting and ending times of live-time intervals. Assuming a number  $N_L$  of live-time intervals, and denoting  $t_k$  the starting time of the  $k^{\text{th}}$  one and  $i_k$  its duration, the expected number of events of the non-homogeneous process that occur during a period of time comprised of the totality of live-time intervals is obtained by summing the expected numbers of events obtained for each live-time interval. According to Section 4.2, this number is equal to

$$\rho \sum_{k=1}^{N_L} (F(t_k + i_k) - F(t_k)).$$

This quantity can as well be interpreted as the expected number of events of the homogeneous Poisson process of rate  $\rho$  that occur during a time equal to the total transformed live time  $\sum_{k=1}^{N_L} (F(t_k + i_k) - F(t_k))$  in the absence of dead times. As seen in Section 2.1, the number  $N$  of events of this process that are actually counted during the transformed total time can then be considered as an estimate of this expected number of events. It follows that an estimate of the rate  $\rho$  can be obtained from the number of counted events and the transformed measured live-time duration,

$$\hat{\rho} = \frac{N}{\sum_{k=1}^{N_L} (F(t_k + i_k) - F(t_k))} \quad (5)$$

In the same way, the variance of  $\hat{\rho}$  for a given value of the transformed live time  $\sum_{k=1}^{N_L} (F(t_k + i_k) - F(t_k))$  can be derived from the variance of the  $N$  events of the homogeneous Poisson process in the transformed time scale,

$$\frac{\text{var}(\hat{\rho})}{\hat{\rho}^2} = \frac{\text{var}(N)}{N^2} \approx \frac{1}{N} \quad (6)$$

As in Section 2.1, when events other than those of the measured process are present and can trigger dead times, this makes the number of live-time intervals  $N_L$  larger than  $N$ . The applicability of relations (5) and (6) then relies on the possibility of selectively counting the events due to the measured process of rate  $\rho$ , and thus of discriminating them between the other events. Provided that this condition is also satisfied for any other Poisson process of rate  $\rho' f(t)$  superimposed to the preceding one, the relations (5) and (6) can be applied to that process, replacing  $F$  with  $F'$  in the transformation of time and  $N$  with the number  $N'$  of counted events of that process.

Furthermore, to make the comparison with the rate given by relation (1) for homogeneous Poisson processes, it may be noted that the expression given by (5) can still be read as the quotient of a number of counted events divided by a live time, this one being corrected for decay. In this view, the correction factor to be applied to the  $k^{\text{th}}$  live-time interval of duration  $i_k$  is equal to  $(F(t_k + i_k) - F(t_k)) / i_k$ . This expression turns out to be equal to the mean value of  $f(t)$  over this live-time interval. The correction factor to be applied to the total live time  $T_L$  is then the mean value of  $f(t)$  taken over the  $N_L$  live-time intervals.

## 5.2. Application of the maximum likelihood method

To confirm them, the preceding results can be obtained in an alternative way. To this aim, the maximum-likelihood method is applied to a given number of superimposed non-homogeneous Poisson processes. The calculation is detailed hereafter.

A sequence of events that originates from a number  $p$  of independent superimposed non-homogeneous Poisson processes is measured. The problem is to derive the rates of the different processes corrected for dead time, from the  $N$  events counted during the total time  $T_{\text{tot}}$  including  $N_L$  live-time intervals of known starting and ending times.

Each process  $j$  is defined at time  $t$  by its rate  $\rho_j f_j(t)$ . The superimposition of these processes results in a non-homogeneous process of rate  $\rho(t)$  at time  $t$  equal to the sum of their respective rates,

$$\rho(t) = \sum_{j=1}^p \rho_j f_j(t) .$$

In what follows, the different functions  $f_j$  are supposed to be known. Applying relation (4) to the  $p$  superimposed processes, the probability  $dp(t, t+i)$  that, given a live time starting at time  $t$ , the next counted event occurs within  $di$  after the live-time interval  $i$  can now be written,

$$dp(t, i) = \sum_{j=1}^p \left[ \rho_j f_j(t+i) \cdot \exp \left( - \sum_{j=1}^p \left[ \rho_j \left( F_j(t+i) - F_j(t) \right) \right] \right) \right] di \quad (7)$$

The probability for an event arriving at time  $t+i$  to originate from process  $n$  is equal to  $\rho_n f_n(t+i) / \sum_{j=1}^p \rho_j f_j(t+i)$ . Then the probability  $dp_n(t, t+i)$  that, given a live time starting at time  $t$ , the next counted event occurs within  $di$  after the time interval  $i$  and originates from process  $n$  is

$$dp_n(t, i) = \rho_n f_n(t+i) \cdot \exp \left( - \sum_{j=1}^p \left[ \rho_j \left( F_j(t+i) - F_j(t) \right) \right] \right) \cdot di$$

To include the possibility for a dead time to be triggered by other events than those from the  $p$  processes measured (i.e.  $N_L$  larger than  $N$ ), one has to account for the probabilities of non-occurrence of the measured events in the live-time intervals that precede such dead times. For a live-time of duration  $i$  such a probability is equal to

$$\exp \left( - \sum_{j=1}^p \left[ \rho_j \left( F_j(t+i) - F_j(t) \right) \right] \right) .$$

Then one can express the probability  $\prod(N_L, N)$  of occurrence, given the starting times of the  $N_L$  live-time intervals, of the  $N$  counted events according to the measured sequence. It is equal to the product of the  $(N_L - N)$  forementioned non-occurrence probabilities and of the  $N$  probabilities  $dp(t_l, i_l)$  given by relation (7) where the index  $l$  denotes the live-time interval that precedes the  $l^{\text{th}}$  counted event,

$$\prod(N_L, N) = \prod_{l=1}^N \left( \sum_{j=1}^p \rho_j f_j(t_l + i_l) di \right) \cdot \prod_{k=1}^{N_L} \exp \left( - \sum_{j=1}^p \left[ \rho_j \left( F_j(t_k + i_k) - F_j(t_k) \right) \right] \right) \quad (8)$$

After this preliminary calculation, one can apply the maximum-likelihood method as presented in Section 2, based on the maximization of this probability. For this purpose, the partial derivative with respect to the variable  $\rho_n$  of the expression given by relation (8) is calculated. Then the value of  $\rho_n$  which makes this derivative equal to zero gives an estimate of the unknown  $\rho_n$ .

The expression to be derivated is expressed as a product of functions of the variable  $\rho_n$ . Therefore, the product rule for derivatives can be applied. The partial derivative of  $\prod(N_L, N)$  with respect to  $\rho_n$  can then be expressed by

$$\frac{\partial(\prod(N_L, N))}{\partial \rho_n} = (di)^N \cdot \prod_{k=1}^{N_L} \exp \left( - \sum_{j=1}^p \left( \rho_j \left( F_j(t_k + i_k) - F_j(t_k) \right) \right) \right) \cdot \left( \sum_{l=1}^N \left[ f_n(t_l + i_l) \cdot \prod_{h=1, h \neq l}^N \left( \sum_{j=1}^p \rho_j f_j(t_h + i_h) \right) \right] - \sum_{k=1}^{N_L} \left[ \left( F_n(t_k + i_k) - F_n(t_k) \right) \cdot \prod_{l=1}^N \left( \sum_{j=1}^p \rho_j f_j(t_l + i_l) \right) \right] \right)$$

This expression is equal to zero when the following equation is satisfied

$$\sum_{l=1}^N \left[ f_n(t_l + i_l) \cdot \prod_{h=1, h \neq l}^N \left( \sum_{j=1}^p \rho_j f_j(t_h + i_h) \right) \right] = \prod_{l=1}^N \left( \sum_{j=1}^p \rho_j f_j(t_l + i_l) \right) \cdot \sum_{k=1}^{N_L} \left( F_n(t_k + i_k) - F_n(t_k) \right)$$

This equation is equivalent to

$$\sum_{l=1}^N \frac{f_n(t_l + i_l)}{\sum_{j=1}^p \rho_j f_j(t_l + i_l)} = \sum_{k=1}^{N_L} \left( F_n(t_k + i_k) - F_n(t_k) \right)$$

Multiplying each member by the unknown rate  $\rho_n$ , an estimate of  $\rho_n$  can finally be given by the following expression

$$\hat{\rho}_n = \frac{\sum_{l=1}^N \frac{\rho_n f_n(t_l + i_l)}{\sum_{j=1}^p \rho_j f_j(t_l + i_l)}}{\sum_{k=1}^{N_L} \left( F_n(t_k + i_k) - F_n(t_k) \right)}$$

It can be checked that the probability given by relation (8) is maximized for this value of  $\rho_n$ , as its partial derivative with respect to variable  $\rho_n$  is positive for smaller values and negative for higher ones.

The numerator of the right-hand side represents the sum of the  $N$  probabilities for a counted event to be of type  $n$ . If this probability is denoted  $p_{t_l + i_l}(n)$  for the  $l^{\text{th}}$  live-time interval, the relation can now be written

$$\hat{\rho}_n = \frac{\sum_{l=1}^N p_{t_l + i_l}(n)}{\sum_{k=1}^{N_L} \left( F_n(t_k + i_k) - F_n(t_k) \right)} \quad (9)$$

Since  $\rho_n$  is unknown, the  $N$  probabilities of the numerator, taken individually, are unknown. However, an estimated value of their sum is given by the number  $N_n$  of counted events of the process  $n$ . When the number of live-time intervals increases, the deviation between  $N_n$  and this sum of probabilities becomes negligible.

Relation (9) thus becomes

$$\hat{\rho}_n = \frac{N_n}{\sum_{k=1}^{N_L} \left( F_n(t_k + i_k) - F_n(t_k) \right)} \quad (10)$$



This result is identical to relation (5). It can be applied to any of the  $p$  processes involved, replacing index  $n$  with the adequate one.

The variance of  $\hat{\rho}_n$  follows from the estimated variance, equal to  $N_n$ , of the  $N_n$  events of the homogeneous Poisson process of rate  $\rho_n$  in the transformed time scale,

$$\frac{Var(\hat{\rho}_n)}{\hat{\rho}_n^2} = \frac{Var(N_n)}{N_n^2} \approx \frac{1}{N_n} \quad (11)$$

In the calculation of the overall uncertainty of  $\hat{\rho}_n$ , a component due to the transformed (or corrected) live time should be added to account for the uncertainty of  $F_n$ .

### 5.3. Case of absence of discrimination between the types of events counted

When the number of counted events  $N_n$  can be measured, it has been seen that relation (5), or (10), is directly applicable. This implies that the type of each counted event can be identified experimentally and discriminated between the different types of events. For example in nuclear measurements, spectrometry techniques make possible the selection of counted events according to energy. It is then possible to discriminate the counted events according to the radionuclide that emits the detected radiation.

In the opposite case, i.e. when no distinction is made between the different types of events, it is possible to use the former relations to obtain the unknown rate  $\rho_n$  as a function of the rates of the other processes. Contrary to the preceding case, this requires that those rates be known beforehand. In addition, when the total number of counted events  $N$  differs from the number of live-time intervals  $N_L$ , it must be possible to determine  $N$ , and thus to discriminate the counting of an event of any of the  $p$  processes of interest from the other events that can trigger a dead time.

It is assumed in what follows that the number of measured live-time intervals is large enough to consider that relations (9) and (10) give close estimates of the known rates  $\rho_j$ . Therefore, one can write them

$$\rho_j \approx \frac{N_j}{\sum_{k=1}^{N_L} (F_j(t_k + i_k) - F_n(t_k))}$$

Summing the estimated numbers of counted events of the  $p$  processes, and denoting  $\hat{\rho}'_n$  the estimate of  $\rho_n$  obtained in this case, one gets

$$\hat{\rho}'_n \sum_{k=1}^{N_L} (F_n(t_k + i_k) - F_n(t_k)) + \sum_{j=1, j \neq n}^p \left[ \rho_j \sum_{k=1}^{N_L} (F_j(t_k + i_k) - F_n(t_k)) \right] = N$$

The estimate of the number  $N_n$  of counted events of process  $n$  is therefore given by

$$\hat{N}_n = \hat{\rho}'_n \sum_{k=1}^{N_L} (F_n(t_k + i_k) - F_n(t_k)) = N - \sum_{j=1, j \neq n}^p \left[ \rho_j \sum_{k=1}^{N_L} (F_j(t_k + i_k) - F_n(t_k)) \right]$$

It follows that the estimate of the rate  $\rho_n$  can be obtained from the total number of counted events  $N$ , the rates of the other processes and the transformed total live-time durations obtained for all the processes,

$$\hat{\rho}'_n = \frac{N - \sum_{j=1, j \neq n}^p \left[ \rho_j \sum_{k=1}^{N_L} (F_j(t_k + i_k) - F_n(t_k)) \right]}{\sum_{k=1}^{N_L} (F_n(t_k + i_k) - F_n(t_k))} \quad (12)$$

Similarly, the variance of  $\hat{\rho}'_n$  can be expressed from the variance of the  $N_n$  counted events, and more specifically here, of their estimated value  $\hat{N}_n$ . This gives

$$\frac{Var(\hat{\rho}'_n)}{\hat{\rho}'_n^2} \approx \frac{1}{N - \sum_{j=1, j \neq n}^p \left[ \rho_j \sum_{k=1}^{N_L} (F_j(t_k + i_k) - F_n(t_k)) \right]} \quad (13)$$

In the calculation of the overall uncertainty of  $\hat{\rho}'_n$ , one should add components due to the corrected live times for the  $p$  processes to account for the uncertainty of the functions  $F$ , as well as components due to the rates  $\rho_j$  supposed to be known.

In the particular case where all processes are homogeneous Poisson ones, all the previous relations can be applied replacing for each process  $j$ , the quantity  $f_j(t)$  with 1 and the quantity  $F_j(t)$  with  $t$ . In addition, the different probabilities  $p_{t_i+i_l}(n)$  being independent of time can simply be denoted  $p(n)$ .

As expected, relation (9) becomes

$$\hat{\rho}_n = \frac{\sum_{l=1}^N p(n)}{\sum_{k=1}^{N_L} i_k} = p(n) \frac{N}{T_L}$$

and relation (10)

$$\hat{\rho}_n = \frac{\sum_{l=1}^N p(n)}{\sum_{k=1}^{N_L} i_k} = \frac{N_n}{T_L}$$

In the same way, relation (12) becomes

$$\hat{\rho}'_n = \frac{N}{T_L} - \sum_{j=1, j \neq n}^p \rho_j$$

## 5.4. Comments

The relations established in the previous sections are quite general. They apply to any superimposition of Poisson processes, homogeneous or non-homogeneous, associated with any integrable function  $f$ . Provided that the starting and ending times of live-time intervals are known, the formulae do not result from any approximation.

The particular case of short-lived radionuclides, for which the function  $f$  is exponential, is studied in the next section. It is worth noting that the function  $f$  could describe as well any other type of variation of the rates with respect to time, e.g. the movement of a radioactive source in front of the detector, and more generally any variation of the detection efficiency (varying screening, varying detection threshold, etc.), as long as these variations are known. The formulae could also account for parent and/or daughter radionuclides in the measurement of a radionuclide present in a decay chain, provided that their respective decay constants are such that the counts resulting from their disintegrations can be considered as uncorrelated.

## 6. Application to short-lived radionuclides

### 6.1. General case

The disintegration of a radionuclide follows a non-homogeneous Poisson process whose rate at time  $t$  can be given by  $\rho(0) \cdot \exp(-\lambda t)$ , as far as the number of radioactive nuclei remains large enough for the binomial law to be correctly approximated by a Poisson law.

Subject to this prerequisite, the general results presented above can be transposed to the measurement of a radioactive source containing any number of radionuclides to correct their measured count rate for dead time.

In the general case, a radioactive source made of  $p$  different radionuclides is considered. For radionuclide  $n$  of decay constant  $\lambda_n$ , one can write

$$f_n(t) = \exp(-\lambda_n t), \quad F_n(t) = \frac{1 - \exp(-\lambda_n t)}{\lambda_n}, \quad \rho_n(t) = \rho_n(0) \exp(-\lambda_n t)$$

A constant background or a radionuclide  $j$  with negligible decay during the measurement can be introduced in addition using the constant function  $f_j(t) = 1$ , with  $F_j(t) = t$ . In both cases, the following relations are obtained as expected

$$\sum_{k=1}^N (F_j(t_k + i_k) - F_j(t_k)) = \sum_{k=1}^N i_k = T_L, \text{ and } N_j = \rho_j T_L$$

### 6.2. Case of a radionuclide in the presence of background

The case of a single radionuclide  $R$  with an occurrence rate of events  $\rho_R(t) = \rho_R(0) \exp(-\lambda_R t)$ , measured in the presence of a background of constant rate  $\rho_B$ , is presented as an example.

According to relation (5), when the counts due to the radionuclide  $R$  can be identified or discriminated from the background, one can get the true rate at time 0,  $\rho_R(0)$ , directly from the number of counts  $N_R$  due to the radionuclide  $R$ , without having to know the background rate,

$$\hat{\rho}_R(0) = \frac{\lambda_R N_R}{\sum_{k=1}^N (\exp(-\lambda_R t_k) - \exp(-\lambda_R(t_k + i_k)))} \quad (14)$$

According to relation (11), the corresponding variance is then given by

$$\frac{\text{Var}(\hat{\rho}_R(0))}{\hat{\rho}_R(0)^2} = \frac{1}{N_R} \quad (15)$$

In the calculation of the corresponding overall uncertainty, one should add a component due to the uncertainty of the decay constant  $\lambda_R$ .

When there is no discrimination made, the rate  $\rho_R(0)$  can be obtained from relation (12), using the total number of counts  $N$  and the background rate  $\rho_B$  whose value is in that case supposed to be known,

$$\hat{\rho}'_R(0) = \frac{\lambda_R (N - \rho_B T_L)}{\sum_{k=1}^N (\exp(-\lambda_R t_k) - \exp(-\lambda_R(t_k + i_k)))} \quad (16)$$

According to (13), the variance of  $\hat{\rho}'_R(0)$  can then be written

$$\frac{\text{Var}(\hat{\rho}'_R(0))}{\hat{\rho}'_R(0)^2} = \frac{1}{N - \rho_B T_L} \quad (17)$$

The corresponding overall uncertainty should include additional components due to the uncertainties of  $\lambda_R$  and of the background count rate  $\rho_B$ .

## 7. Implementation

The relations obtained in 5 Relations obtained for non-homogeneous Poisson processes, 6 Application to short-lived radionuclides give the rates of the measured events corrected for dead times. They are valid for any type of dead time and, more generally, for any measurement interruption after the counting of an event. For using them, the only requirement is the possibility to get and store the time stamps of the events occurring during the measurement if the data are taken before the generation of dead times, or the time stamps of the live-time intervals if the data are taken after dead time generation. In the field of nuclear measurements, instruments equipped with digital signal processing systems offer this possibility.

Once the dead times have been generated and the live-time intervals defined, their starting and ending times can be transformed according to the change of time scale described in Section 4.2, i.e. changing  $t$  in  $F_j(t)$  for process  $j$  (see Fig. 1). It is worth noting that after this step, the signal processing can be carried out identically to the one applied for activity measurements in usual conditions (i.e. negligible decay, assumption of homogeneous Poisson process). Indeed the same relations can be applied in the transformed time scales.

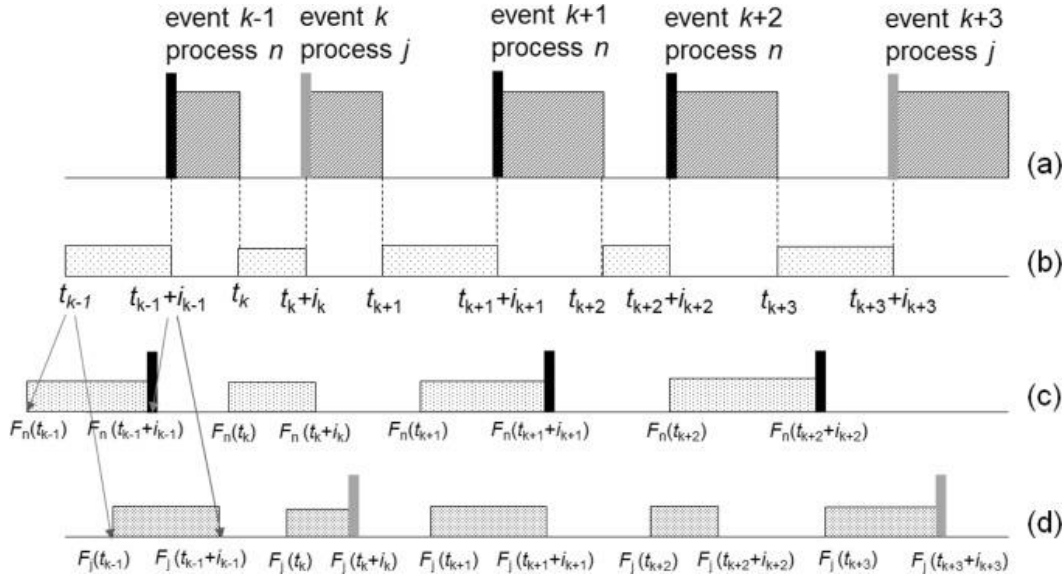


Fig. 1. (a) Chronogram of occurrences of counted events of the non-homogenous Poisson processes  $n$  and  $j$  with associated dead times; (b) live-time intervals with their starting and ending times; (c) and (d) live-time intervals with their starting and ending times after changing the time scales for processes  $n$  and  $j$  respectively, thus transformed into homogeneous Poisson processes.

The rate corrected for dead times can be obtained off-line after the measurement, but also through an on-line process. Indeed, the expressions giving the corrected rate and the associated variance are composed of terms that are incremented at the end of each new live-time interval, by a quantity specific to this interval. Therefore, a new rate can be obtained after each counted event.

In the case of discriminated events, the number of counted events of type  $n$  is incremented by one when an event of type  $n$  is counted. The corrected total live time is incremented by the quantity  $F_n(t_k + i_k) - F_n(t_k)$  after each live-time interval  $[t_k, t_k + i_k]$  regardless of the type of event triggering the dead time. It is possible to acquire additional data to perform statistical tests on the results. For example, one can study the distribution of the corrected live-time intervals between two successive events of type  $n$ . The presence of a tendency with time could highlight an inconsistency in the function  $f_n$  used. The measurement of such a time interval starts after the counting of an event of type  $n$ , it is incremented by the quantity  $F_n(t_k + i_k) - F_n(t_k)$  after each new live-time interval  $[t_k, t_k + i_k]$  up to the next counting of an event of type  $n$ . Then this measurement ends and the next one starts.

When the events are not discriminated, the number of counted events is incremented by one after each new live-time interval  $[t_k, t_k + i_k]$ , and the corrected total live time is incremented by the quantity  $F_j(t_k + i_k) - F_j(t_k)$  for each counted process  $j$ .

## 8. Conclusion

Dead-time correction formulae have been established in the general case of superimposed non-homogeneous Poisson processes. They are based on the possibility to get and store the time stamps of the live-time intervals, i.e. their starting and ending times. Expressions of the estimates of the variances of corrected rates are also presented. The live-time stamping method presented is based on the same principles as conventional live-timed counting, but going further as it fully exploits the additional information made available for each live-time interval. This progress is made possible due to the development of digital signal processing systems, especially in the field of nuclear instrumentation for the measurement of radioactive sources. The general formulae can be transposed to the case of the activity measurement, in the presence of background, of a mixture of short-lived radionuclides regardless of their number. The presented relations are exact within the statistical counting uncertainty, insofar as no explicit approximation is made to obtain them. The measured rate accuracy then depends only on the accuracy of live-time measurements and on the consistency of the measured live-time intervals with the effective ones.

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