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Abstract:

Delaunay triangulations provide a bijection between a set of $N+3$ points in the complex plane, and the set of triangulations with given circumcircle intersection angles. The uniform Lebesgue measure on these angles translates into a Kähler measure for Delaunay triangulations, or equivalently on the moduli space $\mathcal{M}_{0,N+3}$ of genus zero Riemann surfaces with $N+3$ marked points. We study the properties of this measure. First we relate it to the topological Weil-Petersson symplectic form on the moduli space $\mathcal{M}_{0,N+3}$. Then we show that this measure, properly extended to the space of all triangulations on the plane, has maximality properties for Delaunay triangulations. Finally we show, using new local inequalities on the measures, that the volume $\mathcal{V}_N$ on triangulations with $N+3$ points is monotonically increasing when a point is added, $N \to N+1$. We expect that this can be a step towards seeing that the large N limit of random triangulations can tend to the Liouville conformal field theory.

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Contents

1 Introduction

It has been argued by physicists [David, 1985] [Fröhlich, 1985] [Kazakov, 1985] , that the continuous limit of large planar maps, should be the same thing as two dimensional (2D) quantum gravity, i.e. a theory of random metrics (for general references on the subject see e.g. [Ambjørn et al., 2005]). Polyakov had already shown that by choosing a conformal gauge [Polyakov, 1981], 2D quantum gravity can be reformulated as the quantum Liouville theory, which is a 2D conformal field theory (2D-CFT). Together with Kniznik and Zamolodchikov, he showed later that the scaling dimensions of its local operators are encoded into the so called KPZ relations. [Knizhnik et al., 1988] [David, 1988b] [David, 1988a] [Distler and Kawai, 1989]. Another approach is 2D topological gravity, proposed by Witten [Witten, 1990] and notably studied by Kontsevich [Kontsevich, 1992]. Going back to the discrete case, planar maps have been studied since decades by combinatorial and random matrix methods. Many explicit results corroborate the equivalence beetween the continuum limit (large map) of planar map models, quantum Liouville theory and topological gravity (see e.g. [Kostov et al., 2004] [Chekhov et al., 2013]). It has been shown recently, by combinations of combinatorics and probabilistics methods, that the continuous limit (with the Gromov-Hausdorff distance) of large planar maps equipped with the graph distance, exists, and converges as a metric space, towards the so-called "Brownian map" [Le Gall, 2013] [Miermont, 2013] (see the references therein for previous litterature).

The problem which has so far remained elusive, is to prove the general equivalence of this limit (in the GH topology) with the Liouville conformal field theory in the plane. This problem requires methods of embedding planar maps into the Euclidian plane. Many methods are available for planar triangulations. Let us quote the "barycentric" (or Tutte) embedding (see e.g. [Ambjørn et al., 2012]), the "Regge" embedding (see e.g. sect. 6 of [Hamber, 2009]), the exact uniformization embedding, and finally "Circle packing" methods (see e.g. [Benjamini, 2009]).

In [David and Eynard, 2014], two of the authors considered a very natural extension of the circle packing and circle pattern methods, relying on the patterns of circumcircles of Delaunay triangulations. Using the fact that the whole (moduli) space of surfaces is obtained by varying circumcircle intersection angles, they showed that the uniform measure on random planar maps, equipped with the uniform Lebesgue measure on edge angles variables, gets transported by the circle pattern embedding method, to a conformally invariant spatial point process (measure on point distributions) in the plane with many interesting properties. It had an explicit representation in term of geometrical objects (3-rooted trees) on Delaunay triangulations. It is a Kähler metric whose prepotential has a simple formulation in term of 33 hyperbolic geometry. It can be written as a "discrete Fadeev-Popov" determinant, very similar to the conformal gauge fixing Fadeev-Popov determinant of Polyakov. It can also be written as a combination of Chern classes, as in topological 2D gravity.

In this paper we pursue the study of this model in two directions. Firstly, in section 3 we make precise the relation between our model and topological gravity, by showing that our measure is equivalent to the Weil-Peterson volume form on the moduli space of the sphere with marked points (punctures) $\overline{\mathcal{M}}_{0,n}$. This equivalence is non-trival, and requires a precise study of the Chern class formulation of [David and Eynard, 2014] at the boundaries between different domains in moduli space corresponding to different Delaunay triangulations, as well as a study of the relation between our geometrical formulation of the volume form and the so-called λ-length parametrization of $\overline{\mathcal{M}}_{0,n}$. This result proves that, at least as far as topological (i.e. global) observables are concerned, our model is in the same universality class than pure topological gravity ($\gamma = \sqrt{8/3}$ Liouville or (3,2) Liouville CFT). This was a conjecture of [David and Eynard, 2014].

Secondly, we start to study the specific local properties of the model, which are related to the specific conformal embedding of a discrete random metric defined by the Delaunay triangulations. The study of these properties should be crucial to make precise the existence of a local continuum limit and its relation with the Liouville theory. We show in Section 4.1 an interesting property of maximality: our measure can be analytically continued to non-Delaunay triangulations, but is maximal exactly for Delaunay triangulations. This could open the possibility of some convexity properties. Then in Section 4.2 we study local bounds when one considers the process of adding locally a new vertex, thus going from triangulations with N to $N+1$ vertices. We get both local and global lower bounds, and deduce that the partition function $Z_N = V_N/N!$ grows at least like $(\pi^2/8)^N$. These results are encouraging first steps towards the construction of a continuum limit.

2 Reminders

2.1 The model

We recall the notations and definitions of [David and Eynard, 2014]. Let T denote an abstract triangulation of the Riemann sphere $\mathcal{S}_2 = \mathbb{C} \cup \{\infty\}$. $\mathcal{V}(T)$, $\mathcal{E}(T)$ and $\mathcal{F}(T)$ denote respectively the sets of vertices v, edges e and faces (triangles) f of T. Let $\mathcal{T}_N$ be the set of all such T with $N = |\mathcal{V}(T)|$ vertices, hence $|\mathcal{E}(T)| = 3(N-2)$ and $|\mathcal{F}(T)| = 2(N-2)$.

An *Euclidean triangulation* $\widetilde{T} = (T, \boldsymbol{\theta})$ is a triangulation T plus an associated edge angle pattern $\boldsymbol{\theta} = \{\theta(e); e \in \mathcal{E}(T)\}$, such that

$$0 \leq \theta(e) < \pi \ . \tag{2.1}$$

An Euclidean triangulation is *flat* if for each vertex $v \in \mathcal{V}(T)$, the sum of the angles of the adjacent edges satisfy

$$\sum_{e \to v} \theta(e) = 2\pi \tag{2.2}$$

Given a set of N points with complex coordinates z_v in $\mathcal{S}_2$ (with its standard complex structure), the associated Delaunay triangulation in a flat Euclidean triangulation, such that the angle $\theta(e)$ is the angle of intersection between the circumcircles of the oriented triangles (faces) adjacent to e. See fig. **??** The edge angle pattern satisfies in addition the condition that for any closed oriented contour $\mathcal{C}^\star$ on the dual graph $T^\star$ of the triangulation T (the Voronoï diagram), the sum of the angles associated to the edges e dual (orthogonal) to the edges $e^\star$ of $\mathcal{C}^\star$ satisfy

$$\sum_{e\perp\mathcal{C}^\star} \theta(e) \geq 2\pi \tag{2.3}$$

(this condition was not discussed in [David and Eynard, 2014]).

We denote by $\widetilde{\mathcal{T}}_N^f$ the set of Euclidean triangulations $\tilde{T} = (T, \boldsymbol{\theta})$ with N vertices that satisfy 2.1, 2.2 and 2.3.

A theorem by Rivin [Rivin, 1994] states that this is in fact an angle pattern preserving bijection between $\widetilde{\mathcal{T}}_N^f$ and the set of Delaunay triangulations of the complex plane modulo Möbius transformations, which can be identified with $\mathfrak{D}_N = \mathbb{C}^N/SL(2,\mathbb{C})$. It is an extension of the famous theorem by Koebe-Andreev-Thurston [Koebe, 1936] stating that there is a bijection between simple triangulations and circle packings in complex domains, modulo global conformal transformations. The proof relies on the same kind of convex minimization functional, using hyperbolic 3-geometry, than for the original circle packing case (see [Rivin, 1994] and [Bobenko and Springborn, 2004]).

The model of random triangulation considered in [David and Eynard, 2014] is obtained by taking the discrete uniform measure on triangulations, times the flat Lebesgue measure on the angles. Since the PSL(2,$\mathbb{C}$) invariance allows to fix 3 points in the triangulations, from now on we work with triangulations and points ensembles with $M = N + 3$ points. The measure on $\widetilde{\mathcal{T}}_M^f$ is

$$\mu(\widetilde{T}) = \mu(T, d\boldsymbol{\theta}) = \text{uniform}(T) \prod_{e\in\mathcal{E}(T)} d\theta(e) \prod_{v\in\mathcal{V}(T)} \delta\Big(\sum_{e\mapsto v} \theta(e) - 2\pi\Big) \prod_{\mathcal{C}^\star} \Theta\Big(\sum_{e\perp\mathcal{C}^\star} \theta(e) - 2\pi\Big) \tag{2.4}$$

2.2 Kähler form of the measure

One of the main results of [David and Eynard, 2014] is the form of the induced measure on the space $\mathfrak{D}_{N+3}$ of Delaunay triangulations on the plane, i.e on the space of distributions of $N+3$ points on the Riemann sphere. The first three points (z_1, z_2, z_3) being fixed by PSL(2,$\mathbb{C}$), the remaining N coordinates are denoted $\mathbf{z} = (z_4, \cdots, z_{N+3}) \in \mathbb{C}^N$, and $T_{\mathbf{z}} = T$ is the associated abstract Delaunay triangulation, uniquely defined if no subset of 4 points are cocyclic. A simple case is when one of the three fixed point is at ∞.

Theorem 2.1. *[David and Eynard, 2014] The measure* $\mu(T, d\boldsymbol{\theta}) = d\mu(\mathbf{z})$ *on* $\mathbb{C}^D$ *is a Kähler measure of the local form*

$$d\mu(\mathbf{z}) = \prod_{v=4}^{N+3} d^2 z_v \; 2^N \; \det\left[D_{u\bar{v}}\right] \tag{2.5}$$

where D *is the restriction to the* N *lines and columns* $u, \bar{v} = 4, 5, \cdots N+3$ *of the Kähler metric on* $\mathbb{C}^{N+3}$

$$D_{u\bar{v}}(\{z\}) = \frac{\partial}{\partial z_u}\frac{\partial}{\partial \bar{z}_v}\mathcal{A}_T(\{z\}) \tag{2.6}$$

with the prepotential $\mathcal{A}_T$ *given by*

$$\mathcal{A}_T = -\sum_{f \in \mathcal{F}(T)} \mathbf{V}(f) \tag{2.7}$$

where the sum runs over the triangles f *(the faces) of the Delaunay triangulation* T *of the Delaunay triangulation* T *associated to the configuration of points* $\{z\} = \{z_v; v = 1, N+3\}$ *in* $\mathbb{C}^{N+3}$*. For a triangle* f *with (c.c.w. oriented) vertices* (a, b, c)*,* $\mathbf{V}(f)$ *is the hyperbolic volume in the hyperbolic upper half space* $\mathbb{H}^3$ *of the ideal tetraedron with summits* (z_a, z_b, z_c, ∞) *on its boundary at infinity* $\mathbb{C} \cup \{\infty\}$*.*

NB: This definition is a bit loose and some care must be taken in treating the point at ∞ and the three fixed points. See [David and Eynard, 2014] for details.

2.3 Relation with topological gravity

A second result of [David and Eynard, 2014] is that the measure can be written locally (i.e. for a given triangulation T) in term of Chern classes ψ_v of U(1) line bundles $L_v \to \mathcal{M}_{0,N+3}$, (attached to the vertices v) where the space of Delaunay triangulations with N+3 points is identified with the moduli space $\mathcal{M}_{0,N+3}$ of the (conformal structures of the) sphere with $N+3$ marked points. More precisely one has

$$\mu(T, \{d\theta\}) = \frac{1}{N!\, 2^{2N+1}} \left(\sum_v (2\pi)^2 \psi_v\right)^N \tag{2.8}$$

with

$$\psi_v = c_1(L_v) = \frac{1}{(2\pi)^2} \sum_{e' < e \to v} d\theta(e) \wedge d\theta(e') \tag{2.9}$$

where the e's denote the edges adjacent to the vertex v, labelled in c.c.w. order. ψ_v was defined explicitely as the curvature du_v of the global U(1) connection

$$u_v = \frac{1}{(2\pi)^2} \sum_{f \to v} \theta(f_+)\, d\gamma_v(f) \tag{2.10}$$

In 2.10 the sum runs over the faces f adjacent to the vertex v. $\gamma_v(f)$ is the angle between a reference half-line γ_v with endpoint v and the half line starting from v and

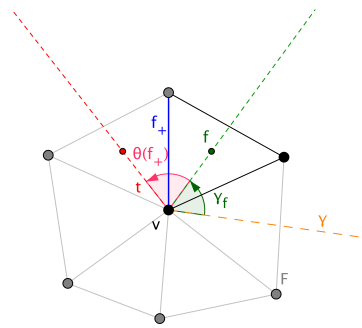

Figure 1: Construction of the connection u_v

passing through the center of the (circumcircle of the) face f. f_+ is the leftmost edge of f adjacent to v (see figure 1). It was stated in [David and Eynard, 2014] that the measure 2.5 is therefore the measure of topological gravity studied in [Kontsevich, 1992]. As we shall see in the next section, this is a local result, but it cannot be extended globally.

3 Relation with Weil-Petersson Metric

3.1 Discontinuities of the Chern Classes

There is a subtle point in establishing the link between the angle measure on Delaunay triangulations (let us denote this measure $\mu_{\mathscr{D}.}$) and the measure of topological gravity, that we shall denote $\mu_{\text{top.}}$ In its general definition through 2.8

$$\mu_{\text{top.}} \propto \psi^N \ , \quad \psi = \sum_v \psi_v = du \ , \quad u = \sum_v u_v \tag{3.1}$$

the curvature 2-form ψ and the 1-form u (the global U(1) connection) depend implicitely on a choice of triangulation T of the marked sphere, which is supposed to be kept fixed, but the final measure μ and its integral over the moduli space does not depend on the choice of triangulation).

In our formulation, the moduli space $\mathcal{M}_{0,N+3}$ is the closure of the union of disjoint domains $\mathcal{M}^{(T)}$ where the triangulation T is combinatorially a given Delaunay triangulation. Two domains $\mathcal{M}^{(T)}$ and $\mathcal{M}^{(T')}$ meet along a face (of codimension 1) where the four vertices of two faces sharing an edge are cocyclic, so that one passes from T to T' by a flip, as depicted on Fig. 2. The relation $\mu_{\mathscr{D}.} = \mu_{\text{top.}}$ will be valid if the form u is continuous along a flip. If this is not the case, there might be some additional boundary terms in du.

Let us therefore compare the 2-form u for a triangulation T and the corresponding 2-form u' for the triangulation T' obtained from T by the flip $(2,4) \to (1,3)$ depicted

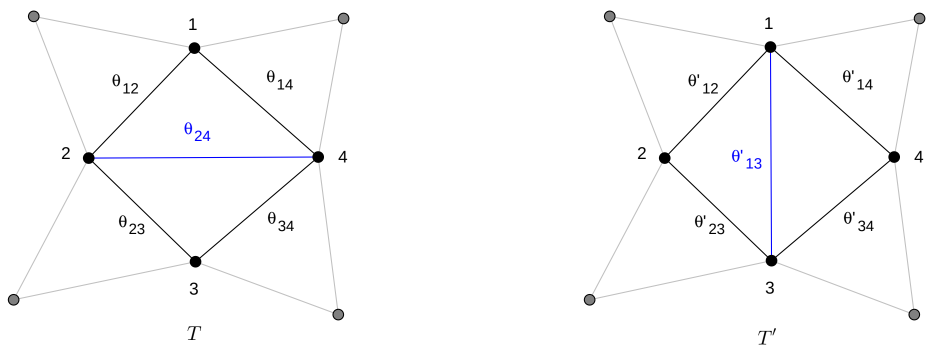

Figure 2:

on Fig. 2. The angles θ of the edges of T and θ' of the edges of T' are a priori different for the five edges depicted here (when the points 1, 2, 3 an 4 are not cocyclic) but only six among the ten angles are independent, since they satisfy the relation at vertex 1

$$\theta_{12} + \theta_{14} = \theta'_{12} + \theta'_{13} + \theta'_{14} \tag{3.2}$$

and the three similar relations for vertex 2, 3 and 4. These relations imply for instance that $\theta_{24} + \theta'_{13} = 0$. From the definition 2.10 of the 1-forms u and u' one computes easily $u - u'$ (which depends a priori on the choice of section angles $\gamma_1, \dots \gamma_4$). However we are interested at the difference at the flip between the Delaunay triangulations T and T', i.e when the 4 points are cocyclic. Then $\theta_{12} = \theta'_{12}$, $\theta_{14} = \theta'_{14}$, $\theta_{23} = \theta'_{23}$, $\theta_{34} = \theta'_{34}$ and $\theta_{24} = \theta_{13} = 0$ and we get

$$u - u'|_{\text{flip}} = (\theta_{14} + \theta_{23} - \theta_{12} - \theta_{34})(d\theta_{12} - d\theta'_{12}) + (\theta_{14} + \theta_{23})d\theta_{24} \tag{3.3}$$

This is clearly non zero. The 1-form of topological gravity $u_{\text{top.}}$ is defined globally as a sum over the triangulations T as

$$u_{\text{top.}} = \sum_T \chi_{(T)} u_{(T)} \tag{3.4}$$

where $\chi_{(T)}$ is the indicator function (hence a 0-form) of the domain $\mathcal{M}^{(T)}$ and $u_{(T)}$ the 1-form for the triangulation T. The 2-form of topological gravity (Chern class) of [Witten, 1990] and [Kontsevich, 1992] is therefore

$$\psi_{\text{top.}} = du_{\text{top.}} = \sum_T \ \chi_{(T)}\, du_{(T)} + d\chi_{(T)} \wedge u_{(T)} \tag{3.5}$$

In [David and Eynard, 2014] it was shown that the Delaunay measure can be written locally (inside each $\mathcal{M}^T$) as the volume form of the Delaunay 2-form

$$\psi_{\mathscr{D}.} = \sum_T \chi_{(T)} du_{(T)} \tag{3.6}$$

and that this measure is continuous at the boundary between two adjacents domains $\mathcal{M}^T$ and $\mathcal{M}^{T'}$, so that the definition 3.6 is global.

The calculation leading to 3.3 shows that the 1-form u is generically not continuous at the boundary between domains, so that the second term in 3.5 is generically non-zero, and localized at the boundaries between domains. Therefore the Delaunay 2-form $\psi_{\mathscr{D}.}$ is different from the topological 2-form $\psi_{\text{top.}}$ (the Chern class) and the associated measures (top forms) are a priori different

$$\mu_{\text{top.}} = c_N\ (\psi_{\text{top.}})^N \ \neq\ \mu_{\mathscr{D}.} = c_N\ (\psi_{\mathscr{D}.})^N \tag{3.7}$$

the difference being localized at the boundaries of the domains $\overline{\mathcal{M}}^{(T)}_{0,N+3}$.

3.2 The angle measure and the Weil-Petersson metric

3.2.1 The Delaunay Kähler form

For a given Delaunay triangulation $\tilde{T}$, the Delaunay Kähler metric form is

$$G_{\mathscr{D}.}(\{z\}) = dz_u d\bar{z}_v\, D_{u\bar{v}}(\{z\}) \tag{3.8}$$

with $D_{u\bar{v}}(\{z\})$ given by 2.6. The associated Delaunay Kähler 2-form is

$$\Omega_{\mathscr{D}.}(\{z\}) = \frac{1}{2\mathrm{i}} dz_u \wedge d\bar{z}_v\, D_{u\bar{v}}(\{z\}) \tag{3.9}$$

$G_{\mathscr{D}.}$ and $\Omega_{\mathscr{D}.}$ are continuous across flips. From [David and Eynard, 2014] the matrix D is

$$D = \frac{1}{4\mathrm{i}} A E A^\dagger \tag{3.10}$$

with A the $(N+3) \times 3(N+1)$ vertex-edge matrix

$$A_{ue} = \begin{cases} \frac{1}{z_u - z_{u'}} & \text{if } u \text{ is an end point of the edge } e = (u,u') \text{ of } T, \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

and E the $3(N+1) \times 3(N+1)$ antisymmetric matrix

$$E_{ee'} = \begin{cases} +1 & \text{if } e \text{ and } e' \text{ consecutives edges of a face } f\text{, in c.w. order,} \\ -1 & \text{if } e \text{ and } e' \text{ consecutives edges of a face } f\text{, in c.c.w. order,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

Then, the 2-form $\Omega_{\mathscr{D}.}$ takes a simple form, as a sum over faces (triangles) f of T. Let us denote (f_1, f_2, f_3) the vertices of a triangle f, in c.c.w. order (this is defined up to a cyclic permutation of the 3 vertices).

$$\Omega_{\mathscr{D}.}(\{z\}) = \sum_{\text{faces}\, f} \omega_{\mathscr{D}.}(z_{f_1}, z_{f_2}, z_{f_3}) \tag{3.13}$$

with, for a face f with vertices labelled $(1,2,3)$ (for simplicity), and denoting $z_{ij} = z_j - z_i$

$$\omega_{\mathscr{D}.}(z_1,z_2,z_3) = \frac{1}{8}\begin{pmatrix} d\log(z_{23})\wedge d\log(\bar z_{31}) + d\log(\bar z_{23})\wedge d\log(z_{31}) \\ + d\log(z_{31})\wedge d\log(\bar z_{12}) + d\log(\bar z_{31})\wedge d\log(z_{12}) \\ + d\log(z_{12})\wedge d\log(\bar z_{23}) + d\log(\bar z_{12})\wedge d\log(z_{23}) \end{pmatrix} \tag{3.14}$$

Reexpressed in term of the log of the modulus and of the argument of the z_{ij}'s

$$\lambda_{ij} = \log(|z_j - z_i|)\ , \quad \vartheta_{ij} = \arg(z_j - z_i) \tag{3.15}$$

we obtain

$$\omega_{\mathscr{D}.} = \omega_{\text{length}} + \omega_{\text{angle}} \tag{3.16}$$

with the length contribution

$$\omega_{\text{length}} = \frac{1}{4}(d\lambda_{12}\wedge d\lambda_{23} + d\lambda_{23}\wedge d\lambda_{31} + d\lambda_{31}\wedge d\lambda_{12}) \tag{3.17}$$

and the angle contribution

$$\omega_{\text{angle}} = \frac{1}{4}(d\vartheta_{12}\wedge d\vartheta_{23} + d\vartheta_{23}\wedge d\vartheta_{31} + d\vartheta_{31}\wedge d\vartheta_{12}) \tag{3.18}$$

Reexpressed in terms of the angles α_1, α_2 and α_3 of the triangle $(1,2,3)$ (using $\alpha_1 = \vartheta_{13} - \vartheta_{12}$, etc.), and using $\alpha_1 + \alpha_2 + \alpha_3 = \pi$, one has

$$\omega_{\text{angle}} = \frac{1}{4}(d\alpha_1\wedge d\alpha_2) = \frac{1}{4}(d\alpha_2\wedge d\alpha_3) = \frac{1}{4}(d\alpha_3\wedge d\alpha_1) \tag{3.19}$$

Using the triangle relation

$$\frac{\sin(\alpha_1)}{\exp(\lambda_{23})} = \frac{\sin(\alpha_2)}{\exp(\lambda_{31})} = \frac{\sin(\alpha_3)}{\exp(\lambda_{12})} \tag{3.20}$$

one gets

$$\begin{aligned} d\alpha_1\cot\alpha_1 - d\lambda_{23} =& d\alpha_2\cot\alpha_2 - d\lambda_{31} = d\alpha_3\cot\alpha_3 - d\lambda_{12} \\ &= (d\alpha_1 + d\alpha_2)\frac{\cot\alpha_1\,\cot\alpha_2 - 1}{\cot\alpha_1 + \cot\alpha_2} - d\lambda_{12} \end{aligned} \tag{3.21}$$

which gives

$$d\alpha_1 = \frac{\csc^2\alpha_2}{\cot\alpha_1 + \cot\alpha_2}(d\lambda_{23} - d\lambda_{12}) + \frac{\cot\alpha_1\,\cot\alpha_2 - 1}{\cot\alpha_1 + \cot\alpha_2}(d\lambda_{31} - d\lambda_{12}) \tag{3.22}$$

$$d\alpha_2 = \frac{\cot\alpha_1\,\cot\alpha_2 - 1}{\cot\alpha_1 + \cot\alpha_2}(d\lambda_{23} - d\lambda_{12}) + \frac{\csc^2\alpha_1}{\cot\alpha_1 + \cot\alpha_2}(d\lambda_{31} - d\lambda_{12}) \tag{3.23}$$

which implies

$$d\alpha_1\wedge d\alpha_2 = d\lambda_{12}\wedge d\lambda_{23} + d\lambda_{23}\wedge d\lambda_{31} + d\lambda_{31}\wedge d\lambda_{12} \tag{3.24}$$

Hence $\omega_{\text{angle}} = \omega_{\text{length}}$. Therefore one has

$$\omega_{\mathscr{D}.} = \frac{1}{2}(d\lambda_{12}\wedge d\lambda_{23} + d\lambda_{23}\wedge d\lambda_{31} + d\lambda_{31}\wedge d\lambda_{12}) \tag{3.25}$$

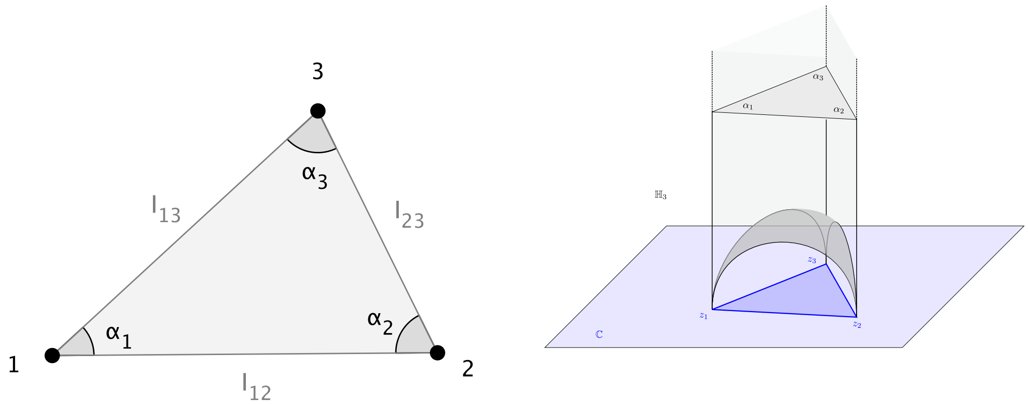

Figure 3: A triangle $f = (1,2,3)$ (left) and the associated ideal spherical triangle $\mathcal{S}_{123}$ in $\mathbb{H}_3$ (right).

3.2.2 Delaunay triangulations and moduli space

We can now compare this structure with the Weil-Petersson Kähler structure on the decorated moduli space $\tilde{\mathcal{M}}_{0,N+3}$ of the punctured sphere, decorated by horocycles. We refer to [Penner, 1987] and to [Penner, 2006], [Thurston, 2012], [Papadopoulos, 2007] (among many references) for a general introduction to the subject.

To any Delaunay triangulation $\tilde{T}$ with $N+3$ points on the complex plane, we can associate an explicit surface $\mathcal{S}$ with constant negative curvature and $N+3$ punctures as follows. Let $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_+^*$ be the upper half-space above $\mathbb{C}$, with coordinates (z,h) embodied with the Poincaré metric $ds^2 = (dzd\bar{z} + dh^2)/h^2$. It makes $\mathbb{H}_3$ the 3-dimensional hyperbolic space, with $\mathbb{C} \cup \{\infty\}$ its asymptotic boundary at infinity.

Consider a triangle f_{123} with vertices $(1,2,3)$ (in c.c.w. order) with complex coordinates (z_1, z_2, z_3) in $\mathbb{C}$. Let $\mathcal{B}_{123}$ be the hemisphere in $\mathbb{H}_3$ whose center is the center of the circumcircle of f_{123} (in $\mathbb{C}$), and which contains the points $(1,2,3)$. $\mathcal{B}_{123}$, embodied with the restriction of the Poincaré metric ds^2 of $\mathbb{H}_3$, is isometric to the 2 dimensional hyperbolic disk $\mathbb{H}_2$. Let $\mathcal{L}_{12}$ be the intersection of $\mathcal{B}_{123}$ with the half plane orthogonal to $\mathbb{C}$ which contains the points 1 and 2, this is a semicircle orthogonal to $\mathbb{C}$. With a similar definition for (23) and (31), the semicircles $\mathcal{L}_{12}$, $\mathcal{L}_{23}$ and $\mathcal{L}_{31}$ delimit a spherical triangle $\mathcal{S}_{123}$ on the hemisphere in $\mathbb{H}_3$. The semicircles $\mathcal{L}_{12}$, $\mathcal{L}_{23}$ and $\mathcal{L}_{31}$ are geodesics in $\mathbb{H}_3$, hence in $\mathcal{B}_{123}$, so that $\mathcal{S}_{123}$ is an ideal triangle in $\mathbb{H}_2$. $\mathcal{S}_{123}$ is nothing but the face (123) of the ideal tetraedra (z_1, z_2, z_3, ∞) in $\mathbb{H}_3$ whose volume $\mathbf{V}(f)$ appears in 2.7.

Now consider a Delaunay triangulation $\tilde{T}$ in the plane, with $N+3$ points, and with one point at infinity for simplicity. The union of the ideal spherical triangles $\mathcal{S}_f$ associated to the faces f of $\tilde{T}$ form surface $\mathcal{S}$ in $\mathbb{H}_3$

$$\mathcal{S} = \bigcup_{f \in \mathcal{F}(T)} \mathcal{S}_f \tag{3.26}$$

See fig. 4. The surface $\mathcal{S}$ embodied with the restriction of the Poincaré metric of $\mathbb{H}_3$, is a constant negative curvature surface. Indeed since the triangles $\mathcal{S}_f$ are glued along geodesics, no curvature is localized along the edges of these triangles. It is easy to see that the endpoints z_i of the triangulations are puncture curvature singularities of $\mathcal{S}$, i.e. points where the metric can be written (in local conformal coordinates with the puncture at the origin)

$$ds^2 = \frac{dw d\bar{w}}{|w|^2 |\log(1/|w|)|^2} \tag{3.27}$$

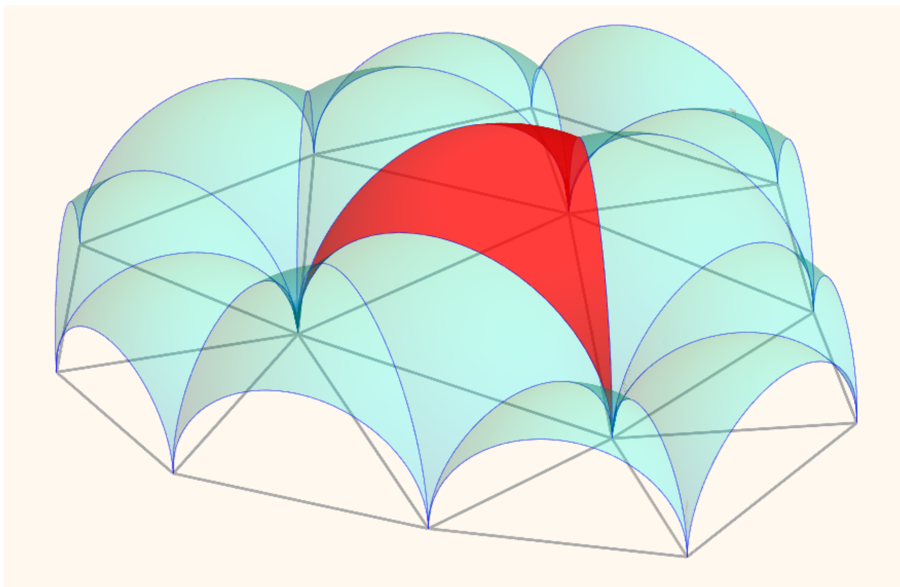

Figure 4: A triangulation and the associated punctured surface

Through the orthogonal projection from $\mathcal{S}$ to the plane $\mathbb{C}$, the metric in each $\mathcal{S}_f$ become the standard Beltrami-Cayley-Klein hyperbolic metric in the triangle f. We recall that it is defined in the unit disk $\mathbb{D}_2 = \{z; |z| < 1\}$ in radial coordinates as

$$ds^2_{\mathscr{B}.} = \frac{dr^2 + r^2 d\theta^2}{(1 - r^2)} + \frac{(r\,dr)^2}{(1 - r^2)^2} \tag{3.28}$$

that it is not conformal, and is such that geodesics are straight lines in the disk. Thus, each Delaunay triangulation – modulo PSL(2,$\mathbb{C}$) tranformations – gives explicitely, the constant curvature surface representative of a point in $\mathcal{M}_{0,N+3}$.

3.2.3 λ-lengths and horospheres

Following [Penner, 1987], decorated surfaces are obtained by supplementing each puncture v by a horocycle $\mathfrak{h}_v$, i.e. a closed curve orthogonal to the geodesics emanating from v (in the constant curvature metric). Horocycles are uniquely characterized by their length ℓ_v. The moduli space of decorated punctured surfaces is simply

$$\tilde{\mathcal{M}}_{g,n} = \mathcal{M}_{g,n} \otimes \mathbb{R}_+^{\otimes v} \tag{3.29}$$

A geodesic triangulation $\mathfrak{T}$ of the abstract surface $\mathcal{S}$ is a triangulation such that the edges are (infinite length) geodesics joining the punctures, and the triangles are c.c.w. oriented (and non-overlapping). For a decorated surface $\tilde{\mathcal{S}}$, for any geodesics $\mathfrak{e}$ joining two punctures u and v (generically one may have $u = v$), its λ-length $\Lambda_\mathfrak{e}(u,v)$ is defined from the (finite, algebraically defined) geodesic distance $d_\mathfrak{e}(u',v')$ along $\mathfrak{e}$ between the intersections u' and v' of e with the horocycles $\mathfrak{h}_u$ and $\mathfrak{h}_v$ by

$$\Lambda_\mathfrak{e}(u,v) = \exp(d_\mathfrak{e}(u',v')/2) \tag{3.30}$$

Fo a given triangulation $\mathfrak{T}$ (of a genus g surface with n punctures), it is known that the set of the independent λ-lengths $\Lambda_e \in \mathbb{R}_+$ for the $6g+3n-6$ edges of $\mathfrak{T}$ provide a complete set of coordinates for the decorated Teichmüller space $\tilde{\mathcal{T}}_{g,n}$ (the universal cover of $\tilde{\mathcal{M}}_{g,n}$). This parametrization is independent of the choice of triangulation, thanks to the Ptolemy's relations between lambda-lengths when one passes from a triangulation $\mathfrak{T}$ to another one $\mathfrak{T}'$ through a flip similar to the ones of fig. 2, namely

$$\Lambda_{13}\Lambda_{24} = \Lambda_{12}\Lambda_{34} + \Lambda_{14}\Lambda_{23} \tag{3.31}$$

In this parametrization, the Weil-Petersson 2-form on $\mathcal{M}_{g,n}$ (through its projection from $\tilde{\mathcal{T}}_{g,n}$) can be written simply as a sum over the $2(2g+n+2)$ oriented faces (triangles) $\mathfrak{f}$ of $\mathfrak{T}$, as

$$\Omega_{\mathscr{WP}} = \sum_{\text{faces}\,\mathfrak{f}} d\log(\Lambda_{12})\wedge d\log(\Lambda_{23}) + d\log(\Lambda_{23})\wedge d\log(\Lambda_{31}) + d\log(\Lambda_{31})\wedge d\log(\Lambda_{12}) \tag{3.32}$$

where (1,2,3) denote the vertices (punctures) $v_1(f)$, $v_2(f)$ and $v_3(f)$ (here in c.c.w. order) of the geodesic triangle $\mathfrak{f}$ of $\mathfrak{T}$, and the Λ_{ij} denote the λ-length of the edges of the triangle.

To compare $\Omega_{\mathscr{WP}}$ to $\Omega_{\mathscr{D}.}$, one simply has to look at horocycles and λ-lengths in Delaunay triangulations. We have an explicit representation of a point in $\mathcal{M}_{0,N+3}$ as the constant curvature surface $\mathcal{S}$ in $\mathbb{H}_3$ constructed above the Delaunay triangulation T for the set of points $\mathbf{z} = \{z_i\}_{i=1,N+3}$ in the complex plane. Horocycles are easily constructed by decorating each point (puncture) z_i by a horosphere $\mathcal{H}_i$, i.e. an Euclidean sphere in $\mathbb{R}^3$, tangent to the complex place $\mathbb{C}$ at the point z_i, and lying above z_i (i.e. in $\mathbb{H}_3$). The intersection (in $\mathbb{H}_3$) of the horosphere $\mathcal{H}_i$ with the union of the ideal spherical triangles $\mathcal{S}_f$ for the faces f which share the vertex i defines the horocircle $\mathfrak{h}_i$ associated to the puncture i of $\mathcal{S}$. See fig. 5.

Let R_i denote the Euclidean radius of the horosphere $\mathcal{H}_i$ above vertex i. The λ-length for the edge joining two vertices (i,j) of the triangulation is easily calculated (applying for instance the formula in the Poincaré half-plane in 2 dimensions) and is

$$\Lambda(i,j) = \frac{|z_i - z_j|}{\sqrt{4\,R_i\,R_j}} \tag{3.33}$$

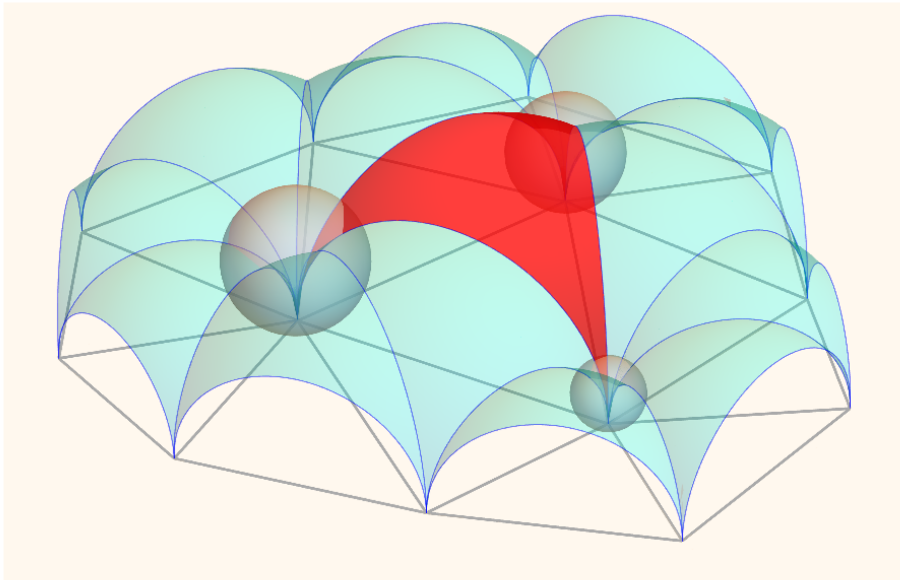

Figure 5: The punctured surface decorated with horospheres

where $|z_i - z_j|$ is the Euclidean distance between the points i and j in the plane $\mathbb{C}$. See fig. 6.

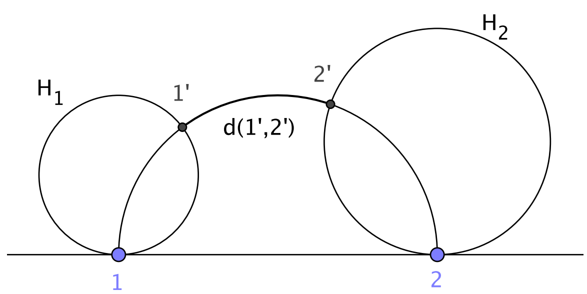

Figure 6: The geodesics between the horospheres H_1 and H_2 at points 1 and 2.

3.2.4 Identity of the Kähler structures

Incorporating this into 3.32, the Petersson-Petersson 2-form 3.32 the takes a form simlar to that of 3.25

$$\Omega_{\mathscr{WP}} = \sum_{\mathfrak{f}} \frac{d|z_1 - z_2]}{|z_1 - z_2|} \wedge \frac{d|z_2 - z_3]}{|z_2 - z_3|} + \frac{d|z_2 - z_3]}{|z_2 - z_3|} \wedge \frac{d|z_3 - z_1]}{|z_3 - z_1|} + \frac{d|z_3 - z_1]}{|z_3 - z_1|} \wedge \frac{d|z_1 - z_2]}{|z_1 - z_2|} \quad (3.34)$$

Weil On one side 3.32 refers to a given geodesic triangulation, but the resulting 2-form $\Omega_{\mathscr{WP}}$ is known to be independent of the triangulation through the Ptolemy's relation. On the other side 3.25 refers to a given Delaunay triangulation, but we know from [David and Eynard, 2014] that since the matrix D is continuous, $\omega_{\mathscr{D}.}$ is continuous when one passes from a Delaunay triangulation to another one through flips when four points are cocyclics. Hence we have the global identity

$$\Omega_{\mathscr{D}.} = \frac{1}{2}\Omega_{\mathscr{W}\mathscr{P}} \tag{3.35}$$

We thus have shown that the Kähler structure constructed out of the circle pattern associated to random Delaunay triangulations of the sphere is equivalent to the Weil-Petersson Kähler structure on the moduli space of the sphere with marked points. This implies of course that the volume measures are identical (up to a factor 2^{-N}) and in particular that the total volume of the space of planer Delaunay triangulations with $N+3$ points and the Weil-Petersson volume of the moduli space of the sphere with $N+3$ punctures are proportional.

3.3 Discussion

Let us firstly mention that the possibility of a relation between the Delaunay triangulation mesure and the Weil-Petersson volume form was already suggested to the authors of [David and Eynard, 2014] by T. Budd, on the basis of calculations of the total volume of the set of Delaunay triangulations for small number of points [Budd, 2014]. His suggestion that the total volumes were identical remained very puzzling to us (and somehow paradoxical), in view of the local form 2.8 for the angle measure in terms of first Chern classes ψ_v (a Weil-Petersson local form for the volume form would have contained a Mumford κ-class), until we became aware of the discontinuity phenomenon derived in sec. 3.1. We have shown here that there is in fact no contradiction, and that not only the total volumes, but that locally the 2-forms are proportional.

This identity shows that the random Delaunay triangulation model is equivalent to the more abstract topological 2D gravity model based on the Weil-Petersson measure. The total volume of the moduli space $\overline{\mathcal{M}}_{g,n}$ of Riemann surfaces of genus g with n punctures is known ([Zograf, 1993] [KMZ, 1996], [MZ, 2000]). They behaves for large n as

$$\mathrm{Vol}\left(\overline{\mathcal{M}}_{g,n}\right) \; = \; C^n \, n^{(5g-7)/2} \left(a_g + \mathcal{O}(1/n)\right) \tag{3.36}$$

(C is a known >0 constant) where we omit the $n!$ factor of [MZ, 2000]) due to the labelling of the punctures, since in our case the punctures are unlabelled. This shows that the Random Delaunay model is in the universality class of pure two-dimensional gravity (Liouville theory with $\gamma = \sqrt{8/3}$ and $c_{\mathrm{matter}} = 0$). This was only conjectured in [David and Eynard, 2014].

Note that it is possible to generalize the random Delaunay model from the planar case (genus $g=0$) to the higher genus $g>0$ case. Since the identification 3.35 between the Delaunay Kähler form and the Weil-Petersson form is local, it should also be valid for the $g>0$ case.

The random Delaunay model remains an interesting model of random two-dimensional geometry since it is an explicit model of a global conformal mapping of an abstract (or intrinsic) but continuous two-dimensional geometry model onto the complex plane. This mapping through Delaunay triangulations is different, and somehow simpler, that the general mapping provided by the Riemann uniformization theorem, which is usually considered. Indeed a local modification of the position of one vertex of the triangulation translates in a local modification of the associated Kähler form, since the Kähler potential $\mathcal{A}_T$ given by 2.7 is a sum over local terms (the hyperbolic volumes $\mathbf{V}(f)$ of the triangles). This is not the case for the uniformization mapping, which leads to a global Kähler potential (a classical Liouville action).

Therefore the model discussed here should allow to study the local properties of the conformal mapping of a random metric onto the plane. We present new, although preliminary, local results in the next section.

4 Local inequalities on the measure

We give here two local results satisfied by the measure $\mu(T, d\boldsymbol{\theta}) = d\mu(\mathbf{z})$ defined in [David and Eynard, 2014] and studied in this paper.

4.1 Maximality property over the Delaunay triangulations

Looking at the measure $d\mu(\mathbf{z})$ on $\mathbb{C}^D$ (the space of distributions of $N+3$ points on the Riemann sphere), theorem 2.5 gives

$$d\mu(\mathbf{z}) = \prod_{v=4}^{N+3} d^2 z_v \; 2^N \; \det\left[D_{u\bar{v}}\right]_{u,\bar{v}\neq\{1,2,3\}} .$$

Let $\{z_v\}$ be a configuration of $N+3$ points on the Riemann sphere, and let T be a planar triangulation associated to these points. Then the Kähler metric $D_{u\bar{v}}(T)$ on $\mathbb{C}^{N+3}$ is still well defined (T is not necessarily a Delaunay triangulation), see equations 2.6 and 2.7. We use a short-hand notation:

$$d_{(ijk)}(T) = \det\Big[(D_{u\bar{v}})_{u,v\neq\{i,j,k\}}(T)\Big]. \tag{4.1}$$

Then the following result stands:

Theorem 4.1. *Given $N+3$ points $z_1, \dots, z_{N+3}$ in $\mathbb{C}$, their Delaunay triangulation $T^D(\{z_v\})$ is the one which maximizes $d_{(ijk)}(T)$ among all possible triangulations T:*

$$d_{(ijk)}(T^D(\{z_v\})) = \max_{T\,\text{triangulation of}\,\{z_v\}} d_{(ijk)}(T). \tag{4.2}$$

In order to prove this assertion, one may look at the transformation $d_{(124)}(T) \xrightarrow[(13)]{(24)} d_{(124)}(T')$ where the triangulation T undergoes the edge flip $(24) \to (13)$ (see figure 7). It leads to the following lemma:

Lemma 4.1. *Denote f the triangle (124), and ω_f, R_f respectively the center and the radius of its circum circle. Then*

$$d_{(124)}(T) - d_{(124)}(T') = \det\left[(D_{u\bar{v}})_{u,v\neq\{1,2,3,4\}}(T)\right] \times \text{Area}(f)\frac{|z_3-\omega_f|^2 - {R_f}^2}{|z_3-z_1|^2\,|z_3-z_2|^2\,|z_3-z_4|^2}. \tag{4.3}$$

Proof. The proof of this lemma is given in appendix A. □

Remark 4.1. *Let us recall that in [David and Eynard, 2014], for a triangulation where all the faces are positively oriented, it has been proved that the Hermitian form $D_{u\bar{v}}$ is positive. The result is true for general planar triangulations, if we impose a positive orientation for the faces (orientation that we enforce here). Hence, the principal minors of $D_{u\bar{v}}$ are positive, so $\det\left[(D_{u\bar{v}})_{u,v\neq\{1,2,3,4\}}(T)\right] \geq 0$.*

Remark 4.2. *From lemma 4.1, one deduces that $d_{(124)}(T) - d_{(124)}(T') \geq 0$ only if z_3 is out of the circumcircle of f.*

In [David and Eynard, 2014], the authors proved that the quantity $d_{ijk}(T)$ change covariantly when changing the points (ijk), a useful property for demonstrating theorem 4.1:

Lemma 4.2. *The quantity*

$$\frac{d_{(ijk)}(T)}{|\Delta_3(i,j,k)|^2}\ ,$$

with $\Delta_3(i,j,k) = (z_i-z_j)(z_i-z_k)(z_j-z_k)$, is independent of the choice of the three fixed points $\{z_i, z_j, z_k\}$.

Proof of the theorem 4.1. Take a triangulation T of the configuration $\{z_v\}$, then apply the recursive Lawson flip algorithm (see [Brévilliers, 2008] or [Lawson, 1972] for details on this algorithm, noted here LFA) to T, one obtains the Delaunay triangulation $T^D(\{z_v\})$ of the $\{z_v\}$. At each step, the LFA applies a single edge flip. Note $(T_i)_{0\leq i\leq n}$ the sequence of successive triangulations obtained by the LFA, with $T_0 = T$ and $T_n = T^D(\{z_v\})$:

$$T_0 = T \xrightarrow[(a^1d^1)]{(b^0c^0)} T_1 \xrightarrow[(a^2d^2)]{(b^1c^1)} \dots \xrightarrow[(a^nd^n)]{(b^{n-1}c^{n-1})} T_n = T^D(\{z_v\}) \tag{4.4}$$

The LFA works in the following way: for T_i, if it is not Delaunay, at least one point, say a^i, is contained in the circum circle of a neighboring face (b^i, c^i, d^i). The situation is depicted in figure 7.

Then the edge (b^i, c^i) is flipped to give (a^i, d^i). It follows that for the two new faces (a^i, c^i, d^i) and (a^i, d^i, b^i), their circum circles do not enclose respectively b^i nor c^i.

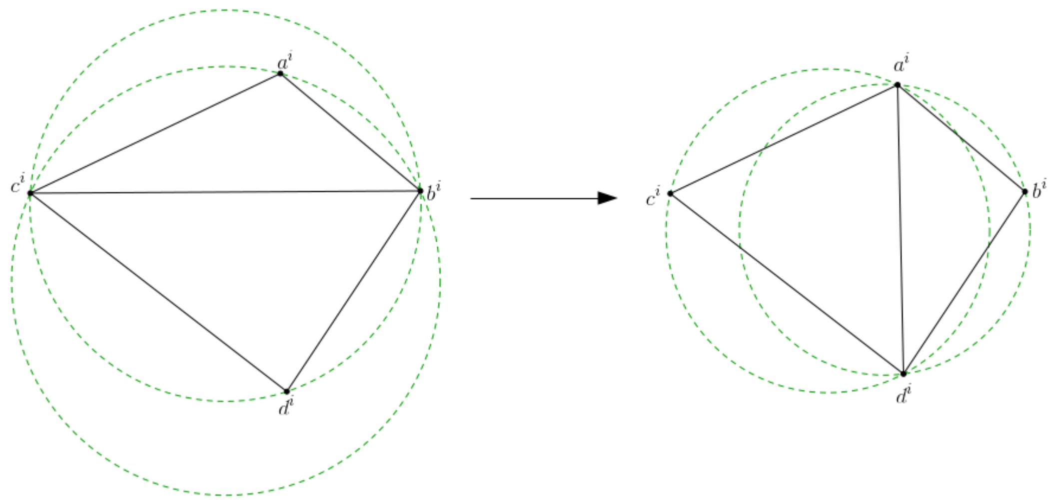

Figure 7: Effect of a flip at one step of the LFA.

Using lemma 4.1 and 4.2:

$$\begin{aligned} d_{(123)}(T^D) - d_{(123)}(T) &= \sum_{i=0}^{n-1} \left[d_{(123)}(T_{i+1}) - d_{(123)}(T_i) \right] \\ &= |\Delta_3(1,2,3)|^2 \sum_{i=0}^{n-1} \left[\frac{d_{(a^i,c^i,d^i)}(T_{i+1}) - d_{(a^i,c^i,d^i)}(T_i)}{|\Delta_3(a^i,c^i,d^i)|^2} \right] \\ &\geq 0 \end{aligned}$$

which ends the proof. □

The measure $d\mu(\mathbf{z})$ used is then maximal over the Delaunay triangulations.

4.2 Growth of the volume

The second result relates to the N dependence of the total volume

$$V_N = \int_{\mathbb{C}^N} \prod_{v=4}^{N+3} d^2 z_v 2^N \det \left[D_{u\bar{v}}(T^D(\{z_v\})) \right]_{u,\bar{v} \neq \{1,2,3\}} \tag{4.5}$$

It is the volume of the space of Delaunay triangulations with $N+3$ vertices with the measure $d\mu(\mathbf{z})$. An underestimate of the growth of the volume when the number of vertices increases is given by the following inequalities:

Theorem 4.2. *If we add a $N+4$'th point to a given triangulation and integrate over its position, the following inequality stands:*

$$\int_{\mathbb{C}} d^2 z_{N+4} \det \left[D_{u\bar{v}}(T^D(\{z_1,\ldots,z_{N+4}\})) \right]_{u,\bar{v} \neq \{1,2,3\}} \geq (N+1) \frac{\pi^2}{8} \det \left[D_{u\bar{v}}(T^D(\{z_1,\ldots,z_{N+3}\})) \right]_{u,\bar{v} \neq \{1,2,3\}} \tag{4.6}$$

It implies the inequality for the total volumes

$$V_{N+1} \geq (N+1)\frac{\pi^2}{8} V_N. \tag{4.7}$$

Before proving the theorem, let us stress that this growth property is global w.r.t. the last point. A similar inequality does not stand locally for the measure $\det\left[D_{u\bar{v}}(T^D(\{z_v\}))\right]_{u,\bar{v}\neq\{1,2,3\}}$ when one adds a vertex at a fixed position to an existing Delaunay triangulation. This has been checked numerically.

Proof. We first focus on the inequality 4.6. The proof follows the following procedure:

- Fix $N+3$ points $\{z_1,\ldots,z_{N+3}\}$ in $\mathbb{C}$, and note $T^D(\{z_v\})$ the Delaunay triangulation constructed on this configuration.
- Pave the Riemann sphere with regions $\mathcal{R}(f)$ (defined above) associated with the faces f of the triangulation.
- Then add a point z_{N+4} in $\mathbb{C}$ to this triangulation. Depending on the region $\mathcal{R}(f)$ where it stands, transform the triangulation to include the new point and compute the measure associated with this triangulation.
- Integrate over z_{N+4}, find a minoration of the integral, and compare the result with the measure associated with $T^D(\{z_v\})$.

For the Delaunay triangulation constructed over $\{z_1,\ldots z_{N+3}\}$, the Riemann sphere can be conformally paved with regions $\mathcal{R}(f)$ associated to each face in the following way. Let us look at the edge e whose neighboring faces are f and f'. The circumcircles of f and f' meet at the vertices located at the ends of e with an angle $\theta'(e) = (\pi - \theta(e))$. Define $\mathcal{C}_e$ the arc of a circle joining the ends of e, and making an angle $\theta'(e)/2 = (\pi - \theta(e))/2$ with each of the circumcircles of f and f' at the vertices of e. See figure 8. The region $\mathcal{R}(f)$ is now defined as the domain enclosed in the three arcs of a circle $\mathcal{C}_{e_1}$, $\mathcal{C}_{e_2}$, $\mathcal{C}_{e_3}$ corresponding to the three edges e_1, e_2, e_3 surrounding f (see figure 9). This domain is now transformed covariantly under a Möbius transformation.

We add the point z_{N+4} in the Riemann sphere. If $z_{N+4} \in \mathcal{R}(f)$, we construct the triangulation $T^D_f(\{z_1,\ldots,z_{N+3}\},z_{N+4})$ by joining the vertex z_{N+4} to the vertices a, b and c of the face f. The triangulation $T^D_f(\{z_1,\ldots,z_{N+3}\},z_{N+4})$ is in general different from the Delaunay triangulation $T^D(\{z_1,\ldots,z_{N+4}\})$. Yet it is still possible to define the measure $\det D_{u\bar{v}}(T^D_f(\{z_v\},z_{N+4}))$, which is still a positive quantity, and which, from theorem 4.1, satisfies:

$$\det\left[D_{u\bar{v}}(T^D_f(\{z_v\},z_{N+4}))\right]_{u,\bar{v}\neq\{1,2,3\}} \leq \det\left[D_{u\bar{v}}(T^D(\{z_1,\ldots,z_{N+4}\}))\right]_{u,\bar{v}\neq\{1,2,3\}} \tag{4.8}$$

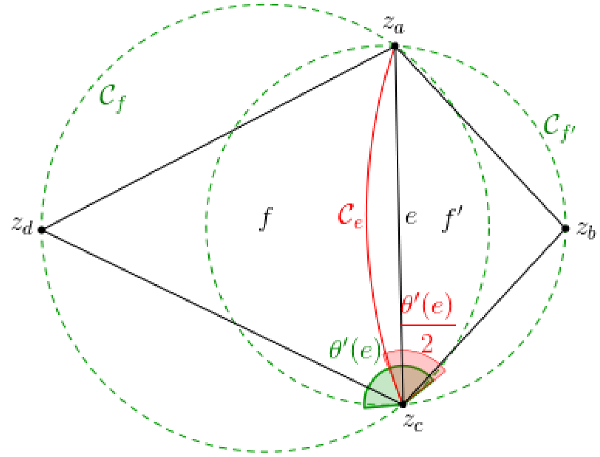

Figure 8: Definition of the arc $\mathcal{C}_e$.

The aim is to find a minorant to the integral over each region $\mathcal{R}(f)$. The interesting result is that we found a minorant that does not depend on the region, although the shapes of the regions depend on the angle $\theta(e)$ between two neighboring circum circles. We take this dependence out by integrating over smaller regions $\mathcal{B}(f) \subseteq \mathcal{R}(f)$. for the face f, $\mathcal{B}(f)$ is the region enclosed by the three arcs or a circle that pass through two of the vertices of f and that are orthogonal to the circum circle of f (see figure 10).

The integration over z_{N+4} thus decomposes in the following way:

$$\begin{aligned}
&\int_{\mathbb{C}} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D(\{z_1,\ldots,z_{N+4}\}))\right]_{u,\bar{v}\neq\{1,2,3\}} \\
&\qquad = \sum_f \int_{\mathcal{R}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D(\{z_1,\ldots,z_{N+4}\}))\right]_{u,\bar{v}\neq\{1,2,3\}} \\
&\qquad \geq \sum_f \int_{\mathcal{R}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D_f(\{z_v\},z_{N+4}))\right]_{u,\bar{v}\neq\{1,2,3\}} \\
&\qquad \geq \sum_f \int_{\mathcal{B}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D_f(\{z_v\},z_{N+4}))\right]_{u,\bar{v}\neq\{1,2,3\}} \qquad (4.9)
\end{aligned}$$

In the last line, the integral can be computed explicitly. If $z_{N+4} \in \mathcal{B}(f)$ with $f = (abc)$), one can compute the integration on $\mathcal{B}(f)$ using lemma 4.2:

$$\begin{aligned}
&\int_{\mathcal{B}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D_f(\{z_v\},z_{N+4}))\right]_{u,\bar{v}\neq\{1,2,3\}} \\
&\qquad = \frac{\Delta_3(z_1,z_2,z_3)}{\Delta_3(a,b,c)} \int_{\mathcal{B}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D_f(\{z_v\},z_{N+4}))\right]_{u,\bar{v}\neq\{a,b,c\}}. \qquad (4.10)
\end{aligned}$$

Then the right term factorizes nicely thanks to the shape of the triangulation around z_{N+4}:

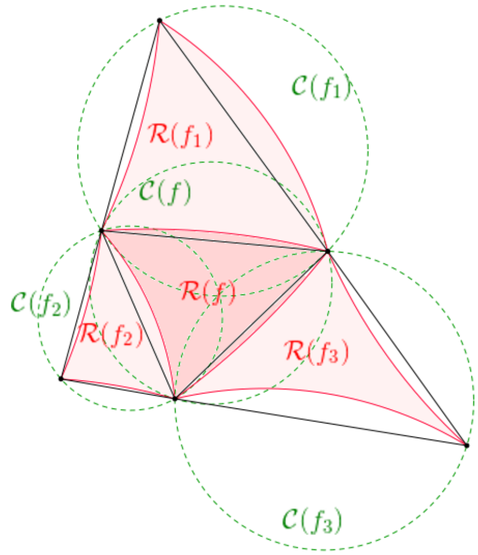

Figure 9: The region $\mathcal{R}(f)$ is enclosed in the bissector arcs.

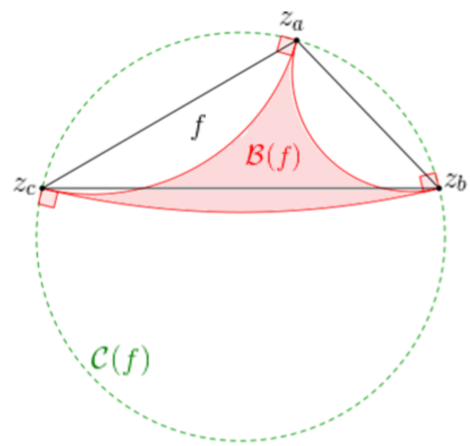

Figure 10: The region $\mathcal{B}(f)$ associated with a face f.

$$\int_{\mathcal{B}(f)} d^2 z_{N+4} \ \det\left[D_{u\bar{v}}(T_f^D(\{z_v\}, z_{N+4}))\right]_{u,\bar{v}\neq\{a,b,c\}}$$
$$= \int_{\mathcal{B}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D(\{z_v\}))\right]_{u,\bar{v}\neq\{a,b,c\}} \times \det\left[D_{u\bar{v}}(T^D(\{a,b,c,z_{N+4}\}, z_{N+4}))\right]_{u,\bar{v}\neq\{a,b,c\}}. \tag{4.11}$$

In the integrand, the term depending on z_{N+4} is the second determinant, so we need to estimate:

$$I = \int_{\mathcal{B}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D(\{a,b,c,z_{N+4}\}, z_{N+4}))\right]_{u,\bar{v}\neq\{a,b,c\}} \tag{4.12}$$

It is the integral of the measure on the Delaunay triangulation made of the 4 points a, b, c and z_{N+4}, where z_{N+4} crosses the region $\mathcal{B}(f)$ (see figure 11). The integral is computable if one considers the measure in terms of the angles (see equation 2.4). With the notations of figure 11:

$$I = \frac{1}{2} \int_{z_{N+4}\in\mathcal{B}(f)} d\theta_1 d\theta_2 \tag{4.13}$$

(We used here a result of the article [David and Eynard, 2014], expressing the measure in term of a basis of angles. Here, the angles θ_1 and θ_2 are a basis of this triangulation). The point z_{N+4} belongs to the region $\mathcal{B}(f)$ if $\theta_i^{\min} \leq \theta_i \leq \theta_i^{\min} + \frac{\pi}{2}$ for $i = 1,2,3$. $\theta_i^{\min}$ corresponds to the angle θ_i for which the point z_{N+4} is on the boundary arc of $\mathcal{B}(f)$ associated with the edge i.

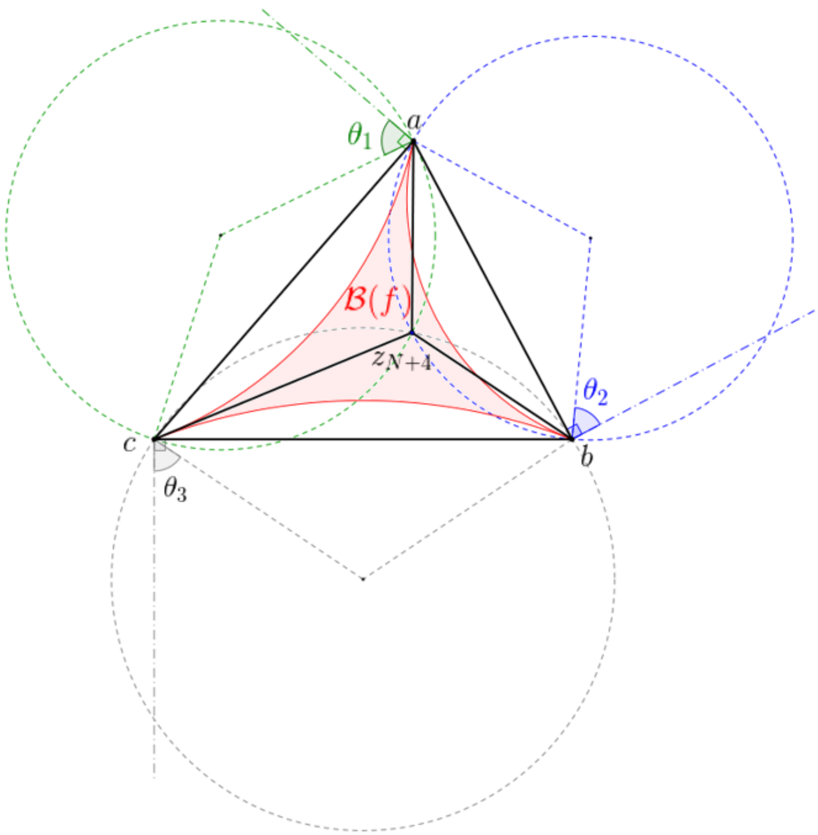

Figure 11: The Delaunay triangulation (in black) with the associated circum circles. The center of the external face is at ∞.

We also have $\theta_1 + \theta_2 + \theta_3 = \pi$ and $\theta_1^{\min} + \theta_2^{\min} + \theta_3^{\min} = \frac{\pi}{2}$, so eventually, $z_{N+4} \in \mathcal{B}(f)$ if:

$$\theta_1^{\min} \leq \theta_1 \leq \theta_1^{\min} + \frac{\pi}{2} \tag{4.14}$$

$$\theta_2^{\min} \leq \theta_2 \leq \theta_2^{\min} + \frac{\pi}{2} \tag{4.15}$$

$$\theta_1^{\min} + \theta_2^{\min} \leq \theta_1 + \theta_2 \leq \theta_1^{\min} + \theta_2^{\min} + \frac{\pi}{2} \tag{4.16}$$

From these conditions we immediately obtain that $I = \frac{1}{2}\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2\right] = \frac{\pi^2}{16}$. Then, one gets in equation 4.11:

$$\begin{aligned}\int_{\mathcal{B}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T_f^D(\{z_v\}, z_{N+4}))\right]_{u,\bar{v}\neq\{a,b,c\}} \\ = \frac{\pi^2}{16} \det\left[D_{u\bar{v}}(T^D(\{z_v\}))\right]_{u,\bar{v}\neq\{a,b,c\}} \\ = \frac{\pi^2}{16} \frac{\Delta_3(a,b,c)}{\Delta_3(z_1,z_2,z_3)} \det\left[D_{u\bar{v}}(T^D(\{z_v\}))\right]_{u,\bar{v}\neq\{z_1,z_2,z_3\}}\end{aligned} \tag{4.17}$$

So in the end:

$$\begin{aligned}\int_{\mathbb{C}} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D(\{z_1,\ldots,z_{N+4}\}))\right]_{u,\bar{v}\neq\{1,2,3\}} \\ \geq \sum_f \frac{\pi^2}{16} \det\left[D_{u\bar{v}}(T^D(\{z_v\}))\right]_{u,\bar{v}\neq\{z_1,z_2,z_3\}} \\ \geq (N+1)\frac{\pi^2}{8} \det\left[D_{u\bar{v}}(T^D(\{z_v\}))\right]_{u,\bar{v}\neq\{z_1,z_2,z_3\}}\end{aligned} \tag{4.18}$$

which gives the result:

$$V_{N+1} \geq (N+1)\frac{\pi^2}{8} V_N \tag{4.19}$$

□

The previous result gives a minorant which does not depend on the shape of the triangle by integrating over a restrained region $\mathcal{B}(f)$. If we do the same calculation and keep the region $\mathcal{R}(f)$, then the minorant is more accurate, but not universal any more. In this case, we then get a refined result:

Theorem 4.3.

$$\begin{aligned}\int_{\mathbb{C}} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D(\{z_1,\ldots,z_{N+4}\}))\right]_{u,\bar{v}\neq\{1,2,3\}} \\ \geq \left[(N+1)\frac{\pi^2}{8} + \frac{1}{8}\sum_{e\in\mathcal{E}} \theta_e(2\pi - \theta_e)\right] \det\left[D_{u\bar{v}}(T^D(\{z_v\}))\right]_{u,\bar{v}\neq\{z_1,z_2,z_3\}}\end{aligned} \tag{4.20}$$

(See appendix B for a proof). We see that the angles associated to the triangulation appear. This angle-dependent term should be related to the kinetic term of the Liouville action in the continuum limit.

5 Conclusion

The final goal would be to understand how the set of Delaunay triangulations equipped with the measure given by the Lebesgue measure of circum-circle crossing angles, introduced in [David and Eynard, 2014], can converge towards the Liouville theory ? In this article we continued to study the properties of this measure.

- In particular, we could the measure, to the Weil-Petersson volume form. This allows to have an hyperbolic representation.

- We found an interesting property of maximality: our measure can be analytically continued to non-Delaunay triangulations, but is maximal exactly for Delaunay triangulations. This could open the possibility of some convexity properties, that need to be further explored, and that could be helpful in studying the continuum limit.

- We found a lower bound on the process of adding a new vertex $N \to N+1$. We have both local and global bounds. We could show that the partition function $V_N/N!$ grows at least like $(\pi^2/8)^N$. In other words, the log of the volume contains at least a term proportional to N, which can be interpreted as the "quantum area". If as we expect the continuum limit exists and is the Liouville theory at $c=0$ ($\gamma=\sqrt{8/3}$), then the log of the volume should converge towards the Liouville action. The Liouville action is made of 2 terms, one is the quantum area, the other is the kinetic energy. The term we have found is compatible with a continuum limit of the quantum area.

  It would be interesting to improve our bound, by taking into account the edges (integrals over $\mathcal{R}(f)-\mathcal{B}(f)$) contributions to see if they can account for the kinetic-term in the Liouville action.

All those results are encouraging steps towards a continuum limit that would be the Liouville pure gravity theory. Consequences need to be further explored.

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A Change of the measure with a flip: proof of lemma 4.1

When the triangulation T undergoes a flip to give the triangulation T', only the two faces surrounding the edge change. So in the prepotentials $\mathcal{A}(T)$ and $\mathcal{A}(T')$, the only terms that differ are those implying the changed faces:

$$\mathcal{A}(T) - \mathcal{A}(T') = \mathbf{V}(124) + \mathbf{V}(234) - \mathbf{V}(123) - \mathbf{V}(134) \tag{A.1}$$

Therefore, the differences between $D(T)$ and $D(T')$ are located in the $D_{i,j}$ with $i, j \in \{1,2,3,4\}$. As we are looking at the quantities $d_{(124)}$, the indices 1, 2 and 4 are not taken into account in the determinant. So the differences between $D(T)$ and $D(T')$ lay in $D_{3,3}$. By expanding the determinant with respect to the third line, we get:

$$d_{(124)}(T) - d_{(124)}(T') = \left[D_{3,3}(T) - D_{3,3}(T')\right] \det\left[(D_{u\bar{v}})_{u,v\neq\{1,2,3,4\}}(T)\right] \tag{A.2}$$

Let us focus on the term $D_{3,3}(T) - D_{3,3}(T')$. Using the form $D = \frac{1}{4i}AEA^\dagger$, and noting $z_{ij} = z_i - z_j$ one gets:

$$\begin{aligned}
D_{3,3}(T) - D_{3,3}(T') &= \frac{1}{4i}\left[\sum_{e\to 3}\sum_{e' \text{ neighbour of } e} A_{3,e}E_{e,e'}\bar{A}_{3,e'}\right] && \text{(A.3)}\\
&= \frac{1}{4i}\left[\frac{1}{z_{31}}\frac{-1}{\bar{z}_{32}} + \frac{1}{z_{32}}\frac{1}{\bar{z}_{31}} + \frac{1}{z_{31}}\frac{1}{\bar{z}_{34}} - \frac{1}{z_{34}}\frac{1}{\bar{z}_{31}} - \frac{1}{z_{32}}\frac{1}{\bar{z}_{34}} + \frac{1}{z_{34}}\frac{1}{\bar{z}_{32}}\right] && \text{(A.4)}\\
&= \frac{1}{4i}\frac{z_{32}z_{34}\bar{z}_{31}\bar{z}_{42} + z_{31}z_{32}\bar{z}_{34}\bar{z}_{21} + z_{31}z_{34}\bar{z}_{32}\bar{z}_{14}}{|z_{31}|^2|z_{32}|^2|z_{34}|^2} && \text{(A.5)}\\
&= \frac{1}{4i}\frac{N(z_3,\bar{z}_3)}{|z_{31}|^2|z_{32}|^2|z_{34}|^2} && \text{(A.6)}
\end{aligned}$$

The coefficient of the term $z_3^2\bar{z}_3$ in $N(z_3,\bar{z}_3)$ gives $\bar{z}_{42} + \bar{z}_{21} + \bar{z}_{14} = 0$. What is more, $\bar{N}(z) = -N(z)$, so N can be written as $N(z_3,\bar{z}_3) = az_3\bar{z}_3 + bz_3 - \bar{b}\bar{z}_3 + c$, with $a \in i\mathbb{R}$, b and $c \in i\mathbb{R}$ functions of z_i, $\bar{z}_i$, $i = 1,2,4$. Setting $\omega = -\frac{\bar{b}}{a}$ and $R^2 = -\frac{c}{a} + |a|^2$,

$$N(z_3,\bar{z}_3) = a\left[(z_3 - \omega)(\bar{z}_3 - \bar{\omega}) - R^2\right] \tag{A.7}$$

$N(z_3,\bar{z}_3) = 0$ is thus the equation of a circle for the point 3. As we have $N(z_i,\bar{z}_i) = 0$ for $i = 1,2,4$, the circle is the circum circle of the face $f = (124)$, of center $\omega_f = \omega$ and radius $R_f = R$. The coefficient a is given by $a = z_{41}\bar{z}_{21} - z_{21}\bar{z}_{41}$, which is the (euclidean) area of the face (124). Eventually we have:

$$D_{3,3}(T) - D_{3,3}(T') = \text{Area}(f)\frac{|z_3 - \omega_f|^2 - R_f^2}{|z_{31}|^2|z_{32}|^2|z_{34}|^2} \tag{A.8}$$

which proves the lemma 4.1.

B Refined minorant for the volume: proof of theorem 4.3

The notations introduced here refer to the figure 11. Each edge of the triangle (abc) is surrounded by two faces. If we remove the point z_{N+4}, we obtain the Delaunay Triangulation for the points $\{z_1, \dots, z_{N+3}\}$, and the triangle (abc) is one of its faces. Let us note $\theta_{(ab)}$, $\theta_{(bc)}$,and $\theta_{(ca)}$ the angles between the face $f = (abc)$ and the other face in contact with the edges (ab), (bc), and (ca) respectively.
Now, in formula 4.12, instead of computing the integral of the measure over the region $\mathcal{B}(f)$, we carry out the integral over the region $\mathcal{R}(f)$. The integrand is not changed: it is the measure of the Delaunay triangulation made of the 4 points a, b, c and z_{N+4}.

$$I_1 = \int_{\mathcal{R}(f)} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D(\{a,b,c,z_{N+4}\}, z_{N+4}))\right]_{u,\bar{v}\neq\{a,b,c\}}. \tag{B.1}$$

Then the computation of I_1 follow the same steps as for I, the only difference being the inequalities 4.14 satisfied by θ_1 and θ_2:

$$\theta_1^{\min} - \frac{\theta_{(ca)}}{2} \leq \theta_1 \leq \theta_1^{\min} + \frac{\pi}{2} \tag{B.2}$$

$$\theta_2^{\min} - \frac{\theta_{(ab)}}{2} \leq \theta_2 \leq \theta_2^{\min} + \frac{\pi}{2} \tag{B.3}$$

$$\theta_1^{\min} + \theta_2^{\min} \leq \theta_1 + \theta_2 \leq \theta_1^{\min} + \theta_2^{\min} + \frac{\theta_{(bc)}}{2} + \frac{\pi}{2} \tag{B.4}$$

The integral is then the area of the red region in figure 12. So we get: $I_1 = \frac{\pi^2}{16} + \frac{1}{16}[\theta_{(ab)}(2\pi - \theta_{(ab)}) + \theta_{(bc)}(2\pi - \theta_{(bc)}) + \theta_{(ca)}(2\pi - \theta_{(ca)})]$. Then, following the same steps as for the previous minorant, the result comes:

$$\int_{\mathbb{C}} d^2 z_{N+4} \det\left[D_{u\bar{v}}(T^D(\{z_1, \dots, z_{N+4}\}))\right]_{u,\bar{v}\neq\{1,2,3\}}$$
$$\geq \left[(N+1)\frac{\pi^2}{8} + \frac{1}{8}\sum_{e\in\mathcal{E}} \theta_e(2\pi - \theta_e)\right] \det\left[D_{u\bar{v}}(T^D(\{z_v\}))\right]_{u,\bar{v}\neq\{z_1,z_2,z_3\}} \tag{B.5}$$

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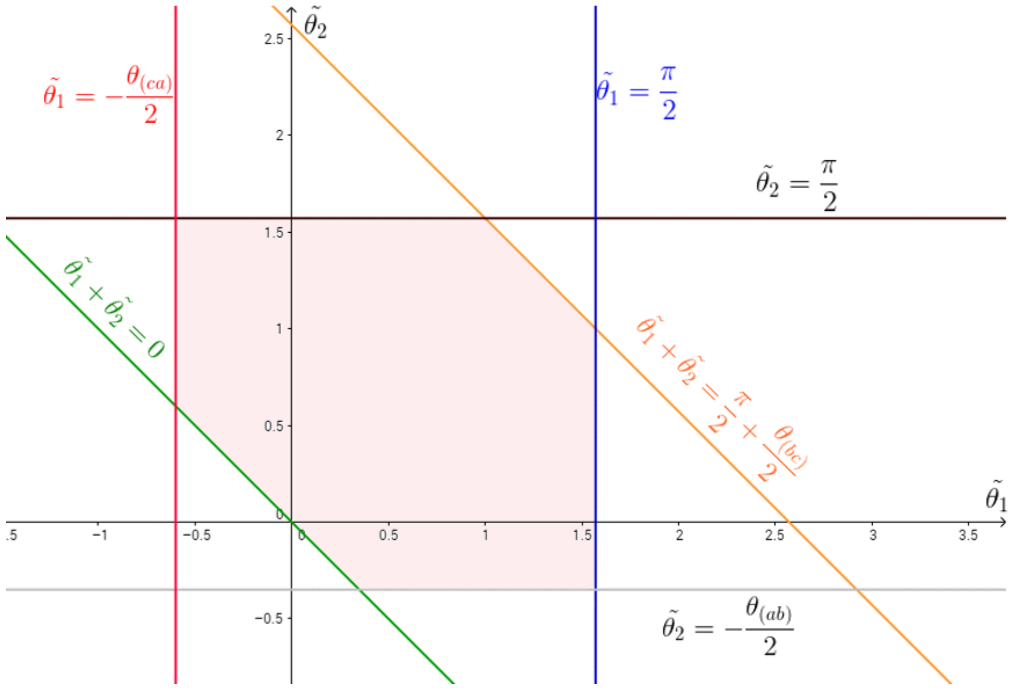

Figure 12: Domain on which $d\tilde{\theta}_1 d\tilde{\theta}_2$ has to be integrated. We take here $\tilde{\theta}_i = \theta_i - \theta_i^{\min}$.

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