# Many-body-localization transition in the strong disorder limit: entanglement entropy from the statistics of rare extensive resonances 

Cécile Monthus

## - To cite this version:

Cécile Monthus. Many-body-localization transition in the strong disorder limit : entanglement entropy from the statistics of rare extensive resonances. Entropy, 2016, 18, pp.122. 10.3390/e18040122 . cea01321700v1

## HAL Id: cea-01321700 <br> https://cea.hal.science/cea-01321700v1

Submitted on 26 May 2016 (v1), last revised 3 Oct 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Many Body Localization Transition in the strong disorder limit : entanglement entropy from the statistics of rare extensive resonances 

Cécile Monthus<br>Institut de Physique Théorique, Université Paris Saclay, CNRS, CEA, 91191 Gif-sur-Yvette, France


#### Abstract

The space of one-dimensional disordered interacting quantum models displaying a Many-BodyLocalization Transition seems sufficiently rich to produce critical points with level statistics interpolating continuously between the Poisson statistics of the Localized phase and the Wigner-Dyson statistics of the Delocalized Phase. In this paper, we consider the strong disorder limit of the MBL transition, where the level statistics at the MBL critical point is close to the Poisson statistics. We analyse a one-dimensional quantum spin model, in order to determine the statistical properties of the rare extensive resonances that are needed to destabilize the MBL phase. At criticality, we find that the entanglement entropy can grow with an exponent $0<\alpha<1$ anywhere between the area law $\alpha=0$ and the volume law $\alpha=1$, as a function of the resonances properties, while the entanglement spectrum follows the strong multifractality statistics. In the MBL phase near criticality, we obtain the simple value $\nu=1$ for the correlation length exponent. Independently of the strong disorder limit, we explain why for the Many-Body-Localization transition concerning individual eigenstates, the correlation length exponent $\nu$ is not constrained by the usual Harris inequality $\nu \geq 2 / d$, so that there is no theoretical inconsistency with the best numerical measure $\nu=0.8(3)$ obtained by D. J. Luitz, N. Laflorencie and F. Alet, Phys. Rev. B 91, 081103 (2015).


## I. INTRODUCTION

The thermalization of isolated many-body quantum systems is nowadays discussed in terms of the Eigenstate Thermalization Hypothesis (E.T.H.) [1-4] (see the review [5] and references therein) : the idea is that each manybody excited eigenstate $\mid \psi>$ is 'thermal', i.e. the reduced density matrix $\rho_{A}$ of a sub-region $A$ corresponds the thermal density matrix

$$
\begin{equation*}
\rho_{A}^{(\beta)}=\frac{e^{-\beta H_{A}}}{\operatorname{Tr}_{A}\left(e^{-\beta H_{A}}\right)} \tag{1}
\end{equation*}
$$

where the inverse temperature $\beta$ selects the correct average energy corresponding to the initial quantum state $\mid \psi>$. Then the Von Neumann entanglement entropy of the region $A$ with the complementary region

$$
\begin{equation*}
S_{A} \equiv-\operatorname{Tr}_{A}\left(\rho_{A} \ln \rho_{A}\right) \tag{2}
\end{equation*}
$$

coincides with the thermal entropy, which is extensive with respect to the volume $L_{A}^{d}$ of the region $A$

$$
\begin{equation*}
S_{A} \underset{E \overline{\bar{T}} H}{\simeq}-\operatorname{Tr}_{A}\left(\rho_{A}^{t h} \ln \rho_{A}^{t h}\right)=s^{t h}(\beta) L_{A}^{d} \tag{3}
\end{equation*}
$$

However the presence of strong disorder can prevent this thermalization and lead to the phenomenon of Many-BodyLocalization (MBL) (see the recent reviews [6, 7] and references therein), where the disorder-averaged entanglement entropy of Eq. 2 follows instead the area-law [8]

$$
\begin{equation*}
\overline{S_{A}} \underset{M B L}{\propto} L_{A}^{d-1} \tag{4}
\end{equation*}
$$

So in the MBL phase, excited states are somewhat similar to ground states, with efficient representation via Density-Matrix-RG or Matrix Product States [9-12] and Tensor Networks 13]. The Fisher Strong Disorder Real Space RG to construct the ground states of random quantum spin models [14-16] has been extended into the Strong Disorder RG procedure for the unitary dynamics [17, 18], and into the RSRG-X procedure in order to construct the whole set of excited eigenstates 19 24]. This construction is actually possible only because the MBL phase is characterized by an extensive number of emergent localized conserved operators [25-32], but breaks down as the MBL transition towards delocalization is approached. As a consequence, the current RG descriptions of the MBL transition are based on different types of RG rules concerning either the entanglement [33] or the resonances [34].

From the point of view of the entanglement entropy of Eq. 2, a natural question is whether at criticality, it is possible to obtain an entanglement power-law growth

$$
\begin{equation*}
\overline{S_{A}} \underset{\text { criti }}{\sim} L_{A}^{\alpha} \tag{5}
\end{equation*}
$$

intermediate between the area-law and the volume law $d-1 \leq \alpha \leq d$, or with possibly some logarithmic corrections. Let us now concentrate on the one-dimensional case $d=1$. If one assumes some standard finite-size-scaling form in the critical region in terms of the diverging correlation length $\xi$

$$
\begin{equation*}
\overline{S_{A}} \underset{F S S}{\simeq} L_{A}^{\alpha} \quad \Phi\left(\frac{L_{A}}{\xi}\right) \tag{6}
\end{equation*}
$$

one obtains that the matching with the area-law of the MBL phase (Eq. 4 in $d=1$ ) requires the divergence as

$$
\begin{equation*}
\overline{S_{A}} \underset{\text { critical,loc }}{\propto} \xi^{\alpha} \tag{7}
\end{equation*}
$$

whereas the matching with the volume-law of the delocalized phase in $d=1$ requires a vanishing coefficient of the volume-law

$$
\begin{equation*}
\overline{S_{A}} \underset{\text { critical,deloc }}{\propto} L_{A}\left(\frac{1}{\xi}\right)^{1-\alpha} \tag{8}
\end{equation*}
$$

As discussed in detail in [35, 36], one should then distinguish two possibilities :
(i) if the transition is directly towards the thermal ergodic phase satisfying E.T.H., the continuous vanishing of the coefficient $\left(\frac{1}{\xi}\right)^{1-\alpha}$ of the volume-law of Eq. 8 is actually forbidden by the strong-subadditivity property [35], and the only possibility is that the critical point is itself thermal, i.e. it should satisfy the volume law $\alpha=1$ with the finite thermal coefficient given by Eq. 3. This scenario was found numerically [37, 38] and via the RG based on entanglement [33] or resonances [34]. Then the difference between the critical point and the delocalized phase is not visible in the disorder-averaged entanglement entropy, but in its variance [33, 37, 38] and in the dynamical properties [33, 34].
(ii) if the transition is towards a delocalized non-ergodic phase, i.e. a phase satisfying the volume law, but not the E.T.H. with the thermal coefficient fixed by Eq. 3 then the continuous vanishing of the coefficient of the volume-law of Eq. 8 is possible, and the exponent $\alpha$ is not a priori fixed by the strong-subadditivity property (see [35] for more details). This scenario is suggested by the point of view that the MBL transition is somewhat similar to an Anderson Localization transition in the Hilbert space of 'infinite dimensionality' as a consequence of the exponential growth of the size of the Hilbert space with the volume [39-45].

In this paper, we consider the strong disorder limit of a one-dimensional quantum spin model in order to analyze the statistical properties of resonances that are needed to destabilize an MBL eigenstate satisfying the area-law $\overline{S_{A}}=O(1)$ and to determine the entanglement properties of the obtained critical state. The paper is organized as follows. In section II, we introduce the one-dimensional quantum spin model and derive the entanglement spectrum in terms of the couplings. In section III, the statistical properties of the entanglement spectrum are studied in terms of Lévy variables. The behavior of the entanglement entropy is analyzed in Section IV Finally in section V, the multifractal statistics of the entanglement spectrum is obtained. Our conclusions are summarized in section VI Two appendices contain complementary computations.

## II. SINGULAR PERTURBATION FROM THE MANY-BODY LOCALIZED PHASE

Before we introduce the MBL model, we need to explain the motivations coming from strong disorder limit of the Anderson Localization Transition.

## A. Motivation : strong disorder limit of the Anderson Localization Transition

At Anderson localization transitions, the critical states can be more or less multifractal (see the review [46] and references therein) :
(i) for the short-ranged tight-binding model in dimension $d$, there is a continuous interpolation between the 'weak multifractality' regime in $d=2+\epsilon$ and the 'strong multifractality' in high dimension $d$.
(ii) for the tight-binding model with long-ranged hopping

$$
\begin{equation*}
V(r)=\frac{V}{r^{a}} \tag{9}
\end{equation*}
$$

criticality is known to occur exactly at $a_{c}=d$, as can be understood from the scaling of the difference between energy levels

$$
\begin{equation*}
\Delta E\left(L^{d}\right) \propto L^{-d} \tag{10}
\end{equation*}
$$

Here there is also a continuous family of critical points as a function of the amplitude $V$, that interpolates between the 'weak multifractality' regime for large $V \rightarrow+\infty$ and the 'strong multifractality' regime for small $V \rightarrow 0$ [46].

From the point of view of the statistics of the energy levels, the 'weak multifractality' of wavefunctions corresponds to a level statistics close to the Wigner Dyson statistics of the delocalized phase, with a strong level repulsion and a vanishing level compressibility $\chi \rightarrow 0$, whereas the 'strong multifractality' of wavefunctions corresponds to a level statistics close to the Poisson statistics of the localized phase, with a very weak level repulsion and a level compressibility close to unity $\chi \simeq 1$ (see more details in [46]).

The 'strong multifractality' regime $V \rightarrow 0$ has been analyzed via the Levitov renormalization method that takes into account the resonances occuring at various scales [47 53] or other methods [54-60].

In a previous work [61], we have described how these strong multifractality results for small aamplitude $V$ in the Long-Ranged-Hopping model can be reproduced via the first-order perturbation theory with respect to the completely localized basis, provided one takes into account the broad Lévy statistics of the terms involved in the perturbative expansion.

## B. Model and notations

In the present paper, we wish to adapt the above strategy for the MBL transition as follows : instead of the standard nearest-neighbor MBL models, we wish to consider a toy model where the Hamiltonian contains direct couplings between any pair of configurations in the Hilbert space, with a vanishing small amplitude and with an appropriate decay as a function of the distance in Hilbert space in order to reach an MBL critical point. This critical point will be then as close as possible to the Many-Body-Localized Phase, and can be studied via singular perturbation theory around the completely localized limit.

So we consider a one-dimensional model involving $L$ quantum spins

$$
\begin{equation*}
H=H_{0}+V \tag{11}
\end{equation*}
$$

with the following properties. The Hamiltonian $H_{0}$ is chosen as the simplest possible Many-Body-Localized Hamiltonian

$$
\begin{equation*}
H_{0}=-\sum_{i=1}^{L} h_{i} \sigma_{i}^{z} \tag{12}
\end{equation*}
$$

where the fields $h_{i}$ are random variables drawn with the Gaussian distribution of variance $W^{2}$

$$
\begin{equation*}
G_{W}(h)=\frac{1}{\sqrt{2 \pi W^{2}}} e^{-\frac{h^{2}}{2 W^{2}}} \tag{13}
\end{equation*}
$$

The $2^{L}$ eigenstates are simply given by the tensor products

$$
\begin{equation*}
\left|\psi_{S_{1}, \ldots, S_{L}}^{(0)}>\equiv\right| S_{1}>\otimes\left|S_{2}>\ldots \otimes\right| S_{L}> \tag{14}
\end{equation*}
$$

with the random energies

$$
\begin{equation*}
E_{S_{1}, . ., S_{L}}^{(0)}=-\sum_{i=1}^{L} h_{i} S_{i} \tag{15}
\end{equation*}
$$

For instance, the ground state corresponds to the choice $S_{i}=\operatorname{sgn}\left(h_{i}\right)$ and has the extensive energy

$$
\begin{equation*}
E_{G S}^{(0)}=-\sum_{i=1}^{L}\left|h_{i}\right| \tag{16}
\end{equation*}
$$

In the following, we focus on the middle of the spectrum, where the density of states follows the Gaussian of zero mean and of variance $\left(L W^{2}\right)$

$$
\begin{equation*}
\rho_{0}(E)=\frac{1}{\sqrt{2 \pi L W^{2}}} e^{-\frac{E^{2}}{2 L W^{2}}} \tag{17}
\end{equation*}
$$

Since there are $2^{L}$ levels, the level spacing near zero energy scales as

$$
\begin{equation*}
\Delta E(L) \propto L^{\frac{1}{2}} 2^{-L} \tag{18}
\end{equation*}
$$

In analogy with Eq. 9, we wish to introduce a small perturbation $V$ that produces a direct coupling to all other $\left(2^{L}-1\right)$ states of the Hilbert space

$$
\begin{equation*}
V=-\sum_{k=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k} \leq L} J_{i_{1}, ., i_{k}} \sigma_{i_{1}}^{x} \sigma_{i_{2}}^{x} \ldots \sigma_{i_{k}}^{x} \tag{19}
\end{equation*}
$$

The couplings $J_{i_{1}, . ., i_{k}}$ are assumed to be of small amplitude, but they should be able to produce resonances at all scales. Their scaling should thus be directly related to the level spacing of Eq. 18, and in particular they should decay exponentially in space. However the precise conditions will be discussed later, and it is clearer to write the perturbation theory in the arbitrary small couplings $J_{i_{1}, . ., i_{k}}$, first for the eigenstates, then for the corresponding reduced density matrices, and finally for the entanglement spectrum.

## C. First order perturbation theory for the eigenstates

At first order in the perturbation $V$, the eigenvalues of Eq. 15 are unchanged

$$
\begin{equation*}
E_{S_{1}, \ldots, S_{L}}^{(1)}=E_{S_{1}, \ldots, S_{L}}^{(0)}+<\psi_{S_{1}, \ldots, S_{L}}^{(0)}|V| \psi_{S_{1}, \ldots, S_{L}}^{(0)}>=E_{S_{1}, \ldots, S_{L}}^{(0)} \tag{20}
\end{equation*}
$$

and the eigenstates read

$$
\begin{equation*}
\left|\psi_{S_{1}, . ., S_{L}}^{(1)}>=\left|\psi_{S_{1}, . ., S_{L}}^{(0)}>+\sum_{\left\{S_{i}^{\prime}\right\}}\right| \psi_{S_{1}^{\prime}, \ldots, S_{L}^{\prime}}^{(0)}>\frac{<\psi_{S_{1}^{\prime}, ., S_{L}^{\prime}}^{(0)}|V| \psi_{S_{1}, ., S_{L}}^{(0)}>}{E_{S_{1}, . ., S_{L}}^{(0)}-E_{S_{1}^{\prime}, . ., S_{L}^{\prime}}^{(0)}}\right. \tag{21}
\end{equation*}
$$

To simplify the notations, let us now focus on the particular state

$$
\begin{equation*}
|0>\equiv| S_{1}=+1, S_{2}=+1 . ., S_{L}=+1> \tag{22}
\end{equation*}
$$

whose energy

$$
\begin{equation*}
E_{0}^{(0)}=-\sum_{i=1}^{L} h_{i} \tag{23}
\end{equation*}
$$

is arbitrary in the spectrum of the unperturbed Hamiltonian $H_{0}$, as a consequence of the random fields of Eq. 13, Let us label the other $\left(2^{L}-1\right)$ states by the $1 \leq k \leq L$ positions $\left(i_{1}, . ., i_{k}\right)$ of flipped spins with respect to this reference configuration

$$
\begin{equation*}
\left|i_{1}, . ., i_{k}>\equiv \sigma_{i_{1}}^{x} \sigma_{i_{2}}^{x} \ldots \sigma_{i_{k}}^{x}\right| 0> \tag{24}
\end{equation*}
$$

Then the perturbed eigenstate of Eq. 21 reads

$$
\begin{equation*}
\left|\psi^{(1)}>=\left|0>+\sum_{k=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k} \leq L}\left(\frac{J_{i_{1}, i_{2}, ., i_{k}}}{2 \sum_{q=1}^{k} h_{i_{q}}}\right)\right| i_{1}, . ., i_{k}>\right. \tag{25}
\end{equation*}
$$

## D. Reduced density matrix of the region $A$

To evaluate the entanglement between the two regions $A=\left[1, L_{A}\right]$ and $B=\left[L_{A}+1, L=L_{A}+L_{B}\right]$, it is convenient to introduce the following basis in each region with the same notations above

$$
\begin{align*}
\mid 0>_{A} & \equiv \mid S_{1}=+1, S_{2}=+1 . ., S_{L_{A}}=+1> \\
\mid i_{1}, . ., i_{k}>_{A} & \equiv \sigma_{i_{1}}^{x} \sigma_{i_{2}}^{x} \ldots \sigma_{i_{k}}^{x} \mid 0>_{A} \quad \text { for } \quad 1 \leq i_{1}<i_{2} . .<i_{k} \leq L_{A} \tag{26}
\end{align*}
$$

and

$$
\begin{align*}
\mid 0>_{B} & \equiv \mid S_{L_{A}+1}=+1, S_{L+2}=+1 . ., S_{L}=+1> \\
\mid j_{1}, . ., j_{k}>_{B} & \equiv \sigma_{j_{1}}^{x} \sigma_{j_{2}}^{x} \ldots \sigma_{j_{k}}^{x} \mid 0>_{B} \quad \text { for } \quad L_{A}+1 \leq j_{1}<j_{2} . .<j_{k} \leq L_{A}+L_{B} \tag{27}
\end{align*}
$$

Then the eigenstate of Eq. 25 can be decomposed into

$$
\begin{align*}
& \left|\psi^{(1)}>=\left|0>_{A} \otimes\right| 0>_{B}\right. \\
& +\sum_{k=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k} \leq L_{A}} \frac{J_{i_{1}, i_{2}, . ., i_{k}}^{2 \sum_{q=1}^{k} h_{i_{q}}}\left|i_{1}, . ., i_{k}>_{A} \otimes\right| 0>_{B}}{+\sum_{k=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<k_{k} \leq L_{A}+L_{B}} \frac{J_{i_{1}, i_{2}, . . i_{k}}^{2 \sum_{q=1}^{k} h_{i_{q}}}}{2}\left|0>_{A} \otimes\right| i_{1}, . ., i_{k}>_{B}} \\
& +\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}} \frac{J_{i_{1}, ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}^{2 \sum_{q=1}^{k_{A} h_{i_{q}}+2 \sum_{p=1}^{k_{B}} h_{j_{p}}}}\left|i_{1}, . ., i_{k_{A}}>_{A} \otimes\right| j_{1}, . ., j_{k_{B}}>_{B}}{}
\end{align*}
$$

It is thus convenient to use instead the following perturbed basis for the region $A$

$$
\begin{align*}
\mid \phi_{0}^{A}> & \equiv \left\lvert\, 0>_{A}+\sum_{k=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k} \leq L_{A}} \frac{J_{i_{1}, i_{2}, . ., i_{k}}^{2 \sum_{q=1}^{k} h_{i_{q}}} \mid i_{1}, . ., i_{k}>_{A}}{}\right. \\
\mid \phi_{i_{1}, . ., i_{k}}^{A}> & \equiv\left|i_{1}, . ., i_{k}>_{A}-\frac{J_{i_{1}, i_{2}, . . i_{k}}}{2 \sum_{q=1}^{k} h_{i_{q}}}\right| 0>_{A} \quad \text { for } \quad 1 \leq i_{1}<i_{2} . .<i_{k} \leq L_{A} \tag{29}
\end{align*}
$$

and the similar basis for the region $B$

$$
\begin{align*}
\mid \phi_{0}^{B}> & \equiv\left|0>_{B}+\sum_{k=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k} \leq L_{A}+L_{B}} \frac{J_{j_{1}, j_{2}, . ., j_{k}}}{2 \sum_{q=1}^{k} h_{j_{q}}}\right| j_{1}, . ., j_{k}>_{B} \\
\mid \phi_{j_{1}, \ldots, j_{k}}^{B}> & \equiv \mid j_{1}, . ., j_{k}>_{B}-\frac{J_{j_{1}, j_{2}, . ., j_{k}}^{2 \sum_{q=1}^{k} h_{j_{q}}} \mid 0>_{B} \quad \text { for } \quad L_{A}+1 \leq j_{1}<j_{2} . .<j_{k} \leq L_{A}+L_{B}}{2 \sum_{i}} \quad l \tag{30}
\end{align*}
$$

Then Eq. 28 can be rewritten at first order in the perturbation as

$$
\begin{align*}
& \left|\psi^{(1)}>=\left|\phi_{0}^{A}>\otimes\right| \phi_{0}^{B}>\right. \\
& +\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}} \frac{J_{i_{1}, . . i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}^{2 \sum_{q=1}^{k_{A}} h_{i_{q}}+2 \sum_{p=1}^{k_{B} h_{j_{p}}}}\left|\phi_{i_{1}, . ., i_{k_{A}}}^{A}>\otimes\right| \phi_{j_{1}, . ., j_{k_{B}}}^{B}>}{}> \tag{31}
\end{align*}
$$

The corresponding density matrix

$$
\begin{align*}
\rho & \equiv \frac{\left|\psi^{(1)}><\psi^{(1)}\right|}{<\psi^{(1)} \mid \psi^{(1)}>}=\frac{1}{1+\Sigma_{1}}\left(\left|\phi_{0}^{A}>\otimes\right| \phi_{0}^{B}>\right. \\
& +\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}} \frac{J_{i_{1}, . . i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}^{\left.2 \sum_{q=1}^{k_{A} h_{i_{q}}+2 \sum_{p=1}^{k_{B} h_{j_{p}}}}\left|\phi_{i_{1}, ., i_{k_{A}}}^{A}>\otimes\right| \phi_{j_{1}, . ., j_{k_{B}}}^{B}>\right)}}{} \begin{array}{l}
\left(<\phi_{0}^{A}\left|\otimes<\phi_{0}^{B}\right|\right. \\
\\
\end{array}+\sum_{k_{A}^{\prime}=1}^{L_{A}} \sum_{1 \leq i_{1}^{\prime}<i_{2}^{\prime} . .<i_{k_{A}^{\prime}}^{\prime} \leq L_{A}} \sum_{k_{B}^{\prime}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}^{\prime}<j_{2}^{\prime} . .<j_{k_{B}^{\prime}}^{\prime} \leq L_{A}+L_{B}} \frac{\left.J_{i_{1}^{\prime}, . . i_{k_{k}}^{\prime}, j_{1}^{\prime}, \ldots j_{k_{B}^{\prime}}^{\prime}}^{2 \sum_{q=1}^{k_{A}^{\prime}} h_{i_{q}^{\prime}}+2 \sum_{p=1}^{k_{B}^{\prime} h_{j_{p}^{\prime}}}}<\phi_{i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}}^{A}\left|\otimes<\phi_{j_{1}^{\prime}, \ldots, j_{k_{B}^{\prime}}^{\prime}}^{B}\right|\right)( }{l}
\end{align*}
$$

with

$$
\begin{equation*}
\Sigma_{1} \equiv \sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}\left(\frac{J_{i_{1}, . . i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}}{2 \sum_{q=1}^{k_{A}} h_{i_{q}}+2 \sum_{p=1}^{k_{B}} h_{j_{p}}}\right)^{2} \tag{33}
\end{equation*}
$$

leads to the reduced density matrix for the region $A$

$$
\begin{align*}
\rho_{A} & \equiv \operatorname{Tr}_{B}(\rho)=<\phi_{0}^{B}|\rho| \phi_{0}^{B}>+\sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}<\phi_{j_{1}, . ., j_{k_{B}}}^{B}|\rho| \phi_{j_{1}, . ., j_{k_{B}}}^{B}> \\
& =\frac{1}{1+\Sigma_{1}}\left|\phi_{0}^{A}><\phi_{0}^{A}\right| \\
& +\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{A}^{\prime}=1}^{L_{A}} \sum_{1 \leq i_{1}^{\prime}<i_{2}^{\prime} . .<i_{k_{A}}^{\prime} \leq L_{A}} \frac{R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)}{1+\Sigma_{1}}\left|\phi_{i_{1}, . ., i_{k_{A}}}^{A}><\phi_{i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}}^{A}\right| \tag{34}
\end{align*}
$$

with

In particular, the diagonal elements involve the positive coefficients

$$
\begin{equation*}
D_{i_{1}, . ., i_{k_{A}}} \equiv R\left(i_{1}, . ., i_{k_{A}} ; i_{1}, . ., i_{k_{A}}\right) \equiv \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}\left(\frac{J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}}{2 \sum_{q=1}^{k_{A}} h_{i_{q}}+2 \sum_{p=1}^{k_{B}} h_{j_{p}}}\right)^{2} \tag{36}
\end{equation*}
$$

and their sum corresponds to Eq. 33

$$
\begin{equation*}
\Sigma_{1} \equiv \sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} D_{i_{1}, . ., i_{k_{A}}} \tag{37}
\end{equation*}
$$

## E. Entanglement spectrum

As we will see in the next section, the diagonal coefficients $D_{i_{1}, . ., i_{k_{A}}}$ of Eq. 36 and their sum $\Sigma_{1}$ of Eq. 37 are very broadly distributed with a Lévy law of index $\mu=1 / 2$. The physical meaning is that they are dominated by the few biggest terms corresponding to the smallest denominators $1 /\left(2 \sum_{q=1}^{k_{A}} h_{i_{q}}+2 \sum_{p=1}^{k_{B}} h_{j_{p}}\right)^{2}$ that can be interpreted as very rare resonances involving spins of both regions $A$ and $B$. The technical consequence is that the diagonal coefficients $D_{i_{1}, . ., i_{k_{A}}}$ of Eq. 36 and their sum $\Sigma_{1}$ give contributions of first order $O(|J|)$ in the couplings. On the contrary, the off-diagonal coefficients of Eq. 35 involving two different denominators have a finite averaged value of second order $O\left(J^{2}\right)$ (see Appendix A).

In conclusion, at first order in the couplings, the off-diagonal coefficients of Eq. 35 do not contribute, so that the diagonal elements directly represent the entanglement spectrum, i.e. the $2^{L_{A}}$ eigenvalues of the reduced density matrix read

$$
\begin{align*}
p_{0} & =\frac{1}{1+\Sigma_{1}} \\
p_{i_{1}, ., i_{k_{A}}} & =\frac{D_{i_{1}, . ., i_{k_{A}}}}{1+\Sigma_{1}} \quad \text { for } 1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A} \tag{38}
\end{align*}
$$

To characterize the statistical properties of these weights, it is convenient to introduce

$$
\begin{equation*}
Y_{q}\left(L_{A}\right)=\operatorname{Tr}_{A}\left(\rho_{A}^{q}\right)=p_{0}^{q}+\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} p_{i_{1}, . ., i_{k_{A}}}^{q}=\frac{1+\Sigma_{q}}{\left(1+\Sigma_{1}\right)^{q}} \tag{39}
\end{equation*}
$$

where the sum

$$
\begin{equation*}
\Sigma_{q} \equiv \sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} D_{i_{1}, . ., i_{k_{A}}}^{q} \tag{40}
\end{equation*}
$$

generalizes Eq. 37 ,
Then the Rényi entanglement entropy of index $q$

$$
\begin{equation*}
S_{q}\left(L_{A}\right) \equiv \frac{\ln Y_{q}\left(L_{A}\right)}{1-q}=\frac{\ln \left(1+\Sigma_{q}\right)-q \ln \left(1+\Sigma_{1}\right)}{1-q} \tag{41}
\end{equation*}
$$

allows to recover the usual entanglement entropy in the limit $q \rightarrow 1$

$$
\begin{equation*}
S_{1}\left(L_{A}\right) \equiv-\operatorname{Tr}_{A}\left(\rho_{A} \ln \rho_{A}\right)=-p_{0} \ln p_{0}-\sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} p_{i_{1}, . ., i_{k_{A}}} \ln p_{i_{1}, . ., i_{k_{A}}} \tag{42}
\end{equation*}
$$

In particular, to obtain the disorder-averaged value of the Renyi entropy of Eq. 41

$$
\begin{equation*}
\overline{S_{q}}=\frac{\overline{\ln \left(1+\Sigma_{q}\right)}-q \overline{\ln \left(1+\Sigma_{1}\right)}}{1-q} \tag{43}
\end{equation*}
$$

one only needs to study separately the probability distributions of $\Sigma_{q}$ and $\Sigma_{1}$, as done in the next section. Other statistical properties involving correlations between $\Sigma_{q}$ and $\Sigma_{1}$ can also be computed (see for instance Appendix B for the disorder-averaged values $\overline{Y_{q}}$ and for the variance of the entanglement entropy $S_{1}$ ).

## III. STATISTICAL PROPERTIES OF THE ENTANGLEMENT SPECTRUM

In this section, we analyze the statistical properties of the entanglement spectrum obtained in Eq. 38

## A. Probability distribution of the variable $D_{i_{1}, \ldots, i_{k_{A}}}$

Let us first consider the probability distribution $P_{i_{1}, . ., i_{k_{A}}}\left(D_{i_{1}, \ldots, i_{k_{A}}}\right)$ of the positive random variable $D_{i_{1}, \ldots, i_{k_{A}}}$ of Eq. 36 by evaluating its Laplace transform at lowest order in the couplings

$$
\begin{aligned}
& \overline{e^{-t D_{i_{1}, \ldots, i_{k_{A}}}}}=\int_{0}^{+\infty} d D_{i_{1}, . ., i_{k_{A}}} P_{i_{1}, . ., i_{k_{A}}}\left(D_{i_{1}, \ldots, i_{k_{A}}}\right) e^{-t D_{i_{1}, \ldots, i_{k_{A}}}}
\end{aligned}
$$

$$
\begin{align*}
& =1-\sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}} \overline{\left(1-e^{-t\left(\frac{\left.J_{i_{1}, \ldots, i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}^{2 \sum_{q=1}^{k_{A} h_{h_{q}}+2 \sum_{p=1}^{k_{B}} h_{h_{p}}}}\right)^{2}}{}\right.}+o(J) .{ }^{2}\right.}+ \tag{44}
\end{align*}
$$

The variable

$$
\begin{equation*}
E=2 \sum_{q=1}^{k_{A}} h_{i_{q}}+2 \sum_{p=1}^{k_{B}} h_{j_{p}} \tag{45}
\end{equation*}
$$

is a sum of $\left(k_{A}+k_{B}\right)$ Gaussian variables (Eq. 13) and is thus distributed with the Gaussian of zero mean and of variance $\left(4 W^{2}\left(k_{A}+k_{B}\right)\right)$

$$
\begin{equation*}
G_{4 W^{2}\left(k_{A}+k_{B}\right)}(E)=\frac{1}{2 \sqrt{2\left(k_{A}+k_{B}\right) \pi W^{2}}} e^{-\frac{E^{2}}{8\left(k_{A}+k_{B}\right) W^{2}}} \tag{46}
\end{equation*}
$$

So using the change of variable $x=t_{i_{1}, . ., i_{k_{A}}} \frac{J_{i_{1}, \ldots . j_{k_{B}}}^{2}}{E^{2}}$, one obtains

$$
\begin{align*}
& 1-e^{-t\left(\frac{J_{i_{1}, \ldots, i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}}{2 \sum_{q=1}^{k_{A} h_{i_{q}}+2 \sum_{p=1}^{k_{B}} h_{j_{p}}}}\right)^{2}} \\
& =\int_{0}^{+\infty} \frac{d E}{\sqrt{2\left(k_{A}+k_{B}\right) \pi W^{2}}} e^{-\frac{E^{2}}{8\left(k_{A}+k_{B}\right) W^{2}}}\left(1-e^{-t\left(\frac{\left.J_{i_{1}, \ldots, i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}^{E}\right)^{2}}{E}\right.}\right) \\
& =\overline{\left|J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}\right| t^{\frac{1}{2}} \frac{1}{2 \sqrt{2\left(k_{A}+k_{B}\right) \pi W^{2}}} \int_{0}^{+\infty} \frac{d x}{x^{\frac{3}{2}}} e^{-\frac{J_{i_{1}, i_{2}, \ldots, i_{k} t_{A}}^{8\left(k_{A}+k_{B}\right) W^{2} x}}{}}\left(1-e^{-x}\right)} \\
& =\overline{\left|J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}\right|} t^{\frac{1}{2}} \frac{1}{2 \sqrt{2\left(k_{A}+k_{B}\right) \pi W^{2}}} \int_{0}^{+\infty} \frac{d x}{x^{\frac{3}{2}}}\left(1-e^{-x}\right)+o\left(J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}\right) \\
& =\overline{\left|J_{i_{1}, . ., i_{k_{A}}}, j_{1}, \ldots j_{k_{B}}\right|} t^{\frac{1}{2}} \frac{1}{\sqrt{2\left(k_{A}+k_{B}\right) W^{2}}}+o\left(J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}\right) \tag{47}
\end{align*}
$$

so that Eq 44 becomes at first order in the couplings $J_{i_{1}, . ., i_{k_{A}}}, j_{1}, \ldots j_{k_{B}}$

$$
\begin{align*}
& e^{-t_{i_{1}, \ldots, i_{k_{A}} D_{i_{1}, \ldots, i_{k_{A}}}}} \\
& =1-t^{\frac{1}{2}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}} \overline{\mid J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}} \frac{1}{\sqrt{2\left(k_{A}+k_{B}\right) W^{2}}}+o(J) \\
& =e^{-t^{\frac{1}{2}} \sum_{k_{B}=1}^{L_{B} \sum_{L_{A}}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}} \overline{\mid J_{i_{1}, . . i_{k_{A}}, j_{1}, \ldots j_{k_{B}} \mid}} \frac{1}{\sqrt{2\left(k_{A}+k_{B}\right) W^{2}}}+o(J)} \tag{48}
\end{align*}
$$

It is thus convenient to introduce the notation

$$
\begin{equation*}
\Omega_{i_{1}, . ., i_{k_{A}}} \equiv \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}} \overline{\left|J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}\right|} \frac{1}{\sqrt{2\left(k_{A}+k_{B}\right) W^{2}}} \tag{49}
\end{equation*}
$$

and the Lévy positive stable-law of index $\mu=1 / 2$ and of parameter $\Omega$

$$
\begin{equation*}
L_{\frac{1}{2} ; \Omega}(D) \equiv \frac{\Omega}{2 \sqrt{\pi} D^{\frac{3}{2}}} e^{-\frac{\Omega^{2}}{4 D}} \tag{50}
\end{equation*}
$$

with its Laplace transform

$$
\begin{equation*}
\hat{L}_{\frac{1}{2}, \Omega}(t) \equiv \int_{0}^{+\infty} d D L_{\frac{1}{2} ; \Omega}(D) e^{-t D}=e^{-\Omega t^{\frac{1}{2}}} \tag{51}
\end{equation*}
$$

The parameter $\Omega$ directly measures the weight of the singularity in $t^{\frac{1}{2}}$ of the Laplace transform, and thus also the weight of the power-law behavior at large $D$ of the distribution of Eq. 50. Lévy variables [62] actually appear very often in the studies of disordered systems (see for instance [63-66] for more details on their properties).

Then Eq. 48 means that at lowest order in the couplings $J$, the variable $D_{i_{1}, . ., i_{k_{A}}}$ is distributed with the Lévy law $L_{\frac{1}{2} ; \Omega_{i_{1}, \ldots, i_{k}}}$

$$
\begin{equation*}
P_{i_{1}, . ., i_{k_{A}}}\left(D_{i_{1}, . ., i_{k_{A}}}\right)=L_{\frac{1}{2} ; \Omega_{i_{1}, \ldots, i_{k_{A}}}}\left(D_{i_{1}, . ., i_{k_{A}}}\right) \tag{52}
\end{equation*}
$$

## B. Probability distribution of $\Sigma_{1}$

The probability distribution $P_{1}\left(\Sigma_{1}\right)$ of the positive random variable $\Sigma 1$ of Eq. 37 can be evaluated similarly via its Laplace transform

$$
\begin{align*}
& \overline{e^{-t \Sigma_{1}}} \equiv \int_{0}^{+\infty} d \Sigma_{1} P_{1}\left(\Sigma_{1}\right) e^{-t \Sigma_{1}} \\
& =e^{-t \sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}\left(\frac{\left.J_{i_{1}, \ldots, i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}^{2 \sum_{q=1}^{k_{A} h_{i_{q}}+2 \sum_{p=1}^{k_{B}} h_{j_{p}}}}\right)^{2}}{}\right.} \\
& =\prod_{k_{A}=1}^{L_{A}} \prod_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \prod_{k_{B}=1}^{L_{L_{B}}} \prod_{1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}\left[1-\left(1-e^{-t\left(\frac{\left.J_{i_{1}, . . i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}^{\left.2 \sum_{q=1}^{k_{A} h_{i_{q}}+2 \sum_{p=1}^{k_{B} h_{j_{p}}}}\right)^{2}}\right)}{}\right]}\right.\right. \\
& =1-\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}\left(1-e^{-t\left(\frac{\left.J_{i_{1}, ., i_{k_{A}, j_{1}, \ldots j_{k_{B}}}}^{\left.2 \sum_{q=1}^{k_{A} h_{i_{q}}+2 \sum_{p=1}^{k_{B} h_{j_{p}}}}\right)^{2}}\right)}{(o(J)}\right) . o n}\right. \tag{53}
\end{align*}
$$

Eq. 47 then yields that

$$
\begin{align*}
\overline{e^{-t \Sigma_{1}}} & =1-t^{\frac{1}{2}} \sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}}^{L_{B}} \sum_{k_{B}=1} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}} \frac{1}{\left\lvert\, J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}} \mid} \frac{1}{\sqrt{2\left(k_{A}+k_{B}\right) W^{2}}}+o(J)\right.} \\
& =e^{-\Omega_{1} t^{\frac{1}{2}}+o(J)} \tag{54}
\end{align*}
$$

with

$$
\begin{align*}
\Omega_{1} & \equiv \sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \Omega_{i_{1}, . ., i_{k_{A}}} \\
& =\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}\left|J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}\right| \frac{1}{\sqrt{2\left(k_{A}+k_{B}\right) W^{2}}} \tag{55}
\end{align*}
$$

Eq. 54 means that at lowest order in the couplings $J$, the variable $\Sigma_{1}$ is distributed with the Lévy law $L_{\frac{1}{2} ; \Omega_{1}}$ (Eqs 50 and 51) of index $\mu=1 / 2$ and of parameter $\Omega_{1}$

$$
\begin{equation*}
P_{1}\left(\Sigma_{1}\right)=L_{\frac{1}{2} ; \Omega_{1}}\left(\Sigma_{1}\right) \tag{56}
\end{equation*}
$$

In particular, using the identity

$$
\begin{equation*}
\ln \left(1+\Sigma_{1}\right)=\int_{0}^{+\infty} \frac{d t}{t} e^{-t}\left(1-e^{-t \Sigma_{1}}\right) \tag{57}
\end{equation*}
$$

one obtains

$$
\begin{align*}
\overline{\ln \left(1+\Sigma_{1}\right)} & =\int_{0}^{+\infty} d \Sigma_{1} P_{1}\left(\Sigma_{1}\right) \ln \left(1+\Sigma_{1}\right) \\
& =\int_{0}^{+\infty} \frac{d t}{t} e^{-t}\left(1-\overline{e^{-t \Sigma_{1}}}\right) \\
& =\int_{0}^{+\infty} \frac{d t}{t} e^{-t} \Omega_{1} t^{\frac{1}{2}}+o(J) \\
& =\Omega_{1} \sqrt{\pi}+o(J) \tag{58}
\end{align*}
$$

## C. Probability distribution of the weight $p_{0}$

The probability distribution $\mathcal{P}\left(p_{0}\right)$ of the weight $0 \leq p_{0} \leq 1$ of the unperturbed state ( Eq (38)

$$
\begin{equation*}
p_{0}=\frac{1}{1+\Sigma_{1}} \tag{59}
\end{equation*}
$$

can be obtained directly from Eq 56 by the change of variable

$$
\begin{equation*}
\mathcal{P}\left(p_{0}\right)=\frac{1}{p_{0}^{2}} L_{\frac{1}{2} ; \Omega_{1}}\left(\frac{1}{p_{0}}-1\right)=\frac{\Omega_{1} p_{0}^{\frac{1}{2}}}{2 \sqrt{\pi}\left(1-p_{0}\right)^{\frac{3}{2}}} e^{-\frac{\Omega_{1}^{2} p_{0}}{4\left(1-p_{0}\right)}} \tag{60}
\end{equation*}
$$

In particular, its averaged value

$$
\begin{equation*}
\int_{0}^{1} d p_{0} p_{0} \mathcal{P}\left(p_{0}\right)=1-\Omega_{1} e^{\frac{\Omega_{1}^{2}}{4}} \int_{\frac{\Omega_{1}}{2}}^{+\infty} d t e^{-t^{2}} \tag{61}
\end{equation*}
$$

is close to unity for small $\Omega_{1}$

$$
\begin{equation*}
\int_{0}^{1} d p_{0} p_{0} \mathcal{P}\left(p_{0}\right)=1-\frac{\sqrt{\pi}}{2} \Omega_{1}+O\left(\Omega_{1}^{2}\right) \tag{62}
\end{equation*}
$$

## D. Probability distribution of $\Sigma_{q}$

Let us now consider the variable $\Sigma_{q}$ of Eq. 40
Here one has to distinguish two regions for the index $q$ :
(i) For $0<q<\frac{1}{2}$, the moment of order $q$ of $D_{i_{1}, ., i_{k_{A}}}$ distributed with the Lévy law of Eq. 50 is finite and reads

$$
\begin{equation*}
\overline{D_{i_{1}, . ., i_{k_{A}}}^{q}}=\int_{0}^{+\infty} d D \frac{\Omega_{i_{1}, . ., i_{k_{A}}}}{2 \sqrt{\pi}} D^{q-\frac{3}{2}} e^{-\frac{\Omega_{i_{1}, \ldots, i_{k_{A}}}^{2}}{4 D}}=\frac{\Omega_{i_{1}, . ., i_{k_{A}}}^{2 q} \Gamma\left(\frac{1}{2}-q\right)}{4 q \sqrt{\pi}} \tag{63}
\end{equation*}
$$

As a consequence, the average of $\Sigma_{q}$ is also finite and given by

$$
\begin{equation*}
\overline{\Sigma_{q}}=\sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \overline{D_{i_{1}, . ., i_{k_{A}}}^{q}}=\frac{\Gamma\left(\frac{1}{2}-q\right)}{4^{q} \sqrt{\pi}} \sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \Omega_{i_{1}, . ., i_{k_{A}}}^{2 q} \tag{64}
\end{equation*}
$$

so that at lowest order in the couplings one has

$$
\begin{equation*}
\overline{\ln \left(1+\Sigma_{q}\right)} \simeq \overline{\Sigma_{q}}=\frac{\Gamma\left(\frac{1}{2}-q\right)}{4^{q} \sqrt{\pi}} \sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \Omega_{i_{1}, . ., i_{k_{A}}}^{2 q}+o\left(|J|^{2 q}\right) \tag{65}
\end{equation*}
$$

(ii) For $q>\frac{1}{2}$, the average of $\Sigma_{q}$ is infinite, and one needs to estimate the Laplace transform of its probability distribution $P_{q}\left(\Sigma_{q}\right)$ as above

$$
\begin{align*}
\overline{e^{-t \Sigma_{q}}} & \equiv \int_{0}^{+\infty} d \Sigma_{q} P_{q}\left(\Sigma_{q}\right) e^{-t \Sigma_{q}}=\prod_{k_{A}=1}^{L} \prod_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L}\left[\overline{e^{-t D_{i_{1}, ., i_{k_{A}}}^{q}}}\right] \\
& =\prod_{k_{A}=1}^{L} \prod_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \overline{1-\left(1-e^{-t D_{i_{1}, ., i_{k_{A}}}^{q}}\right)} \\
& =1-\sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \overline{\left(1-e^{-t D_{i_{1}, . ., i_{k_{A}}}^{q}}\right)}+o(J) \tag{66}
\end{align*}
$$

Using Eq. 50 and the change of variable $x=t D^{q}$, one obtains

$$
\begin{align*}
\overline{\left(1-e^{-t D^{q}}\right)} & =\int_{0}^{+\infty} d D \frac{\Omega}{2 \sqrt{\pi} D^{\frac{3}{2}}} e^{-\frac{\Omega^{2}}{4 D}}\left(1-e^{-t D^{q}}\right) \\
& =\frac{\Omega}{2 q \sqrt{\pi}} t^{\frac{1}{2 q}} \int_{0}^{+\infty} \frac{d x}{x^{1+\frac{1}{2 q}}}\left(1-e^{-x}\right)+o(J) \\
& =\frac{\Omega \sqrt{\pi}}{\sin \left(\frac{\pi}{2 q}\right) \Gamma\left(\frac{1}{2 q}\right)} t^{\frac{1}{2 q}}+o(J) \tag{67}
\end{align*}
$$

so that Eq. 66 becomes

$$
\begin{align*}
\overline{e^{-t \Sigma_{q}}} & =1-\frac{\sqrt{\pi}}{\sin \left(\frac{\pi}{2 q}\right) \Gamma\left(\frac{1}{2 q}\right)} t^{\frac{1}{2 q}} \sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \Omega_{i_{1}, . ., i_{k_{A}}}+o(J) \\
& =e^{-\frac{\sqrt{\pi \Omega_{1}}}{\sin \left(\frac{\pi}{2 q}\right)\left(\frac{1}{2 q}\right)} t^{\frac{1}{2 q}}}+o(J) \tag{68}
\end{align*}
$$

It is thus convenient to introduce the notation

$$
\begin{equation*}
\Omega_{q} \equiv \frac{\sqrt{\pi} \Omega_{1}}{\sin \left(\frac{\pi}{2 q}\right) \Gamma\left(\frac{1}{2 q}\right)} \tag{69}
\end{equation*}
$$

that generalizes Eq. 55, and the Lévy positive stable-law $L_{\mu ; \Omega}$ of index $0<\mu<1$ and of parameter $\Omega$ defined by its Laplace transform

$$
\begin{equation*}
\hat{L}_{\mu, \Omega}(t) \equiv \int_{0}^{+\infty} d D L_{\mu ; \Omega}(D) e^{-t D}=e^{-\Omega t^{\mu}} \tag{70}
\end{equation*}
$$

Eq. 68 means that for $q>\frac{1}{2}$, the variable $\Sigma_{q}$ is distributed with the Lévy stable law $L_{\mu_{q} ; \Omega_{q}}$ of index $\mu_{q}=\frac{1}{2 q}$ and parameter $\Omega_{q}$

$$
\begin{equation*}
P_{q}\left(\Sigma_{q}\right)=L_{\frac{1}{2 q} ; \Omega_{q}}\left(\Sigma_{q}\right) \tag{71}
\end{equation*}
$$

that generalizes Eq 56. In particular, using the identity of Eq. 58 for $\Sigma_{q}$, one obtains

$$
\begin{align*}
\overline{\ln \left(1+\Sigma_{q}\right)} & =\int_{0}^{+\infty} d \Sigma_{q} P_{q}\left(\Sigma_{q}\right) \ln \left(1+\Sigma_{q}\right) \\
& =\int_{0}^{+\infty} \frac{d t}{t} e^{-t}\left(1-\overline{e^{-t \Sigma_{q}}}\right) \\
& =\int_{0}^{+\infty} \frac{d t}{t} e^{-t} \Omega_{q} t^{\frac{1}{2 q}}+o(J) \\
& =\Omega_{q} \Gamma\left(\frac{1}{2 q}\right)+o(J) \\
& =\Omega_{1} \frac{\sqrt{\pi}}{\sin \left(\frac{\pi}{2 q}\right)}+o(J) \tag{72}
\end{align*}
$$

## E. Disorder-averaged values of the Renyi entropies

The disorder-averaged entanglement entropy of Eq. 43 can be obtained from the previous results :
(i) in the region $0<q<1 / 2$, the contribution of Eq. 65 of order $|J|^{2 q}$ dominates over the contribution of Eq. 58, so that the leading order reads

$$
\begin{equation*}
\overline{S_{q}}=\frac{\Gamma\left(\frac{1}{2}-q\right)}{(1-q) 4^{q} \sqrt{\pi}} \sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \Omega_{i_{1}, . ., i_{k_{A}}}^{2 q}+o\left(|J|^{2 q}\right) \tag{73}
\end{equation*}
$$

(ii) in the region $q>\frac{1}{2}$ one obtains using Eq. 58 and Eq. 72

$$
\begin{equation*}
\overline{S_{q}}=\frac{\Omega_{1} \sqrt{\pi}}{1-q}\left[\frac{1}{\sin \left(\frac{\pi}{2 q}\right)}-q\right]+o(J) \tag{74}
\end{equation*}
$$

and in particular in the limit $q \rightarrow 1$

$$
\begin{equation*}
\overline{S_{1}}=\Omega_{1} \sqrt{\pi}+o(J) \tag{75}
\end{equation*}
$$

In conclusion, the parameter $\Omega_{1}$ of Eq. 55 directly represents the scale of the disorder-averaged entanglement entropy of Eq. 75. As shown in Eq. B15 of Appendix B the scale $\Omega_{1}$ also governs the variance of the entanglement entropy $S_{1}$.

## IV. SCALING OF THE ENTANGLEMENT ENTROPY WITH THE LENGTH $L_{A}$

In this section, we study how the scale $\Omega_{1}$ (Eq. 55) of the disorder-averaged entanglement entropy of Eq. 75 depends on the length $L_{A}$ of the region $A$. Here we need to make more precise assumptions on the couplings $J_{i_{1}, \ldots, i_{k}}$.

## A. Statistical properties of the couplings $J_{i_{1}, ., i_{k}}$

As explained in Section IIB within the toy model of Eq. 11 that we consider for the MBL case, the couplings $J_{i_{1}, . ., i_{k}}$ involved in the perturbation of Eq. 19 are the analog of the long-ranged hoppings of Eq. 9 for the Anderson model. In the Anderson case, the hopping between two points depends only on the distance, but in the MBL case, we have to choose the properties for all the couplings $J_{i_{1}, . ., i_{k}}$. To simplify the discussion, let us make the simplest possible choice and assume that the couplings $J_{i_{1}, . ., i_{k}}$ are independent Gaussian random variables

$$
\begin{equation*}
P\left(J_{i_{1}, . ., i_{k}}\right)=\frac{1}{\sqrt{2 \pi \Delta^{2}\left(i_{k}-i_{1}\right)}} e^{-\frac{J_{i_{1}, \ldots, i_{k}}^{2}}{2 \Delta^{2}\left(i_{k}-i_{1}\right)}} \tag{76}
\end{equation*}
$$

of zero mean

$$
\begin{equation*}
\overline{J_{i_{1}, . ., i_{k}}}=0 \tag{77}
\end{equation*}
$$

and of variance depending only on the spatial range $r \equiv i_{k}-i_{1}$, i.e. on the distance between the two extremal spins

$$
\begin{equation*}
\overline{J_{i_{1}, .,, i_{k}}^{2}}=\Delta^{2}\left(r \equiv i_{k}-i_{1}\right) \tag{78}
\end{equation*}
$$

In addition, we consider the following size dependence with three parameters

$$
\begin{equation*}
\Delta(r)=\frac{v}{r^{a}} 2^{-b r} \tag{79}
\end{equation*}
$$

where $b$ governs the exponential decay, $a$ governs the power-law prefactor, and $v$ is the small amplitude of the perturbation theory described in the previous sections. From the exponential decay of the level spacing of Eq. 18, one may already guess that the critical value for the exponential decay will be $b_{c}=1$, as found indeed below.

Since the perturbation $V$ of Eq. 11 is a part of the full Hamiltonian of Eq. 19, we wish to impose that it remains extensive with respect to the number of spins. This means that the local field in the $x$ direction on a given spin, for instance the first one

$$
\begin{equation*}
B_{1}^{x}=-\sum_{k=2}^{L} \sum_{1<i_{2} . .<i_{k} \leq L} J_{1, . ., i_{k}} S_{i_{2}}^{x} \ldots S_{i_{k}}^{x} \tag{80}
\end{equation*}
$$

should remain a finite random variable as $L \rightarrow+\infty$. Its variance may be evaluated as

$$
\begin{equation*}
\overline{\left(B_{1}^{x}\right)^{2}} \simeq \sum_{k=2}^{L} \sum_{1<i_{2} . .<i_{k} \leq L} \overline{J_{1, . ., i_{k}}^{2}}=\sum_{k=2}^{L} \sum_{1<i_{2} . .<i_{k} \leq L} \Delta^{2}\left(i_{k}-1\right) \tag{81}
\end{equation*}
$$

When $i_{k}=1+r$ is fixed, the number of ways to place the remaining $q=(k-2)$ points $\left(i_{2}, . ., i_{k-1}\right)$ into the $\left(i_{k}-2\right)=r-1$ possible positions $\left(i_{1}+1, . ., i_{k}-1\right)$ is given by the binomial coefficient $\binom{r-1}{q}$, so that Eq. 80 yields

$$
\begin{equation*}
\overline{\left(B_{1}^{x}\right)^{2}} \simeq \sum_{r=1} \Delta^{2}(r) \sum_{q=0}^{r-1}\binom{r-1}{q}=\sum_{r=1} \Delta^{2}(r) 2^{r-1} \tag{82}
\end{equation*}
$$

So the extensivity of the perturbation requires the convergence of the sum

$$
\begin{equation*}
\sum_{r=1}^{+\infty} \Delta^{2}(r) 2^{r}<+\infty \tag{83}
\end{equation*}
$$

With the form of Eq. 79, the convergence of the sum

$$
\begin{equation*}
\sum_{r=1}^{+\infty} \Delta^{2}(r) 2^{r}=v^{2} \sum_{r=1}^{+\infty} \frac{1}{r^{2 a}} 2^{(1-2 b) r} \tag{84}
\end{equation*}
$$

requires either (i) $b>1 / 2$, or (ii) $b=1 / 2$ with $a>1 / 2$.

## B. Study of the scale $\Omega_{1}$

The average value of the absolute value of the coupling distributed with Eq. 76

$$
\begin{equation*}
\overline{\left|J_{i_{1}, . ., i_{k} \mid}\right|}=\int_{0}^{+\infty} d J J \frac{2}{\sqrt{2 \pi \Delta^{2}\left(i_{k}-i_{1}\right)}} e^{-\frac{J^{2}}{2 \Delta^{2}\left(i_{k}-i_{1}\right)}}=\sqrt{\frac{2}{\pi}} \Delta\left(i_{k}-i_{1}\right) \tag{85}
\end{equation*}
$$

is also governed by the scale $\Delta(r)$ as a function of the spatial range $r$.
As a consequence, the scale $\Omega_{1}$ of Eq. 55 can be evaluated as above, i.e. once the extreme points $i_{1}$ and $j_{k_{B}}$ have been chosen, one has binomial coefficients to take into account the number of ways to place the $\left(k_{A}-1\right)$ points $\left(i_{2}, . ., i_{k_{A}}\right)$ in the $\left(L_{A}-i_{1}\right)$ remaining possible positions $\left(i_{1}+1, . ., L_{A}\right)$, and to place the $\left(k_{B}-1\right)$ points $\left(j_{1}, . ., j_{k_{B}-1}\right)$ in the $\left(j_{k_{B}}-L_{A}-1\right)$ remaining possible positions $\left(L_{A}+1, . ., j_{k_{B}}-1\right)$

$$
\begin{align*}
\Omega_{1} & =\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}^{\mid J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}} \mid}} \frac{1}{\sqrt{2\left(k_{A}+k_{B}\right) W^{2}}} \\
& =\frac{\sqrt{2}}{\pi W} \sum_{i_{1}=1}^{L_{A}} \sum_{j_{k_{B}}=L_{A}+1}^{L_{A}+L_{B}} \Delta\left(j_{k_{B}}-i_{1}\right) \sum_{k_{A}=1}^{L_{A}-i_{1}+1} \sum_{k_{B}=1}^{j_{k_{B}}-L_{A}} \frac{1}{\sqrt{\left(k_{A}+k_{B}\right)}}\binom{L_{A}-i_{1}}{k_{A}-1}\binom{j_{k_{B}}-L_{A}-1}{k_{B}-1} \\
& =\frac{\sqrt{2}}{\pi W} \sum_{n=0}^{L_{A}-1} \sum_{m=0}^{L_{B}-1} \Delta(1+n+m) \sum_{k_{A}=1}^{n+1} \sum_{k_{B}=1}^{m+1} \frac{1}{\sqrt{\left(k_{A}+k_{B}\right)}}\binom{n}{k_{A}-1}\binom{m}{k_{B}-1} \tag{86}
\end{align*}
$$

where $n=L_{A}-i_{1}$ and $m=j_{k_{B}}-L_{A}-1$ represent the distances with respect to the frontier between the regions $A$ and $B$.

Of course the scale $\Omega_{1}$ has always a finite contribution $O(1)$ coming from the finite distances $(n, m)$ of order $O(1)$ with respect to the frontier. Now we wish to evaluate the contribution of large distances $(n, m)$. Using the identity

$$
\begin{equation*}
\frac{1}{\sqrt{k_{A}+k_{B}}}=\frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} d u u^{-\frac{1}{2}} e^{-\left(k_{A}+k_{B}\right) u} \tag{87}
\end{equation*}
$$

one obtains

$$
\begin{align*}
& \sum_{k_{A}=1}^{n+1} \sum_{k_{B}=1}^{m+1} \frac{1}{\sqrt{\left(k_{A}+k_{B}\right)}}\binom{n}{k_{A}-1}\binom{m}{k_{B}-1} \\
& =\frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} d u u^{-\frac{1}{2}}\left[\sum_{k_{A}=1}^{n+1}\binom{n}{k_{A}-1} e^{-k_{A} u}\right]\left[\sum_{k_{B}=1}^{m+1}\binom{m}{k_{B}-1} e^{-k_{B} u}\right] \\
& =\frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} d u u^{-\frac{1}{2}} e^{-2 u}\left[1+e^{-u}\right]^{1+n+m} \\
& =\frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} d u u^{-\frac{1}{2}} e^{-2 u}\left[2-\left(1-e^{-u}\right)\right]^{1+n+m} \\
& =2^{1+n+m} \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} d u u^{-\frac{1}{2}} e^{-2 u}\left[1-\frac{\left(1-e^{-u}\right)}{2}\right]^{1+n+m} \\
& \simeq  \tag{88}\\
& n+m \rightarrow+\infty
\end{align*} 2^{1+n+m} \frac{\sqrt{2}}{\sqrt{n+m}} 1
$$

As could have been anticipated, this leading behavior means that the sum is dominated by the regions $k_{A} \sim \frac{n}{2}$ and $k_{B} \sim \frac{m}{2}$ that maximize the binomial coefficients, i.e. the dominant resonances on the large distance $(n+m)$ are extensive resonances involving a number of spins of order $k_{A}+k_{B} \simeq \frac{n+m}{2}$.

As a consequence, after the multiplication by $\Delta(1+n+m)$ with Eq. 79 , one obtains the asymptotic behavior

$$
\begin{equation*}
\Delta(1+n+m) \sum_{k_{A}=1}^{n+1} \sum_{k_{B}=1}^{m+1} \frac{1}{\sqrt{\left(k_{A}+k_{B}\right)}}\binom{n}{k_{A}-1}\binom{m}{k_{B}-1} \quad \underset{n+m \rightarrow+\infty}{\propto} \frac{v}{(n+m)^{a+\frac{1}{2}}} 2^{-(1-b)(n+m)} \tag{89}
\end{equation*}
$$

that we need to integrate over $n$ and $m$ to obtain the scale $\Omega_{1}$ of Eq. 86,
So we arrive at the following conclusions :
(i) for $b>1$, there is an exponential convergence $2^{-(1-b)(n+m)}$ at large distance, so the Many-Body-Localized phase is stable and displays a finite entanglement entropy.
(ii) for $b_{c}=1$, there is no exponential factor anymore in Eq. 89, but only the power-law $\frac{1}{(n+m)^{a+\frac{1}{2}}}$. We wish that the integral over $m$ converges in order to have a well-defined thermodynamic limit $L_{B} \rightarrow+\infty$ for the region $B$ : this is the case for $a>1 / 2$. Then for $1 / 2<a<\frac{3}{2}$, the scale $\Omega_{1}$ is dominated by the contribution of the long distance

$$
\begin{align*}
\Omega_{1}\left(b_{c}=1 ; 1 / 2<a<\frac{3}{2}\right) & \propto v \sum_{n=1}^{L_{A}} \frac{1}{\left(a-\frac{1}{2}\right) n^{a-\frac{1}{2}}} \\
& \propto \frac{v}{\left(\frac{3}{2}-a\right)\left(a-\frac{1}{2}\right)} L_{A}^{\frac{3}{2}-a} \tag{90}
\end{align*}
$$

## C. Entanglement growth at criticality

In summary, for the value $b_{c}=1$ in Eq. 79 for the coupling that matches exactly the exponential decay of the level spacing of Eq. 18, we have obtained that, as the parameter $a$ of the power-law of Eq. 79 varies in the interval $1 / 2<a<\frac{3}{2}$, it is possible to construct a critical state with an entanglement growth governed by the exponent

$$
\begin{equation*}
0<\alpha=\frac{3}{2}-a<1 \tag{91}
\end{equation*}
$$

anywhere between the area law $\alpha=0$ and the volume law $\alpha=1$.
Of course within the present approach, we have to stop at the critical point and we have no access to the delocalized phase, but as recalled in the Introduction near Eq. 8 the values $0<\alpha<1$ are only possible if the transition is towards a delocalized non-ergodic phase 35].

## D. Correlation length exponent $\nu$ of the MBL phase

In the MBL phase $b<b_{c}=1$, the exponential convergence $2^{-(1-b)(n+m)}$ in Eq. 89 corresponds to the correlation length

$$
\begin{equation*}
\xi=\frac{1}{b-1} \tag{92}
\end{equation*}
$$

that diverges near the transition $b \rightarrow b_{c}=1$ with the simple correlation length exponent

$$
\begin{equation*}
\nu=1 \tag{93}
\end{equation*}
$$

This value simply reflects the crossing of the exponential decay of the level spacing of Eq. 18 and of exponential decay of the couplings of Eq. 79, It coincides with the exact correlation length exponent $\nu_{l o c}=1$ for the Anderson localization on the Bethe lattice [69 71], where the Hilbert space also grows exponentially with the distance.

Since this value $\nu=1$ does not satisfy the usual Harris [67] or Chayes-Chayes-Fisher-Spencer [68] inequality

$$
\begin{equation*}
\nu \geq \frac{2}{d} \tag{94}
\end{equation*}
$$

that has been rediscussed recently for the specific case of the MBL transition [36], it is important to understand why. The Harris inequality of Eq. 94 is based on the fact that in a volume $L^{d}$ of a disordered sample, there are of order $L^{d}$ random variables defining the disorder, so that there will be fluctuations of order $L^{\frac{d}{2}}$ as a consequence of the Central Limit Theorem. In the present model for instance, the system of $L$ spins contains $L$ random fields $h_{i}$ (Eq. 13). However, the MBL transition is very different from other types of phase transitions, because it is a transition concerning an individual arbitrary eigenstate in the middle of the spectrum, in our case the state $\mid 0>$ of Eq. 22. Then the full enumeration of possible resonances involve the other $\left(2^{L}-1\right)$ states of energies

$$
\begin{equation*}
E_{S_{1}, . ., S_{L}}^{(0)}=-\sum_{i=1}^{L} h_{i} S_{i} \tag{95}
\end{equation*}
$$

So in some sense, the eigenstate $\mid 0>$ does not see only the $L$ random variables $h_{i}$, but it sees effectively the $\left(2^{L}-1\right)$ other random energies that are build from the $L$ variables $h_{i}$ by the various choices of the spin values $S_{i}$, as can be seen also in the exponentially small level spacing of Eq. 18

As a consequence, independently of the strong disorder limit considered in the present paper, we believe that the correlation length exponent $\nu$ has no reason to satisfy the inequality of Eq. 94 so that there is no theoretical inconsistency in the numerical works where the correlation length exponent $\nu$ is found to violate the Harris inequality [37, 38].

## V. MULTIFRACTAL STATISTICS OF THE ENTANGLEMENT SPECTRUM

As in Anderson transitions where the critical point is characterized by multifractal eigenfunctions [46], one expects that the MBL transition is related to some multifractal properties [45, 74-79]. Besides the entanglement entropy described above, it is thus interesting to characterize the statistics of the whole entanglement spectrum of Eq. 38 , both in the Many-Body-Localized phase $b>1$ and at the critical point $b_{c}=1$.

## A. Multifractal exponents $\tau(q)$

In the region $0<q<1 / 2$, the disorder-averaged value of the Renyi entropy of Eq. 73 can be evaluated by generalizing the previous calculations of Section IV B to obtain

$$
\begin{align*}
\overline{S_{q}} & =\frac{\Gamma\left(\frac{1}{2}-q\right)}{(1-q) 4^{q} \sqrt{\pi}} \sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \Omega_{i_{1}, . ., i_{k_{A}}}^{2 q}+o\left(|J|^{2 q}\right) \\
& \propto \sum_{n=0}^{L_{A}-1} \sum_{m=0}^{L_{B}-1} \Delta^{2 q}(1+n+m) \frac{2^{n+m}}{\sqrt{n+m}} \\
& \propto \sum_{n=0}^{L_{A}-1} \sum_{m=0}^{L_{B}-1}\left[\frac{v}{(n+m)^{a}} 2^{-b(n+m)}\right]^{2 q} \frac{2^{n+m}}{\sqrt{n+m}} \\
& \propto v^{2 q} \sum_{n=0}^{L_{A}-1} \sum_{m=0}^{L_{B}-1} \frac{2^{(n+m)(1-2 q b)}}{(n+m)^{\left(\frac{1}{2}+2 q a\right)}} \tag{96}
\end{align*}
$$

For the bipartite case $L_{A}=L_{B}=L$, this yields the following power-laws in terms of the size $\mathcal{M}=2^{2 L}$ of the Hilbert space

$$
\begin{equation*}
\overline{S_{q}} \propto v^{2 q} \mathcal{M}^{-\tau_{b}(q)} \tag{97}
\end{equation*}
$$

with the multifractal exponents

$$
\begin{align*}
& \tau_{b}(q)=2 q b-1 \quad \text { for } \quad q \leq \frac{1}{2 b} \\
& \tau_{b}(q)=0 \quad \text { for } \quad \frac{1}{2 b} \leq q \leq \frac{1}{2} \tag{98}
\end{align*}
$$

In the region $q>\frac{1}{2}$, Eq. 74 yields

$$
\begin{equation*}
\tau_{b}(q)=0 \quad \text { for } \quad q \geq \frac{1}{2} \tag{99}
\end{equation*}
$$

in continuity with the result of Eq. 98 ,

## B. Multifractal spectrum $f^{c r i t i}(\alpha)$ at the critical point $b_{c}=1$

At the critical point $b_{c}=1$, the multifractal exponents of Eqs 98 and 99

$$
\begin{array}{llrl}
\tau^{c r i t i}(q) & =2 q-1 & \text { for } \quad q \leq \frac{1}{2} \\
\tau^{c r i t i}(q) & =0 & \text { for } \quad \frac{1}{2} \leq q \tag{100}
\end{array}
$$

correspond to the Legendre transform of the multifractal spectrum

$$
\begin{equation*}
f^{c r i t i}(\alpha)=\frac{\alpha}{2} \quad \text { for } \quad 0 \leq \alpha \leq 2 \tag{101}
\end{equation*}
$$

that is well-known as the 'strong-multifractality' spectrum in the context of Anderson Localization Transition in the limit of infinite dimension $d \rightarrow+\infty$ [46] or in the long-ranged power-law Anderson model in one-dimension of Eq. 9 [47 61].

More recently, the strong multifractality spectrum of Eq. 101 has been found to describe the statistics of matrix elements of local operators and the statistics of hybridization Ratios in nearest-neighbors Many-Body-Localized models at criticality [45]. Our conclusion is thus that the present toy model is not so far from more realistic short-ranged models, since it is able to reproduce the same critical multifractal spectrum.

## C. Multifractal spectrum in the Many-Body-Localized Phase $b>b_{c}=1$

In the many-Body-Localized phase $b>b_{c}=1$, the multifractal exponents of Eqs 98 and 99

$$
\begin{array}{lll}
\tau_{b}^{l o c}(q) & =2 q b-1 & \\
\text { for } & q \leq \frac{1}{2 b}  \tag{102}\\
\tau_{b}^{l o c}(q)=0 & \text { for } & \frac{1}{2 b} \leq q
\end{array}
$$

is the Legendre transform of the multifractal spectrum

$$
\begin{equation*}
f_{b}^{l o c}(\alpha)=\frac{\alpha}{2 b} \quad \text { for } \quad 0 \leq \alpha \leq 2 b \tag{103}
\end{equation*}
$$

that is a very simple deformation of the critical result of Eq. 101. The fact that multifractality occurs also in the Many-Body-Localized Phase is in agreement with the analysis of matrix elements of local operators and the statistics of hybridization Ratios in nearest-neighbors Many-Body-Localized models [45].

## VI. CONCLUSIONS

In this paper, in analogy with the strong disorder limit of the Anderson transition for a single particle (recalled in II A), we have considered the strong disorder limit of the MBL transition, defined as the limit where the level statistics at the MBL critical point is close to the Poisson statistics of the MBL phase. For a quantum one-dimensional toy model, we have analyzed the statistical properties of the rare extensive resonances that are needed to destabilize the Many-Body Localized phase. At criticality, we have found that the entanglement entropy can grow with an exponent $0<\alpha=3 / 2-a<1$ anywhere between the area law $\alpha=0$ and the volume law $\alpha=1$, as a function of the power-law exponent $a$ of the couplings (Eq. 79), while the entanglement spectrum follows the strong multifractality statistics of Eq. 101, well-known as the 'strong-multifractality' spectrum in the context of Anderson Localization Transition [46], and found recently for nearest-neighbor MBL models at criticality [45].

For an initial short-ranged model, we thus expect that the important extended rare resonances are described by some effective renormalized couplings $J_{i_{1}, \ldots, i_{k}}$ and it would be interesting to compute their properties in terms of the initial parameters, either via the forward approximation [32, 39, 43, 72, 73] or via some RG procedure.

The main difference between the present approach and the existing RG on resonances [34] is as follows :
(i) here, in the strong disorder limit, we have considered the resonances concerning a single eigenstate in the middle of the spectrum.
(ii) on the contrary, Ref. [34] is based on the notion of resonant blocks, so that when two blocks are declared to be resonant, all eigenstates of the two blocks are strongly mixed and exhibit level repulsion. So we feel that this assumption should be valid in the opposite weak-disorder limit of the transition, defined as the limit where the level statistics at the critical point is close to the Wigner-Dyson statistics of the delocalized phase.

So we believe that (i) and (ii) are actually the two extreme theories of a continuous family of MBL critical points, where the level statistics interpolates between the Poisson statistics and the Wigner-Dyson statistics.

Finally, in the MBL phase near criticality, we have obtained the simple value $\nu=1$ for the correlation length exponent, that simply reflects the crossing of the exponential decay of the level spacing and of the exponential decay of the couplings. More generally, and independently of the strong disorder limit, we have explained why the correlation length exponent $\nu$ of the MBL transition has no reason to satisfy the usual Harris inequality $\nu \geq 2 / d$, so that there is actually no theoretical inconsistency in the numerical works where the correlation length exponent $\nu$ is found to violate the Harris inequality [37, 38].

## Acknowledgments

It is a pleasure to thank Romain Vasseur for an illuminating discussion on various aspects of the MBL transition, and in particular for his explanations concerning the RG procedure on resonances [34].

## Appendix A: Off-diagonal elements of the reduced density matrix

In this Appendix, we explain why the off-diagonal elements of Eq. 35 of the reduced density matrix of Eq. 34 can be neglected at first order in the couplings.

Let us consider the probability distribution $P_{i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, \ldots, i_{k_{A}^{\prime}}^{\prime}}\left(R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)\right)$ of the variable $R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)$ of Eq. 35 when the indices $\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)$ are all distincts (when some indices are the same, the generalization is straighforward). The Fourier transform at lowest order in the couplings reads

$$
\begin{aligned}
& e^{\overline{e^{i t R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)}}}=\underline{\int_{-\infty}^{+\infty} d R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right) P_{i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, ., i_{k_{A}^{\prime}}^{\prime}}\left(R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)\right) e^{i t R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)}} \\
& =e^{i t \sum_{k_{B}=1}^{L_{B}} \sum_{L_{A}+1 \leq j_{1}<j_{2} . .<j_{k_{B}} \leq L_{A}+L_{B}}\left(\frac{J_{i_{1}, \ldots, i_{k_{A}}, j_{1}, \ldots j_{k_{B}}}}{\left.2 \sum_{q=1}^{k_{A} h_{i_{q}}+2 \sum_{p=1}^{k_{B} h_{j_{p}}}}\right)\left(\frac{J_{i_{1}^{\prime}, \ldots, i_{k_{A}}^{\prime}, j_{1}, \ldots j_{k_{B}}}^{k_{1}}}{2 \sum_{q=1}^{k_{A}^{\prime} h_{i_{q}^{\prime}}+2 \sum_{p}^{k_{B}} h_{1} h_{j_{p}}}}\right)}\right.}
\end{aligned}
$$

The variables

$$
\begin{align*}
& E_{A}=2 \sum_{q=1}^{k_{A}} h_{i_{q}} \\
& E_{A}^{\prime}=2 \sum_{q=1}^{k_{A}^{\prime}} h_{i_{q}^{\prime}} \\
& E_{B}=2 \sum_{p=1}^{k_{B}} h_{j_{p}} \tag{A2}
\end{align*}
$$

are three independent Gaussian variables (Eq. 13) of zero mean and of variances $4 W^{2} k_{A}, 4 W^{2} k_{A}^{\prime}$ and $4 W^{2} k_{B}$ respectively. As a consequence, the probability distribution $Q(z)$ of the variable $z=\left(E_{A}+E_{B}\right)\left(E_{A}^{\prime}+E_{B}\right)$ involved in Eq. A1 reads using the change of variables $y=E_{A}+E_{B}$ and $y^{\prime}=E_{A}+E_{B}$

$$
\begin{align*}
& Q(z)=\int_{-\infty}^{+\infty} \frac{d E_{B} e^{-\frac{E_{B}^{2}}{8 k_{B} W^{2}}}}{2 \sqrt{2 k_{B} \pi W^{2}}} \int_{-\infty}^{+\infty} \frac{d E_{A} e^{-\frac{E_{A}^{2}}{8 k_{A} W^{2}}}}{2 \sqrt{2 k_{A} \pi W^{2}}} \int_{-\infty}^{+\infty} \frac{d E_{A}^{\prime} e^{-\frac{\left(E_{A}^{\prime}\right)^{2}}{8 k_{A}^{\prime} W^{2}}}}{2 \sqrt{2 k_{A}^{\prime} \pi W^{2}}} \delta\left(z-\left(E_{A}+E_{B}\right)\left(E_{A}^{\prime}+E_{B}\right)\right) \\
& =\int_{-\infty}^{+\infty} \frac{d E_{B} e^{-\frac{E_{B}^{2}}{8 k_{B} W^{2}}}}{2 \sqrt{2 k_{B} \pi W^{2}}} \int_{-\infty}^{+\infty} \frac{d y e^{-\frac{\left(y-E_{B}\right)^{2}}{8 k_{A} W^{2}}}}{2 \sqrt{2 k_{A} \pi W^{2}}} \int_{-\infty}^{+\infty} \frac{d y^{\prime} e^{-\frac{\left(y^{\prime}-E_{B}\right)^{2}}{8 k_{A}^{\prime} W^{2}}}}{2 \sqrt{2 k_{A}^{\prime} \pi W^{2}}} \delta\left(z-y y^{\prime}\right) \\
& \left.=\int_{-\infty}^{+\infty} \frac{d y e^{-\frac{y^{2}}{8 k_{A} W^{2}}}}{\sqrt{8 k_{A} \pi W^{2}}} \int_{-\infty}^{+\infty} \frac{d y^{\prime} e^{-\frac{\left(y^{\prime}\right)^{2}}{8 k_{A}^{\prime} W^{2}}}}{\sqrt{8 k_{A}^{\prime} \pi W^{2}}} \frac{\delta\left(y^{\prime}-\frac{z}{y}\right)}{|y|} \int_{-\infty}^{+\infty} \frac{d E_{B}}{\sqrt{8 k_{B} \pi W^{2}}} e^{-\frac{E_{B}^{2}}{8 W^{2}}\left(\frac{1}{k_{B}}+\frac{1}{k_{A}}+\frac{1}{k_{A}^{\prime}}\right.}\right) e^{\frac{\left(y+y^{\prime}\right) E_{B}}{4 k_{A} W^{2}}} \\
& =e^{\frac{1 W^{2}\left(k_{A}+k_{A}^{\prime}+\frac{k_{A} k_{A}^{\prime}}{k_{B}}\right.}{4 \pi}} \frac{1}{4 \pi W^{2} \sqrt{k_{A} k_{A}^{\prime}+k_{B}\left(k_{A}+k_{A}^{\prime}\right)}} K_{0}\left(|z| \frac{\sqrt{\left(k_{B}+k_{A}\right)\left(k_{B}+k_{A}^{\prime}\right)}}{4 W^{2}\left(k_{A}+k_{A}^{\prime}+\frac{k_{A} k_{A}^{\prime}}{k_{B}}\right)}\right) \tag{A3}
\end{align*}
$$

in terms of the Bessel function $K_{0}(u)$ displaying the logarithmic singularity near the origin

$$
\begin{equation*}
K_{0}(u)=-\ln u+\left(\ln 2-\gamma_{E u l e r}\right)+O\left(u^{2}\right) \tag{A4}
\end{equation*}
$$

and an exponential decay for large $u$.

As a consequence, the average of the inverse of the variable $z$ remains finite

$$
\begin{align*}
\int_{-\infty}^{+\infty} \frac{d z}{z} Q(z) & =\int_{0}^{+\infty} \frac{d z}{z}[Q(z)-Q(-z)] \\
& =\int_{0}^{+\infty} \frac{d z}{z} \frac{\sinh \left(\frac{z}{4 W^{2}\left(k_{A}+k_{A}^{\prime}+\frac{k_{A} k_{A}^{\prime}}{k_{B}}\right)}\right)}{2 \pi W^{2} \sqrt{k_{A} k_{A}^{\prime}+k_{B}\left(k_{A}+k_{A}^{\prime}\right)}} K_{0}\left(|z| \frac{\sqrt{\left(k_{B}+k_{A}\right)\left(k_{B}+k_{A}^{\prime}\right)}}{4 W^{2}\left(k_{A}+k_{A}^{\prime}+\frac{k_{A} k_{A}^{\prime}}{k_{B}}\right)}\right) \\
& =\frac{1}{4 W^{2} \sqrt{k_{A} k_{A}^{\prime}+k_{B}\left(k_{A}+k_{A}^{\prime}\right)}} \arcsin \left(\frac{1}{\left(k_{B}+k_{A}\right)\left(k_{B}+k_{A}^{\prime}\right)}\right) \tag{A5}
\end{align*}
$$

So Eq. A1 yields that the variable $R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)$ has the finite average value

$$
\begin{align*}
\overline{R\left(i_{1}, . ., i_{k_{A}} ; i_{1}^{\prime}, . ., i_{k_{A}^{\prime}}^{\prime}\right)} & =\sum_{k_{B}=1 L_{A}+1 \leq j_{1}<.<j_{k_{B}} \leq L_{A}+L_{B}}^{L_{B}} \frac{J_{i_{1}, . ., i_{k_{A}}, j_{1}, \ldots j_{k_{B}}} J_{i_{1}^{\prime}, ., i_{k_{A}}^{\prime}, j_{1}, \ldots j_{k_{B}}}}{4 W^{2} \sqrt{k_{A} k_{A}^{\prime}+k_{B}\left(k_{A}+k_{A}^{\prime}\right)}} \arcsin \left(\frac{1}{\left(k_{B}+k_{A}\right)\left(k_{B}+k_{A}^{\prime}\right)}\right) \\
& +o\left(J^{2}\right) \tag{A6}
\end{align*}
$$

which is of second order $O\left(J^{2}\right)$ with respect to the couplings. As a consequence, at the first order in the couplings considered in the main text, the off-diagonal elements are negligible.

## Appendix B: Disorder-averaged values $\overline{Y_{q}}$

Using the identity

$$
\begin{equation*}
\frac{1}{a^{q}}=\frac{1}{\Gamma(q)} \int_{0}^{+\infty} d t t^{q-1} e^{-a t} \tag{B1}
\end{equation*}
$$

the disorder-averaged value of Eq. 39 can be decomposed into the two contributions

$$
\begin{align*}
\overline{Y_{q}} & =\frac{\overline{1+\Sigma_{q}}}{\left(1+\Sigma_{1}\right)^{q}}=\left.\overline{Y_{q}}\right|_{\text {first }}+\left.\overline{Y_{q}}\right|_{\text {second }} \\
\left.\overline{Y_{q}}\right|_{\text {first }} & =\frac{1}{\left(1+\Sigma_{1}\right)^{q}}=\frac{1}{\Gamma(q)} \int_{0}^{+\infty} d t t^{q-1} e^{-t} \overline{e^{-t \Sigma_{1}}} \\
\left.\overline{Y_{q}}\right|_{\text {second }} & =\frac{\Sigma_{q}}{\left(1+\Sigma_{1}\right)^{q}}=\frac{1}{\Gamma(q)} \int_{0}^{+\infty} d t t^{q-1} e^{-t} \overline{\Sigma_{q} e^{-t \Sigma_{1}}} \tag{B2}
\end{align*}
$$

Eq. 54 yields that the first contribution reads

$$
\begin{align*}
\left.\overline{Y_{q}}\right|_{\text {first }} & =\frac{1}{\Gamma(q)} \int_{0}^{+\infty} d t t^{q-1} e^{-t}\left(1-\Omega_{1} t^{\frac{1}{2}}+o(J)\right) \\
& =1-\Omega_{1} \frac{\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(q)}+o(J) \tag{B3}
\end{align*}
$$

To evaluate the second contribution, one needs to consider separately the two cases $0<q<\frac{1}{2}$ and $q>\frac{1}{2}$.

1. Region $0<q<\frac{1}{2}$

For $0<q<\frac{1}{2}$, the average of $\Sigma_{q}$ is finite (Eq. 64) so that the second contribution at leading order

$$
\begin{equation*}
\left.\overline{Y_{q<1 / 2}}\right|_{\text {second }}=\overline{\Sigma_{q}}=\frac{\Gamma\left(\frac{1}{2}-q\right)}{4^{q} \sqrt{\pi}} \sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \Omega_{i_{1}, . ., i_{k_{A}}}^{2 q}+o\left(|J|^{2 q}\right) \tag{B4}
\end{equation*}
$$

is bigger than the correction of order $\Omega_{1}=O(J)$ of the first contribution of Eq. B3, so that the sum of the two contributions reads

$$
\begin{equation*}
\overline{Y_{q<1 / 2}}=1+\frac{\Gamma\left(\frac{1}{2}-q\right)}{4^{q} \sqrt{\pi}} \sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \Omega_{i_{1}, . ., i_{k_{A}}}^{2 q}+o\left(|J|^{2 q}\right) \tag{B5}
\end{equation*}
$$

This disorder-averaged value thus coincides at leading order with the typical value computed in Eq. 73

$$
\begin{equation*}
Y_{q<1 / 2}^{t y p} \equiv e^{\overline{\ln Y_{q<1 / 2}}}=e^{(1-q) \overline{S_{q}}}=1+\frac{\Gamma\left(\frac{1}{2}-q\right)}{4^{q} \sqrt{\pi}} \sum_{k_{A}=1}^{L} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L} \Omega_{i_{1}, . ., i_{k_{A}}}^{2 q}+o\left(|J|^{2 q}\right) \tag{B6}
\end{equation*}
$$

## 2. Region $q>\frac{1}{2}$

For $q>\frac{1}{2}$, the average of $\Sigma_{q}$ is infinite, so one needs to evaluate the divergence for small $t$ of

$$
\begin{align*}
\overline{\Sigma_{q} e^{-t \Sigma_{1}}} & =\overline{\sum_{k_{A}=1}^{L_{A}} \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} D_{i_{1}, . ., i_{k_{A}}}^{q} e^{-t \sum_{k_{A}^{\prime}=1}^{L_{A}} \sum_{1 \leq i_{1}^{\prime}<i_{2}^{\prime} . .<i_{k_{A}}^{\prime} \leq L_{A} D_{i_{1}^{\prime}, \ldots, i_{k_{A}}^{\prime}}}}} \begin{aligned}
\sum_{k_{A}=1}^{L_{A}} & \sum_{1 \leq i_{1}<i_{2} . .<i_{k_{A}} \leq L_{A}} \\
D_{i_{1}, . . i_{k_{A}}}^{q} e^{-t D_{i_{1}, . ., i_{k_{A}}}} & {\left[1+O\left(t^{\frac{1}{2}}\right)\right] }
\end{aligned}
\end{align*}
$$

The Lévy distribution of Eqs 50, 51 for the variable $D_{i_{1}, . ., i_{k_{A}}}$ yields the singularity

$$
\begin{align*}
\overline{D_{i_{1}, ., i_{k_{A}}}^{q} e^{-t D_{i_{1}, \ldots, i_{k_{A}}}}} & =\frac{\Omega_{i_{1}, . ., i_{k_{A}}}}{2 \sqrt{\pi}} \int_{0}^{+\infty} d D D^{q-\frac{3}{2}} e^{-\frac{\Omega_{i_{1}, \ldots, i_{k_{A}}}^{2}}{4 D}} e^{-t D} \\
& =\frac{\Omega_{i_{1}, . ., i_{k_{A}}}}{2 \sqrt{\pi}} t^{\frac{1}{2}-q} \int_{0}^{+\infty} d x x^{q-\frac{3}{2}} e^{-\frac{\Omega_{i_{1}, \ldots, i_{k_{A}}}^{4 x}}{4 x}} e^{-x} \\
& =\frac{\Omega_{i_{1}, ., i_{k_{A}}}}{2 \sqrt{\pi}} t^{\frac{1}{2}-q} \Gamma\left(q-\frac{1}{2}\right)+o(|J|) \tag{B8}
\end{align*}
$$

So Eq. B7 displays the same singularity in terms of the parameter of Eq. 55

$$
\begin{equation*}
\overline{\Sigma_{q} e^{-t \Sigma_{1}}}=\frac{\Omega_{1}}{2 \sqrt{\pi}} t^{\frac{1}{2}-q} \Gamma\left(q-\frac{1}{2}\right)+o(|J|) \tag{B9}
\end{equation*}
$$

and the second contribution of Eq. $\overline{\mathrm{B} 2}$ reads at leading order

$$
\begin{align*}
\left.\overline{Y_{q>\frac{1}{2}}}\right|_{\text {second }} & =\frac{1}{\Gamma(q)} \int_{0}^{+\infty} d t t^{q-1} e^{-t} \overline{\Sigma_{q} e^{-t \Sigma_{1}}} \\
& =\frac{\Omega_{1}}{2} \frac{\Gamma\left(q-\frac{1}{2}\right)}{\Gamma(q)}+o(|J|) \tag{B10}
\end{align*}
$$

The sum with the first contribution of Eq. B3 yields the disorder-average of $Y_{q}$ in the region $q>1 / 2$

$$
\begin{align*}
\overline{Y_{q>\frac{1}{2}}} & =1+\frac{\Omega_{1}}{2 \Gamma(q)}\left[\Gamma\left(q-\frac{1}{2}\right)-2 \Gamma\left(q+\frac{1}{2}\right)\right]+o(J) \\
& =1-\Omega_{1} \frac{\Gamma\left(q-\frac{1}{2}\right)}{\Gamma(q-1)}+o(J) \tag{B11}
\end{align*}
$$

This disorder-averaged value is thus different from the typical value computed in Eq. 74

$$
\begin{equation*}
Y_{q>1 / 2}^{t y p} \equiv e^{\overline{\ln Y_{q>1 / 2}}}=e^{(1-q) \overline{S_{q}}}=1+\Omega_{1} \sqrt{\pi}\left[\frac{1}{\sin \left(\frac{\pi}{2 q}\right)}-q\right]+o(J) \tag{B12}
\end{equation*}
$$

## 3. Consequence for the variance of the entanglement entropy

In the vicinity of $q=1$, the difference found above between the averaged value of Eq. B11

$$
\begin{equation*}
\overline{Y_{q>\frac{1}{2}}}=\overline{e^{(q-1) S_{q}}}=1+(q-1) \overline{S_{q}}+\frac{(q-1)^{2}}{2} \overline{S_{q}^{2}}+O(q-1)^{3} \tag{B13}
\end{equation*}
$$

and the typical value of Eq. B12

$$
\begin{equation*}
Y_{q>1 / 2}^{t y p}=e^{(1-q) \overline{S_{q}}}=1+(q-1) \overline{S_{q}}+\frac{(q-1)^{2}}{2}\left(\overline{S_{q}}\right)^{2}+O(q-1)^{3} \tag{B14}
\end{equation*}
$$

yields the variance of the entanglement entropy $S_{1}$

$$
\begin{align*}
\overline{S_{1}^{2}}-\left(\overline{S_{1}}\right)^{2} & =2 \lim _{q \rightarrow 1}\left[\frac{\overline{Y_{q>\frac{1}{2}}}-Y_{q>1 / 2}^{t y p}}{(q-1)^{2}}\right] \\
& =2 \Omega_{1} \lim _{q \rightarrow 1}\left[\frac{\sqrt{\pi}\left[q-\frac{1}{\sin \left(\frac{\pi}{2 q}\right)}\right]-\frac{\Gamma\left(q-\frac{1}{2}\right)}{\Gamma(q-1)}}{(q-1)^{2}}\right]+o(J) \\
& =\Omega_{1} \sqrt{\pi}\left(4 \ln 2-\frac{\pi^{2}}{4}\right)+o(J) \tag{B15}
\end{align*}
$$

So the scale $\Omega_{1}$ that governs the scale of the disorder-averaged entanglement entropy (Eq. 75) also determines its variance (Eq. B15).
[1] J.M. Deutsch, Phys. Rev. A 43, 2046 (1991).
[2] M. Srednicki, Phys. Rev. E 50, 888 (1994).
[3] M. Rigol, V. Dunjko and M. Olshanii, Nature 452, 854 (2008)
[4] S. Goldstein, D.A. Huse, J.L. Lebowitz and R. Tumulka, Phys. Rev. Lett. 115, 100402 (2015).
[5] L. D'Alessio, Y. Kafri, A. Polkovnikov and M. Rigol, arXiv:1509.06411.
[6] R. Nandkishore and D. A. Huse, Ann. Review of Cond. Mat. Phys. 6, 15 (2015).
[7] E. Altman and R. Vosk, Ann. Review of Cond. Mat. Phys. 6, 383 (2015).
[8] B. Bauer and C. Nayak, J. Stat. Mech. P09005 (2013).
[9] D. Pekker and B.K. Clark, arXiv:1410.2224.
[10] X. Yu, D. Pekker and B.K. Clark, arXiv:1509.01244.
[11] M. Friesdorf, A.H. Werner, W. Brown, V. B. Scholz and J. Eisert, Phys. Rev. Lett. 114, 170505 (2015).
[12] V. Khemani, F. Pollmann and S. L. Sondhi, arXiv:1509.00483.
[13] A. Chandran, J. Carrasquilla, I.H. Kim, D.A. Abanin and G. Vidal, Phys. Rev. B 92, 024201 (2015).
[14] D. S. Fisher, Phys. Rev. B 50, 3799 (1994).
[15] D. S. Fisher, Phys. Rev. Lett. 69, 534 (1992);
D. S. Fisher, Phys. Rev. B 51, 6411 (1995).
[16] D. S. Fisher, Physica A 263, 222 (1999).
[17] R. Vosk and E. Altman, Phys. Rev. Lett. 110, 067204 (2013).
[18] R. Vosk and E. Altman, Phys. Rev. Lett. 112, 217204 (2014).
[19] D. Pekker, G. Refael, E. Altman, E. Demler and V. Oganesyan, Phys. Rev. X 4, 011052 (2014).
[20] Y. Huang and J.E. Moore, Phys. Rev. B 90, 220202(R) (2014).
[21] R. Vasseur, A. C. Potter and S.A. Parameswaran, Phys. Rev. Lett. 114, 217201 (2015).
[22] M. Pouranvari and K. Yang, Phys. Rev. B 92, 245134 (2015).
[23] Y.Z. You, X.L. Qi and C. Xu, arXiv:1508.036035
[24] C. Monthus, J. Stat. Mech. 033101 (2016).
[25] B. Swingle, arXiv:1307.0507.
[26] M. Serbyn, Z. Papic and D.A. Abanin, Phys. Rev. Lett. 111, 127201 (2013).
[27] D.A. Huse, R. Nandkishore and V. Oganesyan, Phys. Rev. B 90, 174202 (2014).
[28] A. Nanduri, H. Kim and D.A. Huse, Phys. Rev. B 90, 064201 (2014).
[29] J. Z. Imbrie, arXiv:1403.7837
[30] M. Serbyn, Z. Papic and D.A. Abanin, Phys. Rev. B 90, 174302 (2014).
[31] A. Chandran, I.H. Kim, G. Vidal and D.A. Abanin, Phys. Rev. B 91, 085425 (2015).
[32] V. Ros, M. Müller and A. Scardicchio, Nucl. Phys. B 891, 420 (2015).
[33] R. Vosk, D.A. Huse, and E. Altman, Phys. Rev. X 5, 031032 (2015).
[34] A. C. Potter, R. Vasseur and S.A. Parameswaran, Phys. Rev. X 5, 031033 (2015).
[35] T. Grover, arXiv:1405.1471.
[36] A. Chandran, C.R. laumann and V. Oganesyan, arXiv:1509.04285.
[37] J. A. Kjäll, J. H. Bardarson and F. Pollmann, Phys. Rev. Lett. 113, 107204 (2014).
[38] D. J. Luitz, N. Laflorencie and F. Alet, Phys. Rev. B 91, 081103 (2015).
[39] B.L. Altshuler, Y. Gefen, A. Kamenev and L.S. Levitov, Phys. Rev. Lett. 78, 2803 (1997).
[40] I.V. Gornyi, A.D. Mirlin and D.G. Polyakov, Phys. Rev. Lett. 95, 206603 (2005).
[41] V. Oganesyan and D.A. Huse, Phys. Rev. B 75, 155111 (2007).
[42] C. Monthus and T. Garel, Phys. Rev. B 81, 134202 (2010).
[43] A. De Luca, B.L. Altshuler, V.E. Kravtsov and A. Scardicchio, Phys. Rev. Lett. 113, 046806 (2014).
[44] I.V. Gornyi, A.D. Mirlin, and D.G. Polyakov, Phys. Rev. B 93, 125419 (2016).
[45] C. Monthus, arXiv:1603.04701
[46] F. Evers and A.D. Mirlin, Rev. Mod. Phys. 80, 1355 (2008).
[47] L.S. Levitov, Europhys. Lett. 9, 83 (1989).
[48] L.S. Levitov, Phys. Rev. Lett. 64, 547 (1990).
[49] B.L. Altshuler and L.S. Levitov, Phys. Rep. 288, 487 (1997).
[50] L.S. Levitov, Ann. Phys. (Leipzig) 8, 5, 507 (1999).
[51] F. Evers and A. D. Mirlin Phys. Rev. Lett. 84, 3690 (2000); A.D. Mirlin and F. Evers, Phys. Rev. B 62, 7920 (2000).
[52] Y.V. Fyodorov, A. Ossipov and A. Rodriguez, J. Stat. Mech. L12001 (2009).
[53] Y.V. Fyodorov, A. Kupiainen and C. Webb. arXiv:1509.01366
[54] O. Yevtushenko and V. E. Kratsov, J. Phys. A 36, 8265 (2003).
[55] O. Yevtushenko and A. Ossipov, J. Phys. A 40, 4691 (2007).
[56] S. Kronmüller, O. M. Yevtushenko and E. Cuevas, J. Phys. A 43, 075001 (2010).
[57] V. E. Kratsov, A. Ossipov, O. M. Yevtushenko and E. Cuevas, Phys. Rev. B 82, 161102(R) (2010).
[58] E. Bogomolny and O. Giraud, Phys. Rev. E 84, 036212 (2012).
[59] E. Bogomolny and O. Giraud, Phys. Rev. E 84, 046208 (2012).
[60] E. Bogomolny and O. Giraud, Phys. Rev. Lett. 106, 044101 (2011).
[61] C. Monthus and T. Garel, J. Stat. Mech. (2010) P09015.
[62] P. Lévy, "Théorie de l'addition des variables aléatoires", Gauthier-Villars, Paris (1937).
[63] J.P. Bouchaud and A. Georges, Phys. Rep. 195 , 127 (1990).
[64] B. Derrida, Physica D 107, 186 (1997).
[65] B. Derrida and H. Flyvbjerg, J. Phys. A Math. Gen. 20, 5273 (1987).
[66] C. Monthus and T. Garel, Phys. Rev. E 75, 051119 (2007).
[67] A.B. Harris, J. Phys. C 7, 1671 (1974).
[68] J.T. Chayes, L. Chayes, D.S. Fisher and T. Spencer, Phys. Rev. Lett. 57, 2999 (1986).
[69] H. Kunz and B. Souillard, J. Phys. Lett. 44, L411 (1983).
[70] A.D. Mirlin and Y.V. Fyodorov, Nucl. Phys. B 366, 507 (1991).
[71] C. Monthus and T. Garel, J. Phys. A Math. Theor. 42, 075002 (2009).
[72] F. Pietracaprina, V. Ros, A. Scardicchio, Phys. Rev. B 93, 054201 (2016).
[73] C.R. Laumann, A. Pal and A. Scardicchio, Phys. Rev. Lett. 113, 200405 (2014);
C.L. Baldwin, C.R. Laumann, A. Pal and A. Scardicchio, arXiv:1509.08926.
[74] A. de Luca and A. Scardicchio, EPL 101, 37003 (2013).
[75] E.J. Torres-Herrera and L.S. Santos, Phys. Rev. B 92, 014208 (2015).
[76] X. Chen et al., Phys. Rev. B 92, 214204 (2015).
[77] M. Serbyn, Z. Papic and D.A. Abanin, Phys. Rev. X 041047 (2015).
[78] M. Serbyn and J.E. Moore, Phys. Rev. B 93, 041424 (2016).
[79] C. Monthus, J. Stat. Mech. 033113 (2016).

