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**Abstract.** - An ideal measurement on a system S by an apparatus A is approached in a minimalist, statistical formulation of quantum mechanics, where states encode properties of ensembles. The required final state of S+A is shown to have a Gibbsian thermodynamic equilibrium form, not only for a large ensemble of runs, but also for arbitrary subensembles. This outcome is justified dynamically in quantum statistical mechanics as result of relaxation for models with suitably chosen interactions within A. The quantum ambiguity that precludes the interpretation of a mixed state in terms of physical subensembles is overcome due to a specific type of microcanonical relaxation. The resulting structure for the states describing subsets of runs affords an explanation for the standard properties of ideal measurements, in particular the uniqueness of the result for each individual run, thus offering a statistical solution to the quantum measurement problem.

One of the main foundational challenges of quantum theory is the so-called measurement problem: Why does each individual run of an ideal measurement yield a well-defined outcome, in spite of the existence of quantum coherences? Does measurement theory require a specific principle or interpretation of quantum mechanics? Already raised by the founding fathers, this crucial question has witnessed a revival [1–7]. Many authors explore models, others propose alternative interpretations of quantum mechanics or go beyond it.

As in [8, 9], we will approach the problem through a standard minimalist quantum approach, by analysing the dynamics of the isolated system constituted by the tested system S coupled to the measuring apparatus A. A key point is the *macroscopic size* of the apparatus, which forces us to rely on non-equilibrium quantum statistical mechanics. Moreover, being irreducibly probabilistic, quantum physics does not describe individual objects; we must deal with *statistical ensembles* (§1), then attempt to infer the properties of a single run of the measurement from those of the subensembles of runs in which it may embedded.

We deal with ideal, non demolishing measurements. Their purpose is to test a single observable $\hat{s} = \sum_i s_i \hat{\Pi}_i$ of S characterised by its eigenvalues $s_i$ and the associated eigenprojectors $\hat{\Pi}_i$, while perturbing S minimally. Although ideal measurements are not currently performed in real experiments, their theoretical elucidation is a necessary step towards a full understanding of actual measurement processes, since any general quantum measurement (POVM) can be represented as a partial trace over an ideal measurement [10]. We denote by $\hat{\mathcal{D}}(t)$ the joint density operator of S+A for a large ensemble of runs, and by $\hat{r}(t) = \mathrm{tr}_{\mathrm{A}}\hat{\mathcal{D}}(t)$ and $\hat{\mathcal{R}}(t) = \mathrm{tr}_{\mathrm{S}}\hat{\mathcal{D}}(t)$ the marginal density operators of S and A, respectively. At the initial time $t = 0$, S and A are uncorrelated, S lies in some state $\hat{r}(0)$ and A in a metastable state $\hat{\mathcal{R}}(0)$, so that $\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0)$. Triggered by its coupling with S, A may relax towards one or another among its stable states $\hat{\mathcal{R}}_i$, which should have equal entropies and energies so as to avoid bias in the measurement. These states can be distinguished from one another through observation or registration of the pointer variable $A_i$, identified with the expectation value $\mathrm{tr}_A \hat{\mathcal{R}}_i \hat{A}$ of some collective observable $\hat{A}$ of A. The final indication $A_i$ of the pointer must be fully correlated with the eigenvalue $s_i$ of the tested observable $\hat{s}$ and with the production of the final state $\hat{r}_i$ for S. The analysis of the process should therefore explain how S+A, starting from the state $\hat{\mathcal{D}}(0)$, reaches at a final time $t_{\mathrm{f}}$, *for each run of the measurement*, one among the states [11]

$$\hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i, \qquad p_i \hat{r}_i = \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i, \tag{1}$$

with Born's probability $p_i = \langle \hat{\Pi}_i \rangle = \mathrm{tr}_S \hat{r}(0)\hat{\Pi}_i$.

Some parts of this task have been achieved for various models. Here we deal with arbitrary ideal measurement processes, gathering the key ideas that underlie their full dynamical solution. Due to the generality of our scope, we can describe only qualitatively this solution, but we will demonstrate the feasibility of the program by recalling in footnotes the outcomes of the detailed dynamical solution [6, 12] of the Curie–Weiss (CW) model[1].

*1. Statistical formulation of the principles of quantum mechanics.* We tackle the measurement problem within a statistical formulation of quantum mechanics, also called "statistical interpretation" or "*ensemble* interpretation" [1]. It is a *minimalist*, formal description suited to both microscopic and macroscopic systems. Its spirit is the same as in the $C^*$-algebraic approach [7], although we deal with finite non relativistic systems. It does not prejudge any specific interpretation of quantum oddities [10]. Physical interpretations should emerge at the macroscopic scale, in experimental contexts, as will be discussed in §5.

Physical quantities pertaining to a system (or rather to an ensemble of similar systems) are described as "observables" represented by Hermitean matrices in a Hilbert space. Observables behave as random objects, but, unlike ordinary random variables, their randomness arises from their non-commutative nature and is inescapable.

A "quantum state", whether pure or not, encompasses the probabilistic predictions that one may make about the various observables. It is characterised by a correspondence between observables $\hat{O}$ and real numbers, implemented as $\hat{O} \mapsto \langle \hat{O} \rangle = \mathrm{tr}\, \hat{\mathcal{D}}\hat{O}$ by means of a Hermitean, normalised and nonnegative density operator $\hat{\mathcal{D}}$. Such a definition looks analogous to that of a state in classical statistical mechanics, encoded in a density in phase space. However, "quantum expectation values" $\langle \hat{O} \rangle$, "quantum probabilities" such as $\langle \hat{\Pi}_i \rangle$ and "quantum correlations" such as $\langle \hat{s}\hat{A} \rangle$ present a priori only a formal similarity with standard expectation values, probabilities and correlations; fluctuations arise not only from some ignorance but also from the operator character of physical quantities.

As a standard probability distribution, a quantum state gathers information and refers, implicitly or not, to a statistical ensemble $\mathcal{E}$: We should regard an individual system as embedded in a large, real or virtual, ensemble $\mathcal{E}$ of systems produced under the same conditions. So a state does not "belong to a system", it is not an intrinsic property. Information may be updated as for ordinary probabilities by acknowledging and sorting the outcomes of measurements so as to split $\mathcal{E}$ into subensembles, to each of which a new state will be assigned (§5).

*2. System plus apparatus in thermodynamic equilibrium.* We suppose that the compound system S+A is isolated, including in A a thermal bath or an environment if present. The properties of S+A are governed by the Hamiltonian $\hat{H} = \hat{H}_S + \hat{H}_A + \hat{H}_{SA}$, which must have specific features. If A is decoupled from S, the part $\hat{H}_A$ governing the macroscopic apparatus should produce an initial metastable state $\hat{\mathcal{R}}(0)$ with lifetime longer than the duration of the measurement and several thermodynamic equilibrium states $\hat{\mathcal{R}}_i$, the possible final states. A typical example[1] is given by spontaneously broken discrete invariance, the pointer variable $A_i$ being the order parameter.

As we wish to deal with ideal measurements, the process should perturb S as little as possible: any observable compatible with $\hat{s}$, i.e., commuting with its eigenprojectors $\hat{\Pi}_i$, should remain unaffected. The conservation of all these observables [11] is expressed by the fact that $\hat{H}$ depends on S only through the projectors $\hat{\Pi}_i$. In particular the coupling must have the form $\hat{H}_{SA} = \sum_i \hat{\Pi}_i \otimes \hat{h}_i$, where $\hat{h}_i$ are operators of A. Moreover, if $\hat{s}$ takes the value $s_i$, that is, $\hat{\Pi}_i$ the value 1, A should end up near its stable state $\hat{\mathcal{R}}_i$, the pointer variable being close to $A_i$. This can be achieved if each $\hat{h}_i$ behaves as a source that breaks explicitly the invariance[2] by energetically favouring $A_i$.

Before analysing the dynamics of the measurement process (§3 and §4), we determine for orientation the general form $\hat{\mathcal{D}}_{eq}$ of the thermodynamic equilibrium states associated with the Hamiltonian $\hat{H}$ of S+A. We rely on the maximum von Neumann entropy criterion for assigning a state to a system in equilibrium [13, 14]. We thus maximise $-\mathrm{tr}\hat{\mathcal{D}} \ln \hat{\mathcal{D}}$ under constraints accounting for the conserved quantities. As usual for isolated macroscopic systems, thermal equilibrium is implemented by specifying the average energy $\langle \hat{H} \rangle$. The other constants of the motion, to wit, the expectation values of all observables of S that commute with the projections $\hat{\Pi}_i$, are accounted for by associating a Lagrange multiplier with each of them.

This yields for the equilibrium states a Gibbsian form where the exponent involves an arbitrary linear combination of all the conserved observables, i.e., $\hat{\mathcal{D}}_{eq} \propto \exp(-\beta\hat{H} + \sum_i \hat{y}_i)$ where $\hat{y}_i$ is any operator of S acting

[1] In the CW model [6, sect. 3], S is a spin $\frac{1}{2}$, the measured observable being its $z$-component $\hat{s}_z$, with outcomes $i = \uparrow$ or $\downarrow$. The apparatus simulates a magnetic dot, including $N \gg 1$ spins $\hat{\sigma}^{(n)}$, which interact through the Ising coupling $J$, and a phonon thermal bath at temperature $T < J$; these spins and the phonons are coupled through a dimensionless weak coupling $\gamma$. Initially prepared in its metastable paramagnetic state, A may switch to one or the other stable ferromagnetic state. The pointer observable $\hat{A} = N\hat{m} = \sum_{n=1}^{N} \hat{\sigma}_z^{(n)}$ is the total magnetisation in the $z$-direction of the $N$ Ising spins. The coupling between S and A is $\hat{H}_{SA} = -\sum_{n=1}^{N} g\hat{s}_z\hat{\sigma}_z^{(n)}$, while $\hat{H}_S = 0$.

[2] The interaction Hamiltonian $\hat{H}_{SA}$ allows to describe not only ideal measurements involving well separated eigenvalues $s_i$ of $\hat{s}$, but also more general measurements for which the projectors $\hat{\Pi}_i$, still associated through $\hat{h}_i$ with the pointer indications $A_i$, are no longer in one-to-one correspondence with the eigenvalues of $\hat{s}$. For instance, if some $\hat{\Pi}_i$ encompasses the eigenspaces of several different neighbouring eigenvalues, selecting the outcome $A_i$ will not discriminate them, and the final state $\hat{r}_i = \hat{\Pi}_i\hat{r}(0)\hat{\Pi}_i/p_i$ of S will not be associated with a single eigenvalue of $\hat{s}$ as in an ideal measurement. As another example, consider two orthogonal rank-one projectors $\hat{\Pi}_1$ and $\hat{\Pi}_2$, coupled with sources $\hat{h}_1$ and $\hat{h}_2$ that produce different outcomes $A_1$ and $A_2$, and assume that $\hat{\Pi}_1 + \hat{\Pi}_2$ spans the two-dimensional eigenspace associated with a degenerate eigenvalue of $\hat{s}$; reading the outcome $A_1$ (or $A_2$) then provides more information than this eigenvalue.

inside the diagonal block $i$ (so that $\hat{y}_i = \hat{\Pi}_i \hat{y}_i \hat{\Pi}_i$). Noting that the exponent, which commutes with the projections $\hat{\Pi}_i$, has a block diagonal structure, we find for these equilibrium states the related block diagonal structure

$$\hat{\mathcal{D}}_{\rm eq} = \sum_i q_i \hat{x}_i \otimes \hat{\mathcal{R}}_i^h, \qquad \sum_i q_i = 1. \tag{2}$$

Each factor $q_i\hat{x}_i$, which arises from $\exp(-\beta\hat{H}_{\rm S} + \hat{y}_i)$, is an arbitrary block diagonal operator of S, where $\hat{x}_i = \hat{\Pi}_i \hat{x}_i \hat{\Pi}_i$, $\mathrm{tr}_{\rm S}\hat{x}_i = 1$ behaves as a density operator of S. (If the eigenvalue $s_i$ is non degenerate, $\hat{x}_i$ reduces to $\hat{\Pi}_i$.) Each factor $\hat{\mathcal{R}}_i^h \propto \exp[-\beta(\hat{H}_A + \hat{h}_i)]$ in (2) can be interpreted as a canonical equilibrium density operator in the space of A. Here the invariance is *explicitly broken* by adding to the Hamiltonian of A the source term $\hat{h}_i$ arising from $\hat{H}_{\rm SA}$.

This term should be sufficiently large so that the distribution $\mathrm{tr}_{\rm A}\hat{\mathcal{R}}_i^h \delta(A - \hat{A})$ of $\hat{A}$ has a single narrow peak, and sufficiently small so that this peak lies close to $A_i$, so as to let $\hat{A}$ behave as a pointer observable. These properties are easy to satisfy for a macroscopic apparatus[3]. Thermodynamic equilibrium (2) thus entails a complete correlation between the eigenvalue $s_i$ of $\hat{s}$ and the value $A_i$ of the pointer variable. The moderate size of $\hat{h}_i$ also ensures that the state $\hat{\mathcal{R}}_i^h$ lies in the basin of attraction of the state $\hat{\mathcal{R}}_i$ with broken symmetry, so that $\hat{\mathcal{R}}_i^h$ will relax smoothly to $\hat{\mathcal{R}}_i$ at the end of the measurement process when the coupling $\hat{H}_{\rm SA}$ is switched off[4].

Let us return to measurements. We wish to explain how a final state $\hat{\mathcal{D}}_i$ of the form (1) should be assigned with probability $p_i$ to each run. It is thus necessary (but not sufficient) to prove, by studying the dynamics of a large statistical ensemble $\mathcal{E}$ of runs for which S+A lies initially in the state $\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0)$, that its final state is

$$\hat{\mathcal{D}}(t_{\rm f}) = \sum_i p_i \hat{\mathcal{D}}_i = \sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i. \tag{3}$$

We can identify (3) with a thermodynamic equilibrium state (2), for which relaxation of $\hat{\mathcal{R}}_i^h$ to $\hat{\mathcal{R}}_i$ has taken place[4] after switching off $\hat{H}_{\rm SA}$. Dynamics and conservation laws will determine the free parameters of $\hat{\mathcal{D}}_{\rm eq}$ from the initial state $\hat{\mathcal{D}}(0)$ as $q_i\hat{x}_i = \hat{\Pi}_i\hat{r}(0)\hat{\Pi}_i \equiv p_i\hat{r}_i$. i.e., $q_i = p_i$, $\hat{x}_i = \hat{r}_i$.

We can also identify the state $\hat{\mathcal{D}}_i$ defined in (1), expected to describe the subensemble $\mathcal{E}_i$ obtained by selecting within $\mathcal{E}$ the runs tagged by the value $A_i$ of the pointer, with an equilibrium state for which all $q_j$ with $j \neq i$ vanish. More generally, for an *arbitrary subset* $\mathcal{E}_{\rm sub}$ of runs[5] having yielded a proportion $q_i$ of individual runs with outcomes $A_i$, the expected final state

$$\hat{\mathcal{D}}_{\rm sub}(t_{\rm f}) = \sum_i q_i \hat{\mathcal{D}}_i \tag{4}$$

is a thermodynamic equilibrium state (2) with $\hat{x}_i = \hat{r}_i$.

Thus, an ideal measurement process appears as a mere relaxation of S+A to equilibrium, for the full ensemble $\mathcal{E}$ of runs and for arbitrary subensembles $\mathcal{E}_{\rm sub}$. In quantum mechanics, relaxation of $\hat{\mathcal{D}}(t)$ and $\hat{\mathcal{D}}_{\rm sub}(t)$ towards Gibbsian thermodynamic equilibrium states (3) and (4) is not granted [15]. We must therefore justify these properties within the quantum statistical dynamics framework. We sketch the main steps of such a proof in §3 and §4.

*3. Dynamics of S+A for the full set of runs.* Our first task [8] consists in deriving the final state (3) by solving the Liouville–von Neumann equation $i\hbar \mathrm{d}\hat{\mathcal{D}}(t)/\mathrm{d}t = [\hat{H}, \hat{\mathcal{D}}(t)]$ with initial condition $\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0)$. Taking into account the above form of $\hat{H}$ and the commutation $[\hat{H}_{\rm S}, \hat{r}(0)] = 0$ which ensures that the marginal state $\hat{r}(t)$ of S is perturbed only by the interaction $\hat{H}_{\rm SA}$ during the process, we check that $\hat{\mathcal{D}}(t)$ can be parameterised as

$$\hat{\mathcal{D}}(t) = \sum_{i,j} \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_j \otimes \hat{\mathcal{R}}_{ij}(t) \tag{5}$$

in terms of a set $\hat{\mathcal{R}}_{ij}(t) = \hat{\mathcal{R}}_{ji}^\dagger(t)$ of operators in the Hilbert space of A, to be determined by the equations

$$i\hbar \frac{\mathrm{d}\hat{\mathcal{R}}_{ij}(t)}{\mathrm{d}t} = (\hat{H}_{\rm A} + \hat{h}_i)\hat{\mathcal{R}}_{ij}(t) - \hat{\mathcal{R}}_{ij}(t)(\hat{H}_A + \hat{h}_j), \tag{6}$$

with the initial conditions $\hat{\mathcal{R}}_{ij}(0) = \hat{\mathcal{R}}(0)$. The dynamics thus involves *solely the apparatus*, irrespective of the tested system, a specific property of ideal measurements.

Though macroscopic, A should be treated as a finite system so as to ensure a better control of the dynamics and to discuss under which conditions the process can be regarded as an ideal measurement. We must then explain how the expected irreversible relaxation from $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}(t_{\rm f})$ can be governed by the reversible equations (6), so that we run into the old paradox of irreversibility. As usual in statistical mechanics, it is legitimate for finite but large systems to disregard events having an extremely small probability, to forget about recurrences that would occur after large, unattainable times, and to neglect physically irrelevant correlations between a macroscopic number of degrees of freedom. Such approximations, although not exact mathematically, are fully justified when their outcome is physically indistinguishable from the exact solution. A large apparatus, and a suitable choice of parameters in $\hat{H}_{\rm A}$ and $\hat{H}_{\rm SA}$ will therefore be needed, for each model, to explain the required relaxations and to estimate their time scales, as will be illustrated by the CW model treated extensively in [6].

[3] In the CW model[1], the factors $\hat{h}_\downarrow = -\hat{h}_\uparrow = \sum_{n=1}^N g\hat{\sigma}_z^{(n)}$ that occur in the coupling $\hat{H}_{\rm SA}$ behave as a magnetic field applied to A. The conditions for $\hat{h}_i$ are satisfied if $Ng \gg T$ (which lets the probability of the states with $m < 0$ vanish for $s_z = 1$), and $g < T$ [6, sect. 9.4].

[4] In the CW model $g < T$ ensures this relaxation [6, sect. 7.2].

[5] Subsets obtained by extracting runs at random from $\mathcal{E}$ would be described by the same state $\hat{\mathcal{D}}(t)$ as the full set $\mathcal{E}$. If $\hat{r}_i$ is a mixed state, the runs described by (4) are picked up at random within $\mathcal{E}_i$.

Two types of relaxation arise independently from the dynamical equations (6). (*i*) For $i \neq j$, the coherent contributions $\hat{\mathcal{R}}_{ij}(t)$ decay owing to the difference between $\hat{h}_i$ and $\hat{h}_j$ and eventually vanish. The off-diagonal blocks of the density matrix $\hat{\mathcal{D}}(t)$ are thus truncated as regards the physically attainable observables. Depending on the model, this decay may be governed by different mechanisms[6]. (*ii*) For $i = j$, the evolution of $\hat{\mathcal{R}}_{ii}(t)$ governed by (6) is a mere relaxation from the metastable state $\hat{\mathcal{R}}(0)$ to the equilibrium state $\hat{\mathcal{R}}_i^h$ in the presence of the source $\hat{h}_i$, and then to $\hat{\mathcal{R}}_i$ after $\hat{H}_{\rm SA}$ is switched off. The correlation between $s_i$ and $A_i$ needed to register the outcome is thereby established[7]. Thus, microscopic dynamics confirms the thermodynamic surmise (3) for the final state of S+A in the ensemble $\mathcal{E}$.

*4. Final states for arbitrary subensembles.* We have shown how $\hat{\mathcal{D}}(t)$ evolves from $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}(t_{\rm f}) = \sum_i p_i \hat{\mathcal{D}}_i$ for the large set $\mathcal{E}$ of runs. Our next task consists in proving, again dynamically, that S+A ends up in a state of the form (4), $\hat{\mathcal{D}}_{\rm sub}(t) = \sum_i q_i \hat{\mathcal{D}}_i$, *for all possible subsets*[5] $\mathcal{E}_{\rm sub}$ of $\mathcal{E}$. If the density operator $\hat{\mathcal{D}}(t_{\rm f})$ did behave as an ordinary probability distribution, this result would be obvious, as the form $\sum_i p_i \hat{\mathcal{D}}_i$ of $\hat{\mathcal{D}}(t_{\rm f})$ would mean that $\mathcal{E}$ contains a proportion $p_i$ of individual runs having ended up in the state $\hat{\mathcal{D}}_i$; the form (4) of $\hat{\mathcal{D}}_{\rm sub}$ would follow for any subset gathering a proportion $q_i$ of runs $i$. However, in quantum mechanics, contrary to standard probability theory, specifying the state $\hat{\mathcal{D}}$ for an ensemble $\mathcal{E}$ (with $\mathcal{N}$ elements), that is, the correspondence $\hat{O} \mapsto \langle \hat{O} \rangle = {\rm tr}\, \hat{\mathcal{D}}\hat{O}$, does not imply the existence of probabilities for individual systems (§1) so that the above reasoning is fallacious [16].

Indeed, when an ensemble $\mathcal{E}$ (with $\mathcal{N}$ elements described by $\hat{\mathcal{D}}$) gathers a subensemble $\mathcal{E}_{\rm sub}$ (with $0 < \mathcal{N}_{\rm sub} < \mathcal{N}$ elements described by $\hat{\mathcal{D}}_{\rm sub}$) and its complement, one has

$$\hat{\mathcal{D}}(t) = k\hat{\mathcal{D}}_{\rm sub}(t) + (1-k)\hat{\mathcal{D}}_{\rm C sub}(t), \tag{7}$$

involving the weight $k = \mathcal{N}_{\rm sub}/\mathcal{N}$ and the nonnegative density operator $\hat{\mathcal{D}}_{\rm C sub}(t)$ of the complement of $\mathcal{E}_{\rm sub}$.

However, if we conversely consider some decomposition of a known mixed state $\hat{\mathcal{D}}$ having the form (7), we run into a severe difficulty. Due to the matrix nature of quantum states, nothing tells us that the set $\mathcal{E}$ described by $\hat{\mathcal{D}}$ may be split into two subsets that would be described by the two terms $\hat{\mathcal{D}}_{\rm sub}$ and $\hat{\mathcal{D}}_{\rm C sub}$. We have to face a *quantum ambiguity:* A mixed state possesses $\hat{\mathcal{D}}$ of S+A many *different mathematical decompositions* (7) *which are physically incompatible* [16]. A well known example is the state of an unpolarised ensemble of spins, which can be decomposed both as $\frac{1}{2}|z\rangle\langle z| + \frac{1}{2}|-z\rangle\langle -z|$ and as $\frac{1}{2}|x\rangle\langle x| + \frac{1}{2}|-x\rangle\langle -x|$, where $|a\rangle$ denotes a pure state polarised in the $a$-direction; if these two decompositions were meaningful, there would exist subensembles of spins polarised simultaneously in two orthogonal directions! Likewise, the mixed state $\hat{\mathcal{D}}$ of S+A possesses many mathematical decompositions (7) which are mutually contradictory, so that we cannot infer from its sole knowledge whether a term $\hat{\mathcal{D}}_{\rm sub}$ issued from some decomposition of $\hat{\mathcal{D}}$ is a density operator describing a real subset of runs or not. (We keep the notation $\hat{\mathcal{D}}_{\rm sub}$ also in the latter case.) In particular, the form (3) of $\hat{\mathcal{D}}$, though suggestive, is not sufficient to imply the existence of subensembles of runs that would be described by $\hat{\mathcal{D}}_i$.

In order to overcome this quantum ambiguity, we adopt the following strategy. We start from the state $\hat{\mathcal{D}}(t_{\rm split})$, taken at a time $t_{\rm split}$ earlier than $t_{\rm f}$ but sufficiently late so that $\hat{\mathcal{D}}(t_{\rm split})$ has already reached the form $\sum_i p_i \hat{\mathcal{D}}_i$, after $\hat{H}_{\rm SA}$ has been switched off. We consider *all mathematically allowed decompositions* of $\hat{\mathcal{D}}(t_{\rm split})$ of the form (7), involving two nonnegative Hermitean operators. Although nothing ensures that the operators $\hat{\mathcal{D}}_{\rm sub}(t_{\rm split})$ thus defined are physically meaningful, we are ascertained that *their class includes all physical states* associated with real subsets of runs. Our purpose is then to show, by taking $\hat{\mathcal{D}}(t_{\rm split})$ as initial condition and solving for $t > t_{\rm split}$ the equations of motion for $\hat{\mathcal{D}}(t)$ governed by the Hamiltonian $\hat{H}_{\rm S}+\hat{H}_{\rm A}$, that *any admissible candidate* $\hat{\mathcal{D}}_{\rm sub}(t)$ *for a physical state ends up in the expected form* $\hat{\mathcal{D}}_{\rm sub}(t_{\rm f}) = \sum_i q_i \hat{\mathcal{D}}_i$. We shall thereby have proven the relaxation towards the required equilibrium form (4) for all physical subsets of runs, although it will be impossible before the reasoning of §5 to know which among the operators $\hat{\mathcal{D}}_{\rm sub}(t_{\rm f})$ thereby constructed are the physical ones.

We begin with the determination of the general form, issued from (7), of the initial operators $\hat{\mathcal{D}}_{\rm sub}(t_{\rm split})$. To simplify the discussion, we assume here the eigenvalues of $\hat{s}$ to be non degenerate[8] so that $\hat{r}_i = \hat{\Pi}_i = |s_i\rangle\langle s_i|$. As A is

[6] Several processes are involved in the CW model[1]. Over the short time scale $\hbar/g\sqrt{N}$, truncation results [6, sect. 5] from the dephasing between the oscillations yielded by the factor $\exp 2it\hbar^{-1}\sum_{n=1}^{N} g\hat{\sigma}_z^{(n)}$ entering $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$, which have different frequencies (due to the randomness of $\sigma_z^{(n)}$ in the initial paramagnetic state of A). Information is thereby lost through a cascade of correlations of higher and higher order, less and less accessible, between the spins of A, and $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ practically tends to zero as regards the accessible observables. Recurrences are wiped out [6, sect. 6], either by the coupling $\gamma$ with the phonon bath (provided $T/J \gg \gamma \gg g/NT$), or by a spread $\delta g$ in the couplings $g$ of $\hat{H}_{\rm SA}$ (provided $\delta g \gg g/\sqrt{N}$).

[7] Authors do not always give the same meaning to the various words used. We term as *truncation* the disappearance of the off-diagonal blocks of the density matrix of S+A under the effect of an arbitrary mechanism, and specialise *decoherence* to the production of this effect by interaction with an environment or a thermal bath. We term as *registration* the process which leads each diagonal block to the correlated state $\hat{r}_i \otimes \hat{\mathcal{R}}_i$, and as *reduction* the transition from $\hat{r}(0)$ to some $\hat{r}_i$ for an individual run. While much attention has been paid to the vanishing of the off-diagonal blocks, the relaxation of the diagonal blocks is too often disregarded, although it produces the correlations that ensure registration. In the CW model [6, sect. 7], this process is triggered by $\hat{h}_i$ which makes $\hat{\mathcal{R}}(0)$ unstable and should be sufficiently large to exclude false registrations ($g \gg J/\sqrt{N}$). Later on, the relaxation of $\hat{\mathcal{R}}_{ii}(t)$ to $\hat{\mathcal{R}}_i^h$, and finally to $\hat{\mathcal{R}}_i$ after $\hat{H}_{\rm SA}$ is switched off, is governed by the dumping of free energy from the magnet to the phonon bath; its characteristic duration is the registration time $\hbar/\gamma(J-T)$.

[8] For degenerate eigenvalues $s_i$, the only change in the forthcom-

macroscopic, the fluctuations of $\hat{H}_A$ around $\langle \hat{H}_\mathrm{A} \rangle$ and of the pointer observable around the macroscopic value $A_i$ are relatively small, and we can replace in $\hat{\mathcal{D}}(t_\mathrm{split})$ the canonical equilibrium states $\hat{\mathcal{R}}_i$ by microcanonical ones, $\hat{\mathcal{R}}_i^\mu$. Within the Hilbert space of A, we denote as $|A_i, \eta\rangle$ a basis of kets characterised by a value of $\hat{A}$ close to $A_i$ (within a small margin $\delta A_i$), and by a value of the energy corresponding to the microcanonical equilibrium state $\hat{\mathcal{R}}_i^\mu$. As the spectrum is dense, the index $\eta$ may take a very large number $G_i$ of values, and $\hat{\mathcal{R}}_i^\mu$ is expressed by

$$\hat{\mathcal{R}}_i^\mu = \frac{1}{G_i} \sum_\eta |A_i, \eta\rangle\langle A_i, \eta|. \tag{8}$$

The state $\hat{\mathcal{D}}(t_\mathrm{split}) \simeq \sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i^\mu$ thus involves only diagonal matrix elements within the Hilbert subspace $\mathcal{H}_\mathrm{corr}$ of S+A spanned by the correlated kets $|s_i\rangle\,|A_i, \eta\rangle$. Hence, since both operators on the right side of (7) must be non-negative, their matrix elements must lie within this Hilbert subspace $\mathcal{H}_\mathrm{corr}$, so that any initial (normalised) operator constrained by (7) has the form

$$\hat{\mathcal{D}}_\mathrm{sub}(t_\mathrm{split}) = \sum_{i,j,\eta,\eta'} |s_i\rangle|A_i, \eta\rangle K(i, \eta; j, \eta')\langle s_j|\langle A_j, \eta'|. \tag{9}$$

The evolution for $t > t_\mathrm{split}$ of the operator $\hat{\mathcal{D}}_\mathrm{sub}(t)$ issued from (9) is governed by $\hat{H}_\mathrm{S} + \hat{H}_\mathrm{A}$, and, as in section 3, $\hat{H}_\mathrm{S}$ is ineffective. The Hamiltonian $\hat{H}_\mathrm{A}$ of the apparatus is chosen so as to involve interactions inducing among the kets $|A_i, \eta\rangle$ rapid transitions[9] that modify $\eta$ without affecting the macroscopic value of $A_i$. Apart from the existence here of *several thermodynamic equilibrium states*[10] labelled by $i$, the mechanism is the same as in the standard microcanonical relaxation [17–20] which leads to the equalisation of the populations of the microstates for a single microcanonical equilibrium. Such a "quantum collisional process" is irreversible for a large apparatus. Acting on both $|A_i, \eta\rangle$ and $\langle A_j, \eta'|$ in (9), it produces over the same time scale $\tau_\mathrm{sub}$ two different effects. (*i*) For $i \neq j$, all contributions to (9) fade out. (*ii*) For $i = j$, all terms such that $\eta \neq \eta'$ disappear, while the coefficients $K(i, \eta; i, \eta)$ of the various terms $\eta$ all tend to one another, their sum remaining constant. The duration $\tau_\mathrm{sub}$ of these relaxations being much shorter[9] than $t_\mathrm{f}$, the mechanism is already effective before $t_\mathrm{split}$, so that anyway $\hat{\mathcal{D}}_\mathrm{sub}$ reaches at the final time $t_\mathrm{f} > t_\mathrm{split} + \tau_\mathrm{sub}$ the microcanonical equilibrium

$$\hat{\mathcal{D}}_\mathrm{sub}(t_\mathrm{f}) = \sum_i q_i \hat{r}_i \otimes \hat{\mathcal{R}}_i^\mu, \quad q_i = \sum_\eta K(i, \eta; i, \eta). \tag{10}$$

Since the above derivation holds for arbitrary operators $\hat{\mathcal{D}}_\mathrm{sub}$ issued from a mathematical decomposition (7) of $\hat{\mathcal{D}}$, it encompasses *all the physical subsets* $\mathcal{E}_\mathrm{sub}$ of runs, which therefore end up in the required form (4). The coefficients $q_i$ of the various physical subensembles are related to one another by a *hierarchic structure*: If two disjoint subensembles $\mathcal{E}_\mathrm{sub}^{(1)}$ and $\mathcal{E}_\mathrm{sub}^{(2)}$ of $\mathcal{E}$, described by $\hat{\mathcal{D}}_\mathrm{sub}^{(1)}$ and $\hat{\mathcal{D}}_\mathrm{sub}^{(2)}$, and having $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ elements, respectively, merge into a new subensemble $\mathcal{E}_\mathrm{sub}$, the above proof implies for the coefficients the standard addition rule

$$\left[\mathcal{N}^{(1)} + \mathcal{N}^{(2)}\right] q_i = \mathcal{N}^{(1)} q_i^{(1)} + \mathcal{N}^{(2)} q_i^{(2)}. \tag{11}$$

*5. Emergence of classical probabilistic interpretation.* In order to elucidate the measurement problem for ideal quantum measurements, it remains to show that the operators $\hat{\mathcal{D}}_i$ are not only the building blocks of the final density operators $\hat{\mathcal{D}}(t_\mathrm{f})$ of S+A (associated with the full ensemble $\mathcal{E}$ of runs) and $\hat{\mathcal{D}}_\mathrm{sub}(t_\mathrm{f})$ (associated with its subensembles $\mathcal{E}_\mathrm{sub}$), but also that they describe the outcomes (1) of individual runs. However, we have stressed (§1) that quantum mechanics, in its statistical formulation, does not deal with individual systems but only provides information on statistical ensembles – possibly gedanken but physically consistent. In the strict framework of quantum statistical mechanics, the most detailed result about ideal measurements that can be derived is the structure (4) of the final density operators of S+A *for all possible subensembles* $\mathcal{E}_\mathrm{sub}$ of $\mathcal{E}$. An essential feature of this result will allow extrapolation to individual runs, to wit, the elimination owing to dynamics of the quantum ambiguity, yielding the hierarchic structure (11) of the states of the subensembles $\mathcal{E}_\mathrm{sub}$.

Indeed, the latter structure is just the same as the one that lies at the core of the definition of ordinary probabilities when they are interpreted as relative frequencies of occurrence of individual events [21]. Once the quantum

ing derivation, if the states $\hat{r}_i \equiv |i\rangle\langle i|$ are pure, is the replacement of $|s_i\rangle$ by $|i\rangle$. If $\hat{r}_i$ is mixed, we note that this operator of S is not modified by the process, while remaining fully coupled with $A_i$ for $t > t_\mathrm{split}$. We should therefore preserve this property when we split $\hat{\mathcal{D}}$ so as to build the candidates $\hat{\mathcal{D}}_\mathrm{sub}$ for states of physical subensembles. The microcanonical relaxation of A then produces again the final state (10), at least for all physical subensembles.

[9] Two different mechanisms achieving such a process have been studied for the CW model [6, §11.2], and it has been shown that they produce the result (10). In the more realistic one [6, Appendices H and I], the transitions that modify $\eta$ are produced by an interaction $\hat{V}$ between the magnet and the bath which has a variance $v^2 = \mathrm{tr}\,\hat{V}^2$; an average delay $\theta$ separates successive transitions. Microcanonical relaxation may take place even if $\hat{V}$ is not macroscopic, with a variance scaling as $v \propto N^a$ $(a < 1)$ for large $N$. For a short $\theta$, scaling as $\theta \propto 1/N^b$ $(a < b < 2a)$, the characteristic time $\tau_\mathrm{sub} = \hbar^2/v^2\theta$ scales as $1/N^c$ where $c = 2a - b$, $0 < c < a < 1$; it is short compared to the registration time, which dominates $t_\mathrm{f}$ because registration involves a macroscopic dumping of energy from the magnet to the bath, in contrast to the present relaxation.

[10] This type of relaxation also occurs in the dynamics of finite-temperature quantum phase transitions with *spontaneously broken invariance*, explaining within quantum theory why the order parameter may take a well-defined value in a single experiment. This analogue of the measurement problem is solved along the same lines for macroscopic systems, so that the community has rightfully not been bothered about it. But the problem remains for phase transitions in finite systems, which require an analysis of time scales.

ambiguity has been removed and the property (11) established, it is natural to complement the inherently probabilistic principles of quantum mechanics with the following interpretation of the final mixed states (4) associated with the various subensembles $\mathcal{E}_{\rm sub}$: A coefficient $q_i$ that arose abstractly from the above relaxation process is identified[11] as the proportion of runs having produced the outcome $A_i$ in the subensemble $\mathcal{E}_{\rm sub}$; each $\mathcal{E}_i$ contains Born's proportion $p_i$ of runs. An individual run then belongs to some subset $\mathcal{E}_i$ and can be assigned the state $\hat{\mathcal{D}}_i$, so that *a solution of the quantum measurement problem is achieved.*

The runs are tagged by the value $A_i$ of the pointer variable, which characterises the factor $\hat{\mathcal{R}}_i$ of $\hat{\mathcal{D}}_i$ and which can macroscopically be observed or registered. By picking out the runs having yielded $A_i$, one extracts from the whole ensemble $\mathcal{E}$ the subensemble $\mathcal{E}_i$ that ends up in the state $\hat{\mathcal{D}}_i$. Two steps are thus necessary to go from the initial state $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}_i$. First, the Hamiltonian evolution (irreversible since A is large) of the coupled system S+A for the full ensemble $\mathcal{E}$ leads to the state $\hat{\mathcal{D}}(t_{\rm f})$ expressed by (3); correlations are created, and information is transferred from S to A with some loss. However, the subsequent reduction of the state from $\hat{\mathcal{D}}(t_{\rm f})$ to one of its components $\hat{\mathcal{D}}_i$ is not a consequence of some evolution, but the mere result of selecting the particular outcome $A_i$. This change of state, associated with the extraction from $\mathcal{E}$ of the subset $\mathcal{E}_i$ is merely an updating of information[12], in the same way as the change of an ordinary probability distribution resulting from some gain of information. A further step is involved in the *reduction of the state of* S from $\hat{r}(0)$ to $\hat{r}_i$, the disregard of A after selection of $A_i$.

A "state" defined by the mapping $\hat{O} \mapsto \langle \hat{O} \rangle = {\rm tr}\hat{\mathcal{D}}\hat{O}$ has only a mathematical nature; the $c$-numbers $\langle \hat{O} \rangle$ present only a formal analogy with ordinary expectation values. However, at the end of a measurement, the coefficients $p_i = \langle \hat{\Pi}_i \rangle = {\rm tr}_{\rm S}\hat{r}(0)\hat{\Pi}_i$ can be identified with ordinary probabilities in the frequency interpretation. Accordingly, $\langle \hat{s} \rangle = {\rm tr}\,\hat{r}(0)\hat{s} = \sum_i p_i s_i$ and $A_i = {\rm tr}_{\rm A}\hat{\mathcal{R}}_i\hat{A}$ appear as ordinary statistical averages, and $\langle \hat{s}^2 \rangle - \langle \hat{s} \rangle^2$ as an ordinary variance; the correlation between $\hat{s}$ and $\hat{A}$ in $\hat{\mathcal{D}}(t_{\rm f})$ has a classical nature. As usual in statistical mechanics, the macroscopic behaviour of the apparatus, in particular the irreversibility of the process, emerges from the underlying reversible microscopic theory. But moreover another remarkable type of emergence occurs at the end of the measurement process: *Classical probabilities* emerge from quantum theory, although the underlying "quantum probabilities" were non-commutative and could not a priori be regarded as frequencies in the absence of an experimental context. Certain properties of S, encoded statistically in the initial state $\hat{r}(0)$ of the ensemble $\mathcal{E}$, are selected by the measurement context and revealed by the outcomes of the individual runs of the process.

Thus all the features of ideal measurements, including the measurement problem and the classical aspects of the outcomes, have been explained by the sole use of a minimalist and abstract formulation of quantum theory, applied to a large apparatus satisfying adequate dynamical properties. Interpretation came out naturally, but only in the end. Since alternative interpretations involve unnecessary assumptions of one kind or another, the usage of the statistical formulation in teaching is advocated.

[11] In other words, there exist subensembles $\mathcal{E}_i$ for which all but one of the coefficients $q_i$ vanish. This property, together with (11), was included in the definition of probabilities as frequencies [21].

[12] Measurements involve both a physical process of interaction between S and A and a selection of outcomes for repeated experiments. If we do not select the indications of A, knowledge about S is updated by replacing $\hat{r}(0)$ by $\sum_i p_i \hat{r}_i$. If the tested observable is not fully specified, the least biased subsequent predictions should rely on a state obtained by averaging over all possible interaction processes. If for instance, one is aware that an ensemble of spins initially prepared in the state $\hat{r}(0)$ have been measured in some direction, but if one knows neither in which direction nor the results obtained, one should assign to the final state the density operator $\frac{1}{3}[\hat{1} + \hat{r}(0)]$ as being the best (but imperfect) description. Indeed, a quantum state is not an intrinsic property of a physical system but it reflects our knowledge about the ensemble in which it is embedded.

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