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## Appendix: Maximal power of heat machines

by Roger Balian

When a motor generates work by exchanging heat with two sources at temperatures  $T_1 > T_2$ , Carnot's maximum efficiency  $1 - T_2/T_1$  can be approached only for reversible processes; but these are too slow to yield significant power. At the other extreme, rapid processes are strongly irreversible and cannot produce much work. A compromise is needed to maximise the power output. Jacques Yvon tackled this problem in 1955, focusing on a major cause of irreversibility, the delay required for heat transport. A simple approach is presented here.

During the first step of a cycle, between the times  $t_0$  and  $t_1 = t_0 + \tau_1$ , the motor M evolves at a temperature  $\Theta_1(t)$ , receiving the heat  $Q_1$  from the hot source  $S_1$  at temperature  $T_1$ . We assume that heat transfers obey Fourier's law, so that the heat flux  $\Phi_1(t)$  from  $S_1$  to M has at times  $t$  ( $t_0 \leq t \leq t_1$ ) the form  $\Phi_1(t) = C_1[T_1 - \Theta_1(t)]$  where  $C_1$  is supposed to be given. During the second step, between  $t_1$  and  $t_2 = t_1 + \tau_A$ , M evolves adiabatically, its temperature going down from  $\Theta_1(t_1)$  to  $\Theta_2(t_2)$ . During the third step, between  $t_2$  and  $t_3 = t_2 + \tau_2$ , M evolves at a temperature  $\Theta_2(t)$ , giving the heat  $Q_2$  to the cold source  $S_2$  at temperature  $T_2$ , through a flux  $\Phi_2(t) = C_2[\Theta_2(t) - T_2]$  ( $t_2 \leq t \leq t_3$ ). Finally, between  $t_3$  and  $t_4 = t_3 + \tau'_A$ , M returns adiabatically to its initial state, with a temperature rising from  $\Theta_2(t_3)$  to  $\Theta_1(t_4) = \Theta_1(t_0)$ . The cycle has the total duration  $\tau = t_4 - t_0 = \tau_1 + \tau_A + \tau_2 + \tau'_A$ . The work produced by M during a cycle is the difference between the heat  $Q_1 = \int_{t_0}^{t_1} dt \Phi_1(t)$  that it received from  $S_1$  and the heat  $Q_2 = \int_{t_2}^{t_3} dt \Phi_2(t)$  that it yielded to  $S_2$ , so that the average power output is

$$P = \frac{1}{\tau} \left\{ C_1 \int_{t_0}^{t_1} dt [T_1 - \Theta_1(t)] - C_2 \int_{t_2}^{t_3} dt [\Theta_2(t) - T_2] \right\}. \quad (1)$$

The system M evolves in a closed cycle, so that its change  $\Delta S$  of entropy between the times  $t_0$  and  $t_4$  vanishes:

$$\Delta S = C_1 \int_{t_0}^{t_1} dt \frac{[T_1 - \Theta_1(t)]}{\Theta_1(t)} - C_2 \int_{t_2}^{t_3} dt \frac{[\Theta_2(t) - T_2]}{\Theta_2(t)} = 0. \quad (2)$$

We wish to maximise  $P$  as function of the temperatures  $\Theta_1(t)$ ,  $\Theta_2(t)$  and of the durations  $\tau_1$ ,  $\tau_A$ ,  $\tau_2$ ,  $\tau'_A$ , for given values of  $T_1$ ,  $T_2$ ,  $C_1$ ,  $C_2$ ,  $\tau$ , under the constraint  $\Delta S = 0$ . Introducing a Lagrange multiplier  $\lambda/\tau$ , and writing that  $P - \lambda\Delta S/\tau$  is stationary with respect to  $\Theta_1(t)$  and  $\Theta_2(t)$ , we obtain

$$\Theta_1(t) = \sqrt{\lambda T_1}, \quad \Theta_2(t) = \sqrt{\lambda T_2}. \quad (3)$$

The optimal cycles for M are therefore Carnot cycles with constant temperatures  $\Theta_1(t) \equiv \Theta_1$  and  $\Theta_2(t) \equiv \Theta_2$ . Maximising  $P$  with respect to  $\tau_A$  and  $\tau'_A$  provides  $\tau_A/\tau \simeq 0$ ,  $\tau'_A/\tau \simeq 0$ : The adiabatic steps should be the shortest possible. It remains to write that  $P - \lambda\Delta S/\tau$  is stationary with respect to  $\tau_1/\tau = 1 - \tau_2/\tau$ , which yields

$$C_1 (T_1 - \Theta_1) (1 - \lambda/\Theta_1) + C_2 (\Theta_2 - T_2) (1 - \lambda/\Theta_2) = 0, \quad (4)$$

or equivalently, using (3),

$$\sqrt{\lambda} = \frac{\sqrt{C_1 T_1} + \sqrt{C_2 T_2}}{\sqrt{C_1} + \sqrt{C_2}}. \quad (5)$$

From the condition  $\Delta S = 0$ , we get the optimal durations of the isothermal steps of the cycle:

$$\frac{\tau_1}{\tau} = \frac{\sqrt{C_2}}{\sqrt{C_1} + \sqrt{C_2}}, \quad \frac{\tau_2}{\tau} = \frac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}}. \quad (6)$$

Altogether, the maximal power is

$$P_{\max} = \frac{C_1 C_2}{(\sqrt{C_1} + \sqrt{C_2})^2} (T_1 - T_2). \quad (7)$$

This power is equivalent to a fraction, at most equal to  $\frac{1}{4}$  (reached for  $C_1 = C_2$ ), of the heat flux  $C(T_1 - T_2)$  that would be transferred from  $S_1$  to  $S_2$  for a coefficient  $C$  equal to the average  $\sqrt{C_1 C_2}$ . Finally, the efficiency at maximal power is obtained from  $\Delta S = Q_1/\theta_1 - Q_2/\theta_2 = 0$  as

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\theta_2}{\theta_1} = 1 - \sqrt{\frac{T_2}{T_1}}. \quad (8)$$