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HAL Id: cea-01384760
https://cea.hal.science/cea-01384760
Submitted on 20 Oct 2016

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Dynamics of quasiparticle trapping in Andreev levels

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(Dated: February 13, 2014)

We present a theory describing the trapping of a quasiparticle in a prototypical Josephson junction, a single-channel superconducting weak link. We calculate the trapping and untrapping rates associated to absorption and emission of both photons and phonons. We show that the presence of an electromagnetic mode with frequency smaller than the gap gives rise to a rather abrupt transition between a fast relaxation regime dominated by coupling to photons and a slow relaxation regime dominated by coupling to phonons. This conclusion is illustrated by the analysis of a recent experiment measuring the dynamics of quasiparticle trapping in a superconducting atomic contact coupled to a Josephson junction. With realistic parameters the theory provides a semi-quantitative description of the experimental results.

I. INTRODUCTION

There are several external mechanisms that undermine the quantum coherence of superconducting circuits being explored for quantum information processing\textsuperscript{2}. Their influence has been reduced over the years by new designs that minimize the coupling with external degrees of freedom. However, a fundamental intrinsic decoherence process arises from the coupling of the qubit variables to superconducting quasiparticles tunneling through the Josephson junctions of the circuits. Although in principle the superconducting gap $\Delta$ provides an inherent protection against low energy excitations at low temperatures, in practice there are residual nonequilibrium quasiparticles that can rule the behavior of the circuits\textsuperscript{3–9}. As shown in a recent experiment\textsuperscript{10}, this is particularly true for weak links containing channels of high transmission, where localized excitations occupying Andreev levels of energy below $\Delta$ become possible. This has important consequences for the corresponding proposed qubits designs\textsuperscript{11–14}. Furthermore, single quasiparticle trapping in localized levels could be detrimental in experiments proposed to detect “Majorana bound states” in condensed matter systems since their topological protection relies on parity conservation\textsuperscript{15}. Understanding the dynamics of relaxation of quasiparticles in superconducting weak links is therefore an important present-day issue.

We report here on a theory highlighting the role of the electromagnetic environment in this dynamics. In experiments, Josephson junctions or weak links are very often embedded in electrical circuits having electromagnetic modes at frequencies lower than the superconducting gap. The environment can be a resonator intentionally coupled to the junction like in Ref. 10, or the plasma mode of another junction placed in parallel like in Ref. 1. We show that if the mode impedance is large enough, it rules the quasiparticle dynamics when the sum of the Andreev level energy and of the energy of the mode exceed the superconducting gap.

The rest of the paper is organized as follows: In Sec. II we describe the model considered for a superconducting one channel contact coupled to a generic electromagnetic environment; Sec. III is devoted to the analysis of the transition rates between different quasiparticles states induced by quantum phase fluctuations; in Sec. IV we focus on the experimental situation of Ref. 1 and compare the theoretical results for the transition rates and the stationary probability for quasiparticles trapped in the subgap states with the corresponding experimental results. In Sec. V we present our main conclusions. The more technical details on our calculations are described in appendices A, B, C and D.

II. THEORETICAL FRAMEWORK

We consider the situation illustrated in Fig. 1(a) with a single superconducting channel (SC) coupled to an arbitrary impedance $Z_{\text{env}}$. The excitation spectrum of the
SC contains a discrete, spin-degenerate Andreev level, with an energy \( E_A(\delta) = \Delta \sqrt{1 - \tau^2 \sin^2 \delta / 2} \), where \( \delta \) is the superconducting phase difference across the contact and \( \tau \) the transmission probability for electrons (see Fig. 1(b)). The Andreev level is completely empty when the channel is in its ground state, which has a phase dependent energy \(-E_A(\delta)\) and carries a supercurrent \( I = -\langle \partial E_A / \partial \delta \rangle / \varphi_0 \), where \( \varphi_0 = h / 2e \) is the reduced flux quantum. The lowest-energy excitations correspond to the occupation of the Andreev level by a single quasiparticle from the Andreev level. These processes allow in particular the relaxation of the lowest-energy excitations, i.e.\( E \leq E_A(\delta) \) (see Eqs. (2,3) but with a quadratic density of states). They consist either in the absorption of an environmental photon and transfer of the trapped one into a Cooper pair while releasing the energy as a photon. These two processes are illustrated in panels (a) and (b) of Fig. 2 and the corresponding rates are denoted by \( \Gamma^{(a,b)} \). The Fermi golden rule for the first process yields

\[
\Gamma^{(a)}_{\text{out}} = \frac{2\pi}{\hbar} \sum_{k} \left| \langle k, \sigma | \varphi_0 I | \bar{A}, \bar{\sigma} \rangle \right|^2 P(E_k - E_A(\delta)) \times (1 - f_{\text{FD}}(E_k, T_{\text{qp}})) ,
\]

where \( f_{\text{FD}}(E_k, T_{\text{qp}}) \) is the Fermi population factor for quasiparticles in the continuum (assumed to be in equilibrium at a temperature \( T_{\text{qp}} \)) and \( P(E) \) is the probability of absorbing a photon of energy \( E \) from the environment. This probability is \( P(E) = D(E) f_{\text{BE}}(E, T_{\text{env}}) \), where \( f_{\text{BE}}(E, T) \) is the Bose population factor, and \( D(E) = \text{Re}(Z_{\text{env}}(E)/E)/R_Q \), with \( R_Q = h / 4e^2 \), is the density of states for the modes in the environment. The environment is assumed to be in equilibrium at a temperature \( T_{\text{env}} \), which can be in general different from \( T_{\text{qp}} \). The numerical evaluation of this rate (and every other) for different transmissions, shows a rather universal dependence in the Andreev level energy position \( E_A \). Simple analytical expressions can be derived in the perfect transmission limit \( \tau \to 1 \) and in the tunnel limit \( \tau \to 0 \), for which the wavefunctions have a considerably simpler form. In this limit one obtains (see Appendix B)

\[
\Gamma^{(a)}_{\text{out}} = \frac{8\Delta}{\hbar} \int_{\Delta}^{\infty} dE(D(E - E_A) g(E, E_A) f_{\text{BE}}(E - E_A, T_{\text{env}})(1 - f_{\text{FD}}(E_k, T_{\text{qp}})),
\]

with \( g(E, E_A) = \sqrt{(E^2 - \Delta^2)(\Delta^2 - E_A^2)} / (\Delta (E - E_A)) \). We also give in Appendix B the expression of \( g(E, E_A) \) in the tunnel limit \( \tau \to 0 \).

When the environment of the SC contains a single mode with infinite quality factor, and at low temperature, this expression simplifies to \( \Gamma^{(a)}_{\text{out}} = \frac{2\Delta}{\hbar} \int_{\Delta}^{\infty} dE \frac{2\Delta}{\hbar} g(E_A + h\nu, E_A) \exp(-\nu / k_B T_{\text{env}}) \), where \( \nu \) is the mode frequency. The function \( g \) is of order 1 when \( \Delta - h\nu < E_A < \Delta \), so that the rate is simply determined by the impedance \( Z_0 \) of the oscillator. For aluminum and for \( Z_0 = 50 \Omega, \frac{2\Delta}{\hbar} \frac{2\Delta}{\hbar} \approx 1 \text{ GHz} \).

In a similar way for the second relevant process we find

\[
\Gamma^{(b)}_{\text{out}} = \frac{8\Delta}{\hbar} \int_{\Delta}^{\infty} dE(D(E + E_A) g(E, -E_A) \times (1 + f_{\text{BE}}(E + E_A, T_{\text{env}})) f_{\text{FD}}(E_k, T_{\text{qp}}). \]

We show in Appendix B that for perfect transmission the matrix elements for electron-phonon coupling have the same functional form in terms of \( E \) and \( E_A \) as those for the coupling with the electromagnetic modes. Therefore its inclusion leads to the same expressions for \( \Gamma^{(a,b)}_{\text{out}} \) as in Eqs. (3) but with a quadratic density of states and \( T_{\text{env}} \) replaced by the phonon temperature \( T_{\text{ph}} \) (see Appendix B for more details).

III. TRANSITION RATES

The coupling of the SC to the environment allows for transitions between different quasiparticle states. We shall first consider processes which permit the removal of a quasiparticle from the Andreev level. These processes allow in particular the relaxation of the lowest-energy excited states with one trapped quasiparticle back to the ground state \( | \Psi_0 \rangle \). They consist either in the absorption of an environmental photon and transfer of the trapped quasiparticle into the continuum states, or in the recombination of a quasiparticle from the continuum with the trapped one into a Cooper pair while releasing the energy as a photon. These two processes are illustrated in panels (a) and (b) of Fig. 2 and the corresponding rates are denoted by \( \Gamma^{(a,b)} \). The Fermi golden rule for the first process yields

\[
\Gamma^{(a)}_{\text{out}} = \frac{2\pi}{\hbar} \sum_{k} \left| \langle k, \sigma | \varphi_0 I | \bar{A}, \bar{\sigma} \rangle \right|^2 P(E_k - E_A(\delta)) \times (1 - f_{\text{FD}}(E_k, T_{\text{qp}})) ,
\]

where \( f_{\text{FD}}(E_k, T_{\text{qp}}) \) is the Fermi population factor for quasiparticles in the continuum (assumed to be in equilibrium at a temperature \( T_{\text{qp}} \)) and \( P(E) \) is the probability of absorbing a photon of energy \( E \) from the environment. This probability is \( P(E) = D(E) f_{\text{BE}}(E, T_{\text{env}}) \), where \( f_{\text{BE}}(E, T) \) is the Bose population factor, and \( D(E) = \text{Re}(Z_{\text{env}}(E)/E)/R_Q \), with \( R_Q = h / 4e^2 \), is the density of states for the modes in the environment. The environment is assumed to be in equilibrium at a temperature \( T_{\text{env}} \), which can be in general different from \( T_{\text{qp}} \). The numerical evaluation of this rate (and every other) for different transmissions, shows a rather universal dependence in the Andreev level energy position \( E_A \). Simple analytical expressions can be derived in the perfect transmission limit \( \tau \to 1 \) and in the tunnel limit \( \tau \to 0 \), for which the wavefunctions have a considerably simpler form. In this limit one obtains (see Appendix B)

\[
\Gamma^{(a)}_{\text{out}} = \frac{8\Delta}{\hbar} \int_{\Delta}^{\infty} dE(D(E - E_A) g(E, E_A) f_{\text{BE}}(E - E_A, T_{\text{env}})(1 - f_{\text{FD}}(E_k, T_{\text{qp}})),
\]

with \( g(E, E_A) = \sqrt{(E^2 - \Delta^2)(\Delta^2 - E_A^2)} / (\Delta (E - E_A)) \). We also give in Appendix B the expression of \( g(E, E_A) \) in the tunnel limit \( \tau \to 0 \).

When the environment of the SC contains a single mode with infinite quality factor, and at low temperature, this expression simplifies to \( \Gamma^{(a)}_{\text{out}} = \frac{2\Delta}{\hbar} \int_{\Delta}^{\infty} dE \frac{2\Delta}{\hbar} g(E_A + h\nu, E_A) \exp(-\nu / k_B T_{\text{env}}) \), where \( \nu \) is the mode frequency. The function \( g \) is of order 1 when \( \Delta - h\nu < E_A < \Delta \), so that the rate is simply determined by the impedance \( Z_0 \) of the oscillator. For aluminum and for \( Z_0 = 50 \Omega, \frac{2\Delta}{\hbar} \frac{2\Delta}{\hbar} \approx 1 \text{ GHz} \).

In a similar way for the second relevant process we find

\[
\Gamma^{(b)}_{\text{out}} = \frac{8\Delta}{\hbar} \int_{\Delta}^{\infty} dE(D(E + E_A) g(E, -E_A) \times (1 + f_{\text{BE}}(E + E_A, T_{\text{env}})) f_{\text{FD}}(E_k, T_{\text{qp}}). \]

We show in Appendix B that for perfect transmission the matrix elements for electron-phonon coupling have the same functional form in terms of \( E \) and \( E_A \) as those for the coupling with the electromagnetic modes. Therefore its inclusion leads to the same expressions for \( \Gamma^{(a,b)}_{\text{out}} \) as in Eqs. (3) but with a quadratic density of states and \( T_{\text{env}} \) replaced by the phonon temperature \( T_{\text{ph}} \) (see Appendix B for more details).
The time reversed processes, illustrated in Fig. 3 and characterized by rates \( \Gamma_{in}^{(a)} \) and \( \Gamma_{in}^{(b)} \) are responsible for the population of the Andreev level, either by trapping a quasiparticle from the continuum or by breaking a pair.

IV. COMPARISON TO EXPERIMENTS

We focus on the recent experiments on superconducting atomic contacts\(^{11} \) that have analyzed in detail the quasiparticle trapping in Andreev levels and its dynamics. In these experiments, an atomic contact was embedded in a superconducting loop containing a Josephson junction, thus forming an asymmetric SQUID. It was found that there is a significant probability for the SC to get trapped in an odd state in which the highest transmitted channel carries no supercurrent. The experiments also showed that the relaxation rates for these states fall into a nearly universal behavior as a function of the energy \( E_A \) regardless of the particular values of the transition and phase difference. Trapping occurred essentially when the Andreev level energy was smaller than half the superconducting gap \( E_A \lesssim 0.5\Delta \), with the lifetime of trapped quasiparticles exceeding 100 \( \mu s \). For larger energies no significant trapping could be detected. The origin of this sharp energy threshold was a puzzle not explained in the paper reporting the experiment.

In Ref.\(^{11} \) the SQUID Josephson junction had a Josephson energy much larger than the charging energy, and it can therefore be described as an harmonic oscillator. Spectroscopy measurements\(^{19, 21} \) on similar circuits as the one used in Ref.\(^{11} \) showed that the plasma frequency of this mode can be significantly renormalized by parallel inductances and approach \( \sim 0.5\Delta \) (see Appendix C and Ref.\(^{19} \)). As explained in the following, our theory shows that the main relaxation mechanism for the trapped quasiparticle states is their excitation into the extended continuum states above the superconducting gap by absorption of photons from the plasma mode. This mechanism becomes inefficient when the energy difference between the Andreev level and the continuum exceeds the plasma energy, \( \Delta - E_A > \hbar\nu_p \), hence providing a simple explanation for the observed behavior.

The results for the transition rates \( \Gamma_{out}^{(a,b)} \) obtained using parameters which are appropriate for the experimental situation of Ref.\(^{11} \) (see Appendix C) are shown by the solid lines in Fig. 2. For \( E_A \gtrsim \Delta - \hbar\nu_p \sim 0.52\Delta \), \( \Gamma_{out}^{(a)} \) is large because photons in the plasma mode can excite the trapped quasiparticle out into the continuum. Similarly, \( \Gamma_{in}^{(a)} \) is large in this energy range because quasiparticles near the gap edge can relax in the Andreev level while emitting a plasma photon (see Fig. 3). For lower energies, the energy of the plasma photons is not sufficient and the rate drops abruptly. Other processes, like phonon absorption or emission start to play a role. Hence, both \( \Gamma_{in} \) and \( \Gamma_{out} \) are determined by phonon processes for \( E_A < \Delta - \hbar\nu_p \) and by photon processes for \( E_A > \Delta - \hbar\nu_p \). It should be noticed that three different temperatures enter the calculation. We assume that the phonons in the Al films of Ref.\(^{11} \) are at equilibrium with the substrate and therefore \( T_{ph} \) is taken equal to the base temperature measured by the thermometers in the experiment (30 mK). The two other temperatures, \( T_{env} \) and \( T_{qp} \), can be significantly larger due to incomplete filtering of radiation. To fit the results we have used \( T_{env} \sim 120\,\text{mK} \), similar to what is deduced from measurements of the switching probability of the SQUID\(^{19, 21} \) and \( T_{qp} \sim 180\,\text{mK} \) which simulates the presence of a few tens of out-of-equilibrium quasiparticles per \( \mu\text{m}^2 \), as typically found in experiments with Al resonators and qubits\(^{13, 14, 17, 22} \).

The different transitions which determine the population and relaxation of the Andreev level are illustrated in the inset of Fig. 4. They also involve the even excited state \( |\text{even}\rangle = \gamma_A |\text{A}_d\rangle |\Psi_0\rangle \). The analysis is further simplified by symmetry relations: the rates connecting the even excited state and the odd states are equal to the ones connecting the odd states and the even ground state. This is indicated by the color code used for the arrows in the inset of Fig. 4. Notice that the full determination of the level populations requires also the evaluation of the rates \( \Gamma_{e\rightarrow e} \) and \( \Gamma_{e\rightarrow o} \). In Ref.\(^{11} \) it was assumed that the relaxation rate \( \Gamma_{e\rightarrow o} \) to the ground state was very fast compared to all other ones and that \( \Gamma_{e\rightarrow e} \) was negligible. For photonic or phononic environments these have been calculated in Refs.\(^{11, 19} \) and \( 23 \) respectively and reproduced with the present formalism, as discussed in Appendix C and D. The calculation does corroborate
different transmissions. As can be observed, the theory qualitatively describes the decrease in $P_\infty$ at fixed $E_A$ which is observed experimentally for increasing transmission in the slow relaxation regime. There is, however, some discrepancy in the quantitative values of $P_\infty$ which is overestimated in our model calculations.

We show in Fig. 5 the comparison of the experimental and theoretical results for the total rates $\Gamma_{\text{in}}$ and $\Gamma_{\text{out}}$ as a function of $E_A$ for different values of the contact transmission. One should remark the quite good agreement which is obtained for $\Gamma_{\text{out}}$ in the fast relaxation regime ($E_A > 0.5\Delta$) and for $\Gamma_{\text{in}}$ in the slow relaxation regime. The drop in $\Gamma_{\text{out}}$ by more than two orders of magnitude at $E_A \sim 0.5\Delta$ is also captured by our model. In the slow relaxation regime the model correctly describes the decrease of $\Gamma_{\text{in}}$ and the increase in $\Gamma_{\text{out}}$ which is observed at fixed $E_A$ with increasing transmission. The largest discrepancies between model and experiment are found for $\Gamma_{\text{in}}$ when $E_A \gtrsim 0.5\Delta$ and for $\Gamma_{\text{out}}$ when $E_A \lesssim 0.5\Delta$. It should be noticed, however, that the experimental determination of $\Gamma_{\text{in}}$ is less precise for $E_A > 0.5\Delta$ where $P_\infty \ll 1$. The decay of $\Gamma_{\text{in}}$ in theory is essentially related to the assumption of a thermal distribution of quasiparticles, which leads to $\Gamma_{\text{in}} \propto \exp(-(E_A + h\nu_p)/k_B T_{qp})$. A weaker dependence on energy would be obtained with a self-consistent description of this distribution. On the other hand, the deviations found for $\Gamma_{\text{out}}$ in the slow relaxation regime are more significant and could indicate that some additional relaxation mechanism, like tunneling to vortices or quasiparticle traps in the vicinity of the contact, could be contributing for small $E_A$. Since in the limit where $\Gamma_{\text{in}} \gg \Gamma_{\text{out}} \gg \Gamma_{\text{e\rightarrow e}}$, the stationary occupation probability $P_\infty$ is simply given by $2\Gamma_{\text{in}}/(3\Gamma_{\text{in}} + \Gamma_{\text{out}})$, the deviations for $\Gamma_{\text{out}}$ explain why the theory overestimates $P_\infty$ in this regime. Further combined experimental and theoretical work would be required to clarify this point.
V. CONCLUSIONS

In conclusion, we have presented a theory which describes the dynamics of trapping and untrapping quasiparticles in phase-biased superconducting weak links. It is shown that in realistic conditions this dynamics can be controlled by the coupling of the weak link to its electromagnetic environment. The results are in semi-quantitative agreement with the experiments of Ref. [1] where the sharp jump observed in the trapping and untrapping rates is associated to the onset of the coupling to the environment plasma mode. The mechanisms described here can be relevant for controlling decoherence in superconducting qubits involving channels with non-negligible transmission. In case of the Andreev qubits discussed in Refs. [11,12] where poisoning by trapped quasiparticles in the ABS would be beneficial. In contrast, for the proposals of Refs. [13,14] which are based on the manipulation of the odd states, a larger lifetime of the trapped quasiparticles is desirable. In this case one would need an electromagnetic environment containing no mode of frequency larger than $\Delta - E_A$. Finally, as discussed in Ref. [6] even in the case of qubits based on tunnel junctions changes in the occupation of the Andreev states make the Josephson coupling and hence the qubit frequency fluctuate thus giving rise to dephasing. Therefore, even in this case of Andreev levels very close to the gap edge, slowing down the dynamics of these occupations could have an influence on the qubit decoherence.

Acknowledgments

The authors acknowledge fruitful discussions with J. Martinis, R. Egger, A. Zazunov, D. Urban, J. C. Cuevas, F. S. Bergeret and A. Martín-Rodero. Financial support by EU FP7 SE2ND project, Spanish Mineco project FIS2011-26516, ANR contracts DOCFLUC and MASH, C’Nano and by the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement no. PIIF-GA-2011-298415 is acknowledged.

Appendix A: Diagonalisation of the SC Hamiltonian and SC wavefunctions

The point contact is modelled as a 1D SNS junction with a Dirac delta barrier in the normal region.

The normal region’s length $L_N$ can be taken to the limit $L_N \to 0$ in the ballistic regime, effectively turning the scattering problem into the problem of a delta barrier in a superconducting system, with a well-defined phase bias between the left and right leads.

**Continuum wavefunctions:** For an homogeneous superconducting system, the Hamiltonian’s eigenfunctions have a momentum $|k| = k_F \pm \kappa_E$, where $\kappa_E = \xi_0^{-1} \sqrt{(E/\Delta)^2 - 1}$ ($\xi_0$ being the superconducting coherence length, given by $\hbar v_F/\Delta$ in the ballistic regime). The eigenfunctions with positive energy $E_k$ and spin-up take the following shape:

$$
\psi^{(1)}_{k,\varphi}(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} u_E \ e^{i\varphi} \\ v_E \ e^{i\varphi} \end{array} \right) e^{+ikx} \quad \text{for } |k| > k_F \quad \text{(quasielectrons)}
$$

$$
\psi^{(2)}_{k,\varphi}(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} v_E \\ u_E \ e^{i\varphi} \end{array} \right) e^{-ikx} \quad \text{for } |k| < k_F \quad \text{(quasiholes)}
$$

where $\varphi$ is the phase of the superconducting order parameter, $L$ is a length over which the freely propagating eigenfunctions are defined and $u_E, v_E$ are, respectively, the electron and hole components of the plane wave:

$$
(u/v)_E = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \pm \sqrt{\frac{E}{\Delta}} \\ \mp \sqrt{\frac{\Delta}{E}} \end{array} \right)
$$

FIG. 6: Quasiparticle energies and two-component eigenfunctions near the Fermi level for spin-up quasiparticles. There is a four-fold degeneracy of states for each energy $E$, with two quasielectron states (blue dots) with a momentum $|k| = k_F + \kappa_E$ and another two quasiholes with an absolute momentum of $|k| = k_F - \kappa_E$.

These plane waves are schematically shown in Fig. [6]

Summations in momenta such as the one in Eq. [A1] may be rewritten, for the sake of convenience, as integrals over quasiparticle energies weighted by the superconducting density of states $\rho_{SC}(E) = \rho_F \sqrt{E - E_F}$, where $\rho_F$ is the normal density of states at the Fermi level.

That being the case we shall brand the wavefunctions (and the states they refer to) not by using their momentum $k$ as an index, but their energy $E_k \to E$, their associated quasielectron/hole character and the direction of their momentum.

With the plane waves from Eq. [A1] it is possible to construct solutions to the BdG equations in an inhomogeneous system following a scattering approach[22].

These wavefunctions are Nambu spinors of the form

$$
\psi^{(n\pm)}_{E_k}(x) = \left( \begin{array}{c} \psi^{(n\pm)}_{E_k}(x) \\ \psi^{(n\pm)}_{E_k}(x) \end{array} \right)
$$

Conclusions...
where \( U_E \) is the electron amplitude, \( V_E \) the hole amplitude, and \( \eta \) denotes the electron/hole character of the quasiparticle state. Spin-down eigenfunctions can be easily obtained from spin-up ones by use of the electron-hole symmetry in the system, through the substitution \( U \rightarrow V^*; \ V \rightarrow -U^* \).

For the sake of simplicity, we shall omit the spin sub-index in the wavefunctions that we discuss next.

Condensing all the phase difference \( \delta \) in the right lead, the wavefunctions take the following shape

\[
\psi_{E}^{(\pm)}(x) = \psi_{\text{Src}}^{(\mp)}(x) + \left( A(\eta^\pm)\psi_{E,0}^{(+)}(x) + B(\eta^\pm)\psi_{E,0}^{(-)}(x) \right) \Theta(-x) + \left( C(\eta^\pm)\psi_{E,\delta}^{(e+)}(x) + D(\eta^\pm)\psi_{E,\delta}^{(e-)}(x) \right) \Theta(x), \quad (A4)
\]

where \( \psi_{\text{Src}}^{(\pm)} \) is a source term of a quasielectron or quasihole impinging the contact from any of the leads

\[
\psi_{\text{Src}}^{(e+)} = \psi_{E,0}^{(e+)}(x) \Theta(-x), \quad \psi_{\text{Src}}^{(h+)} = \psi_{E,0}^{(h+)}(x) \Theta(-x), \quad \psi_{\text{Src}}^{(e-)} = \psi_{E,\delta}^{(e-)}(x) \Theta(x), \quad \psi_{\text{Src}}^{(h-)} = \psi_{E,\delta}^{(h-)}(x) \Theta(x). \quad (A5)
\]

The rest of the contributions to the wavefunctions are outgoing partial waves (as illustrated in Fig. 7).

Imposing continuity for the wavefunction and its derivative (taking into account the effect of the delta barrier) the values of the partial wave coefficients are obtained for each incidence

\[
A_{(e+)} = -\mu_\tau (\mu_\tau + i) \sinh^2 \theta_E \cdot Q^{-1} \quad B_{(e+)} = i\tau \sin \delta/2 \sinh \left( \theta_E + i \frac{\delta}{2} \right) \cdot Q^{-1}
\]

\[
C_{(e+)} = -i\tau (\mu_\tau + i) e^{-i\delta/2} \sinh \theta_E \sinh \left( \theta_E + i \frac{\delta}{2} \right) \cdot Q^{-1} \quad D_{(e+)} = -\mu_\tau e^{-i\delta/2} \sinh \theta_E \sin \frac{\delta}{2} \cdot Q^{-1}
\]

\[
\begin{align*}
A_{(e-)} &= C_{(e+)} (-\delta) & A_{(h+)} &= B_{(e+)}^* (-\delta) & A_{(h-)} &= C_{(h+)} (-\delta) \\
B_{(e-)} &= D_{(e+)} (-\delta) & B_{(h+)} &= A_{(e+)}^* (-\delta) & B_{(h-)} &= D_{(h+)} (-\delta) \\
C_{(e-)} &= A_{(e+)} (-\delta) & C_{(h+)} &= D_{(e+)}^* (-\delta) & C_{(h-)} &= A_{(h+)} (-\delta) \\
D_{(e-)} &= B_{(e+)} (-\delta) & D_{(h+)} &= C_{(e+)}^* (-\delta) & D_{(h-)} &= B_{(h+)} (-\delta)
\end{align*}
\]

(\text{A7})

where \( \sinh \theta_E = \sqrt{\left( \frac{E}{\Delta} \right)^2 - 1}, \mu_\tau = \sqrt{\frac{R}{\tau}}, \tau \) is the normal transmission probability from the potential barrier, \( R = 1 - \tau \) its normal reflection probability, and \( Q = \sinh^2 \theta_E + \tau \sin^2 \frac{\delta}{2} \).

The wavefunctions for the Andreev Bound States (ABS) may be obtained in a similar way than those for states lying at \( E > \Delta \) taking into account that the quasiparticle momentum gains an imaginary component below the gap. The partial waves for the ABS are

\[
\begin{align*}
\psi_{E,\pm}^{(e+)}(x) &= \frac{1}{\sqrt{\delta}} \left( \frac{u_A}{u_A e^{i\phi}} \right) e^{\pm i(k_F + i\kappa_A)x} \\
\psi_{E,\pm}^{(h+)}(x) &= \frac{1}{\sqrt{\delta}} \left( \frac{u_A}{u_A e^{i\phi}} \right) e^{\mp i(k_F - i\kappa_A)x}
\end{align*}
\]

(\text{A8})

These differ from the propagating partial waves because \( u_E, v_E \) coefficients and the quasiparticle momentum
Δ

\begin{align*}
    \Delta & = \begin{pmatrix}
        E_0 & \gamma_\eta \phi_\eta \\
        \gamma_\eta^\dagger & \phi_\eta^\dagger \end{pmatrix} \begin{pmatrix}
        \phi_\eta \\
        \phi_\eta^\dagger \end{pmatrix} \\
    & = \begin{pmatrix}
        \text{Even} & E, \eta_\pm, \sigma \\
        \text{Even}^* & \phi_\eta \\
        \text{Odd} & \phi_\eta^\dagger \\
        \text{Even} & \phi_\eta^\dagger \\
    \end{pmatrix}
\end{align*}

A linear homogeneous system of equations is obtained for the partial wave weights by applying the same conditions as in the case of the continuum states. The system exhibits a nontrivial solution for \( E = E_A (\delta) \). Eliminating the redundant equation and imposing the normalisation condition for the wavefunction, it is finally obtained that, in the bound states

\begin{align*}
    A_A &= -i \tilde{N}_A \sin \left( \theta_A - \frac{\delta}{2} \right) e^{-i\tilde{\beta}_r} e^{i\delta/2} \\
    B_A &= -i \sqrt{\tilde{N}_A} \sin \frac{\delta}{2} e^{i\tilde{\beta}_r} e^{i\delta/2} \\
    C_A &= \sqrt{\tilde{N}_A} \sin \left( \frac{\delta}{2} \right) e^{-i\tilde{\beta}_r} \\
    D_A &= \sigma_\delta \tilde{N}_A \sin \left( \theta_A - \frac{\delta}{2} \right) e^{i\tilde{\beta}_r}
\end{align*}

where \( \sin \tilde{\beta}_r = \sqrt{\tau}, \sigma_\delta = \text{sign} (\delta) \) and

\[ \tilde{N}_A = \sqrt{-\frac{\sigma_\delta \sqrt{\tau}}{2 \cos \theta_A \sin (\theta_A - \frac{\delta}{2})}}. \]

A similar derivation of this result can be found in [19].

The solutions of the BdeG equations allow us to express the electron field operators as

\begin{equation}
    \Psi_\sigma (x) = U_A (x) \gamma_{A\sigma} - \sigma V_A^* (x) \gamma_{A\sigma}^\dagger + \sum_{E, \eta_\pm} \left( U^{(\eta_\pm)}_E (x) \gamma_{E, \eta_\pm, \sigma} - \sigma V^{(\eta_\pm)*}_E (x) \gamma_{E, \eta_\pm, \sigma}^\dagger \right)
\end{equation}

where \( \gamma_{A\sigma} \) and \( \gamma_{E, \eta_\pm, \sigma} \) are the quasiparticle operators which diagonalise the SC Hamiltonian. The excitation spectrum of the system is represented in Fig. [8]

Appendix B: Current operator and transition rates

The current operator in the new basis defined by Eq. [A15] is

\[ \hat{I} (x) = -\frac{\hbar e}{2mi} \sum_{i,j,\sigma} \left( \gamma_{i\sigma}^\dagger \gamma_{i\sigma} \right) \begin{pmatrix}
    U_{i+} \frac{dU_{i+}}{dx} - \frac{dU_{i+}^*}{dx} U_{j-} \\
    \sigma \left( V_{i+} \frac{dV_{i+}}{dx} - \frac{dV_{i+}^*}{dx} V_{j-} \right) \\
    V_{i-} \frac{dV_{i-}}{dx} - \frac{dV_{i-}^*}{dx} V_{j+} \\
    \sigma \left( U_{i-} \frac{dU_{i-}}{dx} - \frac{dU_{i-}^*}{dx} U_{j+} \right)
\end{pmatrix} \gamma_{j\sigma}^\dagger \gamma_{j\sigma} \] (B1)

The sum in \( i, j \) indices are a shorthand notation for all the different contributions appearing in Eq. [A15]. The minus sign in front of some particular wavefunction subindices notes that such a component corresponds to the antiparallel wavefunction (e.g., if \( i = E (e+) \), then \( U_{-i} = U_i^{(e-)} \)), which only applies when the index corresponds to an excitation in the continuum.

1. Transitions involving the odd states

A relevant matrix element in the problem is the one associated to the \( \text{|Odd}; \sigma \rightarrow \text{|Even}; E, e+, \sigma \) process, which is found to be

\[ -\frac{\hbar e k_F}{m \sqrt{\tilde{E}_0}} \left( C_A G^*_{(e+)} - D_A D^*_{(e+)} \right) (u_E u_A + v_E v_A). \] (B2)
The electron-hole symmetry in the field transformations ensures that this matrix element is the complex conjugate of the matrix element associated to the process |Odd, σ⟩ → |Even; E, h−, σ⟩.

In the ballistic limit τ → 1, restricting δ to the interval [0, π] so as to establish a bijection between δ and E_A(δ), one finds

$$|C| \rightarrow \left( \frac{\Delta^2 - E_A^2}{\Delta} \right)^{1/4} D_A \rightarrow 0 \quad \text{(B3)}$$

$$|C(e+)| \rightarrow \sqrt{\frac{E^2 - \Delta^2}{E^2 - E_A^2}} D(e+) \rightarrow 0. \quad \text{(B4)}$$

The squared amplitude of these matrix elements in this limit is

$$\frac{\hbar^2 e^2 k_F^2}{m^2 L_0} \sqrt{\frac{\Delta^2 - E_A^2}{\Delta}} \frac{E^2 - \Delta^2}{E^2 - E_A^2} \left( 1 + \frac{E_A}{E} \right). \quad \text{(B5)}$$

Matrix elements for the |Odd, σ⟩ → |Even; E, e−, σ⟩ and |Odd, σ⟩ → |Even; E, h+, σ⟩ processes vanish in the limit of perfect transmission.

Conversely, the squared amplitude for the quasiparticle recombination processes |Odd, σ; E, e−, σ⟩ → |Even⟩ and |Odd, σ; E, h+, σ⟩ → |Even⟩ in the limit τ → 1 is found to be

$$\frac{\hbar^2 e^2 k_F^2}{m^2 L_0} \sqrt{\frac{\Delta^2 - E_A^2}{\Delta}} \frac{E^2 - \Delta^2}{E^2 - E_A^2} \left( 1 + \frac{E_A}{E} \right). \quad \text{(B6)}$$

Whereas the amplitudes for the other two recombination processes, which are |Odd, σ; E, e+, σ⟩ → |Even⟩ and |Odd, σ; E, h−, σ⟩ → |Even⟩, are zero in this same limit.

2. Transitions between the even states

An analytical expression for the amplitude that links the two even states can be derived for any value of τ

$$\langle \text{Even} | \hat{I} | \text{Even}^* \rangle = \frac{\hbar e k_F}{m \xi_0} C_A D_A (u_A - v_A^2) e^{i \delta} \quad \text{(B10)}$$

$$\langle \text{Even} | \hat{I} | \text{Even}^* \rangle^2 = \frac{e^2 \Delta^4 (1 - \tau^2) \tau^2 \sin^4 \frac{\delta}{2}}{\hbar^2 E_A^2 (\delta)} \quad \text{(B11)}$$

These results coincide with the results from [12].

Appendix C: Modelling the EM environment for the experiment in Ref. [1]

The density of environmental modes D (hv) is, following the formalism presented in Ref. [20],

$$D (hv) = \frac{1}{hv} Re \{ Z_{env} (hv) \} \quad \text{(C1)}$$

where $Z_{env} (hv)$ is the electric impedance as seen from the SC, represented in Fig. 9(c).

In the experiment shown in Ref. 11 a SC is placed in parallel with a Josephson junction having a critical current much larger than that of the atomic contact (see Fig. 9(b)). This junction is perceived by the atomic contact as the parallel combination of an inductor $L_J = \varphi_0 / I_0$ and a capacitor $C_J, I_0$ being the critical current of the junction. The finite quality factor of the
corresponding electromagnetic ("plasma") mode is modeled with a resistance $R_I$ in series with the capacitor. The SQUID loop formed by the contact and the junction is connected to a biasing circuit through an inductor $L_c$ and a capacitance $C_c$. Dissipation in this circuit is modeled by a resistance $R_e$ in series with the capacitor (see Fig. 9(a)).

The total impedance seen by the SC is

$$Z_{env}(\nu) = \frac{4}{\pi} \frac{E_C}{R_Q} \frac{i \nu}{\nu_{p0}} a_e(\nu) b_{p0}(\nu), \quad (C2)$$

with $a_e(\nu) = 1 + i \frac{1}{Q_{p0}} \frac{\nu}{\nu_{p0}}$ and $b_{p0}(\nu) = 1 + i \frac{1}{Q_{p0}} \frac{\nu}{\nu_{p0}}$.

The characteristic frequencies and quality factors inside these expressions are

$$\nu_{p0} = \nu_{p0}^0 (L_J C_J)^{-1/2}, \quad \nu_c = \frac{\nu_{p0}^0}{\nu_{p0}} (L_c C_c)^{-1/2}, \quad (C3)$$

$$Q_{p0} = \frac{1}{R_I} \sqrt{\frac{L_J}{C_J}}, \quad Q_c = \frac{1}{R_I} \sqrt{\frac{L_c}{C_c}.} \quad (C4)$$

The equivalent circuit in Fig. 9 has two modes. A low-frequency mode determined essentially by the on-chip LC filter connecting the SQUID to the outside world, and a high-frequency mode corresponding to the plasma oscillation of the junction “dressed” by the external circuit:

$$\nu_p = \nu_{p0} = \nu_{p0}^0 (L_J C_J)^{-1/2}, \quad (C5)$$

Parameters for this equivalent circuit were obtained in the following way:

The Josephson junction inductance $L_J = 595$ pH is determined by the critical current extracted from the switching probability measurements. The other five parameters of the equivalent circuit of Fig. 9 were adjusted so as to reproduce at best all the available experimental information.

a. The energy gap measured from the IV characteristics is $\Delta = 194 \mu eV$.

b. The dressed plasma frequency must be close to $\Delta/2h$ in order to explain the position of the sharp threshold observed in the rates $\Gamma_{in}$ and $\Gamma_{out}$. This is compatible with what was reported in Ref. 20.

c. A value of $C_c = 60$ pF is expected from a measurement at very low frequency (100 kHz) on a larger test capacitor fabricated on the same run.

d. The DC sub-gap current of the JJ alone is $I_J = 4.4$ nA at $V_J = \Delta/4e$. Imposing the power equality $I_J V_J = R_I I_0$ between the DC injected power and the microwave power absorbed at $\Delta/2h$ by the junction’s environment, we get $R_I = 0.3 \Omega$.

e. The low frequency mode of the environment was measured at 558 MHz in a separate microwave reflectometry experiment.

f. At this resonance frequency, the reflection amplitude $S_{11}$ shows a dip of $-15$ dB, from which we determine $R_I = 0.25 \Omega$.

The two capacitances, $C_J$ and $C_c$, and the environmental inductance $L_e$ were adjusted so as to reproduce the two characteristic frequencies of the circuit. The chosen value $C_J = 168$ fF is a 25% lower than what is expected from the nominal area of the junction (2.8 $\mu m^2$) and the typical specific capacitance for the junctions fabricated usually in our laboratory (75 fF/$\mu m^2$). $C_c = 68$ pF is a 13% higher than what is expected from the test low frequency measurement. Finally, the nominal value $L_e = 600$ pH for the environmental inductance is an 80% of what is expected from a crude geometrical estimation.

With these values we predict a dressed plasma mode frequency of 0.48$\Delta/h$ and a quality factor $Q = 116$.

Appendix D: Relaxation due to phonons

The electron-phonon interaction in real space is

$$\hat{H}_{e-ph} = \gamma \int d\mathbf{r} \sum_{\sigma} \Psi_{\sigma}^\dagger (\mathbf{r}) \Psi_{\sigma} (\mathbf{r}) \hat{\phi} (\mathbf{r}) \quad (D1)$$

where $\hat{\phi}$ is the phonon field operator:

$$\hat{\phi} (\mathbf{r}) = \sum_{q} \sqrt{\frac{h \nu_q}{2V}} (b_q e^{i \mathbf{q} \cdot \mathbf{r}} + b_{q}^\dagger e^{-i \mathbf{q} \cdot \mathbf{r}}). \quad (D2)$$

The electron-phonon coupling constant $\gamma$ is $\frac{2 \hbar^2 \pi^2 \nu_0 \nu_{p0}^2}{m \kappa^2 2}$, with $n_0$ being the atomic density, $B$ the adiabatic bulk modulus and $Z$ the electron valence from the superconductor.

The SC density inside the interaction Hamiltonian takes a form similar to the current operator’s in the quasiparticle basis (Eq. 47). But differently from the coupling with the EM environment, the phonon coupling depends on the geometrical spread of the SC wavefunctions, a feature characteristic of the coupling of phonons with localized states.

After eliminating terms linear in $e^{\pm 2i k_F x}$, which vanish in the spatial integration due to their rapid oscillatory behavior, the matrix element associated to the process $|\text{Odd}, \sigma \rangle \to |\text{Even}; E, e+, \sigma \rangle$ is found to be

$$\langle C_{(e+)} C_A - D_{(e+)}^* D_A \rangle (\nu_{E} U_A - \nu_{E}^* V_A) \frac{e^{-(i \gamma_{AA} + i k_F x)} \cdot \sqrt{L_{0}}}{L_{0}}.$$

In the limit $\tau \to 1$, the squared amplitude of the part of this matrix element that does not depend on $x$, as well as the similar quantity obtained from the matrix element
for the process $|\text{Odd}; \sigma\rangle \rightarrow |\text{Even}; E, h-, \sigma\rangle$, tend to the expression in Eq. (36), except for the factor $(h \epsilon k_F/m)^2$.

It can also be found in the same limit that the analogous $x$-independent quantity for the $|\text{Odd}; \sigma; E, e-, \sigma\rangle \rightarrow |\text{Even} \rangle$ and $|\text{Odd}; \sigma; E, h+, \sigma\rangle \rightarrow |\text{Even} \rangle$ processes is, on the other hand, the same as in Eq. (35).

The spatial integrals are of the form

$$\int_0^\infty dx e^{-(\kappa_A \pm i\kappa_E) x} \sin q_s x \cdot F(q_L, x),$$

where

$$F(q_L, x) = \int_{A_L(x)} d^2 r_L e^{i \mathbf{q}_L \cdot \mathbf{r}_L} |\Psi_L(\mathbf{r}_L)|^2,$$

$\Psi_L(\mathbf{r}_L)$ being the axial spread of the SC wavefunctions on the leads, whose geometric details are enclosed in their cross section $A_L(x)$.

The momentum transfer $Q$ to or from a phonon taking an active role in these relaxation processes is large compared to the inverse penetration length of the ABS: $Q \gg \kappa_A, \kappa_E$ in the region $E_A(\delta) < Q/2$. Following the approximations detailed in 23 in the theoretical description of the phonon-mediated $|\text{Even}^*\rangle \rightarrow |\text{Even} \rangle$ relaxation, the $F(q_L, x)$ factor introduces a cutoff $\tilde{L}$ in the integral in the $x$ direction

$$F(q_L, x) \rightarrow \theta \left( \tilde{L} - |x| \right)$$

and the spatial integration may be easily evaluated in the limit $\tilde{L}^{-1} \gg Q \gg \kappa_A, \kappa_E$:

$$\frac{4Q \sin Q\tilde{L}}{Q^2 + |\kappa_A \pm i\kappa_E|^2} \approx 4\tilde{L}.$$

Combining the different contributions, the squared amplitude for the process $|\text{Odd}; \sigma\rangle \rightarrow |\text{Even}; E, e/h\pm, \sigma\rangle$ mediated by the emission of a phonon is

$$|M_1|^2 = \frac{8 \hbar v_Q \tilde{L}^2 \gamma^2}{V L_\delta^3} \sqrt{1 - \frac{E_A^2 - \Delta^2}{E^2}} \delta (h\nu_Q - E - E_A),$$

where we have used the same notation as in the main text. We may rewrite the integral over momenta in Eq. (D4) as an integral over energies, with a density of states quadratic in $E - E_A$ that appears as a result of this transformation. The resulting total rate is then

$$\Gamma_{\text{out}}^{(a)} = \frac{2\pi}{\hbar} \frac{8}{\pi} \left( \frac{\tilde{L}}{\xi_0} \right)^2 \frac{(x^5)}{\pi^2} \frac{\Delta}{\hbar c_s} \int_{-\Delta}^{\Delta} dE \left( \frac{E - E_A}{\Delta} \right)^3 g(E, -E_A) f_{\text{BE}}(E - E_A, T_{\text{ph}}) (1 - f_{\text{FD}}(E, T_{\text{ph}})).$$

Repeating the same process for the form $|\text{Odd}; \sigma; E, \eta\bar{\sigma}\rangle \rightarrow |\text{Even} \rangle$ yields

$$\Gamma_{\text{out}}^{(b)} = \frac{2\pi}{\hbar} \frac{8}{\pi} \left( \frac{\tilde{L}}{\xi_0} \right)^2 \frac{(x^5)}{\pi^2} \frac{\Delta}{\hbar c_s} \int_{-\Delta}^{\Delta} dE \left( \frac{E + E_A}{\Delta} \right)^3 g(E, E_A) (1 - f_{\text{BE}}(E + E_A, T_{\text{ph}})) f_{\text{FD}}(E, T_{\text{ph}}).$$

Using the material constants for aluminum, the factor $\frac{2\pi}{\hbar} \frac{8}{\pi} \left( \frac{\tilde{L}}{\xi_0} \right)^2 \frac{(x^5)}{\pi^2} \frac{\Delta}{\hbar c_s} \int_{-\Delta}^{\Delta} dE \left( \frac{E - E_A}{\Delta} \right)^3 g(E, -E_A) f_{\text{BE}}(E - E_A, T_{\text{ph}}) (1 - f_{\text{FD}}(E, T_{\text{ph}}))$. If this factor is of the order of $10^{-2}$, the phonon-induced relaxation rates are reduced to around 1 kHz, which coincide with other estimations in analogous systems.

14 C. Paduraru and Yu. V. Nazarov, EPL 100, 57006 (2012).
18 Similar processes but for photons from an external driving field were considered in Ref. 9 based on the results of Ref. 25.
21 See supplemental material of Ref. 11.