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Discretized Keiper/Li approach to the Riemann Hypothesis

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Abstract

The Keiper–Li sequence $\{\lambda_n\}$ is most sensitive to the Riemann Hypothesis asymptotically ($n \rightarrow \infty$), but highly elusive both analytically and numerically. We deform it to fully explicit sequences, simpler to analyze and to compute (up to $n = 5 \cdot 10^5$ by G. Misguich). We extend that to the Davenport–Heilbronn counterexamples, then demonstrate explicit tests that selectively react to zeros *off* the critical line.

The present text develops our computations announced from 2015 [34].

1 Main notations and background

- $\zeta(x)$: the Riemann zeta function, $x \in \mathbb{C} \setminus \{1\}$; (e.g., [30][10, chap. 8, 15])
- $2\xi(x)$: a completed zeta function, obeying Riemann’s Functional Equation,

$$2\xi(x) \stackrel{\text{def}}{=} x(x-1)\pi^{-x/2}\Gamma(x/2)\zeta(x) \equiv 2\xi(1-x), \quad (1)$$

and better normalized for us than Riemann’s ξ -function: $2\xi(0) = 2\xi(1) = 1$. The symmetry in (1) makes us also denote $x = \frac{1}{2} + t + iT$ (t, T real).

- $\{\rho\}$: the set of zeros of ξ (the *nontrivial zeros* of ζ or *Riemann zeros*, counted with multiplicities if any); this set has the symmetry axes \mathbb{R} and

$L \stackrel{\text{def}}{=} \{\text{Re } x = \frac{1}{2}\}$ (the *critical line*), it lies in the *critical strip* $\{0 < \text{Re } x < 1\}$, and its ordinate-counting function $N(T) \stackrel{\text{def}}{=} \#\{\rho \mid 0 \leq \text{Im } \rho \leq T\}$ has the *Riemann–von Mangoldt* asymptotic form

$$N(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - 1 \right) + O(\log T), \quad T \rightarrow +\infty; \quad (2)$$

infinite summations over zeros are to be ordered symmetrically, as

$$\sum_{\rho} \stackrel{\text{def}}{=} \lim_{T \rightarrow \infty} \sum_{|\text{Im } \rho| \leq T} . \quad (3)$$

• **The Riemann Hypothesis (RH):** *for all* Riemann zeros, $\text{Re } \rho \equiv \frac{1}{2}$. [26] This remains a major open problem. Up to *bounded* heights $|\text{Im } \rho| < T_0$ on the other hand, $\text{Re } \rho \equiv \frac{1}{2}$ has been systematically confirmed (by direct checking), with T_0 having currently reached the value [15]

$$T_0 \approx 2.4 \cdot 10^{12} \quad (\text{as of 2004}). \quad (4)$$

Our goal is to present a novel type of *explicit* sequential criteria for RH, and to see how these might work to test RH *further* (i.e., beyond T_0).

We will dub ρ' any zeros for which we assume $\text{Re } \rho' > \frac{1}{2}$ (violating RH).

• $k!!$: the double factorial, used here for odd integers k only, in which case

$$\begin{aligned} k!! &\stackrel{\text{def}}{=} k(k-2) \cdots 1 && \text{for odd } k > 0, \\ &\stackrel{\text{def}}{=} 2^{(k+1)/2} \Gamma(\frac{1}{2}k + 1) / \sqrt{\pi} && \text{for odd } k \geq 0 \quad (\text{e.g., } (-1)!! = 1). \end{aligned} \quad (5)$$

• B_{2m} : Bernoulli numbers ($m = 0, 1, 2, \dots$); γ : Euler's constant.

1.1 The Keiper and Li coefficients

In 1992 Keiper [18] considered a real sequence $\{\lambda_n\}$ of generating function

$$\varphi(z) \stackrel{\text{def}}{=} \log 2\xi(M(z)) \equiv \sum_{n=1}^{\infty} \lambda_n^K z^n, \quad M(z) \stackrel{\text{def}}{=} \frac{1}{1-z}, \quad (6)$$

(we write λ_n^K for *Keiper's* λ_n), deduced that

$$\lambda_n^K \equiv n^{-1} \sum_{\rho} [1 - (1 - 1/\rho)^n], \quad (7)$$

and that $\text{RH} \Rightarrow \lambda_n^K > 0$ ($\forall n$), then asserted (without proof nor elaboration): “if we assume the Riemann hypothesis, and further that the zeros are very evenly distributed, we can show that $\lambda_m \approx \frac{\log m}{2} - \frac{\log(2\pi) + 1 - \gamma}{2}$.” I.e.,

$$\lambda_m^K \approx \frac{1}{2} \log m + c, \quad c = \frac{1}{2}(\gamma - \log 2\pi - 1) \approx -1.130330700754. \quad (8)$$

The conformal mapping $M : z \mapsto x$ in (6) pulls back each zero ρ to

$$z_\rho \stackrel{\text{def}}{=} M^{-1}(\rho) = 1 - 1/\rho, \quad (9)$$

and *the critical line L to the unit circle $\{|z| = 1\}$* , ensuring that (Fig. 1):

$$\text{RH} \iff |z_\rho| \equiv 1 \ (\forall \rho) \iff \varphi \text{ is regular in the full disk } \{|z| < 1\}. \quad (10)$$

In 1997 Li [20] independently introduced another sequence $\{\lambda_n\}$, through

$$\lambda_n^L = \frac{1}{(n-1)!} \frac{d^n}{dx^n} [x^{n-1} \log 2\xi(x)]_{x=1}, \quad n = 1, 2, \dots, \quad (11)$$

(we write λ_n^L for *Li's* λ_n), deduced that

$$\lambda_n^L \equiv \sum_{\rho} [1 - (1 - 1/\rho)^n], \quad (12)$$

and proved the *sharp equivalence (Li's criterion)*:

$$\text{RH} \iff \lambda_n^L > 0 \text{ for all } n. \quad (13)$$

As seen for instance by comparing (7) vs (12),

$$\lambda_n^L \equiv n \lambda_n^K \quad \text{for all } n = 1, 2, \dots; \quad (14)$$

our superscripts K vs L will disambiguate λ_n whenever the factor n matters.

1.2 Probing RH through the Keiper–Li constants λ_n

Li's criterion (13) makes it clear that the Keiper–Li sequence is *RH-sensitive*: does it then provide new and more efficient ways to test RH?

Known results actually entail that, beyond the present frontier (4), the sequence $\{\lambda_n\}$ may effectively probe RH only in its *tail* $n \gg 1$ and via its *asymptotic form* for $n \rightarrow \infty$.

In 2000 Oesterlé proved (but left unpublished) that [25, prop. 2][5, § 2.3]

$$\operatorname{Re} \rho = \frac{1}{2} \text{ for all zeros with } |\operatorname{Im} \rho| \leq T_0 \implies \lambda_n \geq 0 \text{ for all } n \leq T_0^2, \quad (15)$$

and that under RH, [25, § 2]

$$\lambda_n^L = n\left(\frac{1}{2} \log n + c\right) + o(n)_{n \rightarrow \infty} \quad \text{with } c = \frac{1}{2}(\gamma - \log 2\pi - 1), \quad (16)$$

which concurs with Keiper's formula (8) but now assuming *RH alone*.

In 2004 Maślanka [21][22] computed a few thousand λ_n^L -values numerically and inferred asymptotic conjectures on them for the case RH true.

In 2004–2006, inspired by the latter (but unaware of [25]), we used the saddle-point method to draw an *asymptotic criterion* for RH: [31] as $n \rightarrow \infty$,

- RH false: $\lambda_n^L \sim - \sum_{\operatorname{Re} \rho' > \frac{1}{2}} z_{\rho'}^{-n} \pmod{o(r^{-n}) \forall r < 1}$ (17)

(exponentially growing oscillations *with both signs*);¹

- RH true: $\lambda_n^L \sim n\left(\frac{1}{2} \log n + c\right) \pmod{o(n)}$ (18)

(implying asymptotic *positivity*).

Note: the remainder term $o(n)$ in (16) got improved to $O(\sqrt{n} \log n)$ by Lagarias [19] (2007), and to ny_n with $\{y_n\} \in \ell^2$ by Arias de Reyna [1] (2011).

Hence, even with the (fixed) sequence $\{\lambda_n\}$, violations of RH will get increasingly harder to track as the floor height T_0 (currently (4)) gets higher.

First, Li's sign test (13) is, by (15), inactive up to $n = T_0^2$ ($\approx 5 \cdot 10^{24}$ today) at least. As for the asymptotic alternative (17)–(18): if a zero $\rho' = \frac{1}{2} + t + iT$ violates RH then $|T| \geq T_0 \gg 1$, and the effect $z_{\rho'}^{-n}$ in (17) becomes detectable against the background (18) only for $n \gtrsim T^2/|t|$. [32] Now in the z variable, that RH-violation is measured by $\delta|z| = |z_{\rho'}| - 1$ (Fig. 1 right); but $\delta|z|/|z| \approx |t|/T^2$ as $|T| \gg 1$, therefore $n \gtrsim T^2/|t|$ means $n|\delta \log |z|| \gtrsim 1$

¹Erratum: in [31][32] we missed the overall $(-)$ sign (with no effect on our conclusions), which we rectified in [33].

which is no less than the *uncertainty principle*, as $(i \log z)$ and n are Fourier-conjugate variables in (6). Then, as that principle is universal, $n \gtrsim T^2/|t|$ will bound *any* detection of ρ' through λ_n . With $|t| < \frac{1}{2}$, the best possible sensitivity domain of $\{\lambda_n\}$ to RH is, finally,

$$n \gtrsim 2T_0^2, \quad \text{currently implying } n \gtrsim 10^{25}. \quad (19)$$

It is then no surprise that published λ_n -plots (having $n \lesssim 7000$) solely reflect the RH-true pattern (18) (already from $n \approx 30$). [18][21]

So, whether one would take (13) or (17)–(18) to track violations of RH, the Keiper–Li sequence $\{\lambda_n\}$ only matters in its *asymptotic tail* $\{n \gg 1\}$, where the alternative (17)–(18) rules (and enacts Li’s sign property as well).

At the same time, the λ_n are quite elusive analytically [7][9]. Numerically too, (see Maślanka [21][22] and Coffey [8]) their evaluation requires a recursive machinery, of intricacy blowing up with n ; [22, fig. 6] moreover reports a loss of ca. 0.2 decimal place of precision per step n (when working *ex nihilo* - i.e., using no Riemann zeros in input); only λ_n -values up to $n \approx 4000$ were thus accessed until recently, with $n = 10^5$ attained by Johansson [17, § 4.2] (who states a loss of 1 bit ≈ 0.3 decimal place per step n). Even then, the range (19) needed for new tests of RH stays out of reach, motivating quests for more accessible sequences.

But first, as the *asymptotic* sensitivity to RH is the main property we will prove to generalize, we review its mechanism for $\{\lambda_n\}$ itself.

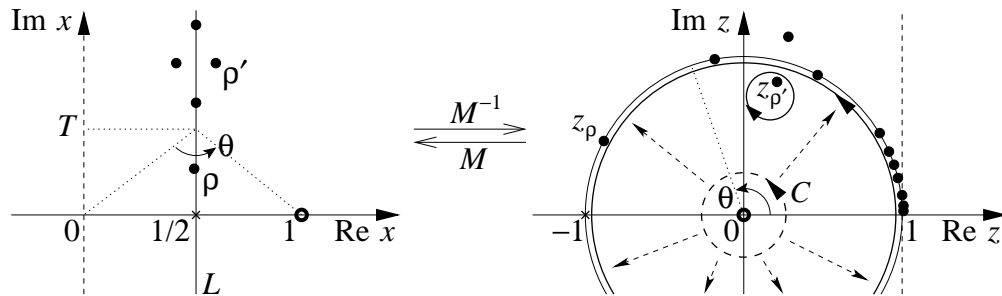


Figure 1: Riemann zeros (\bullet) depicted *schematically* in the x (left) and z (right) upper half-planes (at mock locations, including a putative pair off the critical line L). Symmetrical lower half-planes, and the zeros therein, are implied. In the z -plane (right), we also plot the contour deformation used by Darboux’s method upon the integral (21) for $n \rightarrow \infty$.

1.3 Asymptotic analysis of $\{\lambda_n\}$ (to be generalized)

The forthcoming derivation of the large- n alternative (17)–(18) for λ_n readily settles the RH-false case (17), then needs one more step for the RH-true formula (18). (In [31] we obtained both cases in parallel by the saddle-point method used on a single integral, but this approach does not yet extend.)

Within the broader setting of large-order asymptotics, we basically follow the classic Darboux’s idea [12, § 7.2]: for a sequence like (6), the $n \rightarrow \infty$ form is ruled by the radius of convergence of its generating function $\varphi(z)$, by use of the integral formula (equivalent to (6) by the residue theorem)

$$\lambda_n^K = \frac{1}{2\pi i} \oint_C \frac{dz}{z^{n+1}} \varphi(z), \quad \varphi(z) \equiv \log 2\xi\left(\frac{1}{1-z}\right), \quad (20)$$

where C is a positive contour close to $z = 0$ leaving all other singularities (namely, those of $\varphi(z)$) outside.

1.3.1 Darboux’s method does the RH false case

Solely for this stage, it is worth integrating (20) by parts first, to

$$\lambda_n^L = (2\pi i)^{-1} \oint_C z^{-n} \varphi'(z) dz : \quad (21)$$

φ' , a meromorphic function, will be simpler to use than the multiply-valued φ .

Since the integrand in (21) has the large- n form $e^{\Phi_n(z)}$ where Φ_n tends to ∞ with n ($\Phi_n(z) \sim -n \log z$), we may use the steepest-descent method [13, § 2.5] to deform the contour C toward decreasing $\text{Re}(-n \log z)$, i.e., as a circle of radius r growing toward 1 (Fig. 1 right); then, by the residue theorem, each singularity of φ' swept in turn, namely a simple pole $z_{\rho'}$ per RH-violating zero ρ' on the $\{\text{Re } x > \frac{1}{2}\}$ side, yields an asymptotic contribution ($-z_{\rho'}^{-n}$) in descending order, and these altogether add up to (17). [31]

If now RH is true, then as the radius of the contour attains $r = 1^-$, (17) reaches no better than $\lambda_n^L = o(r^{-n})$ ($\forall r < 1$); only a finer analysis of the limiting integral at $r = 1^-$ pins down an asymptotic form for λ_n , see next.

1.3.2 Oesterlé’s argument for the RH true case [25]

(reworded by us). Its starting point will be the real-integral form (22) below, which comes from letting the contour in (20) go up to $\{|z| = 1^-\}$

(unobstructed, under RH true), making the change of variable $z = e^{i\theta}$, and reducing to an integral over real θ : [25][31]

$$\lambda_n^K = 2 \int_0^\pi \sin n\theta N\left(\frac{1}{2} \cot\left(\frac{1}{2}\theta\right)\right) d\theta. \quad (22)$$

Now, θ real $\iff M(z) = \frac{1}{2} + iT$ with real $T \equiv \frac{1}{2} \cot \frac{1}{2}\theta$, so θ is also the angle subtended by the real vector $(\overrightarrow{O1})$ from the point $\frac{1}{2} + iT$ (Fig. 1 left); the counting function condenses to the critical line: $N(T) \equiv \#\{\rho \in [\frac{1}{2}, \frac{1}{2} + iT]\}$. An alternative validation of (22) is that its (Stieltjes) integral by parts $n^{-1} \int_0^\pi 2(1 - \cos n\theta) dN$ at once yields the sum formula (7) under RH.

The $n \rightarrow \infty$ form mod $o(1)$ of (22) now directly stems from the Riemann–von Mangoldt formula (2), $N(T) = \frac{T}{2\pi} (\log \frac{T}{2\pi} - 1) + O(\log T)_{T \rightarrow +\infty}$:

1) $O(\log T)$ is integrable in θ up to $\theta = 0$, hence the Riemann–Lebesgue lemma says that its integral against $\sin n\theta$ in (22) is $o(1)$, i.e., negligible;

2) we change to the variable $\Theta_n \equiv n\theta$; then, change the resulting upper integration bound $n\pi$ to $+\infty$ and use $T \sim 1/\theta = n/\Theta_n$, all mod $o(1)$, to get

$$\lambda_n^K \sim \int_0^\infty 2 \sin \Theta_n \frac{n}{2\pi\Theta_n} \left[\log \frac{n}{2\pi\Theta_n} - 1 \right] \frac{d\Theta_n}{n} \pmod{o(1)}. \quad (23)$$

This finally reduces, by the classic formulae $\int_0^\infty \sin \Theta d\Theta/\Theta = \pi/2$ and $\int_0^\infty \sin \Theta \log \Theta d\Theta/\Theta = -\pi\gamma/2$ [16, eqs. (3.721(1)) and (4.421(1))], to

$$\lambda_n^K = \frac{1}{2} \log n + c + o(1) \quad (\text{under RH true}), \quad (24)$$

amounting to (16). □

2 An *explicit* variant to the sequence $\{\lambda_n^K\}$

Given those difficulties with the original sequence $\{\lambda_n\}$ (§ 1.2), we propose to *deform* it (specifically, *Keiper's* form (6)) into a simpler one, still RH-sensitive but of *elementary closed form*.

While the original specification (6) for λ_n looks *rigid*, this is due to an extraneous assumption implied on the mapping M : that the x -plane image $x_0 \stackrel{\text{def}}{=} M(z=0)$ has to be the pole of ζ , i.e., $x_0 \equiv 1$. Indeed, (6) at once builds $\{\lambda_n\}$ upon the germ of $\log 2\xi(x)$ at $x_0 = M(0)$ (“basepoint” for $\{\lambda_n\}$), and fixes $M(0) \equiv 1$ but *this* imposition is only optional. To wit, (6) with

other conformal mappings $\widetilde{M} \neq M$ gives RH-sensitivity just as well: the key condition is (10), fulfilled if $\widetilde{M}^{-1}(\{\rho\} \cap L) \subset \{|z| = 1\}$, and this nowhere binds the basepoint x_0 (now: $\widetilde{M}(0)$). Prime such examples are all $\widetilde{M} = M \circ H_{\tilde{z}}$, where $H_{\tilde{z}}$ conformally maps the unit disk onto itself as

$$z \mapsto H_{\tilde{z}}(z) \stackrel{\text{def}}{=} (z - \tilde{z}) / (1 - \tilde{z}^* z) \quad (\text{Möbius transformation}); \quad (25)$$

these \widetilde{M} , for which $x_0 = (1 + \tilde{z})^{-1}$, yield *parametric* coefficients $\lambda_n(x_0)$ in terms of the derivatives $(\log \xi)^{(m)}(x_0)$; those are in fact Sekatskii's [27] "generalized Li's sums" $k_{n,a}$ with $x_0 \equiv (1 - a) \in \mathbb{R} \setminus \{\frac{1}{2}\}$. Independently, different (double-valued) conformal mappings \widetilde{M} yield "centered" λ_n^0 of basepoint precisely $x_0 = \frac{1}{2}$, the symmetry center for $\xi(x)$ ([33, § 3.4], and Appendix).

But to attain truly simpler and explicit results, neither of those alterations goes far enough. As a further step, rather than depending on a single basepoint (except, in a loose sense, $x_0 = \infty$?), we will crucially *discretize* the derivatives of $\log \xi$ within the original λ_n into selected *finite differences* (and likewise in the Appendix for our centered λ_n^0).

2.1 Construction of a new sequence $\{\Lambda_n\}$

The λ_n are quite elusive as they involve *derivatives* of $\log 2\xi$ (and worse, of growing order), cf. (11). On the integral form (20), that clearly ties to the denominator z^{n+1} having its zeros *degenerate* (all at $z = 0$, see fig. 2 left).

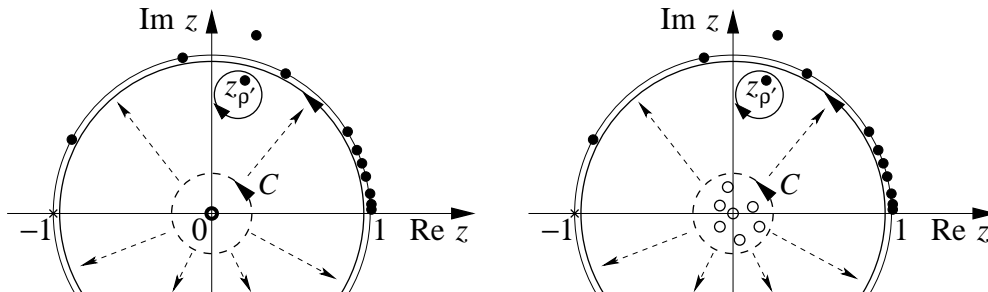


Figure 2: As Fig. 1 right, but now showing the multiple pole $z = 0$ (left plot) split into simple ones (right plot) without disrupting the large- n analysis of § 1.3.

Now at given n , if we *split those zeros apart* as $0, z_1, \dots, z_n$ (all distinct, and still inside the contour: fig. 2 right), then by the residue theorem, (20) will simplify to a linear combination of the $\varphi(z_m)$, i.e., a *finite difference*.

Doing so by plain shifts of the factors $z \mapsto z - z_m$ would also split their unit disks apart, and thus ruin the setting of § 1.3 for asymptotic RH-sensitivity. So to fix $\{|z| = 1\}$, we again use *hyperbolic* translations (25), but now *a different one upon each factor*: $z \mapsto H_{z_m}(z)$, instead of a common one on all of z^{n+1} as done formerly (which left $\log 2\xi$ differentiated, only elsewhere).

The origin $z = 0$ has then lost all special status, hence so does the particular mapping M (selected to make $z = 0$ the preimage of the pole $x = 1$); then the variable x , natural for the ζ -function, is also the simplest to use here. Rewritten in the variable x , (20) reads as

$$\lambda_n^K = \frac{1}{2\pi i} \oint \frac{dx}{x(x-1)} \left(\frac{x}{x-1}\right)^n \log 2\xi(x) \quad (\text{integrated around } x = 1), \quad (26)$$

and the deformations just introduced have the form

$$\frac{1}{2\pi i} \oint_{\mathcal{C}_n} \frac{dx}{x(x-1)} \frac{1}{h_{x_1}(x) \dots h_{x_n}(x)} \log 2\xi(x), \quad h_{\tilde{x}}(x) \equiv \frac{\tilde{x}^*}{\tilde{x}} \frac{x - \tilde{x}}{x + \tilde{x}^* - 1}, \quad (27)$$

where the contour \mathcal{C}_n encircles the set $\{1, x_1, \dots, x_n\}$ positively (and may as well depend on n). Then the integral in (27) readily evaluates to

$$\sum_{m=1}^n \operatorname{Res}_{x=x_m} f_n(x) \log 2\xi(x_m), \quad f_n(x) \stackrel{\text{def}}{=} \frac{1}{x(x-1)} \frac{1}{[h_{x_1} \dots h_{x_n}](x)}, \quad (28)$$

by the residue theorem ($x = 1$ contributes zero thanks to $\log 2\xi(1) = 0$).

Now, we choose $x_m \equiv 2m$ for $m = 1, 2, \dots$ (independently of n), to capitalize on the known values $\zeta(2m)$. That fixes

$$f_n(x) = \frac{g_n(x)}{x(x-1)} \quad (\sim 1/x^2 \quad \text{for } x \rightarrow \infty), \quad (29)$$

$$g_n(x) \stackrel{\text{def}}{=} \prod_{m=1}^n \frac{x + 2m - 1}{x - 2m} \equiv \frac{\Gamma(\frac{1}{2}x - n) \Gamma(\frac{1}{2}(x+1) + n)}{\Gamma(\frac{1}{2}x) \Gamma(\frac{1}{2}(x+1))} \quad (30)$$

$$\equiv g(x) (-1)^n \frac{\Gamma(\frac{1}{2}(x+1) + n)}{\Gamma(1 - \frac{1}{2}x + n)}, \quad g(x) \stackrel{\text{def}}{=} \frac{\sqrt{\pi} 2^{x-1}}{\sin(\pi x/2) \Gamma(x)} \quad (31)$$

(by the duplication and reflection formulae for Γ). The resulting residues are

$$\operatorname{Res}_{x=1} f_n = -\frac{(-1)^n}{A_{n0}}, \quad \operatorname{Res}_{x=2m} f_n = (-1)^{n+m} A_{nm} \quad (m = 0, 1, 2, \dots),$$

with

$$A_{nm} = 2^{-2n} \frac{(n+m, n-m, 2m)!}{2m-1} \equiv \frac{1}{(2m-1)(n-m)!m!} \frac{\Gamma(n+m+1/2)}{\Gamma(m+1/2)}, \quad (32)$$

where $(i, j, k)! \stackrel{\text{def}}{=} \frac{(i+j+k)!}{i!j!k!}$ (multinomial coefficient).

For later use, the partial fraction decomposition itself of f_n is

$$f_n(x) = (-1)^n \left[\frac{-1}{A_{n0}} \frac{1}{x-1} + \sum_{m=0}^n \frac{(-1)^m A_{nm}}{x-2m} \right] : \quad (33)$$

enforcing $f_n(x) \underset{x \rightarrow \infty}{\sim} 1/x^2$ from (29) upon the right-hand side first confirms the constant term therein to be 0 as shown, then yields two more identities:

$$\sum_{m=0}^n (-1)^m A_{nm} \equiv \frac{1}{A_{n0}}, \quad 2 \sum_{m=1}^n (-1)^m A_{nm} m \equiv (-1)^n + \frac{1}{A_{n0}}. \quad (34)$$

Next, for each n we select a contour \mathcal{C}_n that just encircles the real interval $[1, 2n]$ positively (to encircle the subinterval $[2, 2n]$ would suffice, but here it will always be beneficial to *dilate*, not shrink, \mathcal{C}_n). Our final result is then

$$\Lambda_n \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_{\mathcal{C}_n} f_n(x) \log 2\xi(x) dx \quad (\text{with } f_n \text{ from (29)}) \quad (35)$$

$$\equiv (-1)^n \sum_{m=1}^n (-1)^m A_{nm} \log 2\xi(2m), \quad n = 1, 2, \dots, \quad (36)$$

and the latter form is *fully explicit*: A_{nm} are given by (32), and

$$2\xi(2m) = \frac{(-1)^{m+1} B_{2m}}{(2m-3)!!} (2\pi)^m \equiv \frac{2(-1)^{m+1} B_{2m}}{\Gamma(m-\frac{1}{2})} \pi^{m+1/2}, \quad m = 0, 1, \dots \quad (37)$$

$$E.g., \Lambda_1 = \frac{3}{2} \log \frac{\pi}{3}, \quad \Lambda_2 = \frac{5}{24} \log \left[\left(\frac{2}{5} \right)^7 \frac{3^{11}}{\pi^4} \right], \quad \Lambda_3 = \frac{21}{80} \log \left[\frac{5^{25} \pi^8}{2^3 (3^2 \cdot 7)^{11}} \right]. \quad (38)$$

So, we deformed Keiper's $\{\lambda_n^K\}$ by discretizing the derivatives on $\log 2\xi$ to *finite differences* anchored at locations $1, \{2m\}$ where ξ has known values, in a canonical way basically dictated by the *preservation of RH-sensitivity*.² This discretized Keiper sequence $\{\Lambda_n\}$ has the *elementary closed form* (36), which is moreover directly computable at any n (in welcome contrast to the original λ_n , which need an iterative procedure all the way up from $n = 1$).

²One still has the freedom to spread the x_m further out by skipping some locations ($2j$), (or inversely, to keep some residual degeneracy if ever this were to ease computations).

2.2 Remarks.

1) Thanks to the second sum rule (34), the $(\log 2\pi)$ -contributions to (36) from the first expression (37) can be summed, resulting in $\Lambda_n \equiv \frac{1}{2} \log 2\pi + u_n$ with

$$u_n \stackrel{\text{def}}{=} (-1)^n \left[\sum_{m=1}^n (-1)^m A_{nm} \log \frac{|B_{2m}|}{(2m-3)!!} + \frac{1}{2A_{n0}} \log 2\pi \right] : \quad (39)$$

this sequence $\{u_n\}$ is the one used in our first note [34]. Likewise, the right-most expression (37) leads to the partially summed form

$$\Lambda_n \equiv \frac{1}{2} \log \pi + (-1)^n \left[\sum_{m=1}^n (-1)^m A_{nm} \log \frac{|B_{2m}|}{\Gamma(m-\frac{1}{2})} + \left(\frac{1}{A_{n0}} - A_{n0} \right) \log 2 + \left(\frac{1}{A_{n0}} - \frac{A_{n0}}{2} \right) \log \pi \right], \quad (40)$$

suitable for computer routines able to directly deliver $(\log \Gamma)$.

2) If in place of (37) we evaluate $\log 2\xi(2m)$ using (1) plus the expanded logarithm of the Euler product: $\log \zeta(x) \equiv \sum_{p \text{ prime}} \sum_{r=1}^{\infty} \frac{p^{-rx}}{r}$, then (36) gives an *arithmetic* form for Λ_n , like Bombieri–Lagarias’s Thm 2 for λ_n^L . [7, § 3]

3) Báez-Duarte [2] has an equally explicit sequential criterion for RH in terms of the B_{2m} , but in which the critical threshold is inordinately large, $n \gtrsim e^{\pi T_0}$ [23, § 4] ([14, § 7] quotes $n \gtrsim 10^{600,000,000}$); for our Λ_n the analogous n -value will prove considerably lower (§ 4.2).

4) The whole scheme will be extended from the Riemann zeta function ζ to certain *Dirichlet L-functions* in § 3.5, then to some linear combinations thereof, specifically the *Davenport–Heilbronn* functions, in § 4.4.

2.3 Expression of Λ_n in terms of the Riemann zeros

Let the primitive F_n of the function f_n in (29), (33) be defined by

$$\begin{aligned} F_n(x) &\stackrel{\text{def}}{=} \int_{\infty}^x f_n(y) dy && (\Rightarrow F_n(x) \sim -1/x \text{ for } x \rightarrow \infty) \\ &\equiv (-1)^n \left[-\frac{1}{A_{n0}} \log(x-1) + \sum_{m=0}^n (-1)^m A_{nm} \log(x-2m) \right] \end{aligned} \quad (41)$$

and by single-valuedness in the whole x -plane minus the cut $[0, 2n]$: e.g., $F_1(x) = \frac{1}{2} \log [x(x-2)^3/(x-1)^4]$.

Then in terms of (41), the Λ_n are expressible by summations over the zeros (which converge like $\sum_{\rho} 1/\rho$ for any n , hence need the rule (3)):

$$\Lambda_n \equiv \sum_{\rho} F_n(\rho), \quad n = 1, 2, \dots \quad (42)$$

(In the original λ_n^K , (26) uses $[x/(x-1)]^n$ in place of $g_n(x)$, which exceptionally yields *rational* functions $n^{-1}[1 - (1 - 1/(1-x))^n]$ in place of the $F_n(x)$, for which (42) restores (7).)

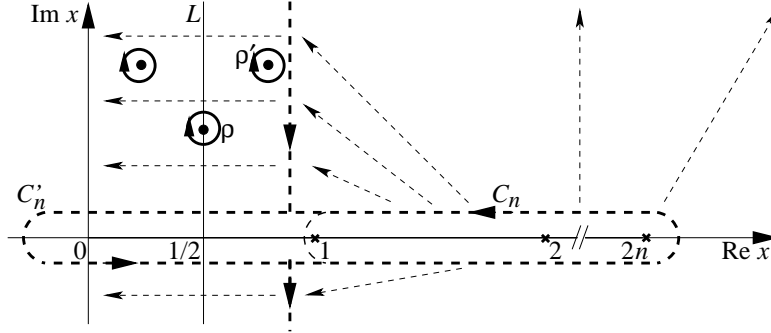


Figure 3: As Fig. 1 left, but superposing the deformation of the integration path for the integral (43) in which the function ξ'/ξ has the Riemann zeros as simple poles. A symmetrical lower half-plane is implied.

Proof of (42) (outlined, see fig. 3): first stretch the contour \mathcal{C}_n in (35) to \mathcal{C}'_n fully enclosing the cut $[0, 2n]$ of F_n (as allowed by $\log 2\xi(0) = 0$). Since F_n is single-valued on \mathcal{C}'_n , the so modified (35) can be integrated by parts,

$$\Lambda_n \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \oint_{\mathcal{C}'_n} F_n(x) \left[\frac{\xi'}{\xi} \right] (x) dx, \quad (43)$$

then the contour \mathcal{C}'_n can be further deformed into a sum of an outer anti-clockwise circle \mathcal{C}_R centered at $\frac{1}{2}$ of radius $R \rightarrow \infty$ (not drawn), and of small clockwise circles around the poles of the meromorphic function ξ'/ξ inside \mathcal{C}_R ; these poles are the Riemann zeros ρ therein, and each contributes $F_n(\rho)$. By the Functional Equation (1), the integral on \mathcal{C}_R is also $\oint_{\mathcal{C}_R} \langle F_n(x) \rangle \left[\frac{\xi'}{\xi} \right] (x) dx$, where $\langle F_n(x) \rangle \stackrel{\text{def}}{=} \frac{1}{2}[F_n(x) + F_n(1-x)] = O(1/x^2)$; then this integral tends

to 0 if $R \rightarrow \infty$ while keeping at a distance from ordinates of Riemann zeros in a classic fashion (so that $|\zeta'/\zeta|(r+iR) < K \log^2 R$ for all $r \in [-1, +2]$, cf. [10, p. 108]), hence (42) results. \square

3 Resulting new sequential criterion for RH

Like the elusive Keiper–Li sequence, the fully explicit one $\{\Lambda_n\}$ proves RH-sensitive (just slightly differently). The argument will use the function

$$\Phi(X) \stackrel{\text{def}}{=} X \log\left(1 - \frac{1}{X^2}\right) + \log \frac{X+1}{X-1} \equiv \int_{\infty}^X \log\left(1 - \frac{1}{Y^2}\right) dY \text{ on } \mathbb{C} \setminus [-1, 1]. \quad (44)$$

3.1 Asymptotic criterion

Our main result is an *asymptotic* sensitivity to RH as $n \rightarrow \infty$, through this alternative for $\{\Lambda_n\}$ which parallels (17)–(18) for $\{\lambda_n\}$:

- RH false: $\Lambda_n \sim \sum_{\rho' - \frac{1}{2} \in 2n \mathcal{D}_{R_0}} F_n(\rho') \pmod{o(R^n) \forall R > R_0 > 1}, \quad (45)$

where $\mathcal{D}_{R_0} \stackrel{\text{def}}{=} \{X \in \mathbb{C} \mid \text{Re } \Phi(X) > \log R_0\}$
 $\subset \{\text{Re } X > 0\}$, so the sum (45) is a truncation of $\sum_{\text{Re } \rho' > \frac{1}{2}}$,

and $F_n(\rho') \sim \frac{g(\rho')}{\rho'(\rho'-1)} (-1)^n \frac{n^{\rho'-\frac{1}{2}}}{\log n} \quad (n \rightarrow \infty)$ for each given ρ' (46)

(giving a power-like growing oscillation *about* 0, cf. (31));

- RH true: $\Lambda_n \sim \log n + C, \quad C = \frac{1}{2}(\gamma - \log \pi - 1) \approx -0.783757110474; \quad (47)$
(implying asymptotic *positivity*);

a comparison with (24) yields $C \equiv c + \frac{1}{2} \log 2$, which testifies to a preserved kinship of Λ_n to λ_n^K .

For the RH false case, the pair (45)–(46) is formally, in the variable $\log n$, an expansion in exponentials multiplied by divergent power series (i.e., a *transseries*), to be interpreted with caveats as detailed in § 3.2.

The derivation scheme will transpose the arguments of § 1.3 to Λ_n expressed in an integral form and now in the x -plane: either (35) (using the

function $g_n(x)$) in place of (20), or its integral by parts (43) (using $F_n(x)$) in place of (21). Two new problems arise here: the large- n forms of $g_n(x)$ and $F_n(x)$ need to be worked out, and the geometry of the integrands is strongly n -dependent hence the relative scales of n and x will matter.

- If we need *uniform* asymptotics in the integrands we must rescale the geometry as, for instance, $x = \frac{1}{2} + 2nX$: this condenses the singularities onto the fixed X -segment $[0, 1]$ as $n \rightarrow \infty$. The Stirling formula applied to the ratio of Γ functions in (30) then yields

$$g_n(\frac{1}{2} + 2nX) \sim \left(\frac{X+1}{X-1}\right)^{1/4} e^{n\Phi(X)} \quad \text{for } n \rightarrow \infty, \quad (48)$$

with the function $\Phi(X)$ from (44).

Next, integrating for $F_n(x) = \int_{\infty}^x \frac{g_n(y)}{y(y-1)} dy$, with $y = \frac{1}{2} + 2nY$ yields

$$F_n(\frac{1}{2} + 2nX) \sim \frac{1}{2n} \int_{\infty}^X \frac{dY}{Y^2} \left(\frac{Y+1}{Y-1}\right)^{1/4} e^{n\Phi(Y)}, \quad (49)$$

which is a Laplace transform in the integration variable Φ , hence it has an asymptotic power series in $(1/n)$ (usually divergent) starting as [13, eq. 2.2(2)]

$$\begin{aligned} F_n(\frac{1}{2} + 2nX) &\sim \frac{1}{2n} \frac{1}{X^2} \left(\frac{X+1}{X-1}\right)^{1/4} \frac{e^{n\Phi(X)}}{n\Phi'(X)} \\ &= \frac{1}{2n^2} \frac{1}{X^2} \left(\frac{X+1}{X-1}\right)^{1/4} \frac{e^{n\Phi(X)}}{\log(1-1/X^2)} \end{aligned} \quad (50)$$

- Whereas if we let $n \rightarrow \infty$ at *fixed* x , (31) at once implies

$$g_n(x) \sim g(x)(-1)^n n^{x-1/2} \sim g(x)(-1)^n e^{\log n(x-1/2)}. \quad (51)$$

Here $(\log n)$ occupies the place of n as large parameter, and the same Laplace argument for the integral $F_n(x) = \int_{\infty}^x \frac{g_n(y)}{y(y-1)} dy$ now yields an asymptotic series in powers of $(1/\log n)$ (usually divergent), starting as

$$F_n(x) \sim \frac{g(x)}{x(x-1)} (-1)^n \frac{n^{x-1/2}}{\log n}. \quad (52)$$

3.2 Details for the case RH false

As in § 1.3.1, we can apply the steepest-descent method to the integral (43) written in the global variable X . We rescale the contour C'_n then deform it toward level contours $\text{Re } \Phi(X) = \Phi_0 \rightarrow 0^+$: these approach the completed critical line in the X -plane, $\{\text{Re } X = 0\} \cup \{\infty\}$, from the $\{\text{Re } X > 0\}$ side (fig. 4). Apart from staying on this side (and being rescaled to the X -plane), the contour deformation is isotopic to that used in § 2.3, hence it likewise yields a contribution $F_n(\rho')$ per RH-violating zero ρ' whose rescaled image $X_{\rho'} \equiv (\rho' - \frac{1}{2})/(2n)$ has $\text{Re } \Phi(X_{\rho'}) > \Phi_0 > 0$; i.e., overall,

$$\sum_{\text{Re } \Phi(X_{\rho'}) > \Phi_0} F_n(\rho').$$

(The novelty vs (42) is that now the terms come asymptotically ordered.) For $X \rightarrow \infty$, $\Phi(X) \sim 1/X$, hence the level curves asymptotically become $\{\text{Re}(1/X) = \Phi_0\}$ (circles tangent to the imaginary X -axis at 0). Thus the above sum over zeros has a natural cutoff $|\text{Im } \rho'| \lesssim \sqrt{n/\Phi_0}$, which is the height above which the disk $\{\text{Re}(1/X) > \Phi_0\}$ parts from the critical strip.

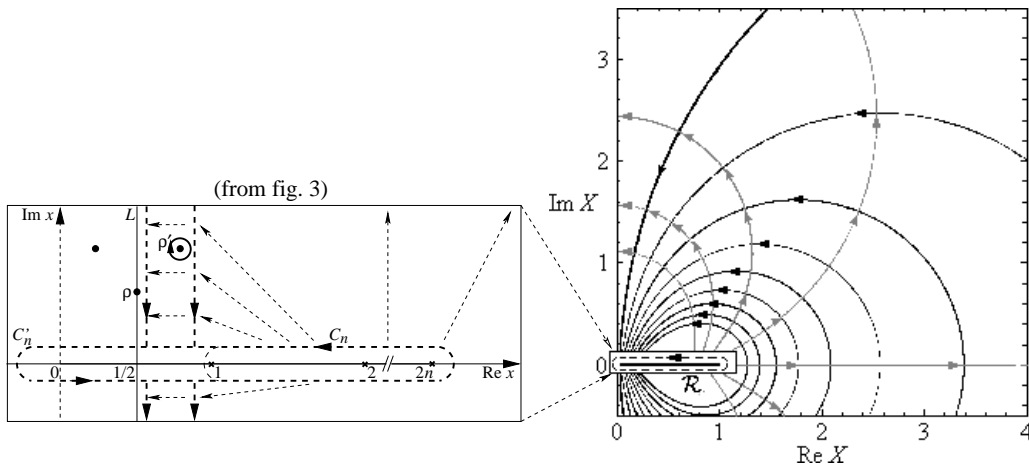


Figure 4: (at right) as fig. 3 but using the global variable $X \stackrel{\text{def}}{=} (x - \frac{1}{2})/(2n)$ in the integral (43) (fig. 3 now fills the rectangle \mathcal{R}). Mathematica [35] contour plots for $\Phi(X)$ of (44): in black, deformed integration contours (level curves $\{\text{Re } \Phi(X) = \Phi_0\}$, $\Phi_0 \downarrow 0^+$); in gray, steepest-descent lines (level curves of $\text{Im } \Phi$).

When zooming in to the $n \rightarrow \infty$, fixed- x regime, then by (52) the level curves turn to parallel lines $\{\text{Re } x - \frac{1}{2} = t_0\}$, and the deformation to $t_0 \rightarrow 0^+$

(fig. 4 left). Only a portion of contour $\mathcal{C}'_n \cap \{\operatorname{Re} x - \frac{1}{2} < \varepsilon\}$ escapes the deformation, but by (52) its contribution to the integral (43) is $O(n^\varepsilon)$, ultimately negligible. Therefore, $\Lambda_n \pmod{o(R^n)} \forall R > e^{\Phi_0}$ equals the above sum $\sum_{\rho'} F_n(\rho')$, which proves (45) under the replacement $R_0 = e^{\Phi_0}$.

This fixed- x regime actually governs *each individual term* of the sum (45), since any given zero lives at fixed x : thus, setting $x = \rho'$ in (52) directly yields the asymptotic form (46) which is *explicit* including the n -dependence, unlike the $F_n(\rho')$ for $n \rightarrow \infty$. Now the sum (45) as a whole cannot be reexpressed that way, because (46) is not uniform in x ; and indeed, a closed asymptotic form for Λ_n is out of reach: just as for λ_n in § 1.3.2 it would encode the asymptotic distribution of the Riemann zeros, but this stays fundamentally undetermined (because *truly 2-dimensional*) if RH is false. Still, (46) correctly specifies the contribution to Λ_n from *any single* RH-violating zero $\rho' \equiv \frac{1}{2} + t + iT$ ($t > 0$): i.e., a term *cycling around zero* of amplitude *growing with n* as

$$|F_n(\rho')| \approx \frac{1}{T^2 \log n} \left(\frac{2n}{|T|} \right)^t \quad \text{for } n \gg |T| \quad (53)$$

(upon using $|g(\rho')| \approx (2/|T|)^t$ for $|T| > T_0 \gg 1$).

If now RH is true then, as Φ_0 attains 0^+ , (45) reaches no better than $\Lambda_n = o(R^n)$ ($\forall R > 1$), and only a finer analysis of the limiting integral on the critical line L will lead to a definite asymptotic form, as follows.

3.3 Details for the case RH true

Here our quickest path is to adapt Oesterlé's argument from §1.3.2.

To deform $\{\lambda_n\}$ to $\{\Lambda_n\}$, we replaced $\left(\frac{x}{x-1}\right)^n$ in (26) by $g_n(x)$ from (30). That changes (22) to

$$\Lambda_n = \int_0^\pi 2 \sin \Theta_n(\theta) N\left(\frac{1}{2} \cot\left(\frac{1}{2}\theta\right)\right) d\theta, \quad (54)$$

where $\Theta_n \in (0, n\pi]$ (previously for $\lambda_n : \Theta_n \equiv n\theta$) is now the sum of the n angles subtended by the real vectors $(\overline{1-2m}, \overline{2m})$ from the point $\frac{1}{2} + iT$, for $m = 1, 2, \dots, n$. The two *endpoint slopes* of the function $\Theta_n(\theta)$ will then play the main (independent) roles:

$$\Theta'_n(0) = \sum_{m=1}^n (4m-1) \equiv n(2n+1), \quad (55)$$

$$\Theta'_n(\pi) = \sum_{m=1}^n (4m-1)^{-1} \equiv \frac{1}{4}[(\Gamma'/\Gamma)(n + \frac{3}{4}) + \gamma + 3 \log 2 - \pi/2]. \quad (56)$$

We then follow the same steps as with λ_n^K in § 1.3.2.

1) $\int_0^\pi 2 \sin \Theta_n(\theta) \delta N(\frac{1}{2} \cot(\frac{1}{2}\theta)) d\theta = o(1)$ if a nonstationary-phase principle applies for the oscillatory function $\sin \Theta_n(\theta)$, i.e., the minimum slope of $\Theta_n(\theta)$ ($\theta \in [0, \pi]$) must go to ∞ with n : previously (with $\Theta_n \equiv n\theta$) that slope was n , now it is $\Theta'_n(\pi) \sim \frac{1}{4} \log n$ which still diverges for $n \rightarrow \infty$ therefore gives that $o(1)$ bound (but due to $\Theta'_n(\pi) \ll n$, this $o(1)$ might decay much more slowly than the corresponding $o(1)$ for λ_n^K).

2) In this step (i.e., $T \rightarrow +\infty$), only $\theta \rightarrow 0$ behaviors enter; here $\Theta_n(\theta) \sim \Theta'_n(0)\theta$, vs $n\theta$ previously, so it suffices to substitute $\Theta'_n(0)$ for n in the asymptotic result (24) for λ_n^K , to get

$$\Lambda_n \sim \frac{1}{2} \log \Theta'_n(0) + c = \frac{1}{2} \log[n(2n+1)] + c \sim \log n + (c + \frac{1}{2} \log 2). \quad (57)$$

□

3.4 Asymptotic or full-fledged Li's criterion for $\{\Lambda_n\}$?

(a heuristic parenthesis).

A full Li's criterion for the new sequence $\{\Lambda_n\}$ would read as

$$\text{RH} \iff \Lambda_n > 0 \text{ for all } n \quad (\text{unproven})$$

but anyway, such a forcible inclusion of *all* n is not vital in our focus on RH-sensitivity: already for the λ_n , only the $n \gg 1$ region counted (§ 1.2); and likewise, our criterion (45)–(47) entails $\Lambda_n > 0$ *asymptotically* if and only if RH holds (fig. 9 will illustrate that on a counterexample to RH).

As for low n , $\Lambda_n > 0$ will prove *numerically* manifest there (see § 4.1).

All those observations make us *conjecture* that Li's criterion fully holds for the sequence $\{\Lambda_n\}$ as well (but appears harder to prove than for $\{\lambda_n\}$).

3.5 Generalized-RH asymptotic alternative

All previous developments carry over from $\zeta(x)$ to *Dirichlet L-functions*, which have the form (using the Hurwitz zeta function $\zeta(x, w)$)

$$L_\chi(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{\chi(k)}{k^x} \equiv d^{-x} \sum_{k=1}^d \chi(k) \zeta(x, k/d), \quad (58)$$

where χ is a special type of d -periodic function on \mathbb{Z} ($d > 1$): a *real primitive Dirichlet character* (which then has a definite parity, even or odd). [10, chaps. 5–6] Such $L_\chi(x)$ have very close properties to Riemann's $\zeta(x)$:

- Functional equations: [10, chap. 9]

$$\xi_\chi(x) \stackrel{\text{def}}{=} (\pi/d)^{-x/2} \Gamma(\frac{1}{2}(x+b)) L_\chi(x) \equiv \xi_\chi(1-x), \quad b = \begin{cases} 0 & (\chi \text{ even}) \\ 1 & (\chi \text{ odd}) \end{cases}. \quad (59)$$

- Explicit values at integers, in terms of Bernoulli polynomials: by the classic identities $(\ell+1)\zeta(-\ell, a) = -B_{\ell+1}(a)$ ($\ell = 0, 1, 2, \dots$), (58) makes $L_\chi(x)$ explicit at all $x = -\ell$; the functional equation (59) then converts that to explicit values for L_χ (and ξ_χ) at all *even, resp. odd, positive integers* according to the *even, resp. odd* parity of χ . And $L_\chi(1)$ (hence $\xi_\chi(1)$) is explicitly computable also for even parity. [10, chap. 1][32, eq.(10.70)]

- Generalized Riemann Hypothesis (GRH): *all* zeros ρ of ξ_χ have $\text{Re } \rho = \frac{1}{2}$.
- A Keiper–Li sequence $\lambda_{\chi,n}$ exists for ξ_χ in full similarity to the case of Riemann's ξ . For GRH, the only change in our asymptotic alternative (17)–(18) affects the constant c in (18) or (24) due to the replacement, in the asymptotics of the counting function $N(T)$ as used in § 1.3.2, of the term $\log(T/2\pi)$ by $\log(Td/2\pi)$, [10, chap. 16] which results in

$$\text{GRH true: } \lambda_{\chi,n}^{\text{K}} \sim \frac{1}{2} \log n + c_d \pmod{o(1)}, \quad c_d = c + \frac{1}{2} \log d. \quad (60)$$

Then as before, the definition of $\{\lambda_{\chi,n}^{\text{K}}\}$ can be discretized to yield an explicit sequence $\{\Lambda_{\chi,n}\}$ involving finite differences of *elementary* $\log \xi_\chi$ -values. For even χ , everything stays as in § 2.1 (where even parity is implied throughout), hence $\Lambda_{\chi,n}$ are given by (36) with $2\xi(2m)$ simply replaced by $\frac{1}{\xi_\chi(1)} \xi_\chi(2m)$. For odd χ on the other hand, the points x_m have to be relocated from $2m$ to $2m+1$, resulting in these main changes:

$$\begin{aligned} [(31) \rightarrow] \quad g_n^{\text{odd}}(x) &\stackrel{\text{def}}{=} \prod_{m=1}^n \frac{x+2m}{x-2m-1} \equiv g^{\text{odd}}(x) (-1)^n \frac{\Gamma(\frac{1}{2}x+1+n)}{\Gamma(\frac{1}{2}(3-x)+n)}, \\ &g^{\text{odd}}(x) \stackrel{\text{def}}{=} \frac{-\sqrt{\pi} 2^{x-1}}{\cos(\pi x/2) \Gamma(x)}, \end{aligned} \quad (61)$$

$$[(32) \rightarrow] \quad A_{nm}^{\text{odd}} = 2^{-2n} (n+m, n-m, 2m+1)! / (2m+1), \quad (62)$$

$$[(41) \rightarrow] \quad F_n^{\text{odd}}(x) = (-1)^n \left[\frac{-1}{A_{n0}^{\text{odd}}} \log x + \sum_{m=0}^n (-1)^m A_{nm}^{\text{odd}} \log(x-2m-1) \right], \quad (63)$$

$$[(36) \rightarrow] \quad \Lambda_{\chi,n} \equiv (-1)^n \sum_{m=1}^n (-1)^m A_{nm}^{\text{odd}} \log \left[\frac{1}{\xi_{\chi}(1)} \xi_{\chi}(2m+1) \right] \quad (\chi \text{ odd}). \quad (64)$$

All in all, the asymptotic alternative (45)–(47) generalizes to one depending on χ , via its parity on the “false” side, and period d on the “true” side:

- GRH false: $\Lambda_{\chi,n} \sim \sum_{\rho' - \frac{1}{2} \in 2n \mathcal{D}_{R_0}} F_n^{\#}(\rho') \pmod{o(R^n) \forall R > R_0 > 1}$ (65)

and, for each given GRH-violating zero ρ' ,

$$F_n^{\#}(\rho') \sim \frac{g^{\#}(\rho')}{\rho'(\rho' - 1)} (-1)^n \frac{n^{\rho' - \frac{1}{2}}}{\log n} \quad (n \rightarrow \infty), \quad (66)$$

$$\text{with } (F_n^{\#}, g^{\#}) \stackrel{\text{def}}{=} \begin{cases} (F_n, g) & \text{for } \chi \text{ even, cf. (41), (31)} \\ (F_n^{\text{odd}}, g^{\text{odd}}) & \text{for } \chi \text{ odd, cf. (63), (61)} \end{cases}; \quad (67)$$

- GRH true: $\Lambda_{\chi,n} \sim \log n + C_d \pmod{o(1)}$, $C_d = C + \frac{1}{2} \log d$ (68)

($\equiv c_d + \frac{1}{2} \log 2$), cf. (47), (60).

4 Quantitative aspects

We now discuss how the discretized Keiper sequence $\{\Lambda_n\}$ vs the Keiper–Li $\{\lambda_n\}$ might serve as a *practical* probe for RH in a complementary perspective to standard tests. (To rigorously (dis)prove RH using $\{\Lambda_n\}$ is also a prospect in theory, but we have not worked on that.)

4.1 Numerical data in the Riemann case

Low- n calculations of Λ_n (fig. 5) agree very early with the *RH-true* behavior (here, (47)), just as was the case for λ_n [18][21]. The remainder term $\delta\Lambda_n \stackrel{\text{def}}{=} \Lambda_n - (\log n + C)$ looks compatible with an $o(1)$ bound (fig. 6), albeit less neatly than the analogous $\delta\lambda_n^K$ [18, fig. 1][21, fig. 6b], (note: even Keiper [18] plotted $\delta\lambda_n^L = n \delta\lambda_n^K$). For the record,

$$\begin{aligned} \Lambda_1 &\approx 0.069176395771, \quad \Lambda_2 \approx 0.22745427267, \quad \Lambda_3 \approx 0.45671413349; \quad (\text{cf. (38)}) \\ n = 2000 : \quad \Lambda_n &\approx 6.815360445451163 \quad (\delta\Lambda_n \approx -0.0017849), \end{aligned}$$

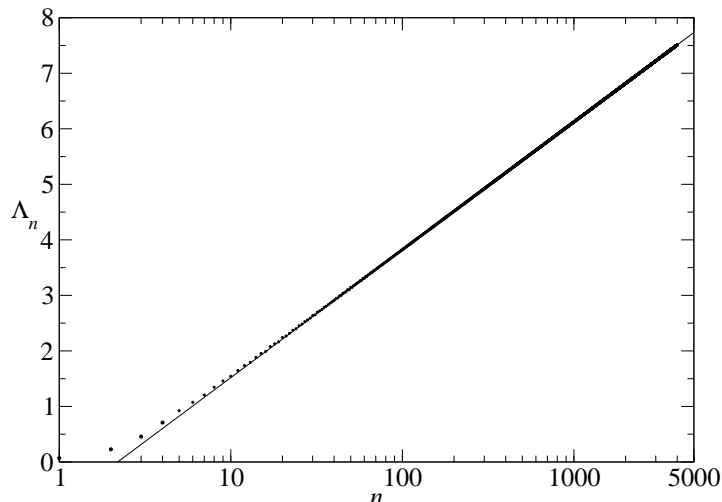


Figure 5: The coefficients Λ_n computed by (36) up to $n = 4000$, on a logarithmic n -scale (straight line: the RH-true form $(\log n + C)$ of (47)).

$$\begin{aligned}
 n = 10000 : & \quad \Lambda_n \approx 8.428662659671506 \quad (\delta\Lambda_n \approx +0.0020794), \\
 n = 20000 : & \quad \Lambda_n \approx 9.119244876955247 \quad (\delta\Lambda_n \approx -0.000485565), \\
 n = 100000 : & \quad \Lambda_n \approx 10.729678153023 \quad (\delta\Lambda_n \approx +0.0005097985), \\
 n = 200000 : & \quad \Lambda_n \approx 11.42244991847 \quad (\delta\Lambda_n \approx +0.0001343834) \\
 n = 500000 : & \quad \Lambda_n \approx 12.33812102688 \quad (\delta\Lambda_n \approx -0.0004852401)
 \end{aligned} \tag{69}$$

($n > 20000$ samples: courtesy of G. Misguich, [24] see § 4.5).

The main oscillation in $(-1)^n \delta\Lambda_n$ (fig. 6) must come from $F_n(\rho_1)$ in (42) for the lowest Riemann zero ρ_1 : the $(\log n)$ -period agrees with $2\pi/\text{Im } \rho_1$ (per the asymptotic form (52)), as $2\pi/14.1347 \approx 0.44$.

4.2 Imprints of putative zeros violating RH

RH-violating zeros ρ' (if any) seem to enter the picture just as for the λ_n : their contributions (46) will asymptotically dominate $\log n$, but numerically they will emerge and take over extremely late. Indeed for such a zero $\rho' = \frac{1}{2} + t + iT$, with $0 < t < \frac{1}{2}$ and $T \gtrsim 2.4 \cdot 10^{12}$ [15], its contribution scales like $T^{-2}(2n/T)^t/\log n$ in modulus, by (53). We then get its crossover threshold (in order of magnitude, neglecting logarithms and constants against powers)

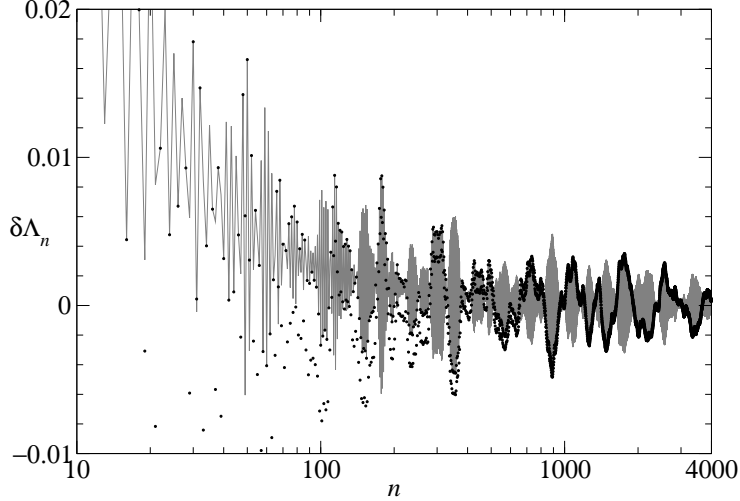


Figure 6: The remainder sequence $\delta\Lambda_n = \Lambda_n - (\log n + C)$ (in gray: the connecting segments are drawn as visual cues only), and a rectified form $(-1)^n \delta\Lambda_n$ (black dots) to cancel the period-2 oscillations.

by solving

$$T^{-2}(n/T)^t \approx 1 \quad (70)$$

$$\implies n \gtrsim T^{1+2/t} \quad (\text{best case: } O(T^{5+0}) \text{ for } t = \frac{1}{2} - 0). \quad (71)$$

This is worse than (19) for λ_n , all the more if $\Lambda_n < 0$ were sought (the right-hand side of (70) should then be $\log^2 n$). There is however room for possible improvement: the core problem is to filter out a weak ρ' -signal from the given background (47), therefore any predictable structure in the latter is liable to boost the gain. For instance, the hyperfine structure of $\delta\Lambda_n$ is oscillatory of period 2 (fig. 6); this suggests to average over that period, which *empirically* discloses a rather neat $(1/n)$ -decay trend (fig. 7):

$$\overline{\delta\Lambda_n} \stackrel{\text{def}}{=} \frac{1}{2}(\delta\Lambda_n + \delta\Lambda_{n-1}) \approx 0.25/n. \quad (72)$$

The same operation on a ρ' -signal $F_n(\rho')$ in (45) roughly applies $\frac{1}{2}(d/dn)$ to the factor n^T therein (again neglecting $t \ll T$ and $\log n$), i.e., multiplies it by $\frac{1}{2}(T/n)$. Thus heuristically, i.e., conjecturing the truth of (72) for $n \rightarrow \infty$ under RH, the crossover condition improves from (70) to

$$\begin{aligned} (T/n) T^{-2}(n/T)^t &\approx \overline{\delta\Lambda_n} \approx 1/n \\ \implies n &\gtrsim T^{1+1/t} \quad (\text{best case: } O(T^{3+0}) \text{ for } t = \frac{1}{2} - 0). \end{aligned} \quad (73)$$

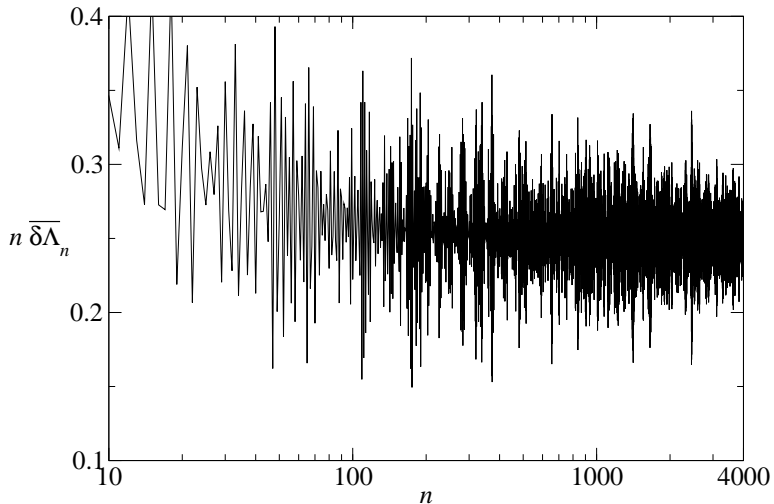


Figure 7: The averaged remainder sequence (72) rescaled by n , namely: $n \overline{\delta \Lambda}_n$. (Some further values: 0.27027 for $n = 10000$, 0.23970 for $n = 20000$, 0.2559 for $n = 100000$, 0.2683 for $n = 200000$, 0.27957 for $n = 500000$. [24])

We can hope that efficient signal-analysis techniques may still lower this detection threshold. Even an empirical mindset may be acceptable there: the $\{\Lambda_n\}$ (like $\{\lambda_n\}$) should anyway work best for *global coarse detection* of a possible RH-violation; this could then guide the classic algorithms [15] (which are local) to concentrate on a smaller region, and then confirm (or disprove) that RH-violating zero in full rigor.

4.3 The uncertainty principle for $\{\Lambda_n\}$

An absolute detection limit is however provided by the *uncertainty principle*, and this has to be quantified here. Near the critical line $\{\text{Re } X = 0\}$ and for large n , Λ_n is an integral of the zeros' distribution against essentially $e^{n\Phi(i\text{Im } X)}$ (by (43), (50)), which is a distorted plane wave. To resolve $t > 0$ for a zero $\rho' = \frac{1}{2} + t + iT \equiv \frac{1}{2} + 2nX$, the uncertainty principle asks its scale ($\text{Re } X = t/(2n)$ in the X -plane) to be at least of the order of the local wavelength which is $1/[n \log(1 + 1/(\text{Im } X)^2)]$ by (44), with $\text{Im } X = T/(2n)$: that means $\frac{1}{2}t \log(1 + 4n^2/T^2) \gtrsim 1$ or, to a fair approximation when $t < \frac{1}{2}$,

$$n \gtrsim \frac{1}{2} |T| e^{1/t}. \quad (74)$$

Against the corresponding bound $n \gtrsim T^2/t$ for $\{\lambda_n\}$ in § 1.2, at any given $t > 0$ this will favor $\{\Lambda_n\}$ once $|T| \gtrsim \frac{1}{2}t e^{1/t}$. E.g., at the current floor height (4), the best possible n -threshold (i.e., for $t = \frac{1}{2} - 0$) gets improved to 10^{13} , from 10^{25} for $\{\lambda_n\}$ in (19). That bound also allows for n to go well below the detection thresholds (71), (73) - *in principle*: there still remains to actually extract a putative ρ' -signal, extremely weak at such decreased n , from the “noise” created by the many more zeros lying nearby on the critical line.

4.4 The Davenport–Heilbronn counterexamples

More tests of interest lie in the generalized setting of § 3.5 for *odd parity*. As Dirichlet L-function we only tested the β -function (of period 4) and saw no difference of behavior patterns in $\{\Lambda_{\beta,n}\}$ vs $\{\Lambda_n\}$. If we then go beyond, there exist special periodic odd Dirichlet series that are *not* Dirichlet L-functions, but obey similar functional equations *and* have many zeros *off* the critical line L . [11][30, § 10.25][29][3][6, § 5] We may then seek a testing ground for the *RH-false branch* of our asymptotic alternative (generalized, as (65)–(66)).

Specifically, for $\phi = \frac{1}{2}(1 + \sqrt{5})$ (the golden ratio), let

$$\tau_{\pm} \stackrel{\text{def}}{=} -\phi \pm \sqrt{1 + \phi^2} \quad (\tau_+ \approx 0.28407904384, \quad \tau_- = -1/\tau_+ \approx -3.52014702134); \quad (75)$$

$$\nu_{\pm}(k) \stackrel{\text{def}}{=} \{1, +\tau_{\pm}, -\tau_{\pm}, -1, 0, \dots\}_{k=1,2,\dots} \quad \text{periodically continued}$$

(an *odd* function on the integers mod 5); and, similarly to (58),

$$f_{\pm}(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{\nu_{\pm}(k)}{k^x} \equiv 5^{-x} \left\{ \zeta\left(x, \frac{1}{5}\right) + \tau_{\pm} \left[\zeta\left(x, \frac{2}{5}\right) - \zeta\left(x, \frac{3}{5}\right) \right] - \zeta\left(x, \frac{4}{5}\right) \right\}. \quad (76)$$

These *Davenport–Heilbronn* (DH) functions f_{\pm} (denoted $f_{\frac{1}{2}}$ in [3], $f(\cdot; \tau_{\pm})$ in [6]) obey *the functional equation of odd Dirichlet L-functions*, namely (59) with $b = 1$ and the period $d = 5$, up to a (\pm) sign: [3][6, § 5]

$$\xi_{\pm}(x) \stackrel{\text{def}}{=} (\pi/5)^{-x/2} \Gamma\left[\frac{1}{2}(1+x)\right] f_{\pm}(x) \equiv \pm \xi_{\pm}(1-x). \quad (77)$$

As in § 3.5, this makes f_{\pm} , ξ_{\pm} *explicit* at the positive *odd* integers:

$$\xi_{\pm}(2m+1) = \mp \frac{2(-1)^m}{(2m+1)!!} \left[B_{2m+1}\left(\frac{1}{5}\right) + \tau_{\pm} B_{2m+1}\left(\frac{2}{5}\right) \right] \sqrt{\pi} (10\pi)^m; \quad (78)$$

$$e.g., \quad \xi_{\pm}(1) = \pm \frac{1}{5}(3 + \tau_{\pm})\sqrt{\pi} \quad \left\{ \begin{array}{l} \xi_+(1) \approx 1.1641757096 \\ \xi_-(1) \approx 0.1843873182 \end{array} \right\}$$

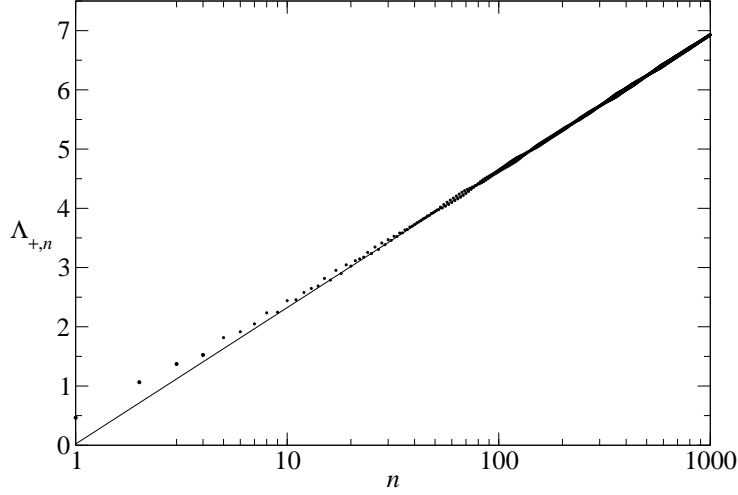


Figure 8: As fig. 5, but for the $\Lambda_{+,n}$ of the Davenport–Heilbronn (DH) function f_+ as given by (79), up to $n = 1000$; e.g., $\Lambda_{+,1} \approx 0.4653858106$, $\Lambda_{+,2} \approx 1.063986745$. Straight line: the GRH-true form $(\log n + C_5)$ of (80).

(we also used $B_{2m+1}(1-a) \equiv -B_{2m+1}(a)$). So, in line with (64) we take

$$\Lambda_{\pm,n} \stackrel{\text{def}}{=} (-1)^n \sum_{m=1}^n (-1)^m A_{mn}^{\text{odd}} \log \left[\frac{1}{\xi_{\pm}(1)} \xi_{\pm}(2m+1) \right] \quad (79)$$

as explicit sequences to probe our asymptotic criterion for GRH (as (65)–(68) with $\# = \text{odd}$ and $d = 5$) upon the DH functions f_{\pm} respectively (as cases with zeros off the critical line L).

Now the numerics break the formal (\pm) -symmetry to a surprising extent.

For ξ_+ , the lowest- T zero off the line L is $\rho'_+ \approx 0.808517 + 85.699348 i$. [29] Then, for its detection through the sequence $\{\Lambda_{+,n}\}$, our predicted threshold (70) gives $n \approx T^{1+2/t} \approx (85.7)^{7.48} \approx 3 \cdot 10^{14}$: indeed, our low- n data (fig. 8) solely reflect the GRH-true pattern (68) for $d = 5$,

$$\Lambda_{+,n} \approx \log n + C_5, \quad C_5 = C + \frac{1}{2} \log 5 \approx +0.020961845743. \quad (80)$$

Whereas for ξ_- , the lowest- T zero off L is $\rho'_- \approx 2.30862 + 8.91836 i$. [3] (Notations therein: $\xi \equiv \tau_+$, $f_2 \equiv f_-$.) Now, not only is this zero well detached from the next higher one ($\approx 1.94374 + 18.8994 i$), but above all it gives a detection threshold $n \approx T^{1+2/t} \approx (8.92)^{2.11} \approx 100$, extremely low!

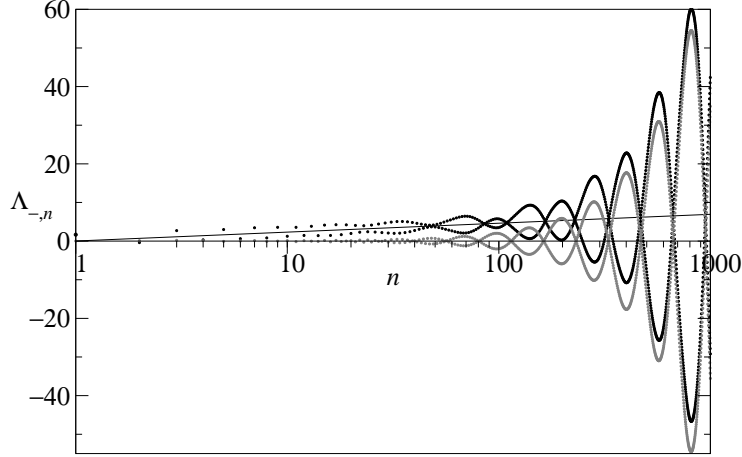


Figure 9: As fig. 8, but for the $\Lambda_{-,n}$ of the DH function f_- (vertical scale shrunk); e.g., $\Lambda_{-,1} \approx 1.661697636$, $\Lambda_{-,2} \approx -0.3913729841$ (already < 0). Straight line: the GRH-true form $(\log n + C_5)$ as in fig. 8; gray dots: the leading GRH-false form $2 \operatorname{Re} F_n^{\text{odd}}(\rho'_-)$ from (63), (81), with $\rho'_- \approx 2.30862 + 8.91836 i$.

Indeed, fig. 9 shows a neat crossover of $\Lambda_{-,n}$ from the GRH-true pattern (80) at low n , toward the dominant GRH-false pattern (65) at higher n , namely

$$F_n^{\text{odd}}(\rho'_-) + F_n^{\text{odd}}(\rho'^*_-) \equiv 2 \operatorname{Re} F_n^{\text{odd}}(\rho'_-). \quad (81)$$

Even though the form (80) is superseded by (81) asymptotically, its numerical contribution to $\Lambda_{-,n}$ fully remains, hence to test (81) we first have to subtract (80) from $\Lambda_{-,n}$; then, the remainder

$$\delta\Lambda_{-,n} \stackrel{\text{def}}{=} \Lambda_{-,n} - (\log n + C_5) \quad (82)$$

shows period-2 oscillations symmetrical about 0, so we rather plot $(-1)^n \delta\Lambda_{-,n}$: now fig. 10 shows that this is very well fitted by $(-1)^n 2 \operatorname{Re} F_n^{\text{odd}}(\rho'_-)$ from (81). In turn, (66) specifies the latter explicitly to first order in $(1/\log n)$, and this also reasonably fits the data (with $(1/\log n)$ not being that small).

Thus, numerical data for the sequences $\{\Lambda_{\pm,n}\}$ support our asymptotic alternative in full. We stress that fig. 9 models how, at some much higher n , $\{\Lambda_n\}$ itself *will* blow up if RH (for $\zeta(x)$) is false. The $\Lambda_{\pm,n}$ also ought to be a testing ground for any ideas to detect RH false earlier with the Λ_n .

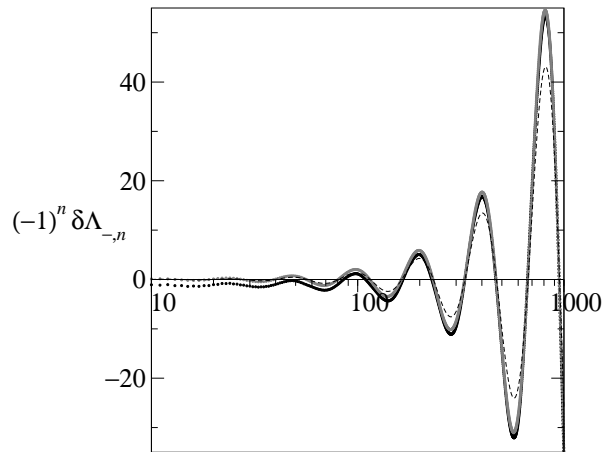


Figure 10: Asymptotics of the deviation (82) from GRH in the case of the DH function f_- . Black dots: the rectified remainder sequence $(-1)^n \delta \Lambda_{-n}$. Gray dots: the rectified asymptotic form $(-1)^n 2 \operatorname{Re} F_n^{\text{odd}}(\rho'_-)$ from (81). Dashed curve: its large- $(\log n)$ explicit form $2 \operatorname{Re} \left[\frac{g^{\text{odd}}(\rho'_-)}{\rho'_-(\rho'_--1)} n^{\rho'_--1/2} \right] (\log n)^{-1}$ from (61), (66).

4.5 The hitch

We now refocus on the numerical quest for $\{\Lambda_n\}$ (the Riemann case), to discuss a major issue. The $(\log n)$ -sized values Λ_n result from alternating summations like (36) over terms growing *much faster* (exponentially) with n : very deep cancellations must then take place, which may explain the sensitivity of Λ_n (to RH) but also create a computational hurdle for $n \gg 1$, namely a loss of precision growing linearly with n . For sums like (36), $\sum s_m$ of order comparable to unity, the *slightest* end accuracy requires each summand s_m to be input with a “base” precision $\approx \log_{10} |s_m|$ (working in decimal digits throughout); plus uniformly adding D to reach $\sum s_m$ accurate to D digits.

We can tune the required precision in (36) for each m -value at large given n by using the Stirling formula, to find that $m_* \approx n/\sqrt{2}$ is where $|s_m|$ is largest and the required base precision $\log_{10} |s_m|$ culminates, reaching $\log_{10} |A_{nm_*} \log 2\xi(2m_*)| \sim \log_{10}(3 + 2\sqrt{2})n \approx 0.76555n$ digits, see fig. 11 (vs $(0.2 \text{ to } 0.3)n$ digits for λ_n [22, fig. 6][17, § 4.2]). Even then, a crude feed of (36) (or (39), (40)) into a mainstream arbitrary-precision system (Mathematica 10 [35]) suffices to reap the Λ_n -values of § 4.1 effortlessly under $n \approx 20000$. Our computing times varied erratically but could go down to ca. 4 min for Λ_{10000} , 43 min for Λ_{20000} using (40) (CPU times on an Intel Xeon

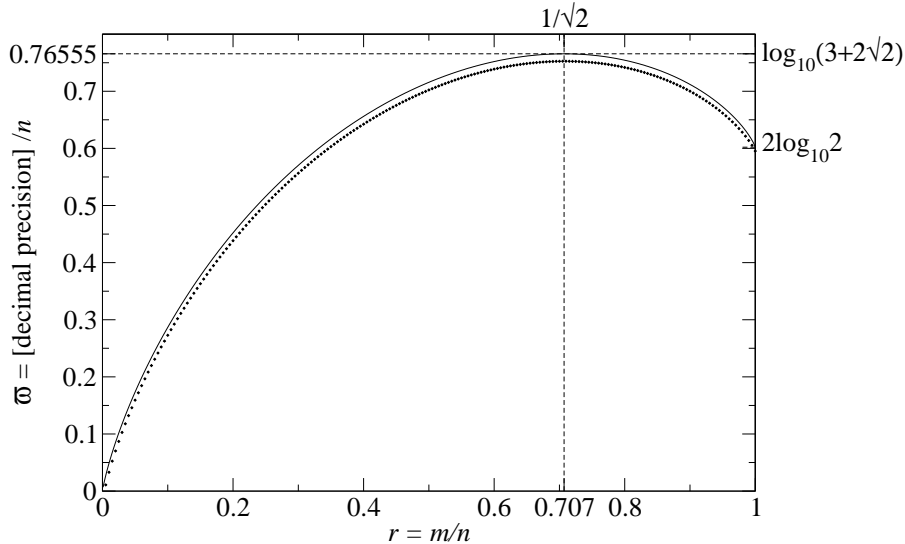


Figure 11: Base decimal precisions needed for the summands of Λ_n in (36), as estimated by $\log_{10} |A_{nm} \log 2\xi(2m)|$ which is plotted against m in axes rescaled by $1/n$. Dotted curve: the case $n = 200$; continuous curve: the $n \rightarrow \infty$ limiting form $\varpi = -2r \log_{10} r + (1+r) \log_{10}(1+r) - (1-r) \log_{10}(1-r)$ ($r = m/n$). (This same form occurs in a seemingly unrelated number-theoretical calculation. [4, § 6.3])

E5-2670 0 @ 2.6 GHz processor).

For higher n -values, G. Misguich kindly developed a much faster parallel code (available on request), based on the multiple-precision GNU MPFR library, [24] and ran it on a 20-core machine still @ 2.6 GHz with 256 Go of memory. He reached CPU times ≈ 97 s for $n = 20000$, 5.6 h for $n = 100000$, 22.7 days for $n = 500000$ (without optimizing the precision by fig. 11; thus n could still be raised, but not so far as to justify the much longer programming and computing times then needed).

Now the true current challenge is to probe $|T| \gtrsim 2.4 \cdot 10^{12}$ by (4), hence to reach $n \gtrsim 2 \cdot 10^{36}$ (assuming the more favorable estimate (73), 10^{60} otherwise), which then needs a working precision $\gtrsim 1.6 \cdot 10^{36}$ decimal places at times. This need for a huge precision already burdened the original λ_n but somewhat less and amidst several steeper complexities; now for the Λ_n , the ill-conditioning increased while other difficulties waned. As current status, the n -range needed for new tests of RH stays beyond reach for the Λ_n too.

Still, $\{\Lambda_n\}$ has assets to win over $\{\lambda_n\}$ in the long run. The Λ_n are fully explicit; their evaluations are recursion-free, thus *very few* samples (at high enough n , for sure) might suffice to signal that RH is violated *somewhere*; and the required working precision peaking at $\approx 0.766 n$ stands as the *only* stumbling block, but this is a *purely logistic* barrier, which might be lowered if (36) ever grew better conditioned variants. Already, (39) needs a much lower precision ($\propto \frac{1}{2} \log_{10} n$) for $\log 2\pi$ as its coefficient $(2A_{n0})^{-1} \sim -\frac{1}{2}\sqrt{\pi n}$ grows negligibly, compared to the $A_{nm} \log(|B_{2m}|/(2m-3)!!)$: thus only the latter *simpler* expressions demand the top precision, and mainly for $m \approx n/\sqrt{2}$. Further such improvements should be pursued in priority to allow for much higher n . The slow growth of the uncertainty-principle threshold (74) is also an aspect favoring $\{\Lambda_n\}$.

Concluding remark and acknowledgments

While other sequences sensitive to RH for large n are known [2][14], not to mention Keiper–Li again, we are unaware of any previous case combining a fully *closed form* like (36) with a practical sensitivity-threshold of *tempered growth* $n = O(T^\nu)$.

We are very grateful to G. Misguich (from our Institute) who wrote and ran a special fast code for numerical calculations reaching $n = 500000$. [24]

Appendix: Centered variant

We sketch a treatment parallel to the main text for our Li-type sequences using the alternative basepoint $x_0 = \frac{1}{2}$, the *center* for the ξ -function [33, §3.4] (and focusing on the Riemann zeros' case, just for the sake of definiteness).

We recall that the Functional Equation $\xi(1-x) \equiv \xi(x)$ allows us, in place of the mapping $z \mapsto x = (1-z)^{-1}$ within ξ as in (6), to use the double-valued map $y \mapsto x_{\tilde{w}}(y) = \frac{1}{2} \pm \sqrt{\tilde{w}} y^{1/2}/(1-y)$ (parametrized by $\tilde{w} > 0$) on the unit disk. That still maps the unit circle $\{|y| = 1\}$ to the completed critical line $L \cup \{\infty\}$, but now minus its interval $\{|\operatorname{Im} x| < \frac{1}{2}\sqrt{\tilde{w}}\}$. As before, we ask all Riemann zeros on L to pull back to $\{|y| = 1\}$, which imposes $\tilde{w} < \tilde{w}_0 \stackrel{\text{def}}{=} 4 \min_\rho |\operatorname{Im} \rho|^2 \approx 799.1618$. We thus define the parametric

sequence $\{\lambda_n^0(\tilde{w})\}$, for $0 < \tilde{w} < \tilde{w}_0$, by

$$\log 2\xi\left(\frac{1}{2} \pm \frac{\sqrt{\tilde{w}} y^{1/2}}{1-y}\right) \equiv \log 2\xi\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{\lambda_n^0(\tilde{w})}{n} y^n \quad (83)$$

([33, §3.4] where only the case $\tilde{w} = 1$ was detailed, [28]); equivalently,

$$\frac{\lambda_n^0(\tilde{w})}{n} \equiv \frac{1}{2\pi i} \oint \frac{dy}{y^{n+1}} \log 2\xi(x_{\tilde{w}}(y)), \quad n = 1, 2, \dots \quad (84)$$

We now build an *explicit* variant for this sequence (84), similar to $\{\Lambda_n\}$ for $\{\lambda_n^K\}$. First, the deformations of (84) analogous to those in § 2.1 read as

$$\frac{1}{2\pi i} \oint \frac{dy}{H_{y_0}(y) \cdots H_{y_n}(y)} \log 2\xi(x) \quad (\text{here } x \equiv x_{\tilde{w}}(y)), \quad (85)$$

cf. (25), for which the simplest analytical form we found, mirroring (27), is

$$\frac{1}{2\pi i} \oint \frac{2 dr}{(r+1)^2} \prod_{m=0}^n \frac{r+r_m}{r-r_m} \log 2\xi(x), \quad r_m \stackrel{\text{def}}{=} \frac{1+y_m}{1-y_m}, \quad (86)$$

where now $x \equiv x(r)$ with the new variable

$$r \stackrel{\text{def}}{=} \frac{1+y}{1-y} \equiv [1 + (2x-1)^2/\tilde{w}]^{1/2} \quad (\text{Re } r > 0). \quad (87)$$

Then with $x_m \equiv 2m$ as before (but now including $m = 0$), the integral (86) evaluated by the residue theorem yields the result (akin to (36))

$$\Lambda_n^0(\tilde{w}) \stackrel{\text{def}}{=} \sum_{m=1}^n \frac{2}{(r_m+1)^2} \frac{\prod_{k=0}^n (r_m+r_k)}{\prod_{k \neq m} (r_m-r_k)} \log 2\xi(2m), \quad r_m \equiv \sqrt{1 + (4m-1)^2/\tilde{w}}. \quad (88)$$

These coefficients, while still explicit, are less tractable than the Λ_n from (36), (32); at the same time their design is more advanced, as it captures the Functional Equation (through the λ_n^0 , and unlike the Λ_n and λ_n); nevertheless we are yet to see any *concrete* benefit to using $\{\Lambda_n^0\}$ over $\{\Lambda_n\}$.

Numerical samples (for $\tilde{w} = 1$, closest case to $\{\Lambda_n\}$; compare with (69)):

$$\Lambda_1^0(1) \approx 0.0881535583, \quad \Lambda_2^0(1) \approx 0.237357366, \quad \dots, \quad \Lambda_{2000}^0(1) \approx 6.815307167.$$

The corresponding *asymptotic alternative* for RH analogous to (45)–(47) reads as

- RH false: $\Lambda_n^0(\tilde{w}) \sim \left\{ \sum_{\operatorname{Re} \rho' > 1/2} \Delta_{\rho'} \Lambda_n^0(\tilde{w}) \right\} \pmod{o(n^\varepsilon) \ \forall \varepsilon > 0}$ (89)

$$\text{with } \log |\Delta_{\rho'} \Lambda_n^0(\tilde{w})| \sim (\rho' - 1/2) \log n, \quad (90)$$

- RH true: $\Lambda_n^0(\tilde{w}) \sim \sqrt{\tilde{w}} (\log n + C)$, $C = \frac{1}{2}(\gamma - \log \pi - 1)$ as in (47). (91)

The latter is proved by extending Oesterlé’s method just as with Λ_n ; whereas the former needs large- n estimations of the product in (86), but our current ones remain crude compared to the full Stirling formula available for (30); that precludes us from reaching the absolute scales of the $\Delta_{\rho'} \Lambda_n^0(\tilde{w})$ and hence the values of n from which any such terms might become detectable.

Numerically though (with all our tests of § 4 admitting centered variants), we saw all changes from the non-centered data (main text) to be slight, especially for $n \gg 1$, aside from the overall factor $\sqrt{\tilde{w}}$ in (91).

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³Erratum: our asymptotic statements for λ_n in the RH false case have wrong sign.

⁴Eqs. (35), (40), (51) have typos, fixed in the present work.