



HAL
open science

Effective Theory of Interacting Dark Energy

Jérôme Gleyzes, David Langlois, Michele Mancarella, Filippo Vernizzi

► **To cite this version:**

Jérôme Gleyzes, David Langlois, Michele Mancarella, Filippo Vernizzi. Effective Theory of Interacting Dark Energy. *Journal of Cosmology and Astroparticle Physics*, 2015, 2015, pp.54 - 54. 10.1088/1475-7516/2015/08/054 . cea-01458940

HAL Id: cea-01458940

<https://cea.hal.science/cea-01458940>

Submitted on 7 Feb 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Effective Theory of Interacting Dark Energy

Jérôme Gleyzes^{a,b}, David Langlois^c,
Michele Mancarella^{a,b} and Filippo Vernizzi^a

^a *CEA, IPhT, 91191 Gif-sur-Yvette cédex, France*
CNRS, URA-2306, 91191 Gif-sur-Yvette cédex, France

^b *Université Paris Sud, 15 rue George Clémenceau, 91405, Orsay, France*

^c *APC, (CNRS-Université Paris 7), 10 rue Alice Domon et Léonie Duquet, 75205 Paris, France*

April 22, 2015

Abstract

We present a unifying treatment of dark energy and modified gravity that allows distinct conformal-disformal couplings of matter species to the gravitational sector. In this very general approach, we derive the conditions to avoid ghost and gradient instabilities. We compute the equations of motion for background quantities and linear perturbations. We illustrate our formalism with two simple scenarios, where either cold dark matter or a relativistic fluid is nonminimally coupled. This extends previous studies of coupled dark energy to a much broader spectrum of gravitational theories.

Contents

1	Introduction	3
2	Unifying description of dark energy with non universal couplings	5
2.1	Gravitational and matter actions	5
2.2	Homogeneous equations	6
2.3	Linear perturbations	7
2.4	Matter couplings and stability conditions	9
2.5	Disformal transformations	10
3	Matter equations of motion	12
3.1	Homogenous equations	13
3.2	Perturbation equations in Newtonian gauge	14
4	Baryons and coupled CDM	16
4.1	Linear perturbations	16
4.2	Quasi-static approximation	17
5	Matter and coupled relativistic fluid	20
6	Conclusions	21
A	Changing frame	22
A.1	Background	23
A.2	Perturbations	24
B	Explicit quadratic action	25
B.1	Matter action	25
B.2	Stability and sound speed of dark energy	26
C	Perturbation equations	27
C.1	Einstein equations	27
C.2	Scalar field equation	28
D	Synchronous gauge	29
E	Definitions of the parameters	31

1 Introduction

The nature of dark energy, responsible for the present cosmological acceleration, is a central topic in theoretical and observational cosmology. One of the main goals of current and future cosmic surveys is to constrain or possibly detect deviations from the standard Λ CDM scenario, induced by the presence of dark energy or modifications of General Relativity (GR) (see e.g. [1]). This is particularly relevant on scales above ~ 10 Mpc, where deviations from GR are not yet well tested. Fortunately, on these scales cosmological perturbations are still in the linear regime today and linear perturbation theory around a FLRW background is thus a valid description.

Given the plethora of existing dark energy and modified gravity models (see for instance [2,3]), it is worth resorting to an effective approach that tries to describe all possible deviations from Λ CDM in a simple and systematic way, relying on a minimal number of parameters. In the linear regime for perturbations, this task has been successfully undertaken for single scalar field models in [4, 5]. Initially inspired by the so-called Effective Field Theory of inflation [6,7] and minimally coupled dark energy [8], this approach relies on the construction of an effective action for linear perturbations. In order to do so, we start from a generic Lagrangian written in terms of Arnowitt-Deser-Misner (ADM) [9] quantities defined with respect to the uniform scalar field hypersurfaces (see also [10, 11] for an analogous approach, [12–14] for recent reviews and e.g. [15–17] for applications). After having been implemented in a public numerical code named EFTCAMB [18], most recently, it has been applied to constrain deviations from the standard cosmological constant scenario by the Planck collaboration [19].

The action developed in [5] contains five free functions of time that parametrize any deviation from Λ CDM. Four of these functions describe cosmological perturbations in Horndeski theories [20–22]. The fifth parameter describes deviations from GR encompassing Horndeski theories. Indeed, the same formalism was also instrumental to uncover the theories beyond Horndeski of [23, 24], which lead to equations of motion higher than second order but are free from Ostrogradski instabilities (see e.g. [25] for an earlier example of theories beyond Horndeski).

The developments described above assume that matter is minimally coupled to a unique metric, which will be called Jordan frame metric for convenience. However, although the universality of couplings is very well tested on Solar System scales [26], on cosmological scales constraints are much weaker. In particular, the scalar field responsible for the current accelerated expansion is known to mediate a fifth force [27], which may lead to violations of the equivalence principle (EP) on large scales [28] (see also [29] for a test of the EP on large scales). Moreover, while fifth force effects on standard matter such as baryons and photons are severely restricted, those on cold dark matter (CDM) or neutrinos could be much larger. This leaves the freedom to consider the case where different matter species¹ couple differently to the scalar field [30, 31].

The goal of the present work is to extend the approach developed in Refs. [4, 5] by relaxing the assumption that all matter species are minimally coupled to the same metric.²

¹By matter species we intend the different components in the Universe (baryons, photons, CDM and neutrinos) but the results derived here could be straightforwardly extended to different types of objects, such as e.g. galaxies of different sizes, behaving differently under the effect of the fifth force.

²Another general parametrisation of theories of single-field dark energy that is explicitly coupled to dark

For simplicity, in the following we restrict our study to effective theories of dark energy or modified gravity that remain within the Horndeski class. This means that we assume that the function α_{H} introduced in [24] to describe theories beyond Horndeski at the level of linear perturbations vanishes here, leaving only four out of the five free independent functions of [5]. We reserve a treatment of theories beyond Horndeski for future work. As shown in [35], the structure of the Horndeski Lagrangians is preserved under a disformal transformation [36] of the metric with coefficients that depend only on the scalar field (not on its gradient), i.e. of the form

$$\tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu} + D(\phi)\partial_\mu\phi\partial_\nu\phi. \quad (1.1)$$

Thus, in the following we assume that each matter species is minimally coupled to a distinct Jordan metric of this form.³ While conformal couplings (i.e. with $D = 0$) have been extensively studied in the literature (see e.g. [1] and references therein), disformal couplings have been investigated only recently (see also e.g. [39–48]). Moreover, the dynamics of the gravitational metric $g_{\mu\nu}$ is usually assumed to be governed by the standard Einstein-Hilbert action. Here, we allow a much more general gravitational sector, based on the effective description given in [5]. In Sec. 2 we review our formalism within the ADM effective approach and the gravitational action in the uniform scalar-field gauge. Apart from the four time-dependent parameters mentioned above, we introduce two extra functions of time for each species, describing the nonminimal coupling to dark energy via an effective metric of the form (1.1). The structure of this action is preserved under transformations of the reference metric of the form (1.1) and the stability conditions for the matter *and* the gravitational sector are shown to be invariant under these transformations. More details on the frame dependence and on the derivation of the stability conditions of gravitational and matter quantities are respectively given in Appendix A and Appendix B.

In Sec. 3 we derive the evolution equations describing the matter sector, which now include the effect of the nonminimal couplings, and in Appendix E we provide the definitions of several parameters introduced in this section. These equations must be supplemented with the Einstein equations describing the gravitational sector, reported in Appendix C. We provide and discuss the perturbation equations using Newtonian gauge but these are also given in synchronous gauge in Appendix D.

The parameters of our effective description can be constrained by observations. As a direct application of our approach, in Sec. 4 we consider the cosmological consequences, for the background evolution and for linear perturbations, of a Universe where the coupling of CDM differs from that of the other species (see e.g. [49–59]). Our analysis extends previous results as we allow gravity itself to be modified, not only the couplings to matter. In Sec. 5 we consider the case where the coupled species is a relativistic fluid. This will allow us to highlight the dependence of conformal and disformal couplings on the equation of state. Finally, we conclude in Sec. 6.

matter has been given in [32], in the framework of the Parameterized Post-Friedmann approach [33, 34].

³Other types of couplings can be found in the literature. For instance, Ref. [37] considers a CDM action that depends on the contraction of the CDM 4-velocity with the normalized space-time gradient of the scalar field, in the context of Lorentz-violating theories. Ref. [38] directly modifies the action for a general perfect fluid.

2 Unifying description of dark energy with non universal couplings

In this section we introduce the gravitational and matter actions within the ADM framework introduced in [5] and recently summarized in [14]. After giving the background equations of motion, we study linear fluctuations and derive the conditions for the linear theory to be healthy, i.e. ghost-free and without gradient instabilities.

2.1 Gravitational and matter actions

In the present work, we assume that the gravitational sector is described by a four-dimensional metric $g_{\mu\nu}$ and a scalar field ϕ . Let us start by choosing a coordinate system such that the constant time hypersurfaces coincide with the uniform scalar field hypersurfaces. In this gauge, referred to as unitary gauge, the metric can be written in the ADM form, which reads

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt) , \quad (2.1)$$

where N is the lapse and N^i the shift. In the following, a dot will stand for a time derivative with respect to t , and D_i will denote the covariant derivative associated with the three-dimensional spatial metric h_{ij} . Spatial indices will be lowered and raised with the spatial metric h_{ij} or its inverse h^{ij} , respectively.

In the unitary gauge, a generic gravitational action can be written in terms of geometric quantities that are invariant under spatial diffeomorphisms [6, 7]. Expressed in ADM coordinates introduced above, these geometric quantities are the lapse N , the extrinsic curvature K_{ij} of the constant time hypersurfaces, whose components are given by

$$K_{ij} = \frac{1}{2N} (\dot{h}_{ij} - D_i N_j - D_j N_i) , \quad (2.2)$$

as well as the 3d Ricci tensor R_{ij} of the constant time hypersurfaces and, possibly, spatial derivatives of all these quantities. Thus, the action is generically of the form

$$S_g = \int d^4x \sqrt{-g} L(N, K_{ij}, R_{ij}, h_{ij}, D_i; t) . \quad (2.3)$$

The gravitational action must be supplemented by a matter action S_m . In order to describe dark energy and modified gravity scenarios with EP violations, we assume that beside the gravitational sector introduced above, the Universe is filled by N_S matter species labelled by an index I , with $I = 1, \dots, N_S$, each minimally coupled to a different metric. For each species I , we denote the corresponding metric by $\check{g}_{\mu\nu}^{(I)}$ and we call this the Jordan frame metric associated with this species. The total matter action is thus given by

$$S_m = \sum_I^{N_S} S_I , \quad S_I = \int d^4x \sqrt{-\check{g}^{(I)}} L_I(\check{g}_{\mu\nu}^{(I)}, \psi_I) , \quad (2.4)$$

with

$$\check{g}_{\mu\nu}^{(I)} = C_I^{(\phi)}(\phi) g_{\mu\nu} + D_I^{(\phi)}(\phi) \partial_\mu \phi \partial_\nu \phi . \quad (2.5)$$

In order to preserve the Lorentzian signature of the Jordan-frame metric of the species I , it is necessary to have $C_I^{(\phi)} > 0$.

There is some arbitrariness in the choice of the gravitational metric $g_{\mu\nu}$ since we work in the context of modified gravity, where the gravitational dynamics cannot be expressed in terms of a standard Einstein-Hilbert term, in general. It is often convenient to choose one particular matter species, say I_* , and define its Jordan metric as the gravitational metric, in which case we have $C_{I_*}^{(\phi)} = 1$ and $D_{I_*}^{(\phi)} = 0$.

2.2 Homogeneous equations

Let us discuss briefly the evolution of the background metric described by a FLRW metric assumed to be spatially flat. In this case the lapse is a function of time only, which we denote $\bar{N}(t)$, the shift vanishes, $N^i = 0$, and the spatial metric reads $g_{ij} = h_{ij} = a^2(t)\delta_{ij}$ where a represents the scale factor. Thus, the metric reads

$$ds^2 = -\bar{N}^2(t)dt^2 + a^2(t)d\mathbf{x}^2. \quad (2.6)$$

The homogeneous dynamics depends on the gravitational Lagrangian L in eq. (2.3), which can be seen as a function $\bar{L}(N, a, \dot{a})$ when the arguments are restricted to their background values, i.e. $N = \bar{N}$, $h_{ij} = a^2(t)\delta_{ij}$, $R_{ij} = 0$, and

$$K_{ij} = \bar{K}_{ij} \equiv \frac{\dot{a}\dot{a}}{\bar{N}}\delta_{ij} = Hh_{ij}, \quad (2.7)$$

where $H \equiv \dot{a}/(a\bar{N})$ denotes the Hubble rate. Here and in the following, barred quantities are evaluated on the background.

The variation of the matter action S_m with respect to the metric $g_{\mu\nu}$ defines the energy-momentum tensor, according to the standard expression

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}. \quad (2.8)$$

This definition applies even if the matter is minimally coupled with respect to a metric $\check{g}_{\mu\nu}$ that differs from $g_{\mu\nu}$, as discussed in Appendix A. In the homogenous case, the energy-momentum tensor depends only on the energy density $\rho_m \equiv -\bar{T}_0^0$ and the pressure $p_m \equiv \bar{T}_i^i/3$. If there are several matter components, the previous quantities simply correspond, respectively, to the sums of the energy densities and pressures associated to each individual species, i.e. $\rho_m = \sum_I \rho_I$ and $p_m = \sum_I p_I$.

The background evolution equations are then obtained by taking the variation of the total homogeneous action $S_g + S_m$ with the respect to \bar{N} and a . As shown in [5], this leads to the equations

$$\bar{L} + \bar{N}L_N - 3H\mathcal{F} = \rho_m \quad (2.9)$$

and

$$\bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} = -p_m, \quad (2.10)$$

where the coefficient \mathcal{F} is defined by

$$\left(\frac{\partial L}{\partial K_{ij}} \right)_{\text{bgd}} \equiv \mathcal{F} h^{ij}. \quad (2.11)$$

Equations (2.9) and (2.10) generalize the usual Friedmann equations. For GR, where the Lagrangian is given by $L = M_P^2(K_{ij}K^{ij} - K^2 + R)/2$, one can check that the standard equations are recovered, since $\bar{L} = -3M_P^2H^2$, $L_N = 0$ and $\mathcal{F} = -2M_P^2H$.

The Friedmann equations eqs. (2.9)–(2.10) can always be written as

$$H^2 = \frac{1}{3M^2} (\rho_m + \rho_{\text{DE}}) , \quad (2.12)$$

$$\dot{H} + \frac{3}{2}H^2 = -\frac{1}{2M^2} (p_m + p_{\text{DE}}) , \quad (2.13)$$

where M denotes the effective Planck mass, which can be in general time-dependent (it will be defined below from the second derivative of L with respect to the intrinsic curvature). The above equations can be interpreted as definitions of the homogeneous energy density and pressure of dark energy, respectively given by

$$\rho_{\text{DE}} \equiv 3M^2H^2 - \rho_m , \quad p_{\text{DE}} \equiv -M^2(2\dot{H} + 3H^2) - p_m . \quad (2.14)$$

These equations can also be shown to be equivalent to the Friedmann equations derived from the Lagrangian [4, 5]

$$L = \frac{M^2}{2} {}^{(4)}R + \frac{c}{N^2} - \Lambda , \quad (2.15)$$

where ${}^{(4)}R$ is the 4d Ricci scalar and $c = c(t)$ and $\Lambda = \Lambda(t)$ are time-dependent functions, respectively given by

$$2c = \rho_{\text{DE}} + p_{\text{DE}} + H(M^2)' - (M^2)'' , \quad (2.16)$$

$$2\Lambda = \rho_{\text{DE}} - p_{\text{DE}} + 5H(M^2)' + (M^2)'' . \quad (2.17)$$

2.3 Linear perturbations

We now expand the gravitational action up to second order in perturbations, in terms of the perturbative quantities

$$\delta N = N - \bar{N}(t) , \quad \delta K_{ij} = K_{ij} - Hh_{ij} , \quad (2.18)$$

as well as R_{ij} , which is already a perturbation since its background value vanishes.

The second-order expansion of the gravitational Lagrangian involves first and second derivatives of L with respect to its arguments K_{ij} , R_{ij} and N . It is convenient to introduce the time-dependent coefficients \mathcal{G} , \mathcal{B}_R , \mathcal{B} , $\hat{\mathcal{A}}_K$, \mathcal{A}_K , $\hat{\mathcal{C}}$, \mathcal{C} , $\hat{\mathcal{A}}_R$ and \mathcal{A}_R respectively as

$$\frac{\partial L}{\partial R_i^j} = \mathcal{G} \delta_i^j , \quad \frac{\partial^2 L}{\partial N \partial R_j^i} = \mathcal{B}_R \delta_i^j , \quad \frac{\partial^2 L}{\partial N \partial K_j^i} = \mathcal{B} \delta_i^j , \quad (2.19)$$

$$\frac{\partial^2 L}{\partial K_i^j \partial K_k^l} = \hat{\mathcal{A}}_K \delta_j^i \delta_l^k + \mathcal{A}_K \left(\delta_l^i \delta_j^k + \delta^{ik} \delta_{jl} \right) , \quad (2.20)$$

$$\frac{\partial^2 L}{\partial R_i^j \partial R_k^l} = \hat{\mathcal{A}}_R \delta_j^i \delta_l^k + \mathcal{A}_R \left(\delta_l^i \delta_j^k + \delta^{ik} \delta_{jl} \right) , \quad (2.21)$$

$$\frac{\partial^2 L}{\partial K_i^j \partial R_k^l} = \hat{\mathcal{C}} \delta_j^i \delta_l^k + \mathcal{C} \left(\delta_l^i \delta_j^k + \delta^{ik} \delta_{jl} \right) , \quad (2.22)$$

where all partial derivatives on the left hand sides are evaluated on the background. The form of the right hand side of these expressions is merely determined by the FLRW symmetries. The first and second derivatives of L with respect to the scalar N are simply denoted as L_N and L_{NN} , respectively.

In the following, for simplicity, we restrict our considerations to Lagrangians that lead to dynamical equations with at most two space derivatives. This is automatically ensured if we impose the conditions [5, 14]⁴

$$\hat{\mathcal{A}}_K + 2\mathcal{A}_K = 0, \quad \hat{\mathcal{C}} + \mathcal{C} = 0, \quad 4\hat{\mathcal{A}}_R + 3\mathcal{A}_R = 0. \quad (2.23)$$

We also impose the further condition

$$\mathcal{B}_R = \frac{1}{\bar{N}}(\mathcal{A}_K - \mathcal{G} - HC), \quad (2.24)$$

which is equivalent to restricting the range of application of the expanded action to Horndeski theories [5].⁵

The second-order gravitational action can then be explicitly written in terms of all the coefficients introduced above. In fact, the quadratic action involves only a few combinations of these coefficients, which are represented by the following dimensionless parameters [14, 17]

$$\alpha_K \equiv \frac{2\bar{N}L_N + \bar{N}^2L_{NN}}{2H^2\mathcal{A}_K}, \quad \alpha_B \equiv \frac{\mathcal{B}\bar{N}}{4H\mathcal{A}_K}, \quad \alpha_T \equiv \frac{\mathcal{G} + \dot{\mathcal{C}}/(2\bar{N}) + HC}{\mathcal{A}_K} - 1. \quad (2.25)$$

The effective Planck mass squared is defined by $M^2 \equiv 2\mathcal{A}_K$. With this definition, M coincides with the time-dependent Planck mass introduced in eqs. (2.9) and (2.10) and in the action (2.15). Its possible time variation is characterized by

$$\alpha_M \equiv \frac{1}{\bar{N}H} \frac{d \ln M^2}{dt}. \quad (2.26)$$

In terms of these parameters, one finds that the second-order gravitational action is given by⁶

$$S_g^{(2)} = \int d^3x dt a^3 \bar{N} \frac{M^2}{2} \left[\delta K_j^i \delta K_i^j - \delta K^2 + R \frac{\delta N}{\bar{N}} + (1 + \alpha_T) \delta_2(\sqrt{h}R/a^3) \right. \\ \left. + \alpha_K H^2 \left(\frac{\delta N}{\bar{N}} \right)^2 + 4\alpha_B H \delta K \frac{\delta N}{\bar{N}} \right], \quad (2.27)$$

where δ_2 denotes taking the expansion at second order in the perturbations. Moreover, we have omitted irrelevant terms that vanish when adding the matter action and imposing the background equations of motion.

⁴Here we have corrected a typo in Ref. [14]. The coefficient in front of $\delta K \delta R$ inside the bracket in eq. (55) (see v2 of the arXiv version) should be $\hat{\mathcal{C}}/2$, so that the condition in the second line of eq. (60) should read $\hat{\mathcal{C}}^* = \hat{\mathcal{C}} + \mathcal{C}$. With this correction, eq. (76) of Ref. [14] is equivalent to eq. (2.23) in this article.

⁵To parametrize deviations from Horndeski theories at the linear level, Ref. [24] introduced the parameter $\alpha_H \equiv (\mathcal{G} + HC + \bar{N}\mathcal{B}_R)/\mathcal{A}_K - 1$. Here we will assume $\alpha_H = 0$.

⁶To write this action, we have not assumed $\bar{N} = 1$ as done in previous references [4, 5, 14]. In such a way the action remains explicitly invariant under a time reparameterization $t \rightarrow \tilde{t}(t)$, which is convenient when changing frame.

To verify that M plays the role of the Planck mass which canonically normalizes the graviton, let us write this action in terms of the tensor fluctuations, defined as the traceless and divergence-free fluctuations of the spatial metric, i.e.

$$h_{ij} = a^2(t) (\delta_{ij} + \gamma_{ij}) , \quad \gamma_{ii} = 0 = \partial_i \gamma_{ij} . \quad (2.28)$$

The above action then yields

$$S_\gamma^{(2)} = \int dx^3 dt a^3 \frac{M^2}{8\bar{N}} \left[\dot{\gamma}_{ij}^2 - c_T^2 \frac{\bar{N}^2}{a^2} (\partial_k \gamma_{ij})^2 \right] , \quad (2.29)$$

where the tensor sound speed squared is given by $c_T^2 \equiv 1 + \alpha_T$. Absence of ghosts and gradient instabilities respectively require that the kinetic and spatial gradient terms are positive, i.e. that

$$M^2 \geq 0 , \quad \alpha_T \geq -1 , \quad (2.30)$$

which will be assumed in the following.

2.4 Matter couplings and stability conditions

To discuss the stability and determine the propagation speed of dark energy perturbations, one must also include quadratic terms that come from the matter action, because the latter depends on the gravitational degrees of freedom. In order to do so, we need to take into account the fact that each matter species I is minimally coupled to a metric $\check{g}_{\mu\nu}^{(I)}$ defined in eq. (2.5). For later convenience, we define, for each matter species, the time-dependent quantity

$$\alpha_{C,I} \equiv \frac{\dot{\phi}}{2HN} \frac{d \ln C_I^{(\phi)}}{d\phi} , \quad (2.31)$$

which parameterizes how the conformal coupling affects physical observables; the impact of the disformal coupling is parameterized by the quantity⁷

$$\alpha_{D,I} \equiv \frac{(\dot{\phi}/N)^2 D_I^{(\phi)}}{C_I^{(\phi)} - (\dot{\phi}/N)^2 D_I^{(\phi)}} , \quad (2.32)$$

and the right-hand side of these equations are to be evaluated on the background. Requiring that the Jordan frame metric is Lorentzian implies $\alpha_{D,I} > -1$ [35].

In unitary gauge, eq. (2.5) reads

$$\check{g}_{\mu\nu}^{(I)} = C_I(t) g_{\mu\nu} + D_I(t) \delta_\mu^0 \delta_\nu^0 , \quad (2.33)$$

with

$$C_I(t) = C_I^{(\phi)}(\phi(t)) , \quad D_I(t) = \dot{\phi}^2(t) D_I^{(\phi)}(\phi(t)) . \quad (2.34)$$

Then the parameters $\alpha_{C,I}$ and $\alpha_{D,I}$ introduced above take the form

$$\alpha_{C,I} = \frac{1}{2H\bar{N}} \frac{d \ln C_I}{dt} , \quad \alpha_{D,I} = \frac{D_I}{\bar{N}^2 C_I - D_I} . \quad (2.35)$$

⁷This parameter coincides with $1/\gamma^2$, where γ is the so-called disformal scalar in the notation of Ref. [47].

Combining the quadratic action for matter with eq. (2.27), one can extract a quadratic action that governs the dynamics of the gravitational scalar degree of freedom and the matter ones. The explicit calculation in the case of perfect fluids is presented in Appendix B. The absence of ghosts is guaranteed by the positivity of the matrix in front of the kinetic terms. For the gravitational scalar degree of freedom, this condition is given by

$$\alpha \equiv \alpha_K + 6\alpha_B^2 + 3 \sum_I \alpha_{D,I} \Omega_I \geq 0, \quad (2.36)$$

where we have introduced the (time-dependent) dimensionless density parameter

$$\Omega_I \equiv \frac{\rho_I}{3M^2 H^2}, \quad (2.37)$$

where we recall that M^2 is in general time dependent. As pointed out already in [43, 60], the presence of a disformal coupling affects the ghost-free condition.

For the matter sector, the analogous condition usually corresponds to the Null Energy Condition [61]. In the Jordan frame of each species I , this can be expressed in terms of the energy density and pressure by $\check{\rho}_I + \check{p}_I \geq 0$ (we use the symbol $\check{\cdot}$ to denote Jordan-frame quantities). In the frame of $g_{\mu\nu}$, this inequality becomes

$$\rho_I + (1 + \alpha_{D,I})p_I \geq 0, \quad (2.38)$$

where we have used that $\check{w}_I = (1 + \alpha_{D,I})w_I$ (see Appendix A for various relations between quantities defined in distinct frames).

The speed of sound for scalar perturbations can be read off from the quadratic action derived in Appendix B. One finds

$$c_s^2 = -\frac{2}{\alpha} \left\{ (1 + \alpha_B) \left[\frac{\dot{H}}{H^2} - \alpha_M + \alpha_T + \alpha_B(1 + \alpha_T) \right] + \frac{\dot{\alpha}_B}{H} + \frac{3}{2} \sum_I \left[1 + (1 + \alpha_{D,I})w_I \right] \Omega_I \right\}, \quad (2.39)$$

where matter appears in the last term in the bracket, proportional to $\sum_I (\check{\rho}_I + \check{p}_I)$. Absence of gradient instabilities is guaranteed provided that

$$c_s^2 \geq 0. \quad (2.40)$$

We also require that the propagation speed for each matter species, in its Jordan frame, is positive, $\check{c}_{s,I}^2 \geq 0$.

2.5 Disformal transformations

As mentioned earlier, there is some arbitrariness in the choice of the metric $g_{\mu\nu}$ that describes the gravitational sector. Let us thus see how the description is modified when the reference metric undergoes a disformal transformation, of the form

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = C^{(\phi)}(\phi)g_{\mu\nu} + D^{(\phi)}(\phi)\partial_\mu\phi\partial_\nu\phi. \quad (2.41)$$

In unitary gauge, this corresponds to the transformation

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = C(t)g_{\mu\nu} + D(t)\delta_\mu^0\delta_\nu^0, \quad (2.42)$$

with $C(t) = C^{(\phi)}(\phi(t))$ and $D(t) = D^{(\phi)}(\phi(t))\dot{\phi}^2(t)$. The effect of this transformation on the ADM quantities, on the background quantities and on the linear perturbations is described in detail in Appendix A. Here, we just present the main consequences on the parametrization of the couplings and of the linear perturbations.

In analogy with (2.35), it is convenient to introduce the dimensionless time-dependent parameters

$$\alpha_C \equiv \frac{\dot{C}}{2H\bar{N}C}, \quad \alpha_D \equiv \frac{D}{\bar{N}^2C - D}, \quad (2.43)$$

which characterize, respectively, the conformal and disformal parts of the above metric transformation.⁸

Let us first see how the gravitational action (2.27) changes under the transformation (2.42). As shown in Ref. [35], the structure of Horndeski Lagrangians is preserved under a disformal transformation. Indeed, using eqs. (A.2) and (A.3), one can check that (2.27) maintains the same structure with the time-dependent coefficients in the action transforming as

$$\tilde{M}^2 = \frac{M^2}{C\sqrt{1 + \alpha_D}} \quad (2.44)$$

and

$$\begin{aligned} \tilde{\alpha}_K &= \frac{\alpha_K + 12\alpha_B[\alpha_C + (1 + \alpha_D)\alpha_D] - 6[\alpha_C + (1 + \alpha_D)\alpha_D]^2 + 3\Omega_m\alpha_D}{(1 + \alpha_C)^2(1 + \alpha_D)^2}, \\ \tilde{\alpha}_B &= \frac{1 + \alpha_B}{(1 + \alpha_C)(1 + \alpha_D)} - 1, \\ \tilde{\alpha}_M &= \frac{\alpha_M - 2\alpha_C}{1 + \alpha_C} - \frac{\dot{\alpha}_D}{2H\bar{N}(1 + \alpha_D)(1 + \alpha_C)}, \\ \tilde{\alpha}_T &= (1 + \alpha_T)(1 + \alpha_D) - 1. \end{aligned} \quad (2.45)$$

We can use these transformations, which depend on the two arbitrary functions α_C and α_D , to set to zero any two of the parameters $\tilde{\alpha}_a$ above.

Finally, the conformal and disformal coefficients associated with the respective matter Jordan frame metrics are modified according to

$$\begin{aligned} \tilde{\alpha}_{D,I} &= \frac{\alpha_{D,I} - \alpha_D}{1 + \alpha_D}, \\ \tilde{\alpha}_{C,I} &= \frac{\alpha_{C,I} - \alpha_C}{1 + \alpha_C}. \end{aligned} \quad (2.46)$$

Alternatively to setting two $\tilde{\alpha}_a$ to zero, it is always possible to choose as the new reference metric $\tilde{g}_{\mu\nu}$ one of the matter Jordan metrics, say $g_{\mu\nu}^{(I_*)}$, which then implies $\tilde{\alpha}_{C,I_*} = \tilde{\alpha}_{D,I_*} = 0$.

One can verify that all the stability conditions are frame independent. In particular, the quantities that appear in the no-ghost conditions, eqs. (2.36) and (2.38), transform as

$$\tilde{\alpha} = \frac{\alpha}{(1 + \alpha_C)^2(1 + \alpha_D)^2}, \quad \tilde{\rho}_I + (1 + \tilde{\alpha}_{D,I})\tilde{p}_I = \frac{\rho_I + (1 + \alpha_{D,I})p_I}{C^2(1 + \alpha_D)^{1/2}}, \quad (2.47)$$

⁸We require $C > 0$ and $\alpha_D > -1$, see discussions respectively in Secs. 2.1 and 2.4.

and since $1 + \alpha_{\text{D}} > 0$ (see discussion in Sec. 2.4), their sign is indeed frame independent. It is also straightforward to check that all the propagation speeds, i.e. of tensor, scalar and matter fluctuations, transform in the same way and that their signs remain unchanged,

$$\tilde{c}_T^2 = (1 + \alpha_{\text{D}})c_T^2, \quad \tilde{c}_s^2 = (1 + \alpha_{\text{D}})c_s^2, \quad \tilde{c}_{s,I}^2 = (1 + \alpha_{\text{D}})c_{s,I}^2. \quad (2.48)$$

In summary, at the level of linear perturbations our gravitational sector is characterized by four time-dependent parameters α_{K} , α_{B} , α_{M} and α_{T} . Each species is characterized by two time-dependent parameters, associated with their conformal and disformal couplings respectively. A priori, for a system of N_S species coupled to different metrics, this gives a total of $2N_S + 4$ parameters. However, the general invariance of the system under an arbitrary change of frame, characterized by two parameters, reduces the number of independent parameters to $2(N_S + 1)$.

In particular, action (2.27) can also be used to describe inflationary perturbations. In this case, matter can be ignored, i.e. $N_S = 0$, and one can always use eq. (2.44) to find a frame where the Planck mass is time-independent and $c_T = 1$, without loss of generality [62]. Thus, inflationary fluctuations can be generically described in the frame where $\alpha_{\text{M}} = 0 = \alpha_{\text{T}}$ by only *two* operators, those proportional to α_{K} and α_{B} , as in Refs. [6, 7].⁹

3 Matter equations of motion

In this section, we leave the unitary gauge description introduced in the previous section, by “covariantizing” the action. This can be done explicitly by performing a time reparametrization of the form

$$t \rightarrow \phi = t + \pi(t, \mathbf{x}), \quad (3.1)$$

where the unitary time t becomes a four-dimensional scalar field ϕ . For convenience, we denote by π the fluctuation of ϕ .

By substituting the above transformation into the total action $S = S_{\text{g}} + S_{\text{m}}$, we then obtain an action that depends on the scalar field ϕ and an arbitrary metric $g_{\mu\nu}$. We will use this more general form for the action to derive the evolution equations for the gravitational and matter sectors.

The equations of motion for the metric are obtained by varying the total action with respect to $g_{\mu\nu}$,

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0. \quad (3.2)$$

which provides the generalized Einstein equations. At linear order, they are explicitly given in Appendix C.

To write the equations of motion for matter, we use the invariance of the matter action S_I under arbitrary diffeomorphisms, $x^\mu \rightarrow x^\mu + \xi^\mu$. This implies

$$\nabla_\mu T_{(I)\nu}^\mu + Q_I \partial_\nu \phi = 0, \quad (3.3)$$

⁹The sound speed of fluctuations in this case is $c_s^2 = -(2/\alpha)[(1 + \alpha_{\text{B}})(\dot{H}/H^2 + \alpha_{\text{B}}) + \dot{\alpha}_{\text{B}}/H]$; see eq. (2.39). Thus, for a constant α_{B} , the usual gradient instability associated with the violation of the Null Energy Condition for $\dot{H} \geq 0$ can be cured by requiring $-1 \leq \alpha_{\text{B}} \leq -\dot{H}/H^2$ [6].

where the function Q_I , which characterizes the coupling between the matter species I and the scalar field, is defined by

$$Q_I \equiv -\frac{1}{\sqrt{-g}} \frac{\delta S_I}{\delta \phi} = -\frac{C'_I}{2C_I} T_{(I)} - \frac{D'_I}{2C_I} T^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \nabla_\mu \left(T^{\mu\nu} \partial_\nu \phi \frac{D_I}{C_I} \right), \quad (3.4)$$

where a prime denotes a derivative with respect to ϕ . The expression on the right hand side is obtained by using the property that the matter action S_I depends on the scalar field only through the Jordan metric eq. (2.5).

Finally, the evolution equation for ϕ can be obtained by variation of the total action with respect to ϕ , $\delta S / \delta \phi = 0$. Thus, from eq. (3.4) we obtain

$$\frac{1}{\sqrt{-g}} \frac{\delta S_g}{\delta \phi} - \sum_I Q_I = 0. \quad (3.5)$$

In the following, we will study the above equations, first in the homogeneous limit and then restricting ourselves to their linearized version.

3.1 Homogenous equations

Let us first consider the homogeneous case, with the flat FLRW metric (2.6) where we set $\bar{N} = 1$. The associated Friedmann equations are given in eqs. (2.9) and (2.10), or (2.12) and (2.13), with $\rho_m = \sum_I \rho_I$ and $p_m = \sum_I p_I$.

For a FLRW background, the definition of Q_I , eq. (3.4), reduces to

$$\bar{Q}_I = \frac{H \rho_I}{1 + \alpha_{D,I}} \left\{ \alpha_{C,I} [1 - 3w_I(1 + \alpha_{D,I})] + \alpha_{D,I} \left(3 + \frac{\dot{\rho}_I}{H \rho_I} \right) + \frac{\dot{\alpha}_{D,I}}{2H(1 + \alpha_{D,I})} \right\}, \quad (3.6)$$

where we recall that the conformal and disformal parameters $\alpha_{C,I}$ and $\alpha_{D,I}$ are respectively defined in eq. (2.35). Substituting the above expression into eq. (3.3), one finds that the homogeneous matter evolution equation can be written in the form

$$\dot{\rho}_I + 3H(1 + w_I - \gamma_I)\rho_I = 0, \quad (3.7)$$

where the dimensionless parameter γ_I is given by

$$\gamma_I \equiv \frac{1}{3} \alpha_{C,I} [1 - 3w_I(1 + \alpha_{D,I})] - w_I \alpha_{D,I} + \frac{\dot{\alpha}_{D,I}}{6H(1 + \alpha_{D,I})}. \quad (3.8)$$

Taking into account (3.7), one can also check that $\bar{Q}_I = 3H \rho_I \gamma_I$. Note that the equation of state in the Jordan frame of the fluid I corresponds to $\check{w}_I = w_I(1 + \alpha_{D,I})$ (see Appendix A.1). Using this relation, one can check that for a relativistic fluid, i.e. $\check{w}_I = 1/3$, the conformal term in (3.8) disappears, as expected from the tracelessness of its stress energy tensor.

Given the Friedmann equations (2.12) and (2.13) as well as the continuity equation for matter, eq. (3.7), the homogenous energy density of dark energy satisfies

$$\dot{\rho}_{\text{DE}} + [3(1 + w_{\text{DE}}) - \alpha_{\text{M}}] H \rho_{\text{DE}} = H \sum_I (\alpha_{\text{M}} - 3\gamma_I) \rho_I, \quad (3.9)$$

where we have introduced the equation of state parameter for the dark energy component $w_{\text{DE}} \equiv p_{\text{DE}} / \rho_{\text{DE}}$.

3.2 Perturbation equations in Newtonian gauge

We now consider a linearly perturbed FLRW metric in Newtonian gauge with only scalar perturbations, i.e.,

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j . \quad (3.10)$$

In this gauge, we decompose the scalar part of the stress-energy tensor for each species, at linear order, as

$$T_{(I)0}^0 \equiv -(\rho_I + \delta\rho_I) , \quad (3.11)$$

$$T_{(I)0}^i \equiv \rho_I(1 + w_I)\partial_i v_I = -a^2 T_{(I)}^i{}_0 , \quad (3.12)$$

$$T_{(I)}^i{}_j \equiv (\rho_I w_I + \delta p_I)\delta_j^i + \left(\partial^i \partial_j - \frac{1}{3}\delta_j^i \partial^2\right)\sigma_I , \quad (3.13)$$

where $\delta\rho_I$ and δp_I are the energy density and pressure perturbations, v_I is the 3-velocity potential and σ_I is the anisotropic stress potential for the species I . As usual, we define the total matter quantities as $\delta\rho_m = \sum_I \delta\rho_I$, $\delta p_m = \sum_I \delta p_I$, $v_m = \sum_I (\rho_I + p_I)v_I/(\rho_m + p_m)$ and $\sigma_m = \sum_I \sigma_I$.

The continuity equation, for each species, can be derived from the time component of eq. (3.3). In Fourier space, at linear order, this reads

$$\delta\dot{\rho}_I + 3H(\delta\rho_I + \delta p_I) - 3\rho_I(1 + w_I)\dot{\Psi} - \rho_I(1 + w_I)\frac{k^2}{a^2}v_I = \bar{Q}_I\dot{\pi} + \delta Q_I , \quad (3.14)$$

where \bar{Q}_I and δQ_I are given respectively by eqs. (3.6) and (C.6). The space components of eq. (3.3) gives the Euler equation, which at linear order reads

$$\rho_I(1+w_I)\dot{v}_I + \rho_I[\dot{w}_I - 3Hw_I(1+w_I)]v_I + \delta p_I + \rho_I(1+w_I)\Phi - \frac{2k^2}{3a^2}\sigma_I = -\bar{Q}_I[\pi + v_I(1+w_I)] . \quad (3.15)$$

We can rewrite the equations above in terms of the density contrast $\delta_I \equiv \delta\rho_I/\rho_I$ and using the explicit expression for δQ_I given in eq. (C.6). This yields

$$\begin{aligned} & \dot{\delta}_I + 3H(1 + \alpha_{C,I})(1 + \alpha_{D,I})\left(\frac{\delta p_I}{\rho_I} - w_I\delta_I\right) - (1 + w_I)\frac{k^2}{a^2}v_I = 3[1 + (1 + \alpha_{D,I})w_I]\dot{\Psi} \\ & + 2(1 + \alpha_{D,I})[\alpha_{C,I}(1 - 3w_I) - 3\gamma_I]H\Phi - \alpha_{D,I}\left(\dot{\Phi} - \ddot{\pi} + w_I\frac{k^2}{a^2}\pi\right) \\ & - [2(1 + \alpha_{D,I})\alpha_{C,I}(1 - 3w_I) - 3w_I\alpha_{D,I} - 3\gamma_I(3 + 2\alpha_{D,I})]H\dot{\pi} \\ & + 3\left[(\gamma_I H)^\cdot + w_I\alpha_{D,I}\dot{H} + (\alpha_{C,I} + \alpha_{D,I}(1 + \alpha_{C,I}))\dot{w}_I H\right]\pi , \end{aligned} \quad (3.16)$$

and

$$\dot{v}_I - 3H\left[w_I - \gamma_I - \frac{\dot{w}_I}{3H(1+w_I)}\right]v_I + \frac{\delta p_I}{(1+w_I)\rho_I} + \Phi - \frac{2}{3(1+w_I)\rho_I}\frac{k^2}{a^2}\sigma_I = -3H\frac{\gamma_I}{1+w_I}\pi . \quad (3.17)$$

As mentioned before, the equation of state parameter in the matter Jordan frame, \check{w}_I , is different from the one in a generic frame, w_I . This means that the relation between pressure and energy density perturbations depends on the frame. Indeed, because of the coupling to the scalar field, there is a non-adiabatic pressure perturbation [63] which appears in frames that are disformally distinct from the Jordan one (see also [64] for a similar remark). For an isentropic perfect fluid with $\check{c}_{s,I}^2 = \check{w}_I = \text{constant}$, this reads (see Appendix A.2)

$$\delta p_{\text{nad},I} \equiv \delta p_I - \frac{\dot{p}_I}{\dot{\rho}_I} \delta \rho_I = p_I \left[2\alpha_{\text{D}}(\Phi - \dot{\pi}) + \frac{\dot{\alpha}_{\text{D}}}{1 + \alpha_{\text{D}}} \left(\frac{\delta \rho_I}{\dot{\rho}_I} - \pi \right) \right]. \quad (3.18)$$

Let us comment on the initial conditions of the above equations. In the simplest case, one can assume that perturbations start in the adiabatic growing solution, which is justified if they have originated from single-field inflation (see e.g. [65]). In this case, their amplitude can be given in terms of the time-independent quantity \mathcal{R}_{in} , defined as the long-wavelength limit ($k \ll aH$) of the total comoving curvature perturbation [66]

$$\mathcal{R} \equiv -\Psi + H \frac{\dot{\Psi} + H\Phi}{\dot{H}}. \quad (3.19)$$

In Ref. [14] it was shown that, in the absence of nonminimal couplings, the generalized Einstein equations and the evolution equations for the matter and field fluctuations admit the adiabatic solution

$$\begin{aligned} \Phi &= -(1 + \alpha_{\text{T}})\mathcal{R}_{\text{in}} + (1 + \alpha_{\text{M}})H\epsilon - \frac{\sigma_{\text{m}}}{M^2}, & \Psi &= -\mathcal{R}_{\text{in}} + H\epsilon, \\ \delta \rho_I &= -\dot{\rho}_I \epsilon, & \delta p_I &= -\dot{p}_I \epsilon, & v_I &= \epsilon, & \pi &= -\epsilon, \end{aligned} \quad (3.20)$$

where

$$\epsilon \equiv \frac{1}{M^2 a} \int a [M^2(1 + \alpha_{\text{T}})\mathcal{R}_{\text{in}} + \sigma_{\text{m}}] dt. \quad (3.21)$$

One can check that these expressions are frame invariant and remain a solution even in the presence of nonminimal couplings, with the same \mathcal{R}_{in} . Note that, for adiabatic initial conditions, the right hand side of (3.18) automatically vanishes and that the matter perturbations are in effect adiabatic in all frames. The nonadiabatic pressure term due to the change of frame manifests itself only for nonadiabatic initial conditions.

Let us point out that the equations written in this section include as a special case (corresponding to $\alpha_{\text{M}} = \alpha_{\text{T}} = \alpha_{\text{B}} = 0$ and $\alpha_{\text{D},I} = 0$) the equations of motion for linear perturbations derived in standard models of dark energy (k -essence) conformally coupled with matter (see e.g. [51]). Our results also include the more recent investigations of disformal couplings between matter, usually CDM, and some standard dark energy (i.e. with $\alpha_{\text{M}} = \alpha_{\text{T}} = \alpha_{\text{B}} = 0$) [41–43, 47, 64].

In the general case, eqs. (3.16)–(3.17) can be directly applied to the usual matter species, i.e. CDM, baryons, photons and neutrinos and implemented in a numerical code. If one wants to study the CMB fluctuations, the fluid approximation is not sufficient for photons and neutrinos and must be replaced by a Boltzmann description. Whereas a nonminimal coupling of photons is constrained to remain tiny [67], one could envisage a nonminimal coupling of neutrinos (see e.g. [68–70]). To deal with this modification, the

simplest method would consist in writing the Boltzmann equation in the Jordan frame of the neutrinos, where it keeps its usual form. The neutrino-frame gravitational potentials appearing in this equation could then be expressed in terms of the gravitational potentials Φ and Ψ associated with the baryon-photon frame, by using explicitly the disformal transformation between the two frames, as given in Appendix A.

4 Baryons and coupled CDM

In this section, we apply the general formalism developed in the previous sections to the cosmological era where the dominant matter species are baryons (denoted by the subscript b) and CDM (subscript c). Whereas there exist very stringent constraints on EP violation for baryons [26, 45], the dark matter sector is much less constrained [19]. For this reason, we now assume that the baryons are minimally coupled, i.e.

$$\alpha_{C,b} = 0, \quad \alpha_{D,b} = 0 \quad \Rightarrow \quad \gamma_b = 0, \quad (4.1)$$

while dark matter is coupled to dark energy via a general metric of the form (2.33).

For both baryons and CDM, one neglects the pressure and anisotropic stress, so that $w_b = w_c = 0$ for the background and $\delta p_b = \delta p_c = \sigma_b = \sigma_c = 0$ for the perturbations¹⁰.

The background equations (3.7) and (3.9) take the form

$$\dot{\rho}_b + 3H\rho_b = 0, \quad (4.2)$$

$$\dot{\rho}_c + 3H(1 - \gamma_c)\rho_c = 0, \quad (4.3)$$

$$\dot{\rho}_{\text{DE}} + [3(1 + w_{\text{DE}}) - \alpha_{\text{M}}]H\rho_{\text{DE}} = -3H\gamma_c\rho_c + H\alpha_{\text{M}}\rho_{\text{m}}. \quad (4.4)$$

According to (3.8), the coupling parameter γ_c is related to the CDM conformal and disformal parameters via

$$\gamma_c = \frac{1}{3}\alpha_{C,c} + \frac{\dot{\alpha}_{D,c}}{6H(1 + \alpha_{D,c})}. \quad (4.5)$$

4.1 Linear perturbations

Let us now consider the linear perturbations. The continuity and Euler equations, (3.16) and (3.17), reduce to

$$\dot{\delta}_b - \frac{k^2}{a^2}v_b = 3\dot{\Psi}, \quad (4.6)$$

$$\dot{v}_b = -\Phi, \quad (4.7)$$

$$\dot{\delta}_c - \frac{k^2}{a^2}v_c = 3(\Psi + \gamma_c H\pi)' + 2(1 + \alpha_{D,c})(\alpha_{C,c} - 3\gamma_c)H(\Phi - \dot{\pi}) - \alpha_{D,c}(\dot{\Phi} - \ddot{\pi}), \quad (4.8)$$

$$\dot{v}_c + 3H\gamma_c v_c = -\Phi - 3H\gamma_c \pi. \quad (4.9)$$

These equations must be supplemented by the Einstein equations, eqs. (C.2)–(C.5) and by the scalar fluctuation equation (C.7).

It is possible to use a combination of the Einstein equations and of (C.7) to eliminate the dependence on π and $\dot{\pi}$ in the above equations in favour of the gravitational potentials.

¹⁰As shown in Appendix A.2, this statement holds in any frame.

The same procedure has been used in the case of minimally coupled matter in Refs. [17] and [14]. In our baryon and coupled CDM system we find a dynamical equation for Ψ of the form:

$$\begin{aligned} \ddot{\Psi} + \frac{\beta_1\beta_2 + \beta_3\alpha_B^2 k_H^2}{\beta_1 + \alpha_B^2 k_H^2} H\dot{\Psi} + \frac{\beta_1\beta_4 + \beta_1\beta_5 k_H^2 + c_s^2\alpha_B^2 k_H^4}{\beta_1 + \alpha_B^2 k_H^2} H^2\Psi = \\ - \sum_I \frac{3}{2} H^2 \Omega_I \left[\frac{\beta_1\beta_{6,I} + \beta_{7,I}\alpha_B^2 k_H^2}{\beta_1 + \alpha_B^2 k_H^2} \delta_I + \frac{\beta_1\beta_{8,I} + \beta_{9,I}\alpha_B^2 k_H^2}{\beta_1 + \alpha_B^2 k_H^2} H v_I \right], \end{aligned} \quad (4.10)$$

where $k_H \equiv k/(aH)$ and the time-dependent coefficients β_a are explicitly given in Appendix E. They generally differ from those given in Refs. [17] and [14] because the disformal coupling to dark matter modifies the evolution equation for π , see eq. (C.7). The relation between Φ and Ψ is given by

$$\begin{aligned} \alpha_B^2 k_H^2 \left(\Phi - \Psi \frac{\xi}{\alpha_B} \right) + \beta_1 \left[\Phi - \Psi(1 + \alpha_T) \left(1 + \hat{\alpha} \frac{\alpha_T - \alpha_M}{2\beta_1} \right) \right] = \\ \frac{\alpha_T - \alpha_M}{2} \left\{ \hat{\alpha} \frac{\dot{\Psi}}{H} + 3 \sum_I \Omega_I \left[\alpha_B \delta_I + \frac{\alpha_K - 6\alpha_B}{2} H v_I \right] \right\}, \end{aligned} \quad (4.11)$$

where $\hat{\alpha} \equiv \alpha_K + 6\alpha_B^2$ and we have introduced the time-dependent combination

$$\xi \equiv \alpha_B(1 + \alpha_T) + \alpha_T - \alpha_M. \quad (4.12)$$

For $\alpha_T = \alpha_M = 0$, this reduces to the familiar relation $\Phi = \Psi$.

We can also eliminate the dependence on π from the continuity and Euler equations for CDM, eqs. (4.8) and (4.9). For simplicity, we give here the explicit expressions only in the case $\alpha_M = \alpha_T = 0$, for which $\Phi = \Psi$, the generalization being straightforward. In this case, the continuity and Euler equations (4.8) and (4.9) become

$$\begin{aligned} \dot{\delta}_c - \frac{k^2}{a^2} v_c = \frac{\beta_1\xi_2 + \xi_3 k_H^2}{\beta_1 + \alpha_B^2 k_H^2} \dot{\Psi} + \frac{\beta_1\xi_4 + \beta_1\xi_5 k_H^2 + c_s^2\alpha_B \frac{\alpha_{D,c}}{1+\alpha_{D,c}} k_H^4}{\beta_1 + \alpha_B^2 k_H^2} H\Psi \\ + \sum_I \frac{3}{2} \Omega_I H \left[\frac{\beta_1\xi_{6,I} + \xi_{7,I} \tilde{k}^2}{\beta_1 + \alpha_B^2 k_H^2} \delta_I + \frac{\beta_1\xi_{8,I} + \xi_{9,I} k_H^2}{\beta_1 + \alpha_B^2 k_H^2} H v_I \right], \end{aligned} \quad (4.13)$$

$$\begin{aligned} \dot{v}_c + 3H\gamma_c v_c = -\Psi - \frac{3\gamma_c}{\beta_1 + \alpha_B^2 k_H^2} \left\{ \hat{\alpha} \frac{\dot{\Psi} + H\Psi}{2H} + k_H^2 \alpha_B \Psi \right. \\ \left. + \sum_I \frac{3}{2} \Omega_I H \left[\alpha_B \delta_I + \frac{\alpha_K - 6\alpha_B \gamma_I}{2} H v_I \right] \right\}, \end{aligned} \quad (4.14)$$

where the time-dependent coefficients ξ_a are given in Appendix E. In this case, where $\Phi = \Psi$, eqs. (4.6), (4.7), (4.10), (4.13) and (4.14) form a closed system of equations.

4.2 Quasi-static approximation

To investigate late-time cosmology, it is convenient to resort, on sufficiently short scales, to the quasi-static limit. This is justified as long as we remain on scales smaller than the

sound horizon of dark energy, i.e. $k \gg aH/c_s$ [71]. In this limit, the conservation and Euler equations for baryons and CDM (eqs. (4.6)–(4.9)) simplify to

$$\dot{\delta}_b - \frac{k^2}{a^2}v_b = 0, \quad (4.15)$$

$$\dot{v}_b = -\Phi, \quad (4.16)$$

$$\dot{\delta}_c - \frac{k^2}{a^2}v_c = 0, \quad (4.17)$$

$$\dot{v}_c + 3H\gamma_c v_c = -\Phi - 3H\gamma_c\pi. \quad (4.18)$$

In these equations, all the modifications are encoded in the single parameter γ_c . Therefore, it is not possible to disentangle the conformal and disformal effects. Note that this is due to the fact that the nonminimally coupled species is pressureless and that we restrict to the quasi-static regime.

We can then use the generalized Einstein equations to derive the Poisson equation for Φ . Combining eqs. (C.2) and (C.4) one finds

$$-\frac{k^2}{a^2}\Phi = \frac{3}{2}H^2\Omega_m \left\{ (1 + \alpha_T + \beta_\xi^2) \omega_b \delta_b + [1 + \alpha_T + \beta_\xi(\beta_\xi + \beta_\gamma)] \omega_c \delta_c \right\}, \quad \omega_I \equiv \frac{\Omega_I}{\Omega_m}, \quad (4.19)$$

where we have defined the dimensionless parameter

$$\beta_\gamma \equiv \frac{3\sqrt{2}\gamma_c}{c_s\alpha^{1/2}}, \quad (4.20)$$

which characterizes the strength of the nonminimal coupling of CDM, as well as the analogous parameter

$$\beta_\xi \equiv \frac{\sqrt{2}\xi}{c_s\alpha^{1/2}}, \quad (4.21)$$

associated with the modified gravity coefficient ξ defined in (4.12). Note that the denominator in the definitions of β_γ and β_ξ is real, since stability requires that $c_s^2\alpha \geq 0$.

Moreover, in the quasi-static limit the evolution equation (C.7) for π reduces to a constraint equation, which reads

$$-\frac{k^2}{a^2}\pi = 3H\Omega_m \frac{\beta_\xi\omega_b\delta_b + (\beta_\xi + \beta_\gamma)\omega_c\delta_c}{\sqrt{2}c_s\alpha^{1/2}}. \quad (4.22)$$

Substituting (4.19) and (4.22) into the matter equations (4.15)–(4.18), we obtain two coupled second-order differential equations for the two density contrasts:

$$\ddot{\delta}_b + 2H\dot{\delta}_b = \frac{3}{2}H^2\Omega_m \left\{ (1 + \alpha_T + \beta_\xi^2)\omega_b\delta_b + [1 + \alpha_T + \beta_\xi(\beta_\xi + \beta_\gamma)] \omega_c\delta_c \right\}, \quad (4.23)$$

$$\ddot{\delta}_c + (2 - 3\gamma_c)H\dot{\delta}_c = \frac{3}{2}H^2\Omega_m \left\{ [1 + \alpha_T + \beta_\xi(\beta_\xi + \beta_\gamma)]\omega_b\delta_b + [1 + \alpha_T + (\beta_\xi + \beta_\gamma)^2] \omega_c\delta_c \right\}. \quad (4.24)$$

In the absence of nonminimal coupling of CDM ($\beta_\gamma = 0$), the gravitational coupling of both species is modified by the same factor $1 + \alpha_T + \beta_\xi^2$. In the absence of modified gravity

($\beta_\xi = 0$ and $\alpha_T = 0$), one finds that the nonminimal coupling of CDM ($\beta_\gamma \neq 0$) modifies the friction term for δ_c , as well as increases the coefficient in front of δ_c in the second equation, whereas all other three coefficients on the right hand sides are unchanged. This is the result obtained in the context of coupled dark energy (see e.g. [51]). By contrast, if one combines modified gravity ($\beta_\xi \neq 0$) with a nonminimal coupling of CDM, all four coefficients on the right hand sides are modified. We leave for the future the detailed study of how these new coefficients parametrize the influence of modified gravity on structure formation.

Let us now turn to the two gravitational potentials Φ and Ψ . When considering the impact of dark energy on observations, it is often convenient to express the new relations between the two potentials Ψ and Φ and the total matter density fluctuations in terms of modifications of the Newton constant. We thus introduce the parameters

$$\mu_\Phi \equiv -\frac{2M^2 k^2 \Phi}{a^2 \rho_m \delta_m}, \quad \mu_\Psi \equiv -\frac{2M^2 k^2 \Psi}{a^2 \rho_m \delta_m}, \quad (4.25)$$

which are equal to one in the standard case. From eq. (4.19) and an analogous Poisson-like equation for Ψ , obtained by combining the Einstein equations, one finds that the above parameters are given by

$$\mu_\Phi = 1 + \alpha_T + \beta_\xi (\beta_\xi + \beta_\gamma \omega_c b_c), \quad (4.26)$$

$$\mu_\Psi = 1 + \beta_B (\beta_\xi + \beta_\gamma \omega_c b_c), \quad (4.27)$$

where we have defined

$$\beta_B \equiv \frac{\sqrt{2}\alpha_B}{c_s \alpha^{1/2}}. \quad (4.28)$$

We have also introduced a time-dependent bias parameter, $b_c \equiv \delta_c / \delta_m$.¹¹

As the gravitational lensing effect depends on the sum of the two potentials, the relevant quantity parametrizing deviations in weak lensing observables (and equal to two in the standard case) is

$$\mu_{\text{WL}} = \mu_\Psi + \mu_\Phi = 2 + \alpha_T + (\beta_B + \beta_\xi) (\beta_\xi + \beta_\gamma \omega_c b_c). \quad (4.29)$$

Thus, the impact of modifications of gravity due to non-vanishing α_B , α_M and α_T affects observable quantities in the perturbations through α_T and the combinations β_B , β_ξ . Analogously, the effect of nonminimal couplings on observations is parameterized by β_γ only (see the next section for the case of a coupled relativistic fluid, where another quantity is needed to parameterize the nonminimal coupling). Note that as a consequence of dropping time derivatives in the fluctuations of π , the parameter α always appears multiplied by c_s^2 . From the definition of the sound speed, eq. (2.39), $c_s^2 \alpha$ is independent of α_K , so that the latter cannot be constrained by observations in the quasi-static limit [16].

When $\beta_\gamma = 0$, i.e. if CDM is minimally coupled and there are no EP violations, the last term inside the parenthesis of eqs. (4.26) and (4.27) drops and these relations simplify to

$$\mu_\Psi = 1 + \beta_\xi \beta_B, \quad \mu_\Phi = 1 + \alpha_T + \beta_\xi^2. \quad (4.30)$$

¹¹In the quasi-static limit, the evolution equations (4.23) and (4.24) are scale independent so that the ratios δ_b / δ_m and δ_c / δ_m do not depend on scales.

In this case, the so-called slip parameter becomes (see for instance [14])

$$\frac{\Psi}{\Phi} = \frac{1 + \beta_\xi \beta_B}{1 + \alpha_T + \beta_\xi^2}. \quad (4.31)$$

By contrast, if there is a non trivial coupling of CDM but gravity itself is not modified, in which case we have $\alpha_B = 0$, $\alpha_M = 0$ and $\alpha_T = 0$ (thus $\xi = 0$), we recover that the Newton constant is not modified, $\mu_\Phi = \mu_\Psi = 1$, and that Φ and Ψ are the same as in GR, *even if* CDM is nonminimally coupled, as is the case in usual scenarios of coupled dark energy. In general, we find that the situation is much richer when both gravity and matter couplings are modified.

5 Matter and coupled relativistic fluid

In this section, as another example we consider nonminimally coupled relativistic particles, in the fluid approximation. They could represent neutrinos, radiation or warm dark matter in the relativistic regime. Baryons and CDM are taken to be minimally coupled, $\alpha_{D,m} = \alpha_{C,m} = 0$. In the Jordan frame of the relativistic fluid, its equation of state parameter is given by $\check{w}_r = 1/3$ (and $\check{c}_{s,r}^2 = 1/3$). Thus, in the frame where baryons and CDM are minimally coupled, the background and perturbed equations of state are

$$w_r = \frac{1}{3(1 + \alpha_{D,r})}, \quad \delta p_r = \frac{\rho_r}{3(1 + \alpha_{D,r})} \left[\delta_r + 2\alpha_{D,r}(\Phi - \dot{\pi}) - \frac{\dot{\alpha}_{D,r}}{1 + \alpha_{D,r}} \pi \right]. \quad (5.1)$$

The second relation has been obtained from eq. (3.18), using eq. (2.48) for the sound speed. To simplify the treatment, we ignore the anisotropic stress, i.e. $\sigma_r = 0$.

We are now going to assume that baryons and CDM dominate the gravitational perturbations, thus neglecting the backreaction of the relativistic fluid. On small scales, we can then resort to the quasi-static approximation. Under these conditions, the evolution equations for matter are

$$\dot{\delta}_m - \frac{k^2}{a^2} v_m = 0, \quad \dot{v}_m = -\Phi. \quad (5.2)$$

For the relativistic fluid, we use eqs. (3.16) and (3.17) and replace γ_r with the expression $\gamma_r = (\dot{\alpha}_{D,r} - 2H\alpha_{D,r})/[6H(1 + \alpha_{D,r})]$. The evolution equations then read

$$\dot{\delta}_r - \frac{4 + 3\alpha_{D,r}}{3(1 + \alpha_{D,r})} \frac{k^2}{a^2} v_r = - \frac{\alpha_{D,r}}{3(1 + \alpha_{D,r})} \frac{k^2}{a^2} \pi, \quad (5.3)$$

$$\dot{v}_r - H \left[1 - 3 \frac{2 + 3\alpha_{D,r}}{1 + 3\alpha_{D,r}} (g_{D,r} - f_{D,r}) \right] v_r + \frac{\delta_r}{4 + 3\alpha_{D,r}} = - (1 + 2f_{D,r})\Phi + 2f_{D,r}\dot{\pi} - 3Hg_{D,r}\pi, \quad (5.4)$$

where we have defined

$$f_{D,r} \equiv - \frac{\alpha_{D,r}}{4 + 3\alpha_{D,r}}, \quad g_{D,r} \equiv f_{D,r} \left[1 - \frac{(1 + 3\alpha_{D,r})\dot{\alpha}_{D,r}}{6(1 + \alpha_{D,r})H\alpha_{D,r}} \right]. \quad (5.5)$$

As expected, for $\alpha_{D,r} = 0$ the effects due to the nonminimal coupling vanish. Note that, even if we are in the quasi-static limit, the term in $\dot{\pi}$ should be kept in the Euler equation, as it is expected to be of the order of $H\pi$ and therefore comparable to the other terms.

Our assumption that the relativistic fluid does not contribute to the gravitational perturbations means that Φ and π are only sourced by CDM and baryons, i.e.

$$-\frac{k^2}{a^2}\Phi = \frac{3}{2}H^2\Omega_m(1 + \alpha_T + \beta_\xi^2)\delta_m, \quad -\frac{k^2}{a^2}\pi = 3H\Omega_m\frac{\beta_\xi}{\sqrt{2}c_s\alpha^{1/2}}\delta_m, \quad (5.6)$$

which correspond to eqs. (4.19) and (4.22) specialized to the case $\beta_\gamma = 0$. Therefore, even when the extra scalar field is not sourced by matter perturbation (e.g. when $\beta_\xi = 0$) and $\pi = 0$, the relativistic particles still feel a force F_r different from that felt by matter, F_m , the relative difference being given by $(F_r - F_m)/F_m = 2f_{D,r}$. This extra force is due to the non-adiabatic pressure perturbation $\delta p_{\text{nad},I}$ in eq. (3.18), induced by the disformal coupling out of the Jordan frame of the fluid (see eq. (5.1)).

To highlight this effect, the Euler and continuity equations can be combined to form a second-order differential equation for the density contrast δ_r , sourced by the matter perturbations according to eq. (5.6). In the simple case where gravity is not modified, i.e. $\beta_\xi = \alpha_T = 0$, we get

$$\ddot{\delta}_r + H\dot{\delta}_r \left(1 + 3\alpha_{D,r}\frac{1 - g_{D,r}/f_{D,r}}{1 + 3\alpha_{D,r}}\right) + \frac{k^2}{3a^2(1 + \alpha_{D,r})}\delta_r = 2H^2\Omega_m\frac{1 + \alpha_{D,r}/4}{1 + \alpha_{D,r}}\delta_m. \quad (5.7)$$

Unlike in the case of the bayons-CDM fluid, the signature of the disformal coupling here is present at the linear level in $\alpha_{D,r}$, while in eqs. (4.23)–(4.24) it appears at the quadratic level through the terms $\beta_\xi\beta_\gamma$ and β_γ^2 .

The main message of this section is that one must define the usual fluid properties (such as the equation of state and the speed of sound) in the Jordan frame, where the species is minimally coupled to gravity.

6 Conclusions

In this work, we have presented an effective description of dark energy and modified gravity, which extends the approach developed in [5] by relaxing the assumption of universal coupling of all matter species. Namely, we have allowed each matter species to be associated with a specific Jordan frame (or metric), conformally and disformally related to the gravitational metric. In this way, we have made connection with a vast sector of the literature devoted to the so-called coupled dark energy, with either a conformal coupling in most works or a disformal coupling for more recent works. However, in contrast with this previous literature, we have considered here a very general description of the gravitational sector, which includes Horndeski's theories (although not their extensions such as G^3) instead of general relativity with a quintessence-like scalar field as usually assumed.

At the level of linear perturbations, the gravitational sector is described by the quadratic action given in eq. (2.27), which depends on four time-dependent parameters α_K , α_B , α_M and α_T . As for matter, each species is characterized by two time-dependent parameters, $\alpha_{C,I}$ and $\alpha_{D,I}$, associated with their conformal and disformal couplings to the gravitational metric. This implies that the whole system depends on a total of $2N_S + 4$ time-dependent parameters, if N_S species are present. However, there is some arbitrariness in the choice of the gravitational metric that is used to define the gravitational and matter sectors. By considering a conformal-disformal transformation (2.42) of this metric, the same physical

system is characterized by $2N_S + 4$ new parameters, which transform according to (2.45) and (2.46). Taking into account this “gauge” redundance, which depends on two arbitrary parameters, one thus finds that the number of physically relevant parameters is reduced to $2(N_S + 1)$.

A very useful result of the present work is the derivation of the linear stability conditions in this very general framework. As the presence of disformal couplings contributes to the kinetic energy of the scalar fluctuations, the condition for the absence of ghosts is modified. This now requires that α defined in eq. (2.36) is positive. We have checked that the stability conditions are invariant under the “gauge transformations” of the parameters discussed above.

We have also written the equations of motion for the linear perturbations and emphasized how the usual equations are modified in the presence of modified gravity and nonminimal (conformal or disformal) couplings. Special care must be taken when the chosen frame does not coincide with the matter Jordan frame as the relations between matter quantities are frame-dependent. For instance, the equation of state parameter, whose natural value (e.g. $1/3$ for radiation) is defined in the Jordan frame associated with the matter species, will be in general different in another frame.

We have illustrated our formalism by considering two types of scenarios, motivated by the already stringent constraints on the nonminimal coupling of ordinary species (baryons and photons) to a scalar field. In the first case, we have focused our attention to the situation where only CDM is nonminimally coupled to the scalar field. For late cosmology, in the quasi-static approximation, we have computed the evolution equations of CDM and baryon density contrasts. In the second case, we have assumed that both baryons and CDM are minimally coupled but allowing for a relativistic fluid (e.g. neutrinos) with nonminimal couplings. These two simple examples illustrate what kind of new effects can be produced by the combination of modified gravity and nonminimal couplings.

It would be interesting to investigate how future observations will be able to constrain simultaneously the parameters describing the deviations from GR and those characterizing the coupling of matter to this generalized gravitational sector.

Acknowledgements. We thank Federico Piazza and Ignacy Sawicki for stimulating discussions. DL and FV acknowledge the participation to the workshop “The Quest for Dark Energy II” at Ringberg Castle, where part of this work was done. JG, MM and FV thank the APC (*AstroParticule et Cosmologie*) and PCCP (Paris Center for Cosmological Physics) for kind hospitality. FV acknowledges partial support from the grant ANR-12-BS05-0002 of the French Agence Nationale de la Recherche.

A Changing frame

We consider a general disformal transformation of the metric (2.41), which in unitary gauge reads

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = C(t)g_{\mu\nu} + D(t)\delta_\mu^0\delta_\nu^0, \quad (\text{A.1})$$

and study how metric and matter quantities change under this transformation. In terms of the two time-dependent parameters C and D , the ADM components of the new metric

$\tilde{g}_{\mu\nu}$ in unitary gauge are given by

$$\tilde{N}^2 = CN^2 - D, \quad \tilde{N}^i = N^i, \quad \tilde{h}_{ij} = Ch_{ij}, \quad (\text{A.2})$$

while the intrinsic Ricci scalar and the extrinsic curvature respectively transform as

$$\tilde{R} = C^{-1}R, \quad \tilde{K}^i_j = \frac{N}{\tilde{N}} \left(K^i_j + \frac{\dot{C}}{2NC} \delta_j^i \right). \quad (\text{A.3})$$

For the matter sector, the stress-energy tensor in the new frame is

$$\tilde{T}_{(I)}^{\mu\nu} \equiv \frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S_I}{\delta \tilde{g}_{\mu\nu}}, \quad (\text{A.4})$$

so that

$$\tilde{T}_{(I)}^{\mu\nu} = \frac{\sqrt{-g}}{\sqrt{-\tilde{g}}} \frac{\delta g_{\alpha\beta}}{\delta \tilde{g}_{\mu\nu}} T_{(I)}^{\alpha\beta} = \frac{N}{C^{5/2} \sqrt{CN^2 - D}} T_{(I)}^{\mu\nu}. \quad (\text{A.5})$$

A.1 Background

Let us now set $\tilde{N} = 1$ and assume a flat FLRW background, $ds^2 = -dt^2 + a^2(t)dx^2$. From eq. (A.2), the background metric in the new frame remains flat FLRW, with line element $d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})d\mathbf{x}^2$, where we have defined

$$\tilde{t} \equiv \int \sqrt{\frac{C}{1 + \alpha_D}} dt, \quad \tilde{a} \equiv \sqrt{C}a. \quad (\text{A.6})$$

From this equation, the Hubble rate in the new frame is given by

$$\tilde{H} \equiv \frac{1}{\tilde{a}} \frac{d\tilde{a}}{d\tilde{t}} = (1 + \alpha_C) \sqrt{\frac{1 + \alpha_D}{C}} H. \quad (\text{A.7})$$

From eq. (A.5), the background energy density and pressure in the two frames are respectively related by

$$\tilde{\rho}_I = \frac{1}{C^2 \sqrt{1 + \alpha_D}} \rho_I, \quad \tilde{p}_I = \frac{\sqrt{1 + \alpha_D}}{C^2} p_I, \quad (\text{A.8})$$

where $\tilde{\rho} \equiv -\tilde{T}_0^0$ and $\tilde{p} \equiv \tilde{T}_i^i/3$. This implies $\tilde{w}_I = w_I(1 + \alpha_D)$.

In terms of these quantities the Friedmann equations (2.12) and (2.13) become

$$\tilde{H}^2 = \frac{1}{3\tilde{M}^2} (\tilde{\rho}_m + \tilde{\rho}_{\text{DE}}), \quad (\text{A.9})$$

$$\frac{d\tilde{H}}{d\tilde{t}} + \frac{3}{2}\tilde{H}^2 = -\frac{1}{2\tilde{M}^2} (\tilde{p}_m + \tilde{p}_{\text{DE}}), \quad (\text{A.10})$$

where \tilde{M}^2 is given by eq. (2.44). Using the expressions above, one can compute the relations between the background energy density and pressure of dark energy in the two frames. One finds, respectively,

$$\tilde{\rho}_{\text{DE}} = \frac{1}{C^2 \sqrt{1 + \alpha_D}} \rho_{\text{DE}} + 3M^2 H^2 \frac{\sqrt{1 + \alpha_D}}{C^2} \left[\alpha_C(2 + \alpha_C) + \frac{\alpha_D}{1 + \alpha_D} \right], \quad (\text{A.11})$$

$$\tilde{p}_{\text{DE}} = \frac{\sqrt{1 + \alpha_D}}{C^2} p_{\text{DE}} + M^2 H^2 \frac{\sqrt{1 + \alpha_D}}{C^2} \left[\alpha_C(4 + \alpha_C) + 2\alpha_C \frac{\dot{H}}{H^2} + 2\frac{\dot{\alpha}_C}{H} + \frac{\dot{\alpha}_D(1 + \alpha_C)}{H(1 + \alpha_D)} \right], \quad (\text{A.12})$$

where as usual a dot denotes a derivative with respect to t .

A.2 Perturbations

Let us now study how perturbations transform under disformal transformations. Due to the invariance of the gravitational action under disformal transformations, the perturbation equations have the same form in both frames. Thus, we just need to derive the relation between perturbation quantities in different frames.

Introducing π in eq. (A.1) via the time reparametrization (3.1), one finds, up to linear order in π ,

$$\tilde{g}_{00} = C \left[g_{00} + \frac{\alpha_D}{1 + \alpha_D} (1 + 2\dot{\pi}) - \frac{2\alpha_C}{1 + \alpha_D} H\pi + \frac{\dot{\alpha}_D}{(1 + \alpha_D)^2} \pi \right], \quad (\text{A.13})$$

$$\tilde{g}_{0i} = C \left[g_{0i} + \frac{\alpha_D}{1 + \alpha_D} \partial_i \pi \right], \quad (\text{A.14})$$

$$\tilde{g}_{ij} = C(1 + 2H\alpha_C\pi)g_{ij}. \quad (\text{A.15})$$

Thus, if we start from a perturbed FLRW metric in Newtonian gauge with $g_{0i} = 0$ we end up with $\tilde{g}_{0i} \neq 0$ after this transformation. To maintain the Newtonian gauge condition $\tilde{g}_{0i} = 0$, we need to supplement the time redefinition (A.6) with a space-dependent shift (see Appendix C of [24]), i.e.

$$\tilde{t} = \int \sqrt{\frac{C}{1 + \alpha_D}} dt - \alpha_D \sqrt{\frac{C}{1 + \alpha_D}} \pi(t, \mathbf{x}). \quad (\text{A.16})$$

Then, the new perturbations result from combination of the field redefinition (A.1) and this change of coordinates. For the metric in Newtonian gauge, this yields

$$\tilde{\Phi} = (1 + \alpha_D)\Phi + [\alpha_C(1 + \alpha_D)H + \dot{\alpha}_D] \pi, \quad (\text{A.17})$$

$$\tilde{\Psi} = \Psi - [\alpha_C(1 + \alpha_D) + \alpha_D]H\pi, \quad (\text{A.18})$$

$$\tilde{\pi} = \sqrt{C(1 + \alpha_D)} \pi. \quad (\text{A.19})$$

For the matter quantities, using eq. (A.5), one finds

$$\tilde{\delta}_I = \delta_I + \alpha_D (\Phi - \dot{\pi}) - \left[3H\alpha_D(1 + w_I - \gamma_I) + 4H\alpha_C(1 + \alpha_D) + \frac{1}{2}\dot{\alpha}_D \right] \pi, \quad (\text{A.20})$$

$$\begin{aligned} \delta\tilde{p}_I/\tilde{\rho}_I = (1 + \alpha_D) \left\{ \delta p_I/\rho_I - w_I\alpha_D (\Phi - \dot{\pi}) \right. \\ \left. - w_I \left[3H\alpha_D(1 + w_I - \gamma_I) + 4H\alpha_C(1 + \alpha_D) - \frac{1}{2}\dot{\alpha}_D - \frac{\dot{w}_I}{w_I}\alpha_D \right] \pi \right\}, \end{aligned} \quad (\text{A.21})$$

$$\tilde{v}_I = \frac{\sqrt{C(1 + \alpha_D)}}{1 + w_I(1 + \alpha_D)} [(1 + w_I)v_I - w_I\alpha_D\pi], \quad (\text{A.22})$$

$$\tilde{\sigma}_I = \frac{\sqrt{1 + \alpha_D}}{C} \sigma_I. \quad (\text{A.23})$$

One can relate the pressure and density perturbations via the speed of sound, which is defined as the ratio between these two quantities in a coordinate system where the fluid is at rest. In the Jordan frame of the fluid, this gives [72]

$$\delta\tilde{p}_I = \tilde{c}_{s,I}^2 \delta\tilde{\rho}_I - (1 + \tilde{w}_I)\tilde{\rho}_I \left[3\tilde{H}(\tilde{c}_{s,I}^2 - \tilde{w}_I) + \frac{d\tilde{w}_I/d\tilde{t}}{1 + \tilde{w}_I} \right] \tilde{v}_I. \quad (\text{A.24})$$

One can then use eqs. (A.20)–(A.22) to rewrite this equation in a generic frame. This yields

$$\begin{aligned} \delta p_I = c_{s,I}^2 \delta \rho_I - \rho_I \left[3H(\tilde{c}_{s,I}^2 - \tilde{w}_I)(1 + \alpha_C) + \frac{\dot{\tilde{w}}_I}{1 + \tilde{w}_I} \right] \left[(1 + w_I)v_I + \frac{\alpha_D}{1 + \alpha_D} \pi \right] \\ - 4H\rho_I(c_{s,I}^2 - w_I)\alpha_C\pi + \rho_I(c_{s,I}^2 + w_I) \left[2\alpha_D(\Phi - \dot{\pi}) - \frac{\dot{\alpha}_D}{1 + \alpha_D} \pi \right], \end{aligned} \quad (\text{A.25})$$

where we recall that, from eqs. (A.8) and (2.48), the equation of state parameters and sound speeds defined in the two frames are respectively related by $\tilde{w}_I = (1 + \alpha_D)w_I$ and $\tilde{c}_{s,I}^2 = (1 + \alpha_D)c_{s,I}^2$.

B Explicit quadratic action

B.1 Matter action

For simplicity, we assume that each matter species can be described by a perfect fluid with vanishing vorticity (this restriction does not affect the analysis of *scalar* linear modes). It is then easy to write an action in terms of a derivatively coupled scalar field with Lagrangian¹²

$$S_m = \sum_I^N S_I, \quad S_I = \int d^4x \sqrt{-\check{g}^{(I)}} P_I(Y_I), \quad Y_I \equiv \check{g}_{(I)}^{\mu\nu} \partial_\mu \sigma_I \partial_\nu \sigma_I. \quad (\text{B.1})$$

The second-order expansion of the action S_I reads

$$\begin{aligned} S_I^{(2)} = \int d^3x dt \bar{N} \frac{a^3}{c_{s,I}^2} \left\{ \frac{1 + \alpha_{D,I} c_{s,I}^2 + (1 + \alpha_{D,I})w_I}{2} \rho_I \left(\frac{\delta N}{\bar{N}} \right)^2 \right. \\ \left. - \frac{1 + (1 + \alpha_{D,I})w_I}{(1 + \alpha_{D,I})^2} \frac{\rho_I}{\dot{\sigma}_I} \left[\delta \dot{\sigma}_I \left(\frac{\delta N}{\bar{N}} - c_{s,I}^2 \delta \sqrt{h} \right) + c_{s,I}^2 N^i \partial_i \delta \sigma_I \right] \right. \\ \left. + \frac{1 + (1 + \alpha_{D,I})w_I}{(1 + \alpha_{D,I})^2} \frac{\rho_I}{2\dot{\sigma}_I^2} \left[\delta \dot{\sigma}_I^2 - \bar{N}^2 c_{s,I}^2 \frac{(\partial_i \delta \sigma)^2}{a^2} \right] \right\}, \end{aligned} \quad (\text{B.2})$$

where we have split the scalar field σ_I into a background value and its perturbations, $\sigma_I = \bar{\sigma}_I(t) + \delta \sigma_I(t, \mathbf{x})$. The fluid quantities are related to the function $P_I(Y_I)$ through

$$\begin{aligned} p_I \equiv \frac{C_I^2}{\sqrt{1 + \alpha_{D,I}}} P_I, \quad \rho_I \equiv C_I^2 \sqrt{1 + \alpha_{D,I}} (2Y_I P_I' - P_I), \\ c_{s,I}^2 \equiv \frac{P_I'}{P_I' + 2Y_I P_I''} (1 + \alpha_{D,I})^{-1}, \end{aligned} \quad (\text{B.3})$$

where a prime denotes a derivative with respect to the variable Y_I . We have omitted in the action irrelevant terms that vanish when imposing the background equations of motion. For $C_I = 1$ and $\alpha_{D,I} = 0$ we recover the usual expressions for a k -essence fluid [73, 74, 76].

¹²The more general k -essence type Lagrangian of Refs. [73, 74] explicitly depends also on the scalar field. Since here we are interested only in the derivative terms, we assume for simplicity that P_I depends only on Y_I and not on σ_I . This description implies that each of the fluids is also barotropic [75], i.e. that its pressure is a function of its energy density, $p_I = p_I(\rho_I)$.

B.2 Stability and sound speed of dark energy

In order to investigate linear stability issues, we need to extract the quadratic action for the propagating degrees of freedom. We concentrate on scalar modes as the stability conditions of tensors are not modified by the nonminimal coupling of matter. To this end, we will expand the total action up to quadratic order in linear scalar fluctuations around a FLRW solution and solve the constraints, generalizing the procedure of Refs. [77] and [24].

The second-order action

$$S^{(2)} = S_{\text{g}}^{(2)} + S_{\text{m}}^{(2)}, \quad (\text{B.4})$$

where the gravitational part $S_{\text{g}}^{(2)}$ is given in eq. (2.27), governs the dynamics of linear scalar fluctuations. Assuming $\bar{N} = 1$ without loss of generality, the scalar modes can be described in unitary gauge by the metric perturbations [78]

$$N = 1 + \delta N, \quad N^i = \delta^{ij} \partial_j \psi, \quad h_{ij} = a^2(t) e^{2\zeta} \delta_{ij}. \quad (\text{B.5})$$

As a consequence, we get

$$\delta\sqrt{h} = 3a^3\zeta, \quad \delta K^i_j = (\dot{\zeta} - H\delta N)\delta_j^i - \delta^{ik}\partial_k\partial_j\psi, \quad (\text{B.6})$$

and

$$\delta_1 R_{ij} = -\delta_{ij}\partial^2\zeta - \partial_i\partial_j\zeta, \quad \delta_2 R = -\frac{2}{a^2} [(\partial\zeta)^2 - 4\zeta\partial^2\zeta]. \quad (\text{B.7})$$

(The metric perturbations δN and ζ and the scalar fluctuation ψ are related to the metric perturbations in Newtonian gauge by $\delta N = \Phi - \dot{\pi}$, $\zeta = -\Psi - H\pi$ and $\psi = a^{-2}\pi$.) Substituting these expressions into (B.4), we obtain the second-order action in terms of the three scalar quantities δN , ψ and ζ . Variation with respect to ψ yields the momentum constraint, whose solution reads

$$\delta N = \frac{1}{1 + \alpha_{\text{B}}} \left(\frac{\dot{\zeta}}{H} + \frac{3}{2}H \sum_I \frac{1 + (1 + \alpha_{\text{D},I})w_I}{1 + \alpha_{\text{D},I}} \Omega_I \frac{\delta\sigma_I}{\dot{\sigma}_I} \right), \quad (\text{B.8})$$

with $\Omega_I = \rho_I/(3M^2H^2)$.

We do not need the solution of the Hamiltonian constraint, as the longitudinal part of the shift ψ only contributes to a boundary term in the action. Replacing the above solution into the second-order action and re-expressing the scalar fields perturbations $\delta\sigma_I$ in terms of the gauge invariant variables

$$\mathcal{Q}_I \equiv \delta\sigma - \frac{\dot{\sigma}_I}{H}\zeta, \quad (\text{B.9})$$

the total second-order action reads, focusing only on the kinetic and spatial gradient parts,

$$S^{(2)} = \int d^3x dt a^3 \frac{M^2}{2} \left[g_{\zeta\zeta} \dot{\zeta}^2 + g_{\partial\zeta\partial\zeta} \frac{(\partial_i\zeta)^2}{a^2} + \sum_I \frac{\kappa_I H^2}{\dot{\sigma}_I^2 c_{s,I}^2} \left(\dot{\mathcal{Q}}_I^2 - c_{s,I}^2 \frac{(\partial_i\mathcal{Q}_I)^2}{a^2} \right) \right. \\ \left. + 2 \sum_I \frac{g_{\text{int},I} H}{\dot{\sigma}_I} \left(\dot{\mathcal{Q}}_I \dot{\zeta} - \frac{c_{s,I}^2}{a^2} \partial_i \mathcal{Q}_I \partial_i \zeta \right) \right], \quad (\text{B.10})$$

with

$$g_{\zeta\zeta} \equiv \frac{1}{(1 + \alpha_B)^2} \left[\alpha + \sum_I \frac{\kappa_I}{c_{s,I}^2} (\alpha_{D,I} - \alpha_B)^2 \right], \quad (\text{B.11})$$

$$g_{\text{int},I} \equiv \frac{1}{1 + \alpha_B} \sum_I \frac{\kappa_I}{c_{s,I}^2} (\alpha_{D,I} - \alpha_B), \quad (\text{B.12})$$

$$g_{\partial\zeta\partial\zeta} \equiv \frac{2}{1 + \alpha_B} \left[\frac{\dot{H}}{H^2} + \frac{\dot{\alpha}_B}{1 + \alpha_B} + \alpha_B(1 + \alpha_T) + \alpha_T - \alpha_M + \sum_I \frac{\kappa_I}{2} (1 + 2\alpha_{D,I} - \alpha_B) \right], \quad (\text{B.13})$$

where we have defined the dimensionless coefficients

$$\alpha \equiv \alpha_K + 6\alpha_B^2 + 3 \sum_I \alpha_{D,I} \Omega_I, \quad \kappa_I \equiv 3 \frac{1 + (1 + \alpha_{D,I})w_I}{(1 + \alpha_{D,I})^2} \Omega_I. \quad (\text{B.14})$$

Absence of ghosts is ensured by requiring that the matrix of the kinetic coefficients is positive definite, which yields the conditions $\alpha \geq 0$ and $\kappa_I \geq 0$. The second condition reads $\rho_I + (1 + \alpha_{D,I})p_I \geq 0$, which is the usual Null Energy Condition written in a disformed frame.

Diagonalization of the kinetic-spatial gradient matrix yields the following speed of propagation for dark energy,

$$c_s^2 = -\frac{2}{\alpha} \left\{ (1 + \alpha_B) \left[\frac{\dot{H}}{H^2} - \alpha_M + \alpha_T + \alpha_B(1 + \alpha_T) \right] + \frac{\dot{\alpha}_B}{H} + \frac{3}{2} \sum_I \left[1 + (1 + \alpha_{D,I})w_I \right] \Omega_I \right\}. \quad (\text{B.15})$$

Absence of gradient instabilities requires $c_s^2 \geq 0$ and $c_{s,I}^2 \geq 0$.

C Perturbation equations

Here we provide the generalized Einstein equations in the presence of dark energy and modifications of gravity. These have been first given in Ref. [5] in terms of the parameters of the Effective Field Theory of dark energy [4] and in Refs. [17] (see also [14]) in terms of the parameters α_a .

C.1 Einstein equations

Let us defined

$$w_m \equiv \sum_I \frac{\rho_I}{\rho_m} w_I, \quad \gamma_m \equiv \sum_I \frac{\rho_I}{\rho_m} \gamma_I, \quad (\text{C.1})$$

where γ_I parametrizes the nonminimal coupling of the species I , see definition in eq. (3.8). The Hamiltonian constraint ((00) component of the Einstein equation) is

$$\begin{aligned} & 6(1 + \alpha_B)H\dot{\Psi} + (6 - \alpha_K + 12\alpha_B)H^2\Phi + 2\frac{k^2}{a^2}\Psi + (\alpha_K - 6\alpha_B)H^2\dot{\pi} \\ & + 6 \left[(1 + \alpha_B)\dot{H} + \frac{3}{2}H^2\Omega_m(1 + w_m - \gamma_m) - \frac{1}{3}\frac{k^2}{a^2}\alpha_B \right] H\pi = -3\Omega_m H^2\delta_m, \end{aligned} \quad (\text{C.2})$$

while the momentum constraint ((0*i*) components of the Einstein equation) reads

$$2\dot{\Psi} + 2(1 + \alpha_B)H\Phi - 2H\alpha_B\dot{\pi} + [2\dot{H} + 3H^2\Omega_m(1 + w_m)]\pi = -3H^2\Omega_m(1 + w_m)v_m. \quad (\text{C.3})$$

The traceless part of the *ij* components of the Einstein equation gives

$$\Phi - (1 + \alpha_T)\Psi + (\alpha_M - \alpha_T)H\pi = -\frac{\sigma_m}{M^2}, \quad (\text{C.4})$$

while the trace of the same components gives, using the equation above,

$$\begin{aligned} & 2\ddot{\Psi} + 2(3 + \alpha_M)H\dot{\Psi} + 2(1 + \alpha_B)H\dot{\Phi} \\ & + 2 \left[\dot{H} - \frac{3}{2}H^2\Omega_m(1 + w_m) + (\alpha_B H)' + (3 + \alpha_M)(1 + \alpha_B)H^2 \right] \Phi \\ & - 2H\alpha_B\ddot{\pi} + 2 \left[\dot{H} + \frac{3}{2}H^2\Omega_m(1 + w_m) - (\alpha_B H)' - (3 + \alpha_M)\alpha_B H^2 \right] \dot{\pi} \\ & + 2 \left\{ (3 + \alpha_M)H\dot{H} + \frac{3}{2}H^2\Omega_m[\dot{w}_m - 3H(1 + w_m - \gamma_m)] + \ddot{H} \right\} \pi = \frac{1}{M^2} \left(\delta p_m - \frac{2k^2}{3a^2}\sigma_m \right). \end{aligned} \quad (\text{C.5})$$

C.2 Scalar field equation

The charge Q_I is defined in eq. (3.4). Its perturbation reads

$$\begin{aligned} \delta Q_I & \equiv 3H \left[\gamma_I \delta_I - \alpha_{C,I} \left(\frac{\delta p_I}{\rho_I} - w_I \delta_I \right) \right] \rho_I - \alpha_{D,I} \frac{1 + w_I}{1 + \alpha_{D,I}} \frac{k^2}{a^2} \rho_I v_I \\ & + 2H \left[\alpha_{C,I} (1 - 3w_I) - 3\gamma_I \right] \rho_I \Phi - \frac{\alpha_{D,I}}{1 + \alpha_{D,I}} \left(\dot{\Phi} + 3\dot{\Psi} - \dot{\delta}_I - \ddot{\pi} + w_I \frac{k^2}{a^2} \pi \right) \rho_I \\ & + \frac{H}{1 + \alpha_{D,I}} \left[-2\alpha_{C,I} (1 - 3w_I) (1 + \alpha_{D,I}) + 3w_I \alpha_{D,I} + 3\gamma_I (2 + \alpha_{D,I}) \right] \rho_I \dot{\pi} \\ & + \frac{3}{1 + \alpha_{D,I}} \left\{ (w_I \alpha_{D,I} + \gamma_I) \dot{H} + [(\alpha_{D,I} + \alpha_{C,I} (1 + \alpha_{D,I})) \dot{w}_I + \dot{\gamma}_I] H \right\} \rho_I \pi. \end{aligned} \quad (\text{C.6})$$

We have checked that this expression agrees with those in the literature (see e.g. [41–43, 47, 64]) in the relevant limits.¹³

The evolution equation for π in the absence of EP violations is given in [5] and can be found in [14] in terms of the parameters used in this article. Including the contribution of $\sum_I \delta Q_I$ using the above equation, and using the continuity equation, eq. (3.16), this

¹³The expressions for the charge Q_I given in eqs. (3.6) and (C.6) are in unitary gauge. To compare to those in the literature, one must rescale by a factor ϕ , i.e. $Q_I \mapsto Q_I \phi$ and $\delta Q_I \mapsto \delta Q_I \phi + \dot{Q}_I (\dot{\phi}/\dot{\phi}) \pi$.

becomes

$$\begin{aligned}
& \left(\alpha_K + 3 \sum_I \alpha_{D,I} \Omega_I \right) H^2 \ddot{\pi} + \left\{ \left[H^2(3 + \alpha_M) + \dot{H} \right] \alpha_K + (H \alpha_K) \cdot \right. \\
& - 3H^2 \sum_I \Omega_I \left[2\alpha_{C,I}(1 - 3w_I)(1 + \alpha_{D,I}) - 3w_I \alpha_{D,I} - 6\gamma_I(1 + \alpha_{D,I}) \right] \left. \right\} H \dot{\pi} \\
& + 3 \left\{ 2\dot{H}^2 + 3\dot{H}H^2 \left[\Omega_m(1 + w_m) + \sum_I w_I \alpha_{D,I} \Omega_I \right] + 2\dot{H} \alpha_B \left[H^2(3 + \alpha_M) + \dot{H} \right] \right. \\
& + 2H(\dot{H} \alpha_B) \cdot + 3H^3 \sum_I \left[\dot{w}_I(\alpha_{D,I} + \alpha_{C,I}(1 + \alpha_{D,I})) + 3H\gamma_I(1 + w_I - \gamma_I) \right] \Omega_I \left. \right\} \pi \\
& - \frac{k^2}{a^2} \left\{ 2\dot{H} + 3H^2 \Omega_m(1 + w_m) + 2H^2 \left[\alpha_B(1 + \alpha_M) + \alpha_T - \alpha_M \right] + 2(H \alpha_B) \cdot \right. \\
& + 3H^2 \sum_I w_I \alpha_{D,I} \Omega_I \left. \right\} \pi + 6H \alpha_B \ddot{\Psi} + \left[H^2(6\alpha_B - \alpha_K) - 3H^2 \sum_I \alpha_{D,I} \Omega_I \right] \dot{\Phi} \\
& + 3 \left[2\dot{H} + 3H^2 \Omega_m(1 + w_m) + 2H^2 \alpha_B(3 + \alpha_M) + 2(\alpha_B H) \cdot + 3H^2 \sum_I w_I \alpha_{D,I} \Omega_I \right] \dot{\Psi} \\
& + \left\{ 6\dot{H} + 9H^2 \Omega_m(1 + w_m) + H^2(6\alpha_B - \alpha_K)(3 + \alpha_M) + 2(9\alpha_B - \alpha_K) \dot{H} + H(6\dot{\alpha}_B - \dot{\alpha}_K) \right. \\
& - 6H^2 \sum_I \left[3\gamma_I - \alpha_{C,I}(1 - 3w_I) \right] (1 + \alpha_{D,I}) \Omega_I \left. \right\} H \Phi + 2 \frac{k^2}{a^2} \{ [H(\alpha_M - \alpha_T)] \Psi - \alpha_B H \Phi \} \\
& + 9H^3 \sum_I \left\{ \gamma_I \delta_I - [\alpha_{C,I}(1 + \alpha_{D,I}) + \alpha_{D,I}] \left(\frac{\delta p_I}{\rho_I} - w_I \delta_I \right) \right\} \Omega_I = 0.
\end{aligned} \tag{C.7}$$

D Synchronous gauge

Here we provide the perturbation equations in synchronous gauge, often employed in numerical codes, where the perturbed FLRW metric has the form

$$ds^2 = -dt^2 + \left[\left(1 + \frac{1}{3}h \right) \delta_{ij} + \left(\frac{k_i k_j}{k^2} - \frac{1}{3} \delta_{ij} \right) (h + 6\eta) \right] dx^i dx^j. \tag{D.1}$$

Defining $\epsilon \equiv a^2(\dot{h} + 6\dot{\eta})/k^2$, one can write Newtonian gauge quantities in terms of synchronous gauge ones using the following relations (see for instance [79]),

$$\begin{aligned}
\Phi &= \dot{\epsilon}, & \Psi &= \eta - H\epsilon, & \pi^{(N)} &= \pi^{(S)} + \epsilon, \\
\delta\rho_I^{(N)} &= \delta\rho_I^{(S)} + \dot{\rho}_I \epsilon, & \delta p_I^{(N)} &= \delta p_I^{(S)} + \dot{p}_I \epsilon, & v_I^{(N)} &= -\theta_I^{(S)}/k^2 - \epsilon,
\end{aligned} \tag{D.2}$$

where we have introduced the divergence of the velocity, $\theta_I \equiv -k^2 v_I/a$. (The anisotropic stress is gauge invariant.) We can then use the above relations to rewrite eqs. (C.2)–(C.5) in synchronous gauge. To do this, we use conformal time, $\tau \equiv \int dt/a$, and denote by a prime the derivative with respect to it. Rescaling the scalar fluctuation π by the conformal

factor, $\pi \rightarrow \pi/a$, and defining the conformal Hubble rate as $\mathcal{H} \equiv a'/a$, one obtains ((00) component)

$$2k^2\eta - \mathcal{H}(1 + \alpha_B)h' - \mathcal{H}^2(6\alpha_B - \alpha_K)\pi' + \left[9\mathcal{H}^2\Omega_m(1 + w_m - \gamma_m) - \mathcal{H}^2(6 - \alpha_K + 12\alpha_B) + 6\mathcal{H}'(1 + \alpha_B) - 2k^2\alpha_B\right]\mathcal{H}\pi = -\frac{a^2}{M^2}\rho_m\delta_m, \quad (\text{D.3})$$

((0i) component)

$$2\eta' - 2\mathcal{H}\alpha_B\pi' + \left[2\mathcal{H}' - 2(1 + \alpha_B)\mathcal{H}^2 + 3\mathcal{H}^2\Omega_m(1 + w_m)\right]\pi = \frac{a^2}{M^2}(\rho_m + p_m)\frac{\theta_m}{k^2}, \quad (\text{D.4})$$

((ij)-traceless)

$$h'' + 6\eta'' + \mathcal{H}(2 + \alpha_M)(h' + 6\eta') - 2k^2(1 + \alpha_T)\eta - 2k^2\mathcal{H}(\alpha_T - \alpha_M)\pi = \frac{2k^2}{M^2}\sigma_m, \quad (\text{D.5})$$

and ((ij)-trace)

$$\begin{aligned} & h'' + \mathcal{H}(2 + \alpha_M)h' - 2k^2(1 + \alpha_T)\eta + 6\alpha_B\mathcal{H}\pi'' \\ & + \left[6\mathcal{H}^2\alpha_B(3 + \alpha_M) + 6(\alpha_B\mathcal{H})' - 9\mathcal{H}^2\Omega_m(1 + w_m) - 6(\mathcal{H}' - \mathcal{H}^2)\right]\pi' \\ & + \left\{6\mathcal{H}^2[2 + \alpha_M + \alpha_B(2 + \alpha_M)] + 6(\alpha_B - \alpha_M)\mathcal{H}' + 6(\alpha_B\mathcal{H})' - 2k^2(\alpha_T - \alpha_M) \right. \\ & \left. - 9\mathcal{H}^2\Omega_m\left[(1 - 3w_m)(1 + w_m) + 3w_m\gamma_m + w'_m/\mathcal{H}\right] - 6\mathcal{H}''\right\}\mathcal{H}\pi = -3\frac{a^2}{M^2}\delta p_m. \end{aligned} \quad (\text{D.6})$$

In synchronous gauge, the evolution equation for the scalar fluctuation, eq. (C.7), reads

$$\begin{aligned} & \mathcal{H}^2\left(\alpha_K + 3\sum_I\alpha_{D,I}\Omega_I\right)\pi'' + \left\{\mathcal{H}^2\alpha_K(2 + \alpha_M) + \mathcal{H}'\alpha_K + (\alpha_K\mathcal{H})' + \right. \\ & \left. - 3\mathcal{H}^2\sum_I\Omega_I\left[2\alpha_{C,I}(1 - 3w_I)(1 + \alpha_{D,I}) - \alpha_{D,I}(1 + 3w_I) - 6\gamma_I(1 + \alpha_{D,I})\right]\right\}\mathcal{H}\pi' \\ & - 2k^2\left\{\mathcal{H}^2[\alpha_B\alpha_M + \alpha_T - \alpha_M - 1] + (\alpha_B\mathcal{H})' + \mathcal{H}' + \frac{3}{2}\sum_I\mathcal{H}^2[1 + w_I(1 + \alpha_{D,I})]\Omega_I\right\}\pi \\ & + \left\{\mathcal{H}^4[6 - 6\alpha_B\alpha_M + \alpha_K(1 + \alpha_M)] + 3\mathcal{H}^2\mathcal{H}'[-4 + \alpha_K - 2\alpha_B(3 - \alpha_M)] \right. \\ & \left. + 6(1 + \alpha_B)\mathcal{H}'^2 - \mathcal{H}^3(6\alpha'_B - \alpha'_K) + 6\mathcal{H}(\alpha_B\mathcal{H}')' \right. \\ & \left. + 3\mathcal{H}^2\sum_I\Omega_I\left[-2\mathcal{H}^2\alpha_{C,I}(1 - 3w_I)(1 + \alpha_{D,I}) + \mathcal{H}'(3 + \alpha_{D,I} + 3w_I(1 + \alpha_{D,I}))\right] \right. \\ & \left. + 3\mathcal{H}^2\left(\gamma_I(1 + 2\alpha_{D,I}) - (1 - 3\gamma_I)(1 + w_I - \gamma_I)\right) + 3\mathcal{H}w'_I(\alpha_{D,I} + \alpha_{C,I}(1 + \alpha_{D,I}))\right\}\pi \\ & - \mathcal{H}\alpha_B h'' - \left\{\mathcal{H}' + \mathcal{H}^2[\alpha_B(1 + \alpha_M) - 1] + (\alpha_B\mathcal{H})' + \frac{3}{2}\sum_I\mathcal{H}^2[1 + w_I(1 + \alpha_{D,I})]\Omega_I\right\}h' \\ & + 2k^2\mathcal{H}(\alpha_M - \alpha_T)\eta = 0. \end{aligned} \quad (\text{D.7})$$

The continuity and Euler equations for matter become, respectively,

$$\begin{aligned}
& \delta'_I + 3\mathcal{H}(1 + \alpha_{C,I})(1 + \alpha_{D,I}) \left(\frac{\delta p_I}{\rho_I} - w_I \delta_I \right) + (1 + w_I)\theta_I - \alpha_{D,I}\pi'' \\
& - \mathcal{H} \left[-2\alpha_{C,I}(1 - 3w_I)(1 + \alpha_{D,I}) + \alpha_{D,I}(1 + 3w_I + 6\gamma_I) + 9\gamma_I \right] \pi' \\
& + \left\{ -2\mathcal{H}^2(1 + \alpha_{D,I})(3\gamma_I - \alpha_{C,I}(1 - 3w_I)) + w_I \alpha_{D,I} k^2 \right. \\
& \left. - \mathcal{H}'(3\gamma_I + \alpha_{D,I}(1 + 3w_I)) - 3\mathcal{H} \left[w'_I(\alpha_{D,I} + \alpha_{C,I}(1 + \alpha_{D,I})) + \gamma'_I \right] \right\} \pi \\
& + \frac{1}{2} \left[1 + w_I(1 + \alpha_{D,I}) \right] h' = 0,
\end{aligned} \tag{D.8}$$

and

$$\theta'_I + \mathcal{H} \left[1 - 3w_I + 3\gamma_I + \frac{w'_I}{\mathcal{H}(1 + w_I)} \right] \theta_I - \frac{k^2 \delta p_I}{\rho_I(1 + w_I)} + \frac{2k^4}{3a^2 \rho_I(1 + w_I)} \sigma_I = \frac{3\mathcal{H}\gamma_I}{(1 + w_I)} k^2 \pi. \tag{D.9}$$

E Definitions of the parameters

The coefficients β_a appearing in eqs. (4.10) and (4.11) are defined as

$$\beta_1 \equiv -\frac{3}{4}\Omega_m \alpha_K - \frac{1}{2}\hat{\alpha} \left(\frac{\dot{H}}{H^2} + \alpha_T - \alpha_M \right) - \frac{9}{2}\alpha_B \gamma_c \Omega_c, \tag{E.1}$$

$$\begin{aligned}
\frac{\beta_1 \beta_2}{\hat{\alpha}} & \equiv \frac{9}{2}\Omega_m \alpha_B \left[\frac{\alpha_B}{\hat{\alpha}} \beta_3 - \frac{4 + \alpha_M + \alpha_T}{6\alpha_B} + \frac{\xi}{\alpha} \right] - \frac{9}{4}\gamma_c \Omega_c \left(c_s^2 - \frac{2\alpha_B \beta_3}{\hat{\alpha}} - 2\alpha_B \frac{3 - 3\gamma_c - \xi}{\alpha} \right) \\
& + \frac{1}{2}(1 + \alpha_M) \left[(\alpha_M - \alpha_T) - \frac{\dot{H}}{H^2} \right] - \frac{1}{2} \left[\frac{\dot{\alpha}_M - \dot{\alpha}_T}{H} + \frac{2\dot{H}(\alpha_M - \alpha_T)}{H^2} - \frac{\ddot{H}}{H^3} \right],
\end{aligned} \tag{E.2}$$

$$\begin{aligned}
\beta_3 & \equiv 3 + \alpha_M \frac{\hat{\alpha}}{\alpha} - 6 \frac{(\alpha_{C,c} - 3\gamma_c)(1 + \alpha_{D,c}) + \alpha_{D,c} \Omega_c}{\alpha} + \frac{\alpha_B^2}{H\alpha} \left(\frac{\alpha_K}{\alpha_B^2} \right) \\
& + 3\Omega_c \frac{\alpha_{D,c}}{\alpha_B \alpha} \left[\frac{\dot{H}}{H^2} - \alpha_M + (1 + \alpha_B)\alpha_T - \frac{(\alpha_B H)}{H^2} + \frac{3}{2}\Omega_m \right],
\end{aligned} \tag{E.3}$$

$$\beta_4 \equiv (1 + \alpha_T)(\beta_2 - 1 - \alpha_M + 2\dot{H}/H^2) + \dot{\alpha}_T/H, \tag{E.4}$$

$$\beta_5 \equiv c_s^2 - \frac{2\alpha_B(\beta_3 - \beta_2)}{\hat{\alpha}} + \frac{\alpha_B^2}{\beta_1}(1 + \alpha_T)(\beta_3 - \beta_2) + \frac{\alpha_B^2 \beta_4}{\beta_1}, \tag{E.5}$$

$$\beta_{6,I} \equiv \beta_{7,I} + 2 \frac{\alpha_B(\beta_2 - \beta_3)}{\hat{\alpha}}, \tag{E.6}$$

$$\beta_{7,I} \equiv c_s^2 + 2 \frac{\alpha_B \xi}{\alpha}, \tag{E.7}$$

$$\beta_{8,I} \equiv \beta_{9,I} - (6\alpha_B - \alpha_K) \frac{\beta_2 - \beta_3}{\hat{\alpha}}, \tag{E.8}$$

$$\beta_{9,I} \equiv -(4 + 3c_s^2 + \alpha_M + \alpha_T) + \beta_3, \tag{E.9}$$

where we remind that $\xi \equiv \alpha_B(1 + \alpha_T) + \alpha_T - \alpha_M$, $\alpha = \alpha_K + 6\alpha_B^2 + 3\alpha_{D,c}\Omega_c$ and we have defined $\hat{\alpha} \equiv \alpha_K + 6\alpha_B^2 = \alpha - 3\alpha_{D,c}\Omega_c$. Setting $\alpha_{D,c} = \alpha_{C,c} = \gamma_c = 0$ in these equations, one recovers the expressions of [17] and [14].

For $\alpha_M = \alpha_T = 0$, the coefficients ξ_a appearing in eqs. (4.13) and (4.14) are defined as

$$\xi_2 \equiv \frac{3}{2} \frac{\hat{\alpha}}{\beta_1} \left[\frac{\alpha (H\gamma)'}{\hat{\alpha} H^2} - \frac{\dot{H}}{H^2} \right] + \frac{9}{2} \frac{\Omega_m}{\beta_1} \left[\alpha_B \Xi - \frac{1}{2} \hat{\alpha} - \alpha_{D,c} \left(\frac{\hat{\alpha}}{\alpha} \alpha_B + \frac{\alpha_K}{2} c_s^2 \right) \right] - \frac{9\alpha_{D,c}\Omega_c}{2H^2\beta_1} (H\gamma)' + \frac{9\gamma_c\Omega_c}{2\beta_1} \left[3\alpha_B\alpha_{D,c}c_s^2 + \Xi + \frac{\alpha_B + 3\gamma_c - 3}{\alpha} \hat{\alpha} \alpha_{D,c} \right], \quad (\text{E.10})$$

$$\xi_3 \equiv \alpha_B \Xi - \frac{\alpha_K \alpha_{D,c}}{2} c_s^2, \quad (\text{E.11})$$

$$\xi_4 \equiv \xi_2 - 3(1 - \gamma_c), \quad (\text{E.12})$$

$$\xi_5 \equiv \frac{\alpha_B}{\beta_1} \Xi + 2 \frac{\alpha_B \xi_2 - \Xi}{\hat{\alpha}} - 3 \frac{\alpha_B^2}{\beta_1} (1 - \gamma_c) - \left[\frac{\alpha_K}{2\beta_1} + 6 \frac{\alpha_B}{\hat{\alpha}} \right] \alpha_{D,c} c_s^2, \quad (\text{E.13})$$

$$\xi_{6,I} \equiv -2\alpha_B \alpha_{D,c} \left(3 \frac{c_s^2}{\hat{\alpha}} + \frac{1}{\alpha} \right) + 2 \frac{\alpha_B \xi_2 - \Xi}{\hat{\alpha}} - 6 \frac{\alpha_{D,c}}{\alpha} \gamma_I, \quad (\text{E.14})$$

$$\xi_{7,I} \equiv -\alpha_B \alpha_{D,c} \left(c_s^2 + 2 \frac{\alpha_B^2}{\alpha} \right) - 6\alpha_B^2 \frac{\alpha_{D,c}}{\alpha} \gamma_I, \quad (\text{E.15})$$

$$\xi_{8,I} \equiv -3 + \frac{6\alpha_B - \alpha_K}{\hat{\alpha}} (3\alpha_{D,c}c_s^2 - \xi_2) + 6 \frac{1 + \alpha_B}{\hat{\alpha}} \Xi, \quad (\text{E.16})$$

$$\xi_{9,I} \equiv -3\alpha_B^2 + \frac{6\alpha_B - \alpha_K}{2} \alpha_{D,c} c_s^2 + \alpha_B \Xi, \quad (\text{E.17})$$

with

$$\Xi \equiv 3\alpha_B + 3\gamma_c - \frac{2\hat{\alpha}}{\alpha} \left[\alpha_{D,c} + \alpha_{C,c}(1 + \alpha_{D,c}) - 3\gamma_c(1 + \alpha_{D,c}) + \alpha_{D,c}\dot{H}/(2H^2) \right] - \frac{\alpha_{D,c}}{\alpha H^2} \left\{ (1 + \alpha_B)\alpha_K H^2 + (\hat{\alpha}H)' - 6\alpha_B [(1 + \alpha_B)H]' - 9\alpha_B\Omega_m H^2 \right\}. \quad (\text{E.18})$$

References

- [1] **Euclid Theory Working Group** Collaboration, L. Amendola *et. al.*, “Cosmology and fundamental physics with the Euclid satellite,” *Living Rev.Rel.* **16** (2013) 6, [1206.1225](#).
- [2] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, “Modified Gravity and Cosmology,” *Phys.Rept.* **513** (2012) 1–189, [1106.2476](#).
- [3] A. Joyce, B. Jain, J. Khoury, and M. Trodden, “Beyond the Cosmological Standard Model,” *Phys.Rept.* **568** (2015) 1–98, [1407.0059](#).
- [4] G. Gubitosi, F. Piazza, and F. Vernizzi, “The Effective Field Theory of Dark Energy,” *JCAP* **1302** (2013) 032, [1210.0201](#).
- [5] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, “Essential Building Blocks of Dark Energy,” *JCAP* **1308** (2013) 025, [1304.4840](#).
- [6] P. Creminelli, M. A. Luty, A. Nicolis, and L. Senatore, “Starting the Universe: Stable Violation of the Null Energy Condition and Non-standard Cosmologies,” *JHEP* **0612** (2006) 080, [hep-th/0606090](#).

- [7] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, “The Effective Field Theory of Inflation,” *JHEP* **0803** (2008) 014, [0709.0293](#).
- [8] P. Creminelli, G. D’Amico, J. Norena, and F. Vernizzi, “The Effective Theory of Quintessence: the w_{i-1} Side Unveiled,” *JCAP* **0902** (2009) 018, [0811.0827](#).
- [9] R. L. Arnowitt, S. Deser, and C. W. Misner, “The Dynamics of general relativity,” *Gen.Rel.Grav.* **40** (2008) 1997–2027, [gr-qc/0405109](#).
- [10] J. K. Bloomfield, É. É. Flanagan, M. Park, and S. Watson, “Dark energy or modified gravity? An effective field theory approach,” *JCAP* **1308** (2013) 010, [1211.7054](#).
- [11] J. Bloomfield, “A Simplified Approach to General Scalar-Tensor Theories,” *JCAP* **1312** (2013) 044, [1304.6712](#).
- [12] F. Piazza and F. Vernizzi, “Effective Field Theory of Cosmological Perturbations,” *Class.Quant.Grav.* **30** (2013) 214007, [1307.4350](#).
- [13] S. Tsujikawa, “The effective field theory of inflation/dark energy and the Horndeski theory,” *Lect.Notes Phys.* **892** (2015) 97–136, [1404.2684](#).
- [14] J. Gleyzes, D. Langlois, and F. Vernizzi, “A unifying description of dark energy,” *Int.J.Mod.Phys.* **D23** (2014) 3010, [1411.3712](#).
- [15] J. Bloomfield and J. Pearson, “Simple implementation of general dark energy models,” *JCAP* **1403** (2014) 017, [1310.6033](#).
- [16] F. Piazza, H. Steigerwald, and C. Marinoni, “Phenomenology of dark energy: exploring the space of theories with future redshift surveys,” *JCAP* **1405** (2014) 043, [1312.6111](#).
- [17] E. Bellini and I. Sawicki, “Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity,” *JCAP* **1407** (2014) 050, [1404.3713](#).
- [18] B. Hu, M. Raveri, N. Frusciante, and A. Silvestri, “Effective Field Theory of Cosmic Acceleration: an implementation in CAMB,” *Phys.Rev.* **D89** (2014), no. 10 103530, [1312.5742](#).
- [19] **Planck** Collaboration, P. Ade *et. al.*, “Planck 2015 results. XIV. Dark energy and modified gravity,” [1502.01590](#).
- [20] G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space,” *Int.J.Theor.Phys.* **10** (1974) 363–384.
- [21] C. Deffayet, X. Gao, D. Steer, and G. Zahariade, “From k-essence to generalised Galileons,” *Phys.Rev.* **D84** (2011) 064039, [1103.3260](#).
- [22] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, “Generalized G-inflation: Inflation with the most general second-order field equations,” *Prog.Theor.Phys.* **126** (2011) 511–529, [1105.5723](#).

- [23] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, “Healthy theories beyond Horndeski,” [1404.6495](#).
- [24] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, “Exploring gravitational theories beyond Horndeski,” *JCAP* **1502** (2015), no. 02 018, [1408.1952](#).
- [25] M. Zumalacárregui and J. García-Bellido, “Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian,” *Phys.Rev.* **D89** (2014), no. 6 064046, [1308.4685](#).
- [26] C. M. Will, “The Confrontation between General Relativity and Experiment,” *Living Rev.Rel.* **17** (2014) 4, [1403.7377](#).
- [27] C. Brans and R. Dicke, “Mach’s principle and a relativistic theory of gravitation,” *Phys.Rev.* **124** (1961) 925–935.
- [28] L. Hui, A. Nicolis, and C. Stubbs, “Equivalence Principle Implications of Modified Gravity Models,” *Phys.Rev.* **D80** (2009) 104002, [0905.2966](#).
- [29] P. Creminelli, J. Gleyzes, L. Hui, M. Simonović, and F. Vernizzi, “Single-Field Consistency Relations of Large Scale Structure. Part III: Test of the Equivalence Principle,” *JCAP* **1406** (2014) 009, [1312.6074](#).
- [30] T. Damour, G. W. Gibbons, and C. Gundlach, “Dark Matter, Time Varying G , and a Dilaton Field,” *Phys.Rev.Lett.* **64** (1990) 123–126.
- [31] L. Amendola, “Coupled quintessence,” *Phys.Rev.* **D62** (2000) 043511, [astro-ph/9908023](#).
- [32] C. Skordis, A. Pourtsidou, and E. Copeland, “The Parameterized Post-Friedmannian Framework for Interacting Dark Energy Theories,” [1502.07297](#).
- [33] T. Baker, P. G. Ferreira, and C. Skordis, “The Parameterized Post-Friedmann framework for theories of modified gravity: concepts, formalism and examples,” *Phys.Rev.* **D87** (2013), no. 2 024015, [1209.2117](#).
- [34] P. G. Ferreira, T. Baker, and C. Skordis, “Testing general relativity with cosmology: a synopsis of the parametrized post-Friedmann approach,” *Gen.Rel.Grav.* **46** (2014) 1788.
- [35] D. Bettoni and S. Liberati, “Disformal invariance of second order scalar-tensor theories: Framing the Horndeski action,” *Phys.Rev.* **D88** (2013), no. 8 084020, [1306.6724](#).
- [36] J. D. Bekenstein, “The Relation between physical and gravitational geometry,” *Phys.Rev.* **D48** (1993) 3641–3647, [gr-qc/9211017](#).
- [37] D. Blas, M. M. Ivanov, and S. Sibiryakov, “Testing Lorentz invariance of dark matter,” *JCAP* **1210** (2012) 057, [1209.0464](#).
- [38] D. Bettoni and S. Liberati, “Dynamics of non-minimally coupled perfect fluids,” [1502.06613](#).

- [39] T. S. Koivisto, “Disformal quintessence,” [0811.1957](#).
- [40] M. Zumalacárregui, T. Koivisto, D. Mota, and P. Ruiz-Lapuente, “Disformal Scalar Fields and the Dark Sector of the Universe,” *JCAP* **1005** (2010) 038, [1004.2684](#).
- [41] T. S. Koivisto, D. F. Mota, and M. Zumalacárregui, “Screening Modifications of Gravity through Disformally Coupled Fields,” *Phys.Rev.Lett.* **109** (2012) 241102, [1205.3167](#).
- [42] C. van de Bruck and G. Sculthorpe, “Modified Gravity and the Radiation Dominated Epoch,” *Phys.Rev.* **D87** (2013), no. 4 044004, [1210.2168](#).
- [43] M. Zumalacárregui, T. S. Koivisto, and D. F. Mota, “DBI Galileons in the Einstein Frame: Local Gravity and Cosmology,” *Phys.Rev.* **D87** (2013) 083010, [1210.8016](#).
- [44] P. Brax, C. Burrage, A.-C. Davis, and G. Gubitosi, “Cosmological Tests of the Disformal Coupling to Radiation,” *JCAP* **1311** (2013) 001, [1306.4168](#).
- [45] P. Brax and C. Burrage, “Constraining Disformally Coupled Scalar Fields,” *Phys.Rev.* **D90** (2014), no. 10 104009, [1407.1861](#).
- [46] J. Sakstein, “Disformal Theories of Gravity: From the Solar System to Cosmology,” *JCAP* **1412** (2014), no. 12 012, [1409.1734](#).
- [47] C. van de Bruck and J. Morrice, “Disformal couplings and the dark sector of the universe,” [1501.03073](#).
- [48] T. Koivisto and H. J. Nyrhinen, “Stability of disformally coupled accretion disks,” [1503.02063](#).
- [49] A. P. Billyard and A. A. Coley, “Interactions in scalar field cosmology,” *Phys.Rev.* **D61** (2000) 083503, [astro-ph/9908224](#).
- [50] G. R. Farrar and P. J. E. Peebles, “Interacting dark matter and dark energy,” *Astrophys.J.* **604** (2004) 1–11, [astro-ph/0307316](#).
- [51] L. Amendola, “Linear and non-linear perturbations in dark energy models,” *Phys.Rev.* **D69** (2004) 103524, [astro-ph/0311175](#).
- [52] O. Bertolami, F. Gil Pedro, and M. Le Delliou, “Dark Energy-Dark Matter Interaction and the Violation of the Equivalence Principle from the Abell Cluster A586,” *Phys.Lett.* **B654** (2007) 165–169, [astro-ph/0703462](#).
- [53] M. Baldi, V. Pettorino, G. Robbers, and V. Springel, “Hydrodynamical N-body simulations of coupled dark energy cosmologies,” *Mon.Not.Roy.Astron.Soc.* **403** (2010) 1684–1702, [0812.3901](#).
- [54] K. Koyama, R. Maartens, and Y.-S. Song, “Velocities as a probe of dark sector interactions,” *JCAP* **0910** (2009) 017, [0907.2126](#).

- [55] J. Valiviita, R. Maartens, and E. Majerotto, “Observational constraints on an interacting dark energy model,” *Mon.Not.Roy.Astron.Soc.* **402** (2010) 2355–2368, [0907.4987](#).
- [56] M. Baldi, “Time dependent couplings in the dark sector: from background evolution to nonlinear structure formation,” *Mon.Not.Roy.Astron.Soc.* **411** (2011) 1077, [1005.2188](#).
- [57] V. Pettorino, L. Amendola, C. Baccigalupi, and C. Quercellini, “Constraints on coupled dark energy using CMB data from WMAP and SPT,” *Phys.Rev.* **D86** (2012) 103507, [1207.3293](#).
- [58] W. Yang and L. Xu, “Testing coupled dark energy with large scale structure observation,” *JCAP* **1408** (2014) 034, [1401.5177](#).
- [59] V. Salvatelli, N. Said, M. Bruni, A. Melchiorri, and D. Wands, “Indications of a late-time interaction in the dark sector,” *Phys.Rev.Lett.* **113** (2014), no. 18 181301, [1406.7297](#).
- [60] J.-P. Bruneton and G. Esposito-Farese, “Field-theoretical formulations of MOND-like gravity,” *Phys.Rev.* **D76** (2007) 124012, [0705.4043](#).
- [61] S. Hawking and G. Ellis, *The Large scale structure of space-time*. Cambridge University Press, 1975.
- [62] P. Creminelli, J. Gleyzes, J. Noreña, and F. Vernizzi, “Resilience of the standard predictions for primordial tensor modes,” *Phys.Rev.Lett.* **113** (2014), no. 23 231301, [1407.8439](#).
- [63] K. A. Malik and D. Wands, “Adiabatic and entropy perturbations with interacting fluids and fields,” *JCAP* **0502** (2005) 007, [astro-ph/0411703](#).
- [64] M. Minamitsuji, “Disformal transformation of cosmological perturbations,” *Phys.Lett.* **B737** (2014) 139–150, [1409.1566](#).
- [65] S. Weinberg, “Adiabatic modes in cosmology,” *Phys.Rev.* **D67** (2003) 123504, [astro-ph/0302326](#).
- [66] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, “Spontaneous Creation of Almost Scale - Free Density Perturbations in an Inflationary Universe,” *Phys.Rev.* **D28** (1983) 679.
- [67] A. Dupays, E. Masso, J. Redondo, and C. Rizzo, “Light scalars coupled to photons and non-newtonian forces,” *Phys.Rev.Lett.* **98** (2007) 131802, [hep-ph/0610286](#).
- [68] R. Fardon, A. E. Nelson, and N. Weiner, “Dark energy from mass varying neutrinos,” *JCAP* **0410** (2004) 005, [astro-ph/0309800](#).
- [69] A. Brookfield, C. van de Bruck, D. Mota, and D. Tocchini-Valentini, “Cosmology with massive neutrinos coupled to dark energy,” *Phys.Rev.Lett.* **96** (2006) 061301, [astro-ph/0503349](#).

- [70] N. Afshordi, M. Zaldarriaga, and K. Kohri, “On the stability of dark energy with mass-varying neutrinos,” *Phys.Rev.* **D72** (2005) 065024, [astro-ph/0506663](#).
- [71] I. Sawicki and E. Bellini, “Limits of Quasi-Static Approximation in Modified-Gravity Cosmologies,” [1503.06831](#).
- [72] R. Bean and O. Dore, “Probing dark energy perturbations: The Dark energy equation of state and speed of sound as measured by WMAP,” *Phys.Rev.* **D69** (2004) 083503, [astro-ph/0307100](#).
- [73] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, “A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration,” *Phys.Rev.Lett.* **85** (2000) 4438–4441, [astro-ph/0004134](#).
- [74] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, “Essentials of k essence,” *Phys.Rev.* **D63** (2001) 103510, [astro-ph/0006373](#).
- [75] L. Boubekeur, P. Creminelli, J. Norena, and F. Vernizzi, “Action approach to cosmological perturbations: the 2nd order metric in matter dominance,” *JCAP* **0808** (2008) 028, [0806.1016](#).
- [76] J. Garriga and V. F. Mukhanov, “Perturbations in k-inflation,” *Phys.Lett.* **B458** (1999) 219–225, [hep-th/9904176](#).
- [77] L. Á. Gergely and S. Tsujikawa, “Effective field theory of modified gravity with two scalar fields: dark energy and dark matter,” *Phys.Rev.* **D89** (2014), no. 6 064059, [1402.0553](#).
- [78] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” *JHEP* **0305** (2003) 013, [astro-ph/0210603](#).
- [79] C.-P. Ma and E. Bertschinger, “Cosmological perturbation theory in the synchronous and conformal Newtonian gauges,” *Astrophys.J.* **455** (1995) 7–25, [astro-ph/9506072](#).