Dissipative random quantum spin chain with boundary-driving and bulk-dephasing: magnetization and current statistics in the Non-Equilibrium-Steady-State

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The Lindblad dynamics with dephasing in the bulk and magnetization-driving at the two boundaries is studied for the quantum spin chain with random fields h_j and couplings J_j (that can be either uniform or random). In the regime of strong disorder in the random fields, or in the regime of strong bulk-dephasing, the effective dynamics can be mapped onto a classical Simple Symmetric Exclusion Process with quenched disorder in the diffusion coefficient associated to each bond. The properties of the corresponding Non-Equilibrium-Steady-State in each disordered sample between the two reservoirs are studied in detail by extending the methods that have been previously developed for the same model without disorder. Explicit results are given for the magnetization profile, for the two-point correlations, for the mean current and for the current fluctuations, in terms of the random fields and couplings defining the disordered sample.

In disordered quantum systems, the phenomenon of Anderson Localization (see the book [1] and references therein) or its generalization with interactions called Many-Body-Localization (see the recent reviews [2–7] and references therein) is due to the coherent character of the unitary dynamics. When these systems are not isolated anymore but become 'open' [8], it is essential to understand whether the dissipation processes that tend to destroy the quantum coherence are able to eliminate the localization phenomenon. This issue has been analyzed recently in the context of random quantum spin chains following some Lindblad dynamics [9–16], where it is very important to distinguish the various types of dissipation : if the dissipation occur only at the boundaries, the coherent dynamics in the bulk is sufficient to maintain the localization properties, while if the dissipation occurs everywhere in the bulk via dephasing, the localization phenomenon will be destroyed and it is interesting to characterize the properties of this dissipative dynamics in the presence of disorder.

In the field of quantum spin chains without disorder, the Lindblad dynamics has been much studied to characterize the non-equilibrium transport properties [9, 17–25] with many exact solutions [26–31]. The Lindblad framework for quantum systems also allows to make the link with the field of non-equilibrium classical stochastic processes described by Master Equations (see the review [32] and references therein) : for instance the relaxation properties can be obtained from the spectrum of the Lindblad operator [33–37], the large deviation formalism has been used to access the full-counting statistics [38–45], the additivity principle has been tested [46] and quantum fluctuation relations have been derived [47].

In the present paper, our goal is to analyze the Lindblad dynamics of the XX quantum spin chain with random fields and couplings that can be either uniform or random, in the presence of dephasing in the bulk and in the presence of magnetization-driving at the two boundaries in order to generate a Non-Equilibrium-Steady-State carrying a current. In the absence of bulk-dephasing, this model has been found to keep its localized nature with a step magnetization profile and an exponentially decaying current with the system size [16]. In the presence of bulk-dephasing, we obtain here that these localization properties are lost, as expected. We use the degenerate second-order perturbative approach in the XX-couplings J_j developed previously either for strong bulk dephasing [33, 34] or for strong disorder in the random fields [11]. The effective dynamics can be then mapped onto a classical Simple Symmetric Exclusion Process with quenched disorder in the local diffusion coefficients. The methods that have been developed previously to study this classical stochastic model without quenched disorder (see the review [32] and references therein) can be then adapted to characterize the Non-Equilibrium-Steady-State in each disordered sample and to obtain explicit results for the magnetizations, the correlations, and the two first cumulants of the integrated current.

The paper is organized as follows. In section I, we introduce the notations for the Lindblad dynamics with boundary-driving and bulk-dephasing. In section II, we focus on the regime of strong-disorder in the random fields or on the regime of strong dephasing where the effective dynamics corresponds to a classical exclusion process with random diffusion coefficients on the links. The properties of the corresponding Non-Equilibrium-Steady-State in each disordered sample are studied in the remaining of the paper, with explicit results for the magnetization profile and the averaged current (section III), for the two-point correlations (section IV) and for the current fluctuations (section V). Our conclusions are summarized in section VI.

I. LINDBLAD DYNAMICS WITH BOUNDARY-DRIVING AND BULK-DEPHASING

A. Lindblad dynamics for the density matrix $\rho(t)$

In this paper, we consider the Lindblad dynamics for the density matrix $\rho(t)$ of the quantum chain of N spins

$$\frac{\partial \rho(t)}{\partial t} = -i[H,\rho] + \mathcal{D}^{Bulk}[\rho(t)] + \mathcal{D}^{Left}[\rho(t)] + \mathcal{D}^{Right}[\rho(t)] \tag{1}$$

The Hamiltonian contains random fields h_j and XX-couplings J_j (that can be either uniform or random)

$$H = \sum_{j=1}^{N} \left[h_j \sigma_j^z + J_j(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \right] = \sum_{j=1}^{N} \left[h_j \sigma_j^z + 2J_j(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) \right] \tag{2}$$

It is possible to add couplings $J_j^z \sigma_i^z \sigma_{j+1}^z$, but these couplings turn out to disappear at leading order in the strong-dephasing approximation [33, 34] or in the strong-disorder approximation [11] that we will consider (see more details in section II A).

The Bulk-dephasing operator acting with some amplitudes γ_j (that can be taken uniform)

$$\mathcal{D}^{Bulk}[\rho] = \sum_{j=1}^{N} \gamma_j \left(\sigma_j^z \rho \sigma_j^z - \rho \right) \tag{3}$$

tends to destroy off-diagonal elements with respect to the σ^z basis.

The Left-Magnetization-driving

$$\begin{aligned} \mathcal{D}^{Left}[\rho] &= \Gamma \frac{1+\mu}{2} \left(\sigma_1^+ \rho \sigma_1^- - \frac{1}{2} \sigma_1^- \sigma_1^+ \rho - \frac{1}{2} \rho \sigma_1^- \sigma_1^+ \right) \\ &+ \Gamma \frac{1-\mu}{2} \left(\sigma_1^- \rho \sigma_1^+ - \frac{1}{2} \sigma_1^+ \sigma_1^- \rho - \frac{1}{2} \rho \sigma_1^+ \sigma_1^- \right) \end{aligned} \tag{4}$$

tends to impose the magnetization (μ) on the first spin σ_1. The Right-Magnetization-driving

$$\begin{aligned} \mathcal{D}^{Right}[\rho] &= \Gamma' \frac{1+\mu'}{2} \left(\sigma_N^+ \rho \sigma_N^- - \frac{1}{2} \sigma_N^- \sigma_N^+ \rho - \frac{1}{2} \rho \sigma_N^- \sigma_N^+ \right) \\ &+ \Gamma' \frac{1-\mu'}{2} \left(\sigma_N^- \rho \sigma_N^+ - \frac{1}{2} \sigma_N^+ \sigma_N^- \rho - \frac{1}{2} \rho \sigma_N^+ \sigma_N^- \right) \end{aligned} \tag{5}$$

tends to impose the magnetization (μ') on the last spin σ_N.

When $\mu \neq \mu'$, the dynamics will converge towards some stationnary non-equilibrium-steady-state that we wish to study.

B. Ladder Lindbladian for the ket $|\rho(t)>$

The density matrix $\rho(t)$ of the chain of N spins can be expanded in the σ^z basis

$$\rho(t) = \sum_{S_1=\pm 1} ... \sum_{S_N=\pm 1} \sum_{T_1=\pm 1} ... \sum_{T_N=\pm 1} \rho_{S_1,..,S_N;T_1,...,T_N}(t) |S_1,...,S_N >< T_1,...,T_N| \tag{6}$$

in terms of the 4^N coefficients

$$\rho_{S_1,..,S_N;T_1,...,T_N}(t) = < S_1,...,S_N|\rho(t)|T_1,...,T_N > \tag{7}$$

It is technically convenient to 'vectorize' the density *matrix* of the *spin chain* [16, 34, 46, 48–50], i.e. to consider that these 4^N coefficients are the components of a *ket* describing the state of a *spin ladder*

$$|\rho(t)>= \sum_{S_1=\pm 1} ... \sum_{S_N=\pm 1} \sum_{T_1=\pm 1} ... \sum_{T_N=\pm 1} \rho_{S_1,..,S_N;T_1,...,T_N}(t) |S_1,...,S_N > \otimes |T_1,...,T_N > \tag{8}$$

The Lindbladian governing the dynamics of the ket $|\rho(t)>$

$$\frac{\partial|\rho(t)>}{\partial t} = \mathcal{L}|\rho(t)> \tag{9}$$

reads in this ladder formulation

$$\begin{aligned}
\mathcal{L} \quad &= -i\sum_{j=1}^{N}\left[h_j\sigma_j^z + 2J_j(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+)\right] + i\sum_{j=1}^{N}\left[h_j\tau_j^z + 2J_j(\tau_j^+\tau_{j+1}^- + \tau_j^-\tau_{j+1}^+)\right] \\
&-\sum_{j=1}^{N}\gamma_j\left(1-\sigma_j^z\tau_j^z\right) \\
&+\Gamma\left(\frac{1+\mu}{2}\sigma_1^+\tau_1^+ + \frac{1-\mu}{2}\sigma_1^-\tau_1^-\right) - \frac{\Gamma}{2} + \frac{\Gamma\mu}{4}(\sigma_1^z+\tau_1^z) \\
&+\Gamma'\left(\frac{1+\mu'}{2}\sigma_N^+\tau_N^+ + \frac{1-\mu'}{2}\sigma_N^-\tau_N^-\right) - \frac{\Gamma'}{2} + \frac{\Gamma'\mu'}{4}(\sigma_N^z+\tau_N^z)
\end{aligned} \tag{10}$$

C. Spectral Decomposition into eigenvalues and eigenstates

It is convenient to use the bra-ket notations to denote the Right and Left eigenvectors associated to the 4^N eigenvalues λ_n

$$\begin{aligned}
\mathcal{L}|\lambda_n^R> &= \lambda_n|\lambda_n^R> \\
<\lambda_n^L|\mathcal{L} &= \lambda_n <\lambda_n^L|
\end{aligned} \tag{11}$$

with the orthonormalization

$$<\lambda_n^L|\lambda_m^R> = \delta_{nm} \tag{12}$$

and the identity decomposition

$$1 = \sum_{n=0}^{4^N-1}|\lambda_n^R><\lambda_n^L| \tag{13}$$

The spectral decomposition of the Lindbladian

$$\mathcal{L} = \sum_{n=0}^{4^N-1}\lambda_n|\lambda_n^R><\lambda_n^L| \tag{14}$$

yields the solution for the dynamics in terms of the initial condition at $t=0$

$$|\rho(t)> = \sum_{n=0}^{4^N-1}e^{\lambda_n t}|\lambda_n^R><\lambda_n^L|\rho(t=0)> \tag{15}$$

The trace of the density matrix $\rho(t)$ corresponds in the Ladder Formulation to

$$Trace(\rho(t)) = \sum_{S_1=\pm1}...\sum_{S_N=\pm1}\rho_{S_1,..,S_N;S_1,...,S_N}(t) = \sum_{S_1=\pm1}...\sum_{S_N=\pm1}<S_1,..,S_N|\otimes<S_1,...,S_N|\rho(t)> \tag{16}$$

The conservation of $Trace(\rho(t))$ by the dynamics means that the eigenvalue

$$\lambda_0 = 0 \tag{17}$$

is associated to the Left eigenvector

$$<\lambda_0^L| = \sum_{S_1=\pm1}...\sum_{S_N=\pm1}<S_1,..,S_N|\otimes<S_1,...,S_N| \tag{18}$$

while the corresponding Right Eigenvector corresponds to the steady state towards which any initial condition will converges

$$|\rho(t\to+\infty)> = |\lambda_0^R> \tag{19}$$

The other (4^N-1) eigenvalues $\lambda_{n\neq0}$ with negative real parts describe the relaxation towards this steady state.

II. EFFECTIVE LINDBLADIAN FOR STRONG DISORDER OR STRONG DEPHASING

A. Perturbation in the couplings J_j

In this section, we consider that the terms of the Lindbladian containing the couplings J_j

$$\mathcal{L}^{per} = i \sum_{j=1}^{N-1} 2J_j(\tau_j^+\tau_{j+1}^- + \tau_j^-\tau_{j+1}^+ - \sigma_j^+\sigma_{j+1}^- - \sigma_j^-\sigma_{j+1}^+) \tag{20}$$

can be treated perturbatively with respect to the other terms of the Lindbladian that do not couple the rungs of the ladder

$$\mathcal{L}^{unper} = \sum_{j=1}^{N} \mathcal{L}_j^{unper} \tag{21}$$

The Lindbladians associated to the rungs of the bulk $j = 2, .., N-1$ read

$$\mathcal{L}_j^{unper} = ih_j(\tau_j^z - \sigma_j^z) - \gamma_j \left(1 - \sigma_j^z\tau_j^z\right) \tag{22}$$

while for the two end-spins, they contain the additional contribution of the boundary-driving

$$\mathcal{L}_{j=1}^{unper} = ih_1(\tau_1^z - \sigma_1^z) - \gamma_1 \left(1 - \sigma_1^z\tau_1^z\right) + \Gamma \left(\frac{1+\mu}{2}\sigma_1^+\tau_1^+ + \frac{1-\mu}{2}\sigma_1^-\tau_1^- \right) - \frac{\Gamma}{2} + \frac{\Gamma\mu}{4}(\sigma_1^z + \tau_1^z) \tag{23}$$

and

$$\mathcal{L}_{j=N}^{unper} = ih_N(\tau_N^z - \sigma_N^z) - \gamma_N \left(1 - \sigma_N^z\tau_N^z\right) + \Gamma' \left(\frac{1+\mu'}{2}\sigma_N^+\tau_N^+ + \frac{1-\mu'}{2}\sigma_N^-\tau_N^- \right) - \frac{\Gamma'}{2} + \frac{\Gamma'\mu'}{4}(\sigma_N^z + \tau_N^z) \tag{24}$$

In the absence of boundary-drivings, this type of perturbation theory has been developed previously for the pure XXZ chain without fields as a 'strong dissipation' approximation [33, 34], and in XXZ-chain with random fields as a 'strong disorder' approximation [11], in order to analyze the relaxation properties towards the trivial maximally mixed steady-state. Note that in both cases, the J^z-coupling turns out to disappear at leading order in this perturbation theory [11, 33], and this is why we have chosen to consider here the case $J_i^z = 0$ from the very beginning (Eq 2) in order to simplify the presentation. In the following, we describe how the perturbation theory developed in [11, 33, 34] has to be adapted to the presence of the boundary-drivings.

B. Spectral decomposition of $\mathcal{L}_1^{unper}$

The four eigenvalues of the Lindbladian $\mathcal{L}_1^{unper}$ of Eq. 23 and the corresponding Left and Right Eigenvectors written in the basis (σ_1^z, τ_1^z) are :

(0) The eigenvalue $\lambda_{1,n=0} = 0$ is associated to

$$\begin{aligned} < \lambda_{1,n=0}^L| &= < ++|+<--| \\ |\lambda_{1,n=0}^R> &= \frac{1+\mu}{2}|++> + \frac{1-\mu}{2}|--> \end{aligned} \tag{25}$$

(1) The eigenvalue $\lambda_{1,n=1} = -\Gamma$ is associated to

$$\begin{aligned} < \lambda_{1,n=1}^L| &= \frac{1-\mu}{2} < ++| - \frac{1+\mu}{2} < --| \\ |\lambda_{1,n=1}^R> &= |++> - |--> \end{aligned} \tag{26}$$

(2) The eigenvalue $\lambda_{1,n=2} = -\frac{\Gamma}{2} - 2\gamma_1 + i2h_1$ is associated to

$$\begin{aligned} < \lambda_{1,n=2}^L| &= < -+| \\ |\lambda_{1,n=2}^R> &= |-+> \end{aligned} \tag{27}$$

(4) The eigenvalue $\lambda_{1,n=3} = -\frac{\Gamma}{2} - 2\gamma_1 - i2h_1$ is associated to

$$\begin{aligned} < \lambda_{1,n=3}^L| &= < +-| \\ |\lambda_{1,n=3}^R> &= |+-> \end{aligned} \tag{28}$$

C. Spectral decomposition of $\mathcal{L}_N^{unper}$

Similarly, the four eigenvalues of the Lindbladian $\mathcal{L}_N^{unper}$ of Eq. 24 and the corresponding Left and Right Eigenvectors written in the basis (σ_N^z, τ_N^z) are :

(0) The eigenvalue $\lambda_{N,m=0} = 0$ is associated to

$$\begin{aligned} < \lambda_{N,m=0}^L | \quad &= < + + | + < - - | \\ | \lambda_{N,m=0}^R > \quad &= \frac{1+\mu'}{2} | + + > + \frac{1-\mu'}{2} | - - > \end{aligned} \tag{29}$$

(1) The eigenvalue $\lambda_{N,m=1} = -\Gamma'$ is associated to

$$\begin{aligned} < \lambda_{N,m=1}^L | \quad &= \frac{1-\mu'}{2} < + + | - \frac{1+\mu'}{2} < - - | \\ | \lambda_{N,m=1}^R > \quad &= | + + > - | - - > \end{aligned} \tag{30}$$

(2) The eigenvalue $\lambda_{N,m=2} = -\frac{\Gamma'}{2} - 2\gamma_N + i2h_N$ is associated to

$$\begin{aligned} < \lambda_{N,m=2}^L | \quad &= < - + | \\ | \lambda_{N,m=2}^R > \quad &= | - + > \end{aligned} \tag{31}$$

(4) The eigenvalue $\lambda_{N,m=3} = -\frac{\Gamma'}{2} - 2\gamma_N - i2h_N$ is associated to

$$\begin{aligned} < \lambda_{N,m=3}^L | \quad &= < + - | \\ | \lambda_{N,m=3}^R > \quad &= | + - > \end{aligned} \tag{32}$$

D. Spectral decomposition of $\mathcal{L}_j^{unper}$ for $j = 2, .., N-1$

The Lindbladian operator $\mathcal{L}_j^{unper}$ (Eq. 22) in the bulk $j = 2, .., N-1$ is diagonal in the (σ_j^z, τ_j^z) basis : the Left and Right Eigenvectors are simply $< S_j, T_j |$ and $| S_j, T_j >$ with the eigenvalues

$$\lambda_{j,S_j,T_j} = ih_j(T_j - S_j) - \gamma_j (1 - S_j, T_j) \tag{33}$$

So the vanishing eigenvalue is degenerate twice

$$\lambda_{j,+,+} = \lambda_{j,-,-} = 0 \tag{34}$$

while the two others eigenvalue read

$$\begin{aligned} \lambda_{j,+,-} &= -2\gamma_j - i2h_j \\ \lambda_{j,-,+} &= -2\gamma_j + i2h_j \end{aligned} \tag{35}$$

E. Spectral decomposition of $\mathcal{L}^{unper}$

The unperturbed Lindbladian of Eq 21 is the sum of the independent Lindbladians discussed above. So its eigenvalues are simply given by the sum of eigenvalues

$$\lambda^{unper}_{(1,n);(j,S_j,T_j);(N,m)} \quad = \lambda_{1,n} + \sum_{j=2}^{N-1} \lambda_{j,S_j,T_j} + \lambda_{N,m} \tag{36}$$

while the left and right eigenvectors are given by the corresponding tensor-products.

In particular the vanishing eigenvalue $\lambda^{unper} = 0$ is very degenerate, and the corresponding subspace of dimension 2^{N-1} is described by the projector

$$\mathcal{P}_0 \quad = \sum_{S_2,..,S_{N-1}} |\lambda_{1,n=0}^R > \otimes_{j=2}^{N-1} | S_j, T_j = S_j > \otimes | \lambda_{N,m=0}^R > < \lambda_{1,n=0}^L | \otimes_{j=2}^{N-1} < S_j, T_j = S_j | \otimes < \lambda_{N,m=0}^L | \tag{37}$$

F. Perturbation theory within the degenerate subspace associated to $\lambda^{unper}=0$

Within the degenerate subspace of dimension 2^{N-1} associated to $\lambda^{unper}=0$, the effective dynamics is described by the operator obtained by the second-order perturbation formula [11, 33, 34]

$$W \equiv \mathcal{L}^{(2^d order)}_{\lambda^{unper}=0} = \mathcal{P}_0 \mathcal{L}^{per}(1-\mathcal{P}_0)\frac{1}{0-\mathcal{L}^{unper}}(1-\mathcal{P}_0)\mathcal{L}^{per}\mathcal{P}_0 \tag{38}$$

The action of the perturbation $\mathcal{L}^{per}$ of Eq. 20 on the Left-Eigenvectors

$$\begin{aligned}
&< \lambda^L_{1,n=0}|\otimes_{j=2}^{N-1} < S_j=\eta_j, T_j=\eta_j|\otimes < \lambda^L_{N,m=0}|\mathcal{L}^{per} \\
&= i2J_1\left(\delta_{\eta_2=+}-\delta_{\eta_2=-}e^{-s}\right)\left(< \lambda^L_{1,n=3}|\otimes < S_2=-,T_2=+|- < \lambda^L_{1,n=2}|\otimes < S_2=+,T_2=-|\right) \\
&\otimes_{j=3}^{N-1} < S_j=\eta_j, T_j=\eta_j|\otimes < \lambda^R_{N,m=0}| \\
&+i\sum_{k=2}^{N-2} 2J_k\delta_{\eta_{k+1}=-\eta_k} < \lambda^L_{1,n=0}|\otimes_{j=2}^{k-1} < S_j=\eta_j, T_j=\eta_j| \\
&(< S_k=\eta_k, T_k=-\eta_k|\otimes < S_{k+1}=-\eta_k, T_{k+1}=\eta_k|- < S_k=-\eta_k, T_k=\eta_k|\otimes < S_{k+1}=\eta_k, T_{k+1}=-\eta_k|) \\
&\otimes_{j=k+2}^{N-1} < S_j=\eta_j, T_j=\eta_j|\otimes < \lambda^L_{N,m=0}| \\
&+i2J_{N-1} < \lambda^R_{1,n=0}|\otimes_{j=2}^{N-2} < S_j=\eta_j, T_j=\eta_j|\otimes \\
&\left(\delta_{\eta_{N-1}=+}-\delta_{\eta_{N-1}=-}\right)\left(< S_{N-1}=+,T_{N-1}=-|\otimes < \lambda^L_{N,m=2}|- < S_{N-1}=-,T_{N-1}=+|\otimes < \lambda^L_{N,m=3}|\right)
\end{aligned} \tag{39}$$

and on the right eigenvectors

$$\begin{aligned}
&\mathcal{L}^{per}|\lambda^R_{1,n=0}> \otimes_{j=2}^{N-1}|S_j=\eta'_j, T_j=\eta'_j> \otimes|\lambda^R_{N,m=0}> \\
&= i2J_1\left(\delta_{\eta'_2=+}\frac{1-\mu}{2}-\delta_{\eta'_2=-}e^{s}\frac{1+\mu}{2}\right)\left(|\lambda^R_{1,n=3}>\otimes|S_2=-,T_2=+> -|\lambda^R_{1,n=2}>\otimes|S_2=+,T_2=->\right) \\
&\otimes_{j=3}^{N-1}|S_j=\eta'_j, T_j=\eta'_j> \otimes|\lambda^R_{N,m=0}> \\
&+i\sum_{k=1}^{N-2} 2J_k\delta_{\eta'_{k+1}=-\eta'_k}|\lambda^R_{1,n=0}> \otimes_{j=2}^{k-1}|S_j=\eta'_j, T_j=\eta'_j> \\
&(|S_k=\eta'_k, T_k=-\eta'_k>\otimes|S_{k+1}=-\eta'_k, T_{k+1}=\eta'_k> -|S_k=-\eta'_k, T_k=\eta'_k>\otimes|S_{k+1}=\eta'_k, T_{k+1}=-\eta'_k>) \\
&\otimes_{j=k+2}^{N-1}|S_j=\eta'_j, T_j=\eta'_j> \otimes|\lambda^R_{N,m=0}> \\
&+i2J_{N-1}|\lambda^R_{1,n=0}> \otimes_{j=2}^{N-2}|S_j=\eta'_j, T_j=\eta'_j> \left(\delta_{\eta'_{N-1}=+}\frac{1-\mu'}{2}-\delta_{\eta'_{N-1}=-}\frac{1+\mu'}{2}\right) \\
&\left(|S_{N-1}=+,T_{N-1}=->\otimes|\lambda^R_{N,m=2}> -|S_{N-1}=-,T_{N-1}=+>\otimes|\lambda^R_{N,m=3}>\right)
\end{aligned} \tag{40}$$

determine the intermediate unperturbed states that appear in the perturbative formula of Eq. 38. Using the corresponding unperturbed eigenvalues of Eq. 36 that appear in the denominators, one finally obtains that the effective operator W reads in terms of Pauli matrices

$$\begin{aligned}
W \quad &= D_{1,2}\left(\frac{1+\mu}{2}\sigma_2^+ + \frac{1-\mu}{2}\sigma_2^- - \frac{1-\mu\sigma_2^z}{2}\right) \\
&+\sum_{k=2}^{N-2} D_{k,k+1}\left(\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+ - \frac{1-\sigma_k^z\sigma_{k+1}^z}{2})\right) \\
&+D_{N-1,N}\left(\frac{1+\mu'}{2}\sigma_{N-1}^+ + \frac{1-\mu'}{2}\sigma_{N-1}^- - \frac{1-\mu'\sigma_{N-1}^z}{2}\right)
\end{aligned} \tag{41}$$

where we have introduced the notations

$$\begin{aligned}
D_{k,k+1} \quad &\equiv \frac{4J_k^2(\gamma_k+\gamma_{k+1})}{(\gamma_k+\gamma_{k+1})^2+(h_k-h_{k+1})^2} \quad \text{for} \quad k=2,..,N-2 \\
D_{1,2} \quad &\equiv \frac{4J_1^2(\Gamma+4(\gamma_1+\gamma_2))}{\left(\frac{\Gamma}{2}+2(\gamma_1+\gamma_2)\right)^2+4(h_1-h_2)^2} \\
D_{N-1,N} \quad &\equiv \frac{4J_{N-1}^2(\Gamma'+4(\gamma_{N-1}+\gamma_N))}{\left(\frac{\Gamma'}{2}+2(\gamma_{N-1}+\gamma_N)\right)^2+4(h_{N-1}-h_N)^2}
\end{aligned} \tag{42}$$

The above approach is consistent if the parameters $D_{k,k+1}$ obtained by the second-order perturbation theory are indeed small, i.e. in particular in the limit of strong bulk dephasing even in the absence of disorder [33, 34] or in the limit of strong disorder in the random fields [11].

G. Summary : mapping onto a classical exclusion process with disorder

Let us now summarize the output of the above calculations. The ket $|\rho(t)>$ of the spin ladder of length N with an Hilbert space of dimension 4^N has been projected onto the ket $|P(t)>$ of a spin chain of $(N-1)$ spins with an Hilbert space of dimension dimension 2^{N-1} that represent the the diagonal elements

$$< S_2, .., S_{N-1}|P(t)>=< \lambda^L_{1,n=0}|\otimes_{j=2}^{N-1} < S_j, T_j = S_j|\otimes < \lambda^L_{N,m=0}|\rho(t)> \tag{43}$$

The Lindbladian that was acting on the ket $|\rho(t)>$ has been projected onto the effective operator W of Eq. 41 that governs the dynamics of the ket $|P(t)>$

$$\frac{\partial |P_t>}{\partial t} = W|P_t> \tag{44}$$

The spectral decomposition

$$W = \sum_{n=0}^{2^{N-1}-1} w_n |w_n^R >< w_n^L| \tag{45}$$

that allows to rewrite the solution of the dynamics as

$$|P_t >= \sum_{n=0}^{2^{N-1}-1} e^{w_n} |w_n^R >< w_n^L|P_{t=0}> \tag{46}$$

has the same properties as the spectral decomposition of the Lindbladian : the vanishing eigenvalue $w_{n=0} = 0$ is associated to the Left Eigenvector

$$< w^L_{n=0}| = \sum_{S_2,..,S_{N-1}} < S_2, .., S_{N-1}| \tag{47}$$

that encodes the conservation of probability

$$\sum_{S_2,..,S_{N-1}} < S_2, .., S_{N-1}|P_t >= 1 \tag{48}$$

while the corresponding Right Eigenvector $|w_n^R>$ corresponds to the non-equilibrium steady state towards which any initial condition converges

$$|P_{t\to+\infty} >= |w^R_{n=0} > \tag{49}$$

The other modes $n \neq 0$ describe the relaxation towards this steady state.

In [11, 33, 34], the operator W of Eq. 41 was written as minus the quantum Heisenberg ferromagnetic Hamiltonian

$$-W \quad = H^{eff} = \sum_{k=1}^{N-1} D_{k,k+1} \left(\frac{1 - \vec{\sigma}_k . \vec{\sigma}_{k+1}}{2} \right) \tag{50}$$

to derive various consequences. In our present case, we will keep the writing of Eq. 41 and interpret it as a classical Master Equation describing a Simple Symmetric Exclusion Process with quenched disorder in the local diffusion coefficients $D_{k,k+1}$ (Eq 42). The pure Simple Symmetric Exclusion Process with uniform $D_{k,k+1} = 1$ is one of the standard model in the field of non-equilibrium classical systems (see the review [32] and references therein). The effects of quenched disorder on *totally or partially asymmetric* exclusion models have been analyzed in [51–55]. In our present case, it is very important to stress that the disorder is in the local diffusion coefficients $D_{k,k+1}$, but that the *symmetry between the jumps from k to $(k+1)$ or from $(k+1)$ to k* is maintained.

H. Dynamics of observables

As a final remark, let us mention how one can study the dynamics of observables. The average at time t of the observable associated to the operator A

$$< A >_t = \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}|A|P_t > = < w_{n=0}^L|A|P_t > \tag{51}$$

evolves in time according to the dynamical equation

$$\frac{\partial < A >_t}{\partial t} = < w_{n=0}^L|AW|P_t > = < w_{n=0}^L|[A,W]|P_t > = < [A,W] >_t \tag{52}$$

where the commutator has been introduced using the property $< w_{n=0}^L|W = 0$ of the Left eigenvector. In the following sections, we analyze the properties of the non-equilibrium-steady-state (NESS) in each disordered sample, via the magnetizations, the correlations and the statistics of the current.

III. LOCAL MAGNETIZATIONS AND LOCAL CURRENTS

A. Dynamics of the local magnetizations

The dynamics of the magnetization on site j is described by Eq. 52 for $A = \sigma_j^z$

$$\frac{\partial < \sigma_j^z >_t}{\partial t} = < [\sigma_j^z, W] >_t = < \mathcal{I}_{j-1,j} - \mathcal{I}_{j,j+1} >_t \tag{53}$$

that involves the current operators associated to the bonds $(j, j+1)$

$$\mathcal{I}_{j,j+1} = -[\sigma_j^z, D_{j,j+1}\left(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+\right)] = 2D_{j,j+1}\left(\sigma_j^-\sigma_{j+1}^+ - \sigma_j^+\sigma_{j+1}^-\right) \tag{54}$$

From the definition of the average in Eq. 51, one obtains that the average of the current simplifies into

$$\begin{aligned}
< \mathcal{I}_{j,j+1} >_t &= \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}|\mathcal{I}_{j,j+1}|P_t > \\
&= 2D_{j,j+1} \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}|\left(\sigma_j^-\sigma_{j+1}^+ - \sigma_j^+\sigma_{j+1}^-\right)|P_t > \\
&= 2D_{j,j+1} < \left(\frac{1+\sigma_j^z}{2}\right)\left(\frac{1-\sigma_{j+1}^z}{2}\right) - \left(\frac{1-\sigma_j^z}{2}\right)\left(\frac{1+\sigma_{j+1}^z}{2}\right) >_t \\
&= D_{j,j+1} < \left(\sigma_j^z - \sigma_{j+1}^z\right) >_t
\end{aligned} \tag{55}$$

i.e locally on each bond, the averaged current is proportional to the difference of magnetizations with a prefactor given by the local diffusion coefficient $D_{j,j+1}$.

B. Current in the Non-Equilibrium-Steady-State

In the non-equilibrium steady state, the magnetizations

$$\mu_j \equiv < \sigma_j^z >_{ness} \tag{56}$$

and the currents

$$I_{j,j+1} \equiv < \mathcal{I}_{j,j+1} >_{ness} \quad = D_{j,j+1}(\mu_j - \mu_{j+1}) \tag{57}$$

are constrained by the conservation of the current along the chain (Eq 53)

$$I = I_{j,j+1} = D_{j,j+1}(\mu_j - \mu_{j+1}) \tag{58}$$

Since the magnetization are fixed at the two boundaries

$$\begin{aligned} \mu_1 &= \mu \\ \mu_N &= \mu' \end{aligned} \tag{59}$$

the current I is simply obtained from the sum of the differences of magnetizations along the chain

$$\mu - \mu' = \sum_{i=1}^{N-1} (\mu_i - \mu_{i+1}) = I \sum_{i=1}^{N-1} \frac{1}{D_{j,j+1}} \tag{60}$$

leading to

$$I = \frac{\mu - \mu'}{\sum_{i=1}^{N-1} \frac{1}{D_{j,j+1}}} \tag{61}$$

The denominator reads more explicitly in terms of the initial variables (Eq 42)

$$\begin{aligned} \sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}} &= \frac{\left(\frac{\Gamma}{2} + 2(\gamma_1 + \gamma_2)\right)^2 + 4(h_1 - h_2)^2}{4J_1^2(\Gamma + 4(\gamma_1 + \gamma_2))} + \frac{\left(\frac{\Gamma'}{2} + 2(\gamma_{N-1} + \gamma_N)\right)^2 + 4(h_{N-1} - h_N)^2}{4J_{N-1}^2(\Gamma' + 4(\gamma_{N-1} + \gamma_N))} \\ &+ \sum_{k=2}^{N-2} \frac{(\gamma_k + \gamma_{k+1})^2 + (h_k - h_{k+1})^2}{4J_k^2(\gamma_k + \gamma_{k+1})} \end{aligned} \tag{62}$$

In the limit of large size $N \to +\infty$, this sum will grow extensively in the size N

$$\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}} \underset{N \to +\infty}{\propto} N \overline{\left(\frac{1}{D_{k,k+1}} \right)} + O(N^{\frac{1}{2}}) \tag{63}$$

as long as the disorder-averaged value of the inverse of the local diffusion coefficient converges

$$\overline{\left(\frac{1}{D_{k,k+1}} \right)} = \overline{\left(\frac{(\gamma_k + \gamma_{k+1})^2 + (h_k - h_{k+1})^2}{4J_k^2(\gamma_k + \gamma_{k+1})} \right)} < +\infty \tag{64}$$

Then the current of Eq. 61 will decay as $1/N$ as in the usual Fourier-Fick law.

C. Magnetization profile in the Non-Equilibrium-Steady-State

The corresponding magnetization profile reads (Eq 58)

$$\mu_j = \frac{\mu \left(\sum_{k=j}^{N-1} \frac{1}{D_{k,k+1}} \right) + \mu' \left(\sum_{k=1}^{j-1} \frac{1}{D_{k,k+1}} \right)}{\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}}} \tag{65}$$

that generalizes the usual linear profile of the pure case without disorder

$$\mu_j^{pure} = \frac{\mu(N-j) + \mu'(j-1)}{(N-1)} \tag{66}$$

IV. TWO-POINT CORRELATIONS

A. Dynamics of the two-point correlations

For the two-point correlation

$$C_{i,j}(t) \equiv < \sigma_i^z \sigma_j^z >_t \tag{67}$$

the dynamical equation (Eq 52) reads for $i < j-1$

$$\begin{aligned} \frac{\partial C_{i,j}(t)}{\partial t} &= < [\sigma_i^z \sigma_j^z, W] >_t \\ &= < (\mathcal{I}_{i-1,i} - \mathcal{I}_{i,i+1}) \sigma_j^z >_t + < \sigma_i^z (\mathcal{I}_{j-1,j} - \mathcal{I}_{j,j+1}) >_t \\ &= D_{i-1,i}(C_{i-1,j}(t) - C_{i,j}(t)) - D_{i,i+1}(C_{i,j}(t) - C_{i+1,j}(t)) \\ &+ D_{j-1,j}(C_{i,j-1}(t) - C_{i,j}(t)) - D_{j,j+1}(C_{i,j}(t) - C_{i,j+1}(t)) \end{aligned} \tag{68}$$

and for two neighbors

$$\begin{aligned} \frac{\partial C_{i,i+1}(t)}{\partial t} &= < [\sigma_i^z \sigma_{i+1}^z, W] >_t = < \mathcal{I}_{i-1,i} \sigma_{i+1}^z >_t - < \sigma_i^z \mathcal{I}_{i+1,i+2} >_t \\ &= D_{i-1,i}(C_{i-1,i+1}(t) - C_{i,i+1}(t)) - D_{i+1,i+2}(C_{i,i+1}(t) - C_{i,i+2}(t)) \end{aligned} \tag{69}$$

B. Two-point correlation in the Non-Equilibrium-Steady-State

In the Non-Equilibrium-Steady-State, the correlations have to satisfy linear interpolation formula for fixed j

$$C_{i,j} = \frac{C_{1,j} \left(\sum_{k=i}^{j-2} \frac{1}{D_{k,k+1}} \right) + C_{j-1,j} \left(\sum_{k=1}^{i-1} \frac{1}{D_{k,k+1}} \right)}{\sum_{k=1}^{j-2} \frac{1}{D_{k,k+1}}} \quad \text{for} \quad 1 \le i \le j-1 \tag{70}$$

and for fixed i

$$C_{i,j} = \frac{C_{i,i+1} \left(\sum_{k=j}^{N-1} \frac{1}{D_{k,k+1}} \right) + C_{i,N} \left(\sum_{k=i+1}^{j-1} \frac{1}{D_{k,k+1}} \right)}{\sum_{k=i+1}^{N-1} \frac{1}{D_{k,k+1}}} \quad \text{for} \quad i+1 \le j \le N \tag{71}$$

where the correlation between two neighbors have to satisfy

$$C_{i,i+1} = \frac{C_{1,i+1} \left(\sum_{k=i+1}^{N-1} \frac{1}{D_{k,k+1}} \right) + C_{i,N} \left(\sum_{k=1}^{i-1} \frac{1}{D_{k,k+1}} \right)}{\left(\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}} \right) - \frac{1}{D_{i,i+1}}} \tag{72}$$

The correlations with the fixed boundary-spins $\sigma_1^z = \mu$ and $\sigma_N^z = \mu'$ can be obtained from the magnetization profile of Eq. 65

$$C_{1,j} = \mu \mu_j = \mu \frac{\mu \left(\sum_{k=j}^{N-1} \frac{1}{D_{k,k+1}} \right) + \mu' \left(\sum_{k=1}^{j-1} \frac{1}{D_{k,k+1}} \right)}{\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}}} \tag{73}$$

and

$$C_{i,N} = \mu' \mu_i = \mu' \frac{\mu \left(\sum_{k=i}^{N-1} \frac{1}{D_{k,k+1}} \right) + \mu' \left(\sum_{k=1}^{i-1} \frac{1}{D_{k,k+1}} \right)}{\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}}} \tag{74}$$

Putting everything together, one finally obtains the connected correlation function for $i < j$

$$C^c_{i,j} \equiv C_{i,j} - \mu_i \mu_j \tag{75}$$
$$= -(\mu - \mu') \frac{\left(\sum_{k=1}^{i-1} \frac{1}{D_{k,k+1}} \right) \left(\sum_{k=j}^{N-1} \frac{1}{D_{k,k+1}} \right)}{\left(\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}} \right)^2} \left[\frac{\mu}{D_{i,i+1} \left(\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}} - \frac{1}{D_{i,i+1}} \right)} - \frac{\mu'}{D_{j-1,j} \left(\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}} - \frac{1}{D_{j-1,j}} \right)} \right]$$

This formula generalizes the known expressions for the connected correlations in the pure case $D_{k,k+1} = 1$ [56, 57]

$$[C^c_{i,j}]^{pure} = -(\mu - \mu')^2 \frac{(i-1)(N-j)}{(N-1)^2 (N-2)} \tag{76}$$

The important property is that any non-equilibrium case $\mu \neq \mu'$ is characterized by correlations that are weak in amplitude but long-ranged (see the review [32] and references therein).

For two neighbors, Eq. 75 simplifies into

$$C^c_{i,i+1} \equiv C_{i,i+1} - \mu_i \mu_{i+1} = -(\mu - \mu')^2 \frac{\left(\sum_{k=1}^{i-1} \frac{1}{D_{k,k+1}} \right) \left(\sum_{k=i+1}^{N-1} \frac{1}{D_{k,k+1}} \right)}{D_{i,i+1} \left(\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}} \right)^2 \left(\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}} - \frac{1}{D_{i,i+1}} \right)} \tag{77}$$

This result will be useful in the next section to compute the fluctuations of the integrated current.

V. CURRENT FLUCTUATIONS

For the Simple Symmetric Exclusion Process without disorder, the current fluctuations have been studied in [58]. In particular, the variance of the integrated current can be obtained from conservation rules [58]. In this section, we describe how this method can be adapted in the presence of disorder.

A. Integrated current Q_k on a given link

In order to keep the information on the integrated current Q_k on the link $(k, k+1)$ during $[0, t]$, we need to decompose the ket at time t into a sum over the possible values of Q_k

$$|P_t> = \sum_{Q_k} |P_t(Q_k)> \tag{78}$$

and to write the dynamics of these components

$$\frac{\partial |P_t(Q_k)>}{\partial t} = W_k^+ |P_t(Q_k - 2)> + W_k^- |P_t(Q_k + 2)> + (W - W_k^+ - W_k^-) |P_t(Q_k)> \tag{79}$$

where W is the full operator of Eq. 41, and where the contributions corresponding to the increase or the decrease of the integrated current Q_k are

$$\begin{aligned} W_k^+ &= D_{k,k+1} \sigma_k^- \sigma_{k+1}^+ \\ W_k^- &= D_{k,k+1} \sigma_k^+ \sigma_{k+1}^- \end{aligned} \tag{80}$$

In particular, the average of the integrated current

$$< Q_k >_t = \sum_{Q_k} Q_k \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}|P_t(Q_k) > \tag{81}$$

evolves according to (using the probability conservation $\sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}|W=0)$

$$\begin{aligned}
&\frac{\partial < Q_k >_t}{\partial t} \\
&= \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}| \left(W_k^+ \sum_{Q_k} Q_k (|P_t(Q_k-2)> - |P_t(Q_k)>) + W_k^- \sum_{Q_k} Q_k (|P_t(Q_k+2)> - |P_t(Q_k)>) \right) \\
&= \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}| (2(W_k^+ - W_k^-) \sum_{Q_k} |P_t(Q_k)> \\
&= < 2D_{k,k+1}(\sigma_k^- \sigma_{k+1}^+ - \sigma_k^+ \sigma_{k+1}^-) >_t = < \mathcal{I}_{k,k+1} >_t
\end{aligned} \tag{82}$$

i.e. one obtains the average of the current operator of Eq. 54 as it should.

The average of the square

$$< Q_k^2 >_t = \sum_{Q_k} Q_k^2 \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}|P_t(Q_k)> \tag{83}$$

evolves according to

$$\begin{aligned}
&\frac{\partial < Q_k^2 >_t}{\partial t} \\
&= \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}| \left(W_k^+ \sum_{Q_k} Q_k^2 (|P_t(Q_k-2)> - |P_t(Q_k)>) + W_k^- \sum_{Q_k} Q_k^2 (|P_t(Q_k+2)> - |P_t(Q_k)>) \right) \\
&= \sum_{Q_k} \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}| \left(4(W_k^+ + W_k^-) + 4(W_k^+ - W_k^-) Q_k \right) |P_t(Q_k)> \\
&= 4D_{k,k+1} < (\sigma_k^- \sigma_{k+1}^+ + \sigma_k^+ \sigma_{k+1}^-) >_t + 2 < \mathcal{I}_{k,k+1} Q_k >_t
\end{aligned} \tag{84}$$

The first term can be written in terms of the two-point magnetization-correlation (using Eq. 51 and Eq. 55)

$$\begin{aligned}
< (\sigma_k^- \sigma_{k+1}^+ + \sigma_k^+ \sigma_{k+1}^-) >_t &= \sum_{S_2,..,S_{N-1}} < S_2,..,S_{N-1}| \left(\sigma_k^- \sigma_{k+1}^+ + \sigma_k^+ \sigma_{k+1}^- \right) |P_t> \\
&= < \left(\frac{1+\sigma_k^z}{2} \right) \left(\frac{1-\sigma_{k+1}^z}{2} \right) + \left(\frac{1-\sigma_k^z}{2} \right) \left(\frac{1+\sigma_{k+1}^z}{2} \right) >_t \\
&= < \frac{1-\sigma_k^z \sigma_{k+1}^z}{2} >_t
\end{aligned} \tag{85}$$

The second term of Eq. 84 involves the correlation between the current $\mathcal{I}_{k,k+1}$ and the integrated current Q_k. Since Eq. 82 yields

$$\frac{\partial < Q_k >_t^2}{\partial t} = 2 < Q_k >_t \frac{\partial < Q_k >_t}{\partial t} = 2 < Q_k >_t < \mathcal{I}_{k,k+1} >_t \tag{86}$$

one obtains from the difference with Eq. 84 that the dynamics of the fluctuation of the integrated current Q_k involves the connected correlation the current $\mathcal{I}_{k,k+1}$ and the integrated current Q_k

$$F_k(t) \equiv \frac{\partial (< Q_k^2 >_t - < Q_k >_t^2)}{\partial t} = 2D_{k,k+1} < (1 - \sigma_k^z \sigma_{k+1}^z) >_t + 2(< \mathcal{I}_{k,k+1} Q_k >_t - < \mathcal{I}_{k,k+1} >_t < Q_k >_t) \tag{87}$$

Using again Eq. 51 and Eq. 55, one may rewrite Eq. 84 in terms of connected correlations between the integrated current Q_k on the link $(k, k+1)$ and the magnetizations $(\sigma_k^z, \sigma_{k+1})$ of the two spins connected to the link

$$\begin{aligned}
F_k(t) &= 2D_{k,k+1} < (1 - \sigma_k^z \sigma_{k+1}^z) >_t \\
&+ 2D_{k,k+1} \left[(< \sigma_k^z Q_k >_t - < \sigma_k^z >_t < Q_k >_t) - (< \sigma_{k+1}^z Q_k >_t - < \sigma_{k+1}^z >_t < Q_k >_t) \right]
\end{aligned} \tag{88}$$

For the special cases of the boundary links $k = 1$ and $k = N-1$ involving the fixed spins $\sigma_1^z \to \mu$ and $\sigma_N^z \to \mu'$, this simplifies into

$$\begin{aligned} F_1(t) &= 2D_{1,2}(1 - \mu < \sigma_2^z >_t) \\ &+2D_{1,2}\left[0 - (< \sigma_2^z Q_1 >_t - < \sigma_2^z >_t < Q_1 >_t)\right] \end{aligned} \tag{89}$$

and $k = N-1$

$$\begin{aligned} F_{N-1}(t) &= 2D_{N-1,N}(1 - < \sigma_{N-1}^z >_t \mu') \\ &+2D_{N-1,N}\left[(< \sigma_{N-1}^z Q_{N-1} >_t - < \sigma_{N-1}^z >_t < Q_{N-1} >_t) - 0\right] \end{aligned} \tag{90}$$

B. Comparison of the fluctuations on the different links

The integrated currents Q_{k-1} and Q_k on two neighboring links $(k-1, k)$ and $(k, k+1)$ are closely related since the total change of magnetization of the spin σ_k between them is given by their difference

$$\sigma_k^z(t) - \sigma_k^z(t=0) = Q_{k-1} - Q_k \tag{91}$$

In particular, since this difference remains bounded, the fluctuations $F_k(t)$ introduced above will become independent of k and independent of time in the steady-state reached at large time

$$F_k(t) \underset{t \to +\infty}{\simeq} F \tag{92}$$

and the goal is to compute this limit from observables in the steady-state.

From the structure of the system (Eqs 88, 89, 90), it is clear that Eq 91 will allow to simplify the following sum

$$\begin{aligned} \sum_{k=1}^{N-1} \frac{F_k(t)}{2D_{k,k+1}} &= \sum_{k=1}^{N-1} < (1 - \sigma_k^z \sigma_{k+1}^z) >_t \\ &+ \sum_{k=2}^{N-1} (< \sigma_k^z (Q_k - Q_{k-1}) >_t - < \sigma_k^z >_t < (Q_k - Q_{k-1}) >_t) \\ &= 1 + \sum_{k=2}^{N-1} < \sigma_k^z >_t^2 - \sum_{k=1}^{N-1} < \sigma_k^z \sigma_{k+1}^z >_t \\ &+ \sum_{k=2}^{N-1} (< \sigma_k^z(t) \sigma_k^z(t=0) > - < \sigma_k^z(t) > < (\sigma_k^z(t=0) >) \end{aligned} \tag{93}$$

C. Fluctuation F in the Non-Equilibrium-Steady-State

In the large-time limit $t \to +\infty$, the time-auto-correlation of the last line of Eq. 93 can be neglected, so that the common value F of the fluctuations (Eq. 92) can be computed from the knowledge of the magnetization and the two-point correlation in the steady state

$$\frac{F}{2}\left(\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}}\right) = 1 + \sum_{k=2}^{N-1} < \sigma_k^z >_{ness}^2 - \sum_{k=1}^{N-1} < \sigma_k^z \sigma_{k+1}^z >_{ness} \tag{94}$$

In terms of the magnetizations μ_j (Eq. 65) with the boundary conditions $\mu_1 = \mu$ and $\mu_N = \mu'$ and of the connected two-point correlation $C_{i,i+1}^c$ (Eq. 77), the fluctuation F reads

$$\frac{F}{2}\left(\sum_{k=1}^{N-1} \frac{1}{D_{k,k+1}}\right) = 1 - \frac{\mu^2 + (\mu')^2}{2} + \frac{1}{2}\sum_{i=1}^{N-1} (\mu_i - \mu_{i+1})^2 - \sum_{i=1}^{N-1} C_{i,i+1}^c \tag{95}$$

Using the difference of magnetizations on consecutive spins (Eq. 58) in terms of the current I of Eq. 61, the first sum simplify into

$$\frac{1}{2}\sum_{i=1}^{N-1}(\mu_i-\mu_{i+1})^2=\frac{I^2}{2}\sum_{i=1}^{N-1}\frac{1}{D_{i,i+1}^2}=\frac{(\mu-\mu')^2}{2\left(\sum_{k=1}^{N-1}\frac{1}{D_{k,k+1}}\right)^2}\sum_{i=1}^{N-1}\frac{1}{D_{i,i+1}^2} \tag{96}$$

while the sum of the connected correlation of Eq. 77 reads

$$-\sum_{i=1}^{N-1}C^c_{i,i+1}=\frac{(\mu-\mu')^2}{\left(\sum_{k=1}^{N-1}\frac{1}{D_{k,k+1}}\right)^2}\sum_{i=2}^{N-2}\frac{\left(\sum_{k=1}^{i-1}\frac{1}{D_{k,k+1}}\right)\left(\sum_{k=i+1}^{N-1}\frac{1}{D_{k,k+1}}\right)}{D_{i,i+1}\left(\sum_{k=1}^{N-1}\frac{1}{D_{k,k+1}}-\frac{1}{D_{i,i+1}}\right)} \tag{97}$$

so that the final result for the fluctuation F reads in terms of the boundary magnetizations (μ,μ') and in terms of the random diffusion coefficients D_k

$$F=\frac{2-\mu^2-(\mu')^2}{\left(\sum_{k=1}^{N-1}\frac{1}{D_{k,k+1}}\right)}+\frac{(\mu-\mu')^2}{\left(\sum_{k=1}^{N-1}\frac{1}{D_{k,k+1}}\right)^3}\left[\sum_{i=1}^{N-1}\frac{1}{D_{i,i+1}^2}+2\sum_{i=2}^{N-2}\frac{\left(\sum_{k=1}^{i-1}\frac{1}{D_{k,k+1}}\right)\left(\sum_{k=i+1}^{N-1}\frac{1}{D_{k,k+1}}\right)}{D_{i,i+1}\left(\sum_{k=1}^{N-1}\frac{1}{D_{k,k+1}}-\frac{1}{D_{i,i+1}}\right)}\right] \tag{98}$$

VI. CONCLUSION

In this paper, we have studied the Lindblad dynamics of the XX quantum chain with random fields h_j in the presence of two types of dissipative processes, namely dephasing in the bulk and magnetization-driving at the two boundaries. We have focused on the regime of strong disorder in the random fields [11], or in the regime of strong bulk-dephasing [33, 34], where the effective dynamics can be mapped via degenerate second-order perturbation theory in the couplings J_j onto a classical Simple Symmetric Exclusion Process with quenched disorder in the diffusion coefficient associated to each bond. We have then studied the properties of the corresponding Non-Equilibrium-Steady-State in each disordered sample between the two reservoirs by extending the methods that have been previously developed for the same model without disorder. We have given explicit results for the magnetization profile, for the two-point correlations, for the mean current and for the current fluctuations in terms of the random fields and couplings defining the disordered sample.

As expected, these results are completely different from the transport properties of the same model in the absence of bulk dephasing [16], where the quantum coherence of the bulk dynamics maintains the localized character via a step-magnetization profile and an exponentially decaying current with the system size [16].

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