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Physics Letters B 758 (2016) 89–92

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

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ARTICLE INFO

Article history:
Received 20 April 2016
Accepted 29 April 2016
Available online 4 May 2016
Editor: M. Cvetič

ABSTRACT

We study the entanglement entropy between the two outgoing particles in an elastic scattering process. It is formulated within an S-matrix formalism using the partial wave expansion of two-body states, which plays a significant role in our computation. As a result, we obtain a novel formula that expresses the entanglement entropy in a high energy scattering by the use of physical observables, namely the elastic and total cross sections and a physical bound on the impact parameter range, related to the elastic differential cross-section.

1. Introduction

Entanglement is a significant concept which appears in various subjects of quantum physics. The quantum entanglement has been attracting much attention of theoretical physicists, since remarkable progress in the entanglement between the systems on two regions was made in quantum field theories [1] and holography [2], and the intriguing conjecture called ER = EPR [3] was suggested. In the context of the ER = EPR conjecture, the entanglements between two particles, which are, for example, a pair of accelerating quark and anti-quark [4] and a pair of scattering gluons [5], have been studied. Then it naturally induces the following primitive question: How does the entanglement entropy of a pair of particles change from an initial state to a final one in an elastic channel of scattering process? It is qualitatively expected that the elastic collision of two initial particles, *e.g.*, in a high energy collider, generates some amount of entanglement between the particles in the final state. We are interested in quantifying the entanglement entropy generated by collision.

By just neglecting inelastic channels in weak coupling perturbation [6], Ref. [7] analyzed such entanglement entropy in a field theory by the use of an S-matrix.[1] In this article we exploit the S-matrix formalism further in order for a non-perturbative understanding of the entanglement entropy in a scattering process with also an inelastic channel to be taken into account. This is especially required in the case of strong interaction scattering at high energy where inelastic multi-particle scattering contributes to a large part of the total cross-section, while elastic scattering is still important. The basic S-matrix formalism of strong interaction, as developed long time ago, *e.g.*, in Refs. [9,10], allows us to find an approach to scattering processes without referring explicitly to an underlying quantum field theory.

Following Refs. [9,10], we consider a scattering process of two incident particles, A and B, whose masses are m_A and m_B respectively, in $1+3$ dimensions. This process is divided [9] into the following two channels:

$$\text{“Elastic” channel:} \quad \mathrm{A}+\mathrm{B} \to \mathrm{A}+\mathrm{B}$$

$$\text{“Inelastic” channel:} \quad \mathrm{A}+\mathrm{B} \to \mathrm{X}$$

where X stands for any possible states except for the two-particle state, A + B. We postpone the study extended to a matrix including more varieties of two-particle channels [10] to a further publication.

The full Hilbert space of states is not usually factorized as $\mathcal{H}_{\rm full} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_X$ in an interacting system. However the Hilbert space of both the initial and final states is factorizable in the S-matrix formalism, because one considers only asymptotic initial and final states long before and after the interaction. We introduce the S-matrix, $\mathcal{S}$, for the overall set of initial and final states. Once we fix an initial state $|\text{ini}\rangle$, the final state $|\text{fin}\rangle$ is determined by the S-matrix. In this article we are interested in the entanglement between two outgoing particles, A + B, in a final

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[1] We quote for completion Ref. [8], where the entanglement entropy is discussed in a low energy decay process using different concept and method.

http://dx.doi.org/10.1016/j.physletb.2016.04.063

state of elastic scattering in the presence of a non-negligible fraction of open inelastic final states. Therefore we additionally introduce a projection operator Q onto the two-particle Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ from $\mathcal{H}_{\text{full}}$. Then the final elastic state, in other words, the state of two outgoing particles, is described as $|\text{fin}\rangle = Q\,\mathcal{S}|\text{ini}\rangle$.

We employ the two-particle Fock space $\{|\vec{p}\,\rangle_A\} \otimes \{|\vec{q}\,\rangle_B\}$ as the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The two-particle state which consists of particle A with momentum $\vec{p}$ and B with $\vec{q}$ is denoted by $|\vec{p},\vec{q}\,\rangle = |\vec{p}\,\rangle_A \otimes |\vec{q}\,\rangle_B$. We define an inner product of the two-particle states in a conventional manner by $\langle \vec{p},\vec{q}\,|\vec{k},\vec{l}\,\rangle = 2E_{A\vec{p}}\delta^{(3)}(\vec{p}-\vec{k})\,2E_{B\vec{q}}\delta^{(3)}(\vec{q}-\vec{l}\,)$, where $E_{I\vec{p}} = \sqrt{p^2+m_I^2}$ $(I=A,B)$ and $p=|\vec{p}|$.

We shall study the entanglement between the two outgoing particles, A and B. When the density matrix of the final state on $\mathcal{H}_A \otimes \mathcal{H}_B$ is denoted by ρ, we define a reduced density matrix as $\rho_A = \text{tr}_B\,\rho$. Then the entanglement entropy is given by $S_{\text{EE}} = -\,\text{tr}_A\,\rho_A \ln \rho_A$. The other way to calculate the entanglement entropy is to use the Rényi entropy, $S_{\text{RE}}(n) = (1-n)^{-1}\ln \text{tr}_A(\rho_A)^n$. It leads to the entanglement entropy described as $S_{\text{EE}} = \lim_{n\to 1} S_{\text{RE}}(n) = -\lim_{n\to 1}\frac{\partial}{\partial n}\text{tr}_A(\rho_A)^n$.

2. Partial wave expansion

The partial wave expansion is often useful to analyze a scattering process. Before starting to study the entanglement entropy, let us recall what Refs. [9,10] studied.

We adopt a center-of-mass frame. The state of the two particles, A + B, which have momenta $\vec{p}$ and $-\vec{p}$, is denoted by $|\vec{p}\,\rangle\rangle := |\vec{p},-\vec{p}\,\rangle$, while the many-particle state of X is denoted by $|X\rangle$. Since the complete set of states is given by the orthogonal basis, $\{|\vec{p}\,\rangle\rangle, |X\rangle\}$, one can describe the identity matrix as

$$\mathbf{1} = \int \frac{d^3\vec{p}}{2E_{A\vec{p}}2E_{B\vec{p}}\delta^{(3)}(0)}|\vec{p}\,\rangle\rangle\langle\langle\vec{p}| + \int dX\,|X\rangle\langle X|\,. \tag{2.1}$$

We notice that $\delta^{(3)}(0)$ comes from $\langle\langle\vec{k}|\vec{l}\,\rangle\rangle = 2E_{A\vec{k}}2E_{B\vec{k}}\delta^{(3)}(\vec{k}-\vec{l}\,)\delta^{(3)}(0)$, due to our definition of the inner product of states.

One can expand the S-matrix elements in term of partial waves. Let us consider the S-matrix and T-matrix defined by $\mathcal{S} = \mathbf{1} + 2i\mathcal{T}$. The unitarity condition is $\mathcal{S}^\dagger\mathcal{S} = \mathbf{1}$, which is equivalent to $i(\mathcal{T}^\dagger - \mathcal{T}) = 2\mathcal{T}^\dagger\mathcal{T}$. Extracting the factor of energy–momentum conservation, we describe the T-matrix elements as

$$\begin{aligned}&\langle\langle\vec{p}|\mathcal{T}|\vec{q}\,\rangle\rangle = \delta^{(4)}(P_{\vec{p}} - P_{\vec{q}})\langle\langle\vec{p}|\mathbf{t}|\vec{q}\,\rangle\rangle\,,\\ &\langle\langle\vec{p}|\mathcal{T}|X\rangle = \delta^{(4)}(P_{\vec{p}} - P_X)\langle\langle\vec{p}|\mathbf{t}|X\rangle\,.\end{aligned} \tag{2.2}$$

$P_{\vec{p}}$ and P_X are the total energy-momenta of $|\vec{p}\,\rangle\rangle$ and $|X\rangle$ respectively, which say $P_{\vec{p}} = (E_{A\vec{p}} + E_{B\vec{p}}, 0, 0, 0)$.

One introduces the overlap matrix $F_{\vec{p}\,\vec{k}}(k,\cos\theta)$,

$$F_{\vec{p}\,\vec{k}} = \frac{2\pi k}{E_{A\vec{k}} + E_{B\vec{k}}}\int dX\langle\langle\vec{p}|\mathbf{t}^\dagger|X\rangle\delta^{(4)}(P_X - P_{\vec{k}})\langle X|\mathbf{t}|\vec{k}\rangle\rangle\,, \tag{2.3}$$

where k and θ are defined by $\vec{p}\cdot\vec{k} = pk\cos\theta$ and $k=p$. This matrix implies the contribution of the inelastic channel at the middle of the scattering process. The T-matrix element in the elastic channel and the overlap matrix are decomposed in terms of partial waves,

$$\frac{\pi k}{E_{A\vec{k}} + E_{B\vec{k}}}\langle\langle\vec{p}|\mathbf{t}|\vec{k}\rangle\rangle = \sum_{\ell=0}^{\infty}(2\ell+1)\tau_\ell(k)P_\ell(\cos\theta)\,, \tag{2.4}$$

$$F_{\vec{p}\,\vec{k}}(k,\cos\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta)\,, \tag{2.5}$$

where $P_\ell(\cos\theta)$ are the Legendre polynomials. Then one can rewrite the unitarity condition as

$$\text{Im}\,\tau_\ell = |\tau_\ell|^2 + \frac{f_\ell}{2}\,. \tag{2.6}$$

Using $s_\ell := 1 + 2i\tau_\ell$, which comes from the partial wave expansion of the S-matrix element,

$$\frac{\pi k}{E_{A\vec{k}} + E_{B\vec{k}}}\langle\langle\vec{p}|\mathbf{s}|\vec{k}\rangle\rangle = \sum_{\ell=0}^{\infty}(2\ell+1)s_\ell P_\ell(\cos\theta)\,, \tag{2.7}$$

the unitarity condition is equivalent to $s_\ell^* s_\ell = 1 - 2f_\ell$. If there is not an inelastic channel, *i.e.* $f_\ell = 0$, then the unitarity condition is reduced to $s_\ell^* s_\ell = 1$. A comment in order [9,10] is that we can define a pseudo-unitary two-body S-matrix with partial wave components, $\omega_\ell^*\omega_\ell = 1$, by rescaling s_ℓ as $\omega_\ell := s_\ell/\sqrt{1-2f_\ell}$.

The partial wave expansion allows us to depict the integrated elastic cross section, the integrated inelastic cross section and the total cross section as

$$\begin{aligned}&\sigma_{\text{el}} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)|\tau_\ell|^2\,, \quad \sigma_{\text{inel}} = \frac{2\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)f_\ell\,,\\ &\sigma_{\text{tot}} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\,\text{Im}\,\tau_\ell\,.\end{aligned} \tag{2.8}$$

The differential elastic cross section is

$$\begin{aligned}\frac{d\sigma_{\text{el}}}{dt} &= \frac{\pi}{k^4}\sum_{\ell,\ell'}(2\ell+1)(2\ell'+1)\tau_\ell\tau_{\ell'}^* P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\\ &= \frac{|A|^2}{64\pi sk^2}\,,\end{aligned} \tag{2.9}$$

where $A(s,t)$ is the scattering amplitude, s and t are the Mandelstam variables, and the scattering angle $\cos\theta = 1 + t/(2k^2)$.

3. Entanglement entropy of two particles

We consider two unentangled particles, A and B, with momenta $\vec{k}$ and $\vec{l}$ as incident particles. That is to say, we choose a single state as an initial state;

$$|\text{ini}\rangle = |\vec{k},\vec{l}\,\rangle = |\vec{k}\rangle_A \otimes |\vec{l}\,\rangle_B\,. \tag{3.1}$$

Here we have not taken the center-of-mass frame yet. Of course the entanglement entropy of the initial state vanishes. In terms of the S-matrix, the final state of two particles, $|\text{fin}\rangle = Q\,\mathcal{S}|\text{ini}\rangle$, is described as

$$|\text{fin}\rangle = \left(\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{d^3\vec{q}}{2E_{B\vec{q}}}|\vec{p}\,,\vec{q}\,\rangle\langle\vec{p}\,,\vec{q}\,|\right)\mathcal{S}|\vec{k},\vec{l}\,\rangle\,. \tag{3.2}$$

Then we can define the total density matrix of the final state by $\rho := \mathcal{N}^{-1}|\text{fin}\rangle\langle\text{fin}|$. The normalization factor $\mathcal{N}$ will be determined later so that ρ satisfies $\text{tr}_A\,\text{tr}_B\,\rho = 1$. Tracing out ρ with respect to the Hilbert space of particle B, we obtain the reduced density matrix, $\rho_A := \text{tr}_B\,\rho$, namely,

$$\begin{aligned}\rho_A = \frac{1}{\mathcal{N}}\int &\frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{d^3\vec{q}}{2E_{B\vec{q}}}\frac{d^3\vec{p}\,'}{2E_{A\vec{p}'}}\\ &\times\left(\langle\vec{p}\,,\vec{q}\,|\mathcal{S}|\vec{k},\vec{l}\,\rangle\langle\vec{k},\vec{l}\,|\mathcal{S}^\dagger|\vec{p}\,',\vec{q}\,\rangle\right)|\vec{p}\,\rangle_{A\,A}\langle\vec{p}\,'|\,.\end{aligned} \tag{3.3}$$

Now let us adopt the center-of-mass frame, which leads to $\vec{k}+\vec{l} = 0$. Then the initial state is $|\text{ini}\rangle = |\vec{k}\rangle\rangle$, and the reduced density matrix becomes

$$\rho_A = \frac{1}{\mathcal{N}}\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{\delta(0)\delta(p-k)}{4k(E_{A\vec{k}} + E_{B\vec{k}})}\left|\langle\langle\vec{p}\,|\mathbf{s}|\vec{k}\rangle\rangle\right|^2|\vec{p}\,\rangle_{A\,A}\langle\vec{p}\,|\,, \tag{3.4}$$

where $\mathbf{s} = \mathbf{1} + 2i\mathbf{t}$, and $\delta(0)$ stems from the modulus equality of the initial and final particles' momenta. By substituting (2.7) into (3.4), the normalization condition, $\mathrm{tr}_A\,\rho_A = 1$, fixes $\mathcal{N}$ as $\mathcal{N} = \delta^{(4)}(0)\mathcal{N}'$ with

$$\mathcal{N}' = \frac{E_{A\vec{k}} + E_{B\vec{k}}}{\pi k}\sum_{\ell=0}^{\infty}(2\ell+1)|s_\ell|^2\,. \tag{3.5}$$

Since $\mathrm{tr}_A(\rho_A)^n$ straightforwardly provides us the Rényi and entanglement entropy, we calculate

$$\mathrm{tr}_A(\rho_A)^n = \int_{-1}^{1} d\zeta\,\mathcal{P}(\zeta)G^{n-1}(\zeta)\,, \quad (\zeta := \cos\theta) \tag{3.6}$$

$$\mathcal{P}(\zeta) = \frac{1}{2}\frac{\left|\sum_\ell(2\ell+1)s_\ell P_\ell(\zeta)\right|^2}{\sum_\ell(2\ell+1)|s_\ell|^2}\,,$$
$$G(\zeta) = \frac{\left|\sum_\ell(2\ell+1)s_\ell P_\ell(\zeta)\right|^2}{\sum_\ell(2\ell+1)\cdot\sum_\ell(2\ell+1)|s_\ell|^2}\,, \tag{3.7}$$

where we used the three-dimensional Dirac delta function in spherical coordinates with azimuthal symmetry, $\delta^{(3)}(\vec{p}-\vec{k}) = (4\pi k^2)^{-1}\delta(p-k)\sum_\ell(2\ell+1)P_\ell(\cos\theta)$, and the partial wave expansion of a delta function, $2\delta(1-\cos\theta) = \sum_\ell(2\ell+1)P_\ell(\cos\theta)$. Due to $s_\ell = 1 + 2i\tau_\ell$ and the unitarity condition (2.6), one can rewrite $\mathcal{P}(\zeta)$ in Eqs. (3.7) as

$$\mathcal{P}(\zeta) = \delta(1-\zeta)\frac{V - 4\sum_\ell(2\ell+1)\,\mathrm{Im}\,\tau_\ell}{V - 2\sum_\ell(2\ell+1)f_\ell} + \frac{2|\sum_\ell(2\ell+1)\tau_\ell P_\ell(\zeta)|^2}{V - 2\sum_\ell(2\ell+1)f_\ell}\,, \tag{3.8}$$

where $V := \sum_\ell(2\ell+1)$. Here $\sum_\ell(2\ell+1)\,\mathrm{Im}\,\tau_\ell$, $\sum_\ell(2\ell+1)f_\ell$ and $|\sum_\ell(2\ell+1)\tau_\ell P_\ell(\zeta)|^2$ correspond to physical observables and thus are necessarily finite, while the infinite sum V diverges. Therefore Eq. (3.8) leads to $\mathcal{P}(\zeta) = \delta(1-\zeta)$. Then one can easily proceed the integration in (3.6) and gets finally

$$\mathrm{tr}_A(\rho_A)^n = K^{n-1}\,, \tag{3.9}$$
$$K := G(1) = \frac{\left|\sum_\ell(2\ell+1)s_\ell\right|^2}{\sum_\ell(2\ell+1)\cdot\sum_\ell(2\ell+1)|s_\ell|^2}\,. \tag{3.10}$$

Obviously Eq. (3.9) for $n = 1$ correctly reproduces the normalization condition, $\mathrm{tr}_A\,\rho_A = 1$.

From Eq. (3.9) the Rényi entropy is $S_{\mathrm{RE}} = -\ln K$ and equals the entanglement entropy,

$$S_{\mathrm{EE}} = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{tr}_A(\rho_A)^n = -\ln K\,. \tag{3.11}$$

Using a Cauchy–Schwarz inequality applied to (3.10), K satisfies $0 \le K \le 1$, that is to say, the entanglement entropy S_{EE} is equal to zero or positive.

When there is no interaction, s_ℓ is equal to one for all ℓ and K becomes one, that is to say, the entanglement entropy S_{EE} vanishes. This is natural, because the final state is same as the initial state without interaction and the initial state (3.1) is not entangled. On the other hand, if the system has interaction, the entanglement entropy is expected to increase in scattering processes.

We have a comment on the elastic case without the inelastic channel, *i.e.*, $f_\ell = 0$ for all ℓ. In this case, one has $s_\ell = \exp(2i\delta_\ell)$, where δ_ℓ are the phase shifts. Hence one obtains the expression of Eq. (3.10) in terms of the phase shifts, $K = V^{-2}(|\sum_\ell(2\ell+1)\cos 2\delta_\ell|^2 + |\sum_\ell(2\ell+1)\sin 2\delta_\ell|^2)$.

Let us rewrite Eq. (3.10) in terms of τ_ℓ and f_ℓ as

$$K = 1 - \frac{4\sum_\ell(2\ell+1)|\tau_\ell|^2 - \frac{4}{V}|\sum_\ell(2\ell+1)\tau_\ell|^2}{V - 2\sum_\ell(2\ell+1)f_\ell}\,. \tag{3.12}$$

Formally the full Hilbert space extends over all partial waves, and thus one has $V = \sum_{\ell=0}^{\infty}(2\ell+1) = \infty$. It causes $K = 1$, in other words, the entanglement entropy vanishes. However, in physical elastic processes, the Hilbert space is essentially limited by energy–momentum conservation, so that the physical Hilbert space provides a meaningful entanglement entropy as we shall see further.

Since the partial wave expansions of the integrated elastic cross section, the integrated inelastic cross section, the total cross section and the differential cross section are shown in Eqs. (2.8) and (2.9), K can be described in terms of these physical observables as

$$K = 1 - \frac{\sigma_{\mathrm{el}} - \frac{4k^2}{V}\frac{d\sigma_{\mathrm{el}}}{dt}\Big|_{t=0}}{\frac{\pi V}{k^2} - \sigma_{\mathrm{inel}}}\,. \tag{3.13}$$

By a power expansion of S_{EE} with respect to $V^{-1} \ll 1$, we obtain $S_{\mathrm{EE}} = (k^2/\pi)\sigma_{\mathrm{el}}V^{-1} + \mathcal{O}(V^{-2})$. The leading term is proportional to the elastic cross section, and this is consistent with the result in Ref. [7], which calculated the entanglement entropy of two outgoing particles in the field theories in weak coupling perturbation.

4. Physical Hilbert space

In an actual scattering process at a given momentum k, too high angular momentum modes are strongly depleted and negligible in the elastic scattering amplitude. In a semi-classical picture using the impact parameter $b = \ell/k$ representation, the limitation can be depicted as a maximal sizable value $b/2 \le R$, where R is interpreted as the mean of incident particle effective radii. In this context the largest relevant angular momentum $\ell_{\max}$ is

$$\ell_{\max} \sim 2kR\,. \tag{4.1}$$

In practice, we shall consider (4.1) as the maximal value of the angular momentum beyond which the summation over partial wave amplitudes τ_ℓ can be neglected. We thus approximate by truncation the sum over ℓ of the Hilbert space states. Note that reasonable values of R may be obtained from experimental determination of the impact-parameter profile of the scattering amplitude, which can be inferred [11] from the elastic differential cross-section $d\sigma_{\mathrm{el}}/dt$.

At high energy, *i.e.*, large momentum k with maximal impact parameter $2R$, $\ell_{\max}$ is large. Although the key point in the derivation of Eqs. (3.9) and (3.10) is that $\mathcal{P}(\zeta)$ is identified with the delta function coming from $\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\zeta)$, it keeps approximately valid for large $\ell_{\max}$. Therefore under this approximation one can conclude the entropy is $S_{\mathrm{RE}} = S_{\mathrm{EE}} = -\ln K$ in replacing $\sum_{\ell=0}^{\infty}$ with $\sum_{\ell=0}^{\ell_{\max}}$. The Hilbert space volume becomes $V = \sum_{\ell=0}^{\ell_{\max}}(2\ell+1) = (1+\ell_{\max})^2 \sim \ell_{\max}^2 \gg 1$. Then Eq. (3.13) remains a good approximation with the parameter $V/k^2 \sim 4R^2$. Finally K is obtained as

$$K \sim 1 - \frac{\sigma_{\mathrm{el}} - \frac{1}{R^2}\frac{d\sigma_{\mathrm{el}}}{dt}\Big|_{t=0}}{4\pi R^2 - \sigma_{\mathrm{inel}}}\,, \tag{4.2}$$

so that one gets a finite value for the Rényi and entanglement entropy. In this expression the explicit V dependence disappears. Note that $4\pi R^2$ can be considered as the classical "geometric" cross section of the scattering. Formula (4.2) implies that, if we measure the cross sections and get an evaluation of the impact parameter profile in a collider experiment, one can give a reliable approximate estimate of the entanglement entropy of the final elastic state of the two outgoing particles.

It is instructive to examine the limiting values of (4.2) in $0 \le K \le 1$. The value $K = 1$, corresponding to zero entanglement entropy, can be met when R^2 reaches its minimal value $(d\sigma_{\rm el}/dt)|_{t=0}/\sigma_{\rm el}$, which is nothing else than the average size of the elastic diffraction peak. The limit $K \to 0$ (*i.e.*, $S_{\rm EE} \to \infty$) may be reached only at a zero of the expression $4\pi R^2 - \sigma_{\rm tot} + (d\sigma_{\rm el}/dt)|_{t=0}/R^2$, whose only solution is $\mathrm{Re}\, A(s, 0) = 0$ and $\sigma_{\rm tot} = 8\pi R^2$, that is twice the geometric cross section. An exception is when both numerator and denominator in (4.2) tend simultaneously to zero, namely $\sigma_{\rm el} = \sigma_{\rm inel} = \sigma_{\rm tot}/2 = 4\pi (d\sigma_{\rm el}/dt)|_{t=0}/\sigma_{\rm el} = 4\pi R^2$. Interestingly enough it corresponds to the so-called "black disk" limit, which happens to be phenomenologically relevant for the high energy asymptotics [12].

5. Conclusion and comments

We have studied the entanglement entropy between two outgoing particles, A and B, in an elastic scattering at high energy, where many inelastic channels are also opened. In the derivation of the entanglement entropy, we used the unitarity condition on the S-matrix (2.6). As a result, we obtained the formula for the entanglement entropy (3.11), $S_{\rm EE} = -\ln K$, with Eq. (3.10).

The Rényi entropy is same as the entanglement entropy, *i.e.*, $S_{\rm RE} = -\ln K$. This implies that the outgoing particles are maximally entangled. This is caused by the fact that the reduced density matrix (3.4) is diagonal due to the momentum conservation of two scattering particles in the center-of-mass frame.

Eq. (3.11) is reminiscent of Boltzmann's entropy formula with the Boltzmann constant $k_B = 1$. In this sense, one can regard $1/K$ as a kind of micro-canonical ensemble of final states. Indeed it can be recast in the following form derived from (3.6):

$$S_{\rm EE} = \ln \frac{V}{2} - \int_{-1}^{1} d\zeta\, \mathcal{P}(\zeta) \ln \mathcal{P}(\zeta)\,, \tag{5.1}$$

due to $\mathcal{P}(\zeta) = \frac{V}{2} G(\zeta)$. $\mathcal{P}(\zeta)$ is positive and of norm one in both cases of the full and physical Hilbert spaces, because $\int_{-1}^{1} d\zeta\, \mathcal{P}(\zeta) = 1$ thanks to the orthogonality of Legendre polynomials. Hence one can identify $\mathcal{P}(\zeta)$ with a well-defined probability measure over the interval $\zeta \in [-1, +1]$. We also see that $\mathcal{P}(\zeta)$ originates from the probability $|\langle\langle \vec{p}|\mathbf{s}|\vec{k}\rangle\rangle|^2$ in Eq. (3.4). Since V can be interpreted as the total number of final two-body quantum states ($2\ell + 1$ at level ℓ), the second term in Eq. (5.1) can be understood as the correction to the total entropy due to entanglement.

The result for K is described as Eqs. (3.12) and (3.13). The subspace volume of elastic states is small in size with respect to the volume of the overall Hilbert space, K is almost equal to one. In other words, the entanglement entropy is negligibly small.

For scattering at high energy, conveniently called "soft scattering", we can employ the physical truncation of the Hilbert space given by Eq. (4.1). We take the limit of large momentum k with a fixed maximal impact parameter $2R$. Then K becomes Eq. (4.2). This implies that the entanglement entropy is described in terms of the cross sections and the maximal impact parameter. Since it is possible to measure these parameters in experiments, *e.g.*, a proton–proton scattering in a collider, the entanglement entropy can be evaluated using (4.2). It would be interesting, in order to confirm the validity of our formula, to confront this result obtained within the S-matrix framework of strong interactions, to a microscopic derivation of the entanglement entropy in a gauge field theory at strong coupling using, *e.g.*, the AdS/CFT correspondence. It would require the holographic study of a QCD-like theory.

Acknowledgements

SS was supported in part by the National Research Foundation of Korea grant No. NRF-2013R1A1A2059434 and No. NRF-2013R1A2A2A05004846. SS is grateful to Institut des Hautes Études Scientifiques (IHÉS) for their hospitality.

References

[1] P. Calabrese, J.L. Cardy, Entanglement entropy and quantum field theory, J. Stat. Mech. 0406 (2004) P06002, arXiv:hep-th/0405152.
[2] S. Ryu, T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. 96 (2006) 181602, arXiv:hep-th/0603001;
S. Ryu, T. Takayanagi, Aspects of holographic entanglement entropy, J. High Energy Phys. 0608 (2006) 045, arXiv:hep-th/0605073.
[3] J. Maldacena, L. Susskind, Cool horizons for entangled black holes, Fortschr. Phys. 61 (2013) 781, arXiv:1306.0533 [hep-th].
[4] K. Jensen, A. Karch, Holographic dual of an Einstein–Podolsky–Rosen pair has a wormhole, Phys. Rev. Lett. 111 (21) (2013) 211602, arXiv:1307.1132 [hep-th].
[5] S. Seki, S.-J. Sin, EPR = ER and scattering amplitude as entanglement entropy change, Phys. Lett. B 735 (2014) 272, arXiv:1404.0794 [hep-th].
[6] V. Balasubramanian, M.B. McDermott, M. Van Raamsdonk, Momentum-space entanglement and renormalization in quantum field theory, Phys. Rev. D 86 (2012) 045014, arXiv:1108.3568 [hep-th].
[7] S. Seki, I.-Y. Park, S.-J. Sin, Variation of entanglement entropy in scattering process, Phys. Lett. B 743 (2015) 147, arXiv:1412.7894 [hep-th].
[8] L. Lello, D. Boyanovsky, R. Holman, Entanglement entropy in particle decay, J. High Energy Phys. 1311 (2013) 116, arXiv:1304.6110 [hep-th].
[9] L. Van Hove, A phenomenological discussion of inelastic collisions at high energies, Nuovo Cimento 28 (1963) 2344.
[10] A. Białas, L. Van Hove, Two-particle and many-particle channels in high-energy collisions, Nuovo Cimento 38 (1965) 1385.
[11] S. Munier, A.M. Stasto, A.H. Mueller, Impact parameter dependent S matrix for dipole proton scattering from diffractive meson electroproduction, Nucl. Phys. B 603 (2001) 427, arXiv:hep-ph/0102291.
[12] M.M. Block, L. Durand, P. Ha, F. Halzen, Comprehensive fits to high energy data for σ, ρ, and B and the asymptotic black-disk limit, Phys. Rev. D 92 (11) (2015) 114021, arXiv:1511.02406 [hep-ph].