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# A discrete duality finite volume discretization of the vorticity-velocity-pressure formulation of the 2D Stokes problem on almost arbitrary two-dimensional grids

Sarah Delcourte<sup>1</sup> and Pascal Omnes<sup>2,3</sup>

## Abstract

We present an application of the discrete duality finite volume method to the numerical approximation of the vorticity-velocity-pressure formulation of the 2D Stokes equations, associated to various non-standard boundary conditions. The finite volume method is based on the use of discrete differential operators obeying some discrete duality principles. The scheme may be seen as an extension of the classical MAC scheme to almost arbitrary meshes, thanks to an appropriate choice of degrees of freedom. The efficiency of the scheme is illustrated by numerical examples over unstructured triangular and locally refined non-conforming meshes, which confirm the theoretical convergence analysis led in the article.

Keywords: Stokes equations; boundary conditions; finite volumes; arbitrary meshes; a priori estimates;

## 1 Introduction

Let  $\Omega$  be a bounded, open, connected but not necessarily simply connected polygon of  $\mathbb{R}^2$ , whose boundary is denoted by  $\Gamma$ ; we consider the numerical approximation by means of finite volumes of the solution  $(\mathbf{u}, p)$  of the Stokes equations:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = g \text{ in } \Omega \tag{1.2}$$

supplemented with one of the following non-standard sets of conditions

$$\mathbf{u} \cdot \mathbf{n} = \sigma \text{ over } \Gamma \text{ , } \nabla \times \mathbf{u} = \omega_d \text{ over } \Gamma \text{ and } \int_{\Omega} p(\mathbf{x}) d\mathbf{x} = 0 \text{ ,} \tag{1.3}$$

$$\mathbf{u} \cdot \mathbf{n} = \sigma \text{ over } \Gamma \text{ , } p = p_d \text{ over } \Gamma \text{ and } \int_{\Omega} \nabla \times \mathbf{u}(\mathbf{x}) d\mathbf{x} = m_{\omega} \text{ ,} \tag{1.4}$$

$$\mathbf{u} \cdot \boldsymbol{\tau} = \sigma \text{ over } \Gamma \text{ , } \nabla \times \mathbf{u} = \omega_d \text{ over } \Gamma \text{ and } \int_{\Omega} p(\mathbf{x}) d\mathbf{x} = 0 \text{ ,} \tag{1.5}$$

$$\mathbf{u} \cdot \boldsymbol{\tau} = \sigma \text{ over } \Gamma \text{ , } p = p_d \text{ over } \Gamma \text{ and } \int_{\Omega} \nabla \times \mathbf{u}(\mathbf{x}) d\mathbf{x} = m_{\omega} \text{ ,} \tag{1.6}$$

where  $\mathbf{f}$ ,  $g$ ,  $\sigma$ ,  $p_d$  and  $\omega_d$  are given functions and  $m_{\omega}$  is a given real number. These conditions are written here in the case of simply connected domains for the sake of simplicity but they will

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<sup>1</sup>Université de Lyon, CNRS UMR 5208, Université Lyon 1, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne cedex, France. E-Mail: delcourte@math.univ-lyon1.fr

<sup>2</sup>CEA-Saclay, DEN, DM2S, STMF, LMSF, F-91191 Gif-sur-Yvette, France. E-Mail: pascal.omnes@cea.fr

<sup>3</sup>Université Paris 13, LAGA, CNRS UMR 7539, Institut Galilée, 99, Avenue J.-B. Clément F-93430 Villetaneuse Cedex, France.

be extended in the core of the article to non-simply connected domains. There are compatibility conditions between the data  $(g, \sigma)$  in (1.3) and (1.4), the data  $(\omega_d, \sigma)$  in (1.5) and the data  $(m_\omega, \sigma)$  in (1.6). They will be discussed in subsection 5.1.

As recognized for example by Dubois *et. al.* [25], these non-standard conditions can be treated in a very general and natural way thanks to the vorticity-velocity-pressure formulation of the Stokes problem (for earlier works based on different approaches, we refer to [6] and [29]). Since

$$-\Delta \mathbf{u} = \nabla \times \nabla \times \mathbf{u} - \nabla \nabla \cdot \mathbf{u}, \quad (1.7)$$

and using (1.2), we may rewrite Eq. (1.1) in the following way

$$\nabla \times \nabla \times \mathbf{u} + \nabla p = \mathbf{f} + \nabla g \quad \text{in } \Omega. \quad (1.8)$$

Further, introducing the vorticity  $\omega$ , Eq. (1.8) may be split as

$$\nabla \times \omega + \nabla p = \mathbf{f} + \nabla g \quad \text{in } \Omega, \quad (1.9)$$

$$\nabla \times \mathbf{u} = \omega \quad \text{in } \Omega. \quad (1.10)$$

The mathematical analysis of system (1.9)-(1.10)-(1.2) with various boundary conditions has been provided in several references, among which [1, 2, 5, 23, 25]. Finite element methods for the vorticity-velocity-pressure formulation have been derived and analyzed in [1, 2, 24]. Spectral methods have been considered in [5, 7] and [42], where a least-square formulation is used.

In the present work, we shall be interested in a finite volume generalization of the Marker and Cell (MAC) scheme on very general meshes (for other approaches with finite volumes, we refer to [9, 26, 27, 28]). The MAC scheme was developed initially in [31] on staggered rectangular grids and extended to the so-called covolume scheme using Delaunay-Voronoi mesh pairs, as reviewed in [40]. We note that the orthogonality property of these mesh pairs might be in certain cases a drawback, in particular in the context of adaptive mesh refinement. The standard MAC scheme discretizes (1.1)-(1.2), while the covolume scheme discretizes (1.9)-(1.10)-(1.2). It was proved in [40] that the MAC discretization may be obtained by the covolume scheme using well chosen triangular meshes. Given a (primal) mesh, the MAC and covolume schemes use as velocity unknowns the normal components of the velocity field with respect to the edges of the control volumes, while the pressure unknowns are located at their circumcenters. Then the normal component of Eqs. (1.1) or (1.9) is integrated on staggered control volumes centered on the edges. As far as the MAC scheme is concerned, a simple finite difference is used to evaluate the normal derivative of the velocity unknown, while in the covolume scheme, vorticity unknowns have to be evaluated at the vertices of the primal mesh. This is performed by integrating Eq. (1.10) on dual control volumes centered on the vertices and obtained by joining the circumcenters of the primal cells that share a common vertex. Due to the orthogonality property of the primal and dual meshes, the tangential components of the velocity with respect to the dual mesh, which are needed to discretize Eq. (1.10), are exactly the normal components on the edges of the primal control volumes. Finally, (1.2) is integrated on each primal control volume, and its discretization uses the normal components of the velocity on the edges of the primal mesh.

The generalization of the MAC scheme we propose is a new application of the ‘‘discrete duality finite volume’’ (DDFV) method [20]. Originally developed for linear diffusion equations [22, 33, 34], the DDFV method has been extended to nonlinear diffusion [4, 10, 15], convection-diffusion [16], electro-cardiology [3, 17], drift-diffusion and energy-transport models [13], electro-

and magnetostatics [21], electromagnetism [35], and Stokes flows [19, 36, 37, 38]. The advantage of this covolume-like method is that it allows the use of almost arbitrary meshes, including very distorted, degenerating, or highly non-conforming meshes (see the numerical tests in [22]). The name of the method comes from the definition of discrete gradient and divergence operators which verify a discrete Green formula, as will be recalled in the core of the article. Note that the works [36, 37, 38] are dedicated to the DDFV discretization of Stokes flows with standard Dirichlet boundary conditions and with velocity unknowns located at the centers and the nodes of the mesh, while pressure is discretized at the faces, which is different from the approach we follow in the present work.

In order to get rid of the orthogonality constraints on the mesh, the price to pay is to discretize *both* velocity components on the edges of the control volumes, while pressure and vorticity unknowns are associated with the centroids of the primal cells *and* to their vertices, i.e. to the dual cells. Then, we integrate *both* components of (1.9) on the cells associated with the edges (the so-called “diamond mesh”) and (1.10) and (1.2) on *both* the primal and dual cells. This process enables us to derive discrete versions of the differential operators divergence, gradient and curl which appear in (1.9), (1.10) and (1.2). These discrete operators are known to satisfy properties which are analogous to properties verified by the continuous operators [21]. With the help of these properties, we show that the solution of the DDFV discretization applied to the Stokes equations with any of the sets of conditions (1.3) to (1.6) can be reduced to the solution of four discrete Laplace equations involving the pressure, the vorticity and the potentials stemming from the discrete Hodge decomposition of the velocity.

When  $g = 0$  in Eq. (1.2), another advantage of this scheme is that it satisfies the notion of “reinforced incompressibility” introduced in a finite volume element context in [32] and in a finite element context in [8] to overcome spurious (non-perfectly divergence free) velocity modes that may appear in unsteady Navier-Stokes simulations performed with Crouzeix-Raviart [18] finite elements of lowest order. Indeed, since these elements involve pressure unknowns located at the triangle barycenters only, the incompressibility constraint is satisfied only around these barycenters and the resulting velocity field may be non-divergence free in the sense that the discrete divergence, when computed *around the vertices of the mesh*, may not vanish or even be small. A possible cure to this problem, proposed in [8] and [32], is to add pressure unknowns at the vertices of the mesh, which introduces incompressibility constraints around these nodes. Thus, if one restricts the discussion to primal triangular meshes, the scheme presented here has exactly the same unknowns as (but is however not equivalent to) those involved in [8] and [32], and incompressibility conditions are written on each triangle (primal cell) and around each vertex (dual cell) of the mesh. The advantage of the scheme we present here is that it handles meshes that are much more general than triangular.

The convergence analysis shows that pressure, vorticity, their gradients and the velocity field are first order accurate on general meshes. Moreover, for families of meshes for which the diamond cells are (almost all) parallelograms, we prove that the pressure and vorticity gradients satisfy the superconvergence order 1.5.

These theoretical results are validated by the numerical tests included in this paper. The results of these tests even go beyond the theoretical findings; indeed, we observe that pressure and vorticity converge on general meshes with the order 2 (although we are able to prove order 1 only), while velocity converges with the order 2 for families of meshes for which the diamond cells are (almost all) parallelograms.

The paper is organized as follows: in Section 2, we introduce the notations associated with the primal, dual and diamond meshes. Then, in Section 3 we define discrete differential operators: the discrete gradient (respectively vector curl) operator is defined on the diamond cells and the corresponding adjoint discrete divergence (resp. scalar curl) operator is defined over the primal and dual cells. In Section 4, we state discrete properties of the discrete differential operators. In Section 5, we write down the finite volume schemes for the steady Stokes problems with the various conditions given by (1.3)–(1.6) extended to non-simply connected domains. Section 6 is devoted to the convergence analysis of the finite volume scheme with the boundary condition (1.3) only. Finally, we present some numerical results of convergence over unstructured and non-conforming meshes.

## 2 Definitions and notations

Let  $\Omega$  be a bounded connected polygon of  $\mathbb{R}^2$ , whose boundary is denoted by  $\Gamma$ . We suppose in addition that the domain has  $Q$  holes. Throughout the paper, we shall assume that  $Q > 0$ , but the results also hold for the case  $Q = 0$ . Let  $\Gamma_0$  denote the exterior boundary of  $\Omega$  and let  $\Gamma_q$ , with  $q \in [1, Q]$ , be the interior polygonal boundaries of  $\Omega$ , so that  $\Gamma = \Gamma_0 \bigcup_{q \in [1, Q]} \Gamma_q$ .

### 2.1 Construction of the primal mesh

We consider a first partition of  $\Omega$  (named primal mesh) composed of elements  $T_i$ , with  $i \in [1, I]$ , supposed to be convex polygons. With each element  $T_i$  of the mesh, we associate a node  $G_i$  located at the barycentre of  $T_i$ . The area of  $T_i$  is denoted by  $|T_i|$ . We shall denote by  $J$  the total number of edges of this mesh and by  $J^\Gamma$  the number of these edges which are located on the boundary  $\Gamma$  and we associate with each of these boundary edges its midpoint, also denoted by  $G_i$  with  $i \in [I + 1, I + J^\Gamma]$ . By a slight abuse of notations, we shall write  $i \in \Gamma_q$  if and only if  $G_i \in \Gamma_q$ .

### 2.2 Construction of the dual mesh

Further, we denote by  $S_k$ , with  $k \in [1, K]$ , the nodes of the polygons of the primal mesh. With each of these points, we associate a polygon denoted by  $P_k$ , obtained by joining the points  $G_i$  associated to the elements of the primal mesh (and possibly to the boundary edges) of which  $S_k$  is a node to the midpoints of the edges of which  $S_k$  is a node. The area of  $P_k$  is denoted by  $|P_k|$ . The family of cells  $(P_k)_{k \in [1, K]}$  constitutes a second partition of  $\Omega$ , which we name dual mesh. Figure 1 displays an example of a non-conforming primal mesh and its associated dual mesh. Moreover, we suppose that the set  $[1, K]$  is ordered so that when  $S_k$  is not on  $\Gamma$ , then  $k \in [1, K - J^\Gamma]$ , and when  $S_k$  is on  $\Gamma$ , then  $k \in [K - J^\Gamma + 1, K]$ . We shall also write  $k \in \Gamma_q$  if and only if  $S_k \in \Gamma_q$ .

### 2.3 Construction of the diamond mesh

With each edge of the primal mesh, denoted by  $A_j$  (whose length is  $|A_j|$ ), with  $j \in [1, J]$ , we associate a quadrilateral named “diamond cell” and denoted by  $D_j$ . When  $A_j$  is not on the

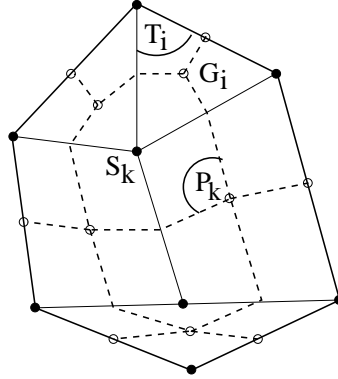


Figure 1: An example of a primal mesh and its associated dual mesh.

boundary, this cell is obtained by joining the points  $S_{k_1(j)}$  and  $S_{k_2(j)}$ , which are the two nodes of  $A_j$ , with the points  $G_{i_1(j)}$  and  $G_{i_2(j)}$  associated with the elements of the primal mesh which share this edge. When  $A_j$  is on the boundary  $\Gamma$ , the cell  $D_j$  is obtained by joining the two nodes of  $A_j$  with the point  $G_{i_1(j)}$  associated with the only element of the primal mesh of which  $A_j$  is an edge and to the point  $G_{i_2(j)}$  associated with  $A_j$  (*i.e.* by convention  $i_2(j)$  is element of  $[I + 1, I + J^\Gamma]$  when  $A_j$  is located on  $\Gamma$ ). The cells  $D_j$  constitute a third partition of  $\Omega$ , which we name “diamond-mesh”. The area of the cell  $D_j$  is denoted by  $|D_j|$ . Such cells are displayed on Fig. 2. Moreover, we suppose that the set  $[1, J]$  is ordered so that when  $A_j$  is not on  $\Gamma$ , then  $j \in [1, J - J^\Gamma]$ , and when  $A_j$  is on  $\Gamma$ , then  $j \in [J - J^\Gamma + 1, J]$ . We shall also write  $j \in \Gamma_q$  if and only if  $A_j \subset \Gamma_q$ .

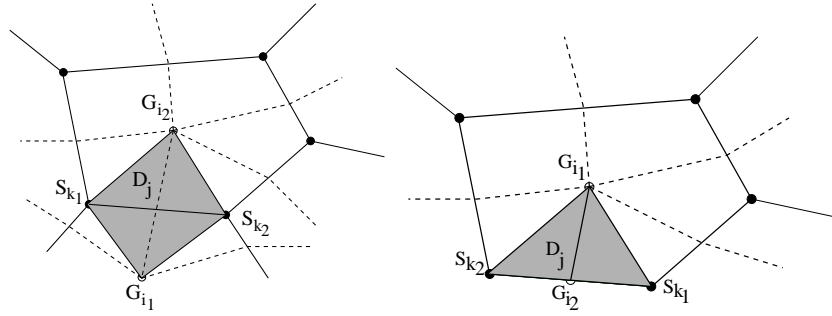


Figure 2: Examples of diamond cells.

## 2.4 Definitions of geometrical elements

The following geometrical elements are displayed on Fig. 3. The unit vector normal to  $A_j$  is denoted by  $\mathbf{n}_j$  and is oriented so that its dot product with  $\mathbf{G}_{i_1(j)}\mathbf{G}_{i_2(j)}$  is positive. We further denote by  $A'_j$  the segment  $[G_{i_1(j)}G_{i_2(j)}]$  (whose length is  $|A'_j|$ ) and by  $\mathbf{n}'_j$  the unit vector normal to  $A'_j$  oriented so that  $\mathbf{S}_{k_1(j)}\mathbf{S}_{k_2(j)} \cdot \mathbf{n}'_j \geq 0$ . We denote by  $M_j$  the midpoint of  $A_j$  and by  $A'_{j_1}$  (respectively  $A'_{j_2}$ ) the segment  $[G_{i_1(j)}M_j]$  (resp.  $[M_jG_{i_2(j)}]$ ) and by  $\mathbf{n}'_{j_1}$  (resp.  $\mathbf{n}'_{j_2}$ ) the unit normal vector to  $A'_{j_1}$  (resp.  $A'_{j_2}$ ) oriented so that

$$|A'_j|\mathbf{n}'_j = |A'_{j_1}|\mathbf{n}'_{j_1} + |A'_{j_2}|\mathbf{n}'_{j_2}. \quad (2.1)$$

When  $S_k \in \Gamma$  ( $k \in [K - J^\Gamma + 1, K]$ ), we define  $\tilde{A}_k$  as the part of the boundary  $\Gamma$  which consists

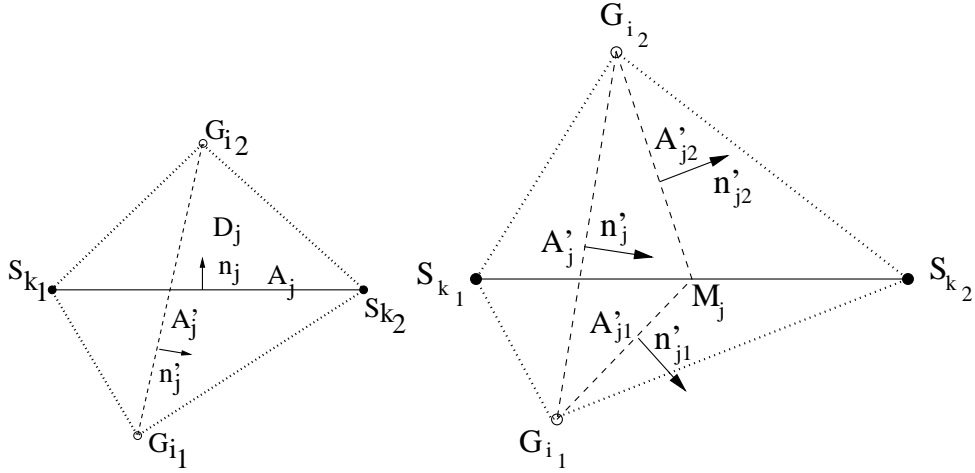


Figure 3: Notations for the diamond cell.

of the union of the halves of the two segments  $A_j$  located on  $\Gamma$  and of which  $S_k$  is a node, and by  $\tilde{\mathbf{n}}_k$  the exterior unit normal vector to  $\tilde{A}_k$  (see figure 4). We define for each  $i \in [1, I]$  the set  $V(i)$  of integers  $j \in [1, J]$  such that  $A_j$  is an edge of  $T_i$  and for each  $k \in [1, K]$  the set  $E(k)$  of integers  $j \in [1, J]$  such that  $S_k$  is a node of  $A_j$ .

We define for each  $j \in [1, J]$  and each  $k$  such that  $j \in E(k)$  (resp. each  $i$  such that  $j \in V(i)$ ) the real-valued number  $s'_{jk}$  (resp.  $s_{ji}$ ) whose value is  $+1$  or  $-1$  whether  $\mathbf{n}'_j$  (resp.  $\mathbf{n}_j$ ) points outward or inward  $P_k$  (resp.  $T_i$ ). We define  $\mathbf{n}'_{jk} := s'_{jk}\mathbf{n}'_j$  (resp.  $\mathbf{n}_{ji} := s_{ji}\mathbf{n}_j$ ) and remark that  $\mathbf{n}'_{jk}$  (resp.  $\mathbf{n}_{ji}$ ) always points outward  $P_k$  (resp.  $T_i$ ). In the same way, we set  $\mathbf{n}'_{jk_1} := s'_{jk}\mathbf{n}'_{j_1}$  and  $\mathbf{n}'_{jk_2} := s'_{jk}\mathbf{n}'_{j_2}$ .

For  $j \in [1, J - J^\Gamma]$ , as indicated on Figure 5, we also denote by  $D_{j,1}$  and  $D_{j,2}$ , the triangles  $S_{k_1(j)}G_{i_1(j)}S_{k_2(j)}$  and  $S_{k_2(j)}G_{i_2(j)}S_{k_1(j)}$ . In the same way, we denote by  $D'_{j,1}$  and  $D'_{j,2}$ , the triangles  $G_{i_2(j)}S_{k_1(j)}G_{i_1(j)}$  and  $G_{i_1(j)}S_{k_2(j)}G_{i_2(j)}$ .

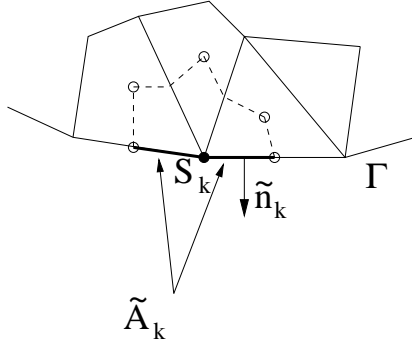


Figure 4: Definition of  $\tilde{A}_k$  and  $\tilde{\mathbf{n}}_k$  for the boundary nodes

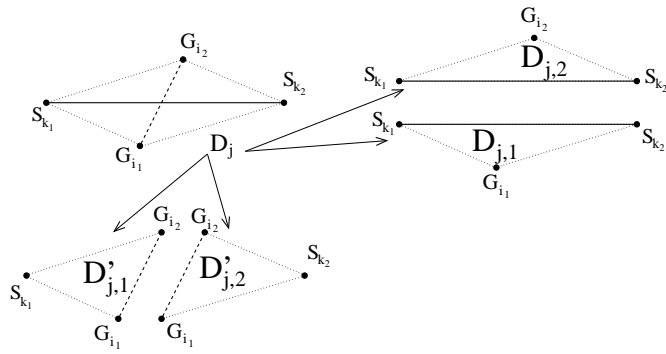


Figure 5: A diamond-cell may be split into two triangles in two distinct ways



## 2.5 A hypothesis on the mesh regularity

In the sequel of this article, we shall obtain error estimates under the following hypothesis (see Figure 6 for the notations).

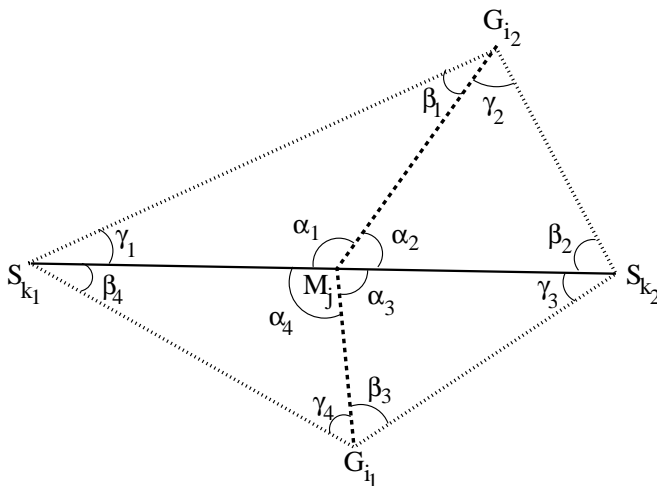


Figure 6: Notations for Hypothesis 2.1.

**Hypothesis 2.1.** The angles of the subtriangulation  $G_i M_j S_k$  of the diamond-cells  $D_j$  are greater than an angle  $\theta^*$  which is strictly positive and independent of the mesh:

$$\exists \theta^*, \quad 0 < \theta^* < \frac{\pi}{2} \quad \text{s.t.} \quad \min\{\alpha_m, \beta_m, \gamma_m\} \geq \theta^*, \quad m = 1, 2, 3, 4.$$

We can easily verify that Hypothesis 2.1 implies various weaker hypotheses: [22, Hypotheses 5.5 and 5.6], [21, Hypothesis 5.5] and [39, Definition 2.1] under which results obtained in these articles, and used in the present work, are valid.

## 3 The discrete operators

We may approach the gradient operator  $\nabla \bullet = \left( \frac{\partial \bullet}{\partial x}, \frac{\partial \bullet}{\partial y} \right)^T$  by a discrete gradient operator on the diamond cells  $D_j$  (see [22]).

**Definition 3.1.** Given any  $\phi = (\phi_i^T, \phi_k^P) \in \mathbb{R}^{I+J^T} \times \mathbb{R}^K$ , the discrete gradient  $\nabla_h^D$  is defined by its values over the diamond cells  $D_j$  (see Fig. 3):

$$(\nabla_h^D \phi)_j := \frac{1}{2|D_j|} \left\{ [\phi_{k_2}^P - \phi_{k_1}^P] |A'_j| \mathbf{n}'_j + [\phi_{i_2}^T - \phi_{i_1}^T] |A_j| \mathbf{n}_j \right\}. \quad (3.1)$$

Note that formula (3.1) is exact for any affine function  $\phi$  if we set  $\phi_k^P := \phi(S_k)$  and  $\phi_i^T := \phi(G_i)$ , for any  $(i, k)$ .

In the very same way, we may approach the vector curl operator  $\nabla \times \bullet = \left( \frac{\partial \bullet}{\partial y}, -\frac{\partial \bullet}{\partial x} \right)^T$  by a discrete vector curl operator:

**Definition 3.2.** Given any  $\phi = (\phi_i^T, \phi_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ , the discrete vector curl operator  $\nabla_h^D \times$  is defined by its values over the diamond cells  $D_j$ :

$$(\nabla_h^D \times \phi)_j := -\frac{1}{2|D_j|} \left\{ [\phi_{k_2}^P - \phi_{k_1}^P] |A'_j| \boldsymbol{\tau}'_j + [\phi_{i_2}^T - \phi_{i_1}^T] |A_j| \boldsymbol{\tau}_j \right\}, \quad (3.2)$$

where the unit vectors  $\boldsymbol{\tau}_j$  and  $\boldsymbol{\tau}'_j$  are such that  $(\mathbf{n}_j, \boldsymbol{\tau}_j)$  and  $(\mathbf{n}'_j, \boldsymbol{\tau}'_j)$  are orthogonal positively oriented bases of  $\mathbb{R}^2$ .

Next, we define the discrete divergence of a vector field  $\mathbf{u}$  by its values both on the primal and dual cells of the mesh. Supposing that the vector field  $\mathbf{u}$  is given by its discrete values  $\mathbf{u}_j$  on the cells  $D_j$ , we state the definition of the discrete divergence  $\nabla_h^T \cdot$  on each  $T_i$  and the discrete divergence  $\nabla_h^P \cdot$  on each  $P_k$ .

**Definition 3.3.** Given any  $\mathbf{u} = (\mathbf{u}_j) \in (\mathbb{R}^2)^J$ , the discrete divergence  $\nabla_h^{T,P} \cdot := (\nabla_h^T \cdot, \nabla_h^P \cdot)$  is defined by its values over the primal cells  $T_i$  and the dual cells  $P_k$  (see Fig. 7)

$$\begin{aligned} (\nabla_h^T \cdot \mathbf{u})_i &:= \frac{1}{|T_i|} \sum_{j \in V(i)} |A_j| \mathbf{u}_j \cdot \mathbf{n}_{ji}, \\ (\nabla_h^P \cdot \mathbf{u})_k &:= \frac{1}{|P_k|} \left( \sum_{j \in E(k)} (|A'_{j1}| \mathbf{n}'_{jk1} + |A'_{j2}| \mathbf{n}'_{jk2}) \cdot \mathbf{u}_j \right. \\ &\quad \left. + \sum_{j \in E(k) \cap [J - J^\Gamma + 1, J]} \frac{1}{2} |A_j| \mathbf{u}_j \cdot \mathbf{n}_j \right), \end{aligned} \quad (3.3)$$

where we recall that  $V(i)$  (resp.  $E(k)$ ) is the set of integers  $j \in [1, J]$  such that  $A_j$  is an edge of  $T_i$  (resp.  $S_k$  is a node of  $A_j$ ) and that  $\mathbf{n}_{ji}$  (resp.  $\mathbf{n}'_{jk1}$  and  $\mathbf{n}'_{jk2}$ ) is the unit vector orthogonal to  $A_j$  (resp.  $A'_{j1}$  and  $A'_{j2}$ ) pointing outward  $T_i$  (resp.  $P_k$ ).

Remark that if the node  $S_k$  is not on the boundary  $\Gamma$  (i.e. if  $k \in [1, K - J^\Gamma]$ ), then the set  $E(k) \cap [J - J^\Gamma + 1, J]$  is empty. On the contrary, if  $P_k$  is a boundary dual cell, then the set  $E(k) \cap [J - J^\Gamma + 1, J]$  is composed of the two boundary edges which have  $S_k$  as a vertex. In this case, the quantity  $\sum_{j \in E(k) \cap [J - J^\Gamma + 1, J]} \frac{1}{2} |A_j| \mathbf{u}_j \cdot \mathbf{n}_j$  is an approximation of  $\int_{\tilde{A}_k} \mathbf{u} \cdot \tilde{\mathbf{n}}_k(\xi) d\xi$  (see

figure 4). Note also that we could have replaced  $(|A'_{j1}| \mathbf{n}'_{jk1} + |A'_{j2}| \mathbf{n}'_{jk2})$  by  $|A'_j| \mathbf{n}'_{jk}$  since these two quantities are equal.

For a given vector field  $\mathbf{u}$ , it is straightforward to check that formulae (3.3) are the exact mean-values of  $\nabla \cdot \mathbf{u}$  over  $T_i$ , respectively over an inner  $P_k$ , if

$$|A_j| \mathbf{u}_j \cdot \mathbf{n}_{ji} = \int_{A_j} \mathbf{u} \cdot \mathbf{n}_{ji} ds,$$

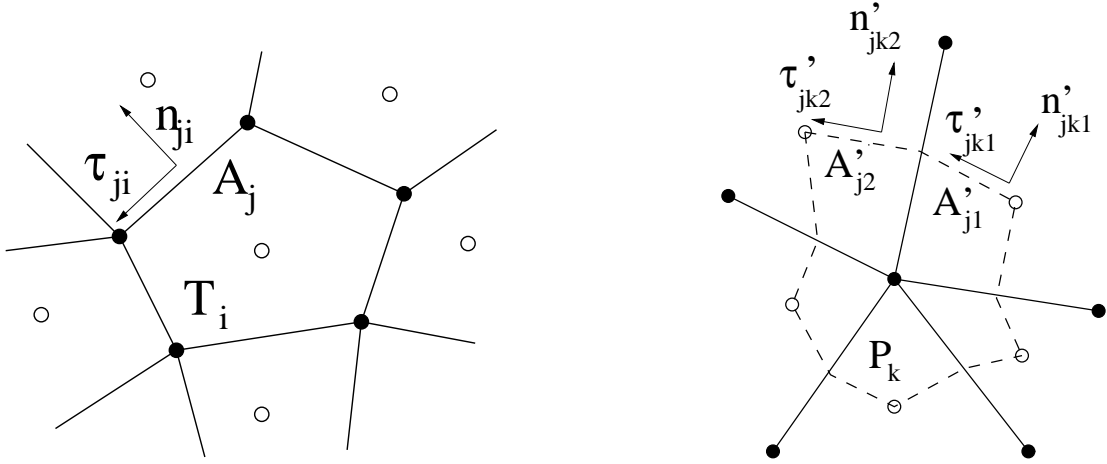


Figure 7: Edges and unit vectors for the discrete divergence and curl

resp. if

$$(|A'_{j1}| \mathbf{n}'_{jk1} + |A'_{j2}| \mathbf{n}'_{jk2}) \cdot \mathbf{u}_j = \int_{A'_{j1}} \mathbf{u} \cdot \mathbf{n}'_{jk1} ds + \int_{A'_{j2}} \mathbf{u} \cdot \mathbf{n}'_{jk2} ds.$$

In the very same way, we may approach the scalar curl operator  $\nabla \times \bullet = \left( \frac{\partial \bullet_y}{\partial x} - \frac{\partial \bullet_x}{\partial y} \right)$  by a discrete scalar curl operator:

**Definition 3.4.** Given any  $\mathbf{u} = (\mathbf{u}_j) \in (\mathbb{R}^2)^J$ , the discrete scalar curl operator  $\nabla_h^{T,P} \times := (\nabla_h^T \times, \nabla_h^P \times)$  is defined by its values over the primal cells  $T_i$  and the dual cells  $P_k$ :

$$\begin{aligned} (\nabla_h^T \times \mathbf{u})_i &:= \frac{1}{|T_i|} \sum_{j \in V(i)} |A_j| \mathbf{u}_j \cdot \boldsymbol{\tau}_{ji}, \\ (\nabla_h^P \times \mathbf{u})_k &:= \frac{1}{|P_k|} \left( \sum_{j \in E(k)} (|A'_{j1}| \boldsymbol{\tau}'_{jk1} + |A'_{j2}| \boldsymbol{\tau}'_{jk2}) \cdot \mathbf{u}_j \right. \\ &\quad \left. + \sum_{j \in E(k) \cap [J-J^\Gamma+1, J]} \frac{1}{2} |A_j| \mathbf{u}_j \cdot \boldsymbol{\tau}_j \right). \end{aligned} \quad (3.4)$$

## 4 Properties of the operators

In this section, we state properties of the discrete operators which are analogues to properties of the continuous operators in two dimensions. Their proofs are skipped, since they may be found in [21] and [22].

**Definition 4.1.** (*The discrete scalar products*) Let  $(\phi, \psi) \in (\mathbb{R}^I \times \mathbb{R}^K)^2$  and  $(\mathbf{u}, \mathbf{v}) \in ((\mathbb{R}^2)^J)^2$ , then we define the following scalar products:

$$(\mathbf{u}, \mathbf{v})_D := \sum_{j \in [1, J]} |D_j| \mathbf{u}_j \cdot \mathbf{v}_j, \quad (4.1)$$

$$(\phi, \psi)_{T,P} := \frac{1}{2} \left( \sum_{i \in [1, I]} |T_i| \phi_i^T \psi_i^T + \sum_{k \in [1, K]} |P_k| \phi_k^P \psi_k^P \right). \quad (4.2)$$

The associated norms will be denoted by  $\|\cdot\|_D$  and  $\|\cdot\|_{T,P}$ .

We also define discrete scalar products on the boundaries  $\Gamma_q$ , for sets of values defined on the boundary edges. For any  $(u_j, v_j)_{j \in \Gamma_q}$ , we set

$$(u, v)_{\Gamma_q, h} := \sum_{j \in \Gamma_q} |A_j| u_j v_j, \quad (4.3)$$

and for any  $(u_j, v_j)_{j \in \Gamma}$ , we set

$$(u, v)_{\Gamma, h} := \sum_{q \in [0, Q]} (u, v)_{\Gamma_q, h}. \quad (4.4)$$

In particular, we shall often consider these boundary scalar product for the trace  $\tilde{\phi}$  of a given  $\phi \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ , defined by

**Definition 4.2.** (*Trace operator*) For any  $\phi \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ , and for any  $j \in \Gamma$ , we set

$$\tilde{\phi}_j := \frac{1}{4} \left( \phi_{k_1(j)}^P + 2\phi_{i_2(j)}^T + \phi_{k_2(j)}^P \right). \quad (4.5)$$

The scalar products are built such that the discrete operators, defined in section 3, verify some discrete duality principles, expressed in the following proposition.

**Proposition 4.1.** (*The discrete Green formulae*) The following discrete analogues of the Green formulae hold:

$$(\nabla_h^{T,P} \cdot \mathbf{u}, \phi)_{T,P} = -(\mathbf{u}, \nabla_h^D \phi)_D + (\mathbf{u} \cdot \mathbf{n}, \tilde{\phi})_{\Gamma, h}, \quad (4.6)$$

$$(\nabla_h^{T,P} \times \mathbf{u}, \phi)_{T,P} = (\mathbf{u}, \nabla_h^D \times \phi)_D + (\mathbf{u} \cdot \boldsymbol{\tau}, \tilde{\phi})_{\Gamma, h}, \quad (4.7)$$

for all  $\mathbf{u} \in (\mathbb{R}^J)^2$  and all  $\phi = (\phi^T, \phi^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ .

As far as the continuous operators are concerned, there holds  $\nabla \cdot (\nabla \times) = 0$ ,  $\nabla \times \nabla = 0$  and  $\nabla \times (\nabla \times) = -\nabla \cdot \nabla$  (in two dimensions). The following two propositions state analogous properties verified by the discrete operators (note however the hypothesis needed on the boundary dual cells):

**Proposition 4.2.** For all  $\phi = (\phi^T, \phi^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ , the following equalities hold:

$$\left( \nabla_h^{T,P} \cdot (\nabla_h^D \times \phi) \right)_{i,k} = 0, \quad \forall i \in [1, I], \forall k \in [1, K - J^\Gamma], \quad (4.8)$$

$$\left( \nabla_h^{T,P} \times (\nabla_h^D \phi) \right)_{i,k} = 0, \quad \forall i \in [1, I], \forall k \in [1, K - J^\Gamma]. \quad (4.9)$$

Moreover, on the boundary dual cells  $P_k$  ( $k \notin \Gamma$ ), these formulae still hold if for each boundary  $\Gamma_q$ , with  $q \in [0, Q]$ , there exist two real numbers  $c_q^T$  and  $c_q^P$  such that  $\phi_i^T = c_q^T$  and  $\phi_k^P = c_q^P$  uniformly over  $\Gamma_q$ .

**Proposition 4.3.** For all  $\phi = (\phi^T, \phi^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ , the following equality holds:

$$(\nabla_h^{T,P} \times \nabla_h^D \times \phi)_{i,k} = -(\nabla_h^{T,P} \cdot \nabla_h^D \phi)_{i,k}, \quad \forall i \in [1, I], \forall k \in [1, K]. \quad (4.10)$$

Finally, in the continuous case, the Hodge decomposition for non simply connected domains reads: Let  $V = \{\phi \in H^1(\Omega) : \int_\Omega \phi = 0\}$  and  $W = \{\psi \in H^1(\Omega) : \psi|_{\Gamma_0} = 0, \psi|_{\Gamma_q} = c_q \in \mathbb{R}, \forall q \in [1, Q]\}$ , then

$$(L^2)^2 = \nabla V \oplus \nabla \times W.$$

As far as the discrete operators are concerned, an analogous property holds:

**Proposition 4.4.** Let  $(\mathbf{u}_j)_{j \in [1, J]}$  be a discrete vector field defined by its values on the diamond cells  $D_j$ . Then, there exist unique  $\phi = (\phi_i^T, \phi_k^P)$  and  $\psi = (\psi_i^T, \psi_k^P)$  both in  $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$  and  $(c_q^T, c_q^P)$  in  $\mathbb{R}^Q \times \mathbb{R}^Q$  such that:

$$\mathbf{u}_j = (\nabla_h^D \phi)_j + (\nabla_h^D \times \psi)_j, \quad \forall j \in [1, J], \quad (4.11)$$

$$\text{with } \sum_{i \in [1, I]} |T_i| \phi_i^T = \sum_{k \in [1, K]} |P_k| \phi_k^P = 0, \quad (4.12)$$

$$\psi_i^T = 0, \quad \forall i \in \Gamma_0, \quad \psi_k^P = 0, \quad \forall k \in \Gamma_0, \quad (4.13)$$

$$\forall q \in [1, Q], \quad \psi_i^T = c_q^T, \quad \forall i \in \Gamma_q, \quad \psi_k^P = c_q^P, \quad \forall k \in \Gamma_q. \quad (4.14)$$

Moreover, the decomposition (4.11) is orthogonal, in the sense that

$$(\nabla_h^D \phi, \nabla_h^D \times \psi)_D = 0, \quad (4.15)$$

a property that is true for all  $\phi$  and all  $\psi$  verifying (4.13) and (4.14).

Formulae (4.12) are discrete analogues (respectively stated on the primal mesh and on the dual mesh) of the condition  $\int_\Omega \phi = 0$  that appears in the definition of the space  $V$ , while formulae (4.13) and (4.14) are discrete analogues of the boundary conditions that appear in the definition of  $W$ .

**Remark 4.1.** We may also write a similar decomposition by changing the conditions (4.12)–(4.14) in the following way

$$\sum_{i \in [1, I]} |T_i| \psi_i^T = \sum_{k \in [1, K]} |P_k| \psi_k^P = 0, \quad (4.16)$$

$$\phi_i^T = 0, \quad \forall i \in \Gamma_0, \quad \phi_k^P = 0, \quad \forall k \in \Gamma_0, \quad (4.17)$$

$$\forall q \in [1, Q], \quad \phi_i^T = c_q^T, \quad \forall i \in \Gamma_q, \quad \phi_k^P = c_q^P, \quad \forall k \in \Gamma_q. \quad (4.18)$$

Finally, we mention discrete analogues of the so-called Poincaré inequalities, which were proved in [39] (respectively Theorem 3.3 and Theorem 3.9):

**Proposition 4.5.** *Let  $\phi = (\phi_i^T, \phi_k^P)$  and  $\psi = (\psi_i^T, \psi_k^P)$  both in  $\mathbb{R}^{I+J^T} \times \mathbb{R}^K$  and  $(c_q^T, c_q^P)$  in  $\mathbb{R}^Q \times \mathbb{R}^Q$  be such that (4.12), (4.13) and (4.14) hold; then, there exists a constant  $C(\theta^*)$ , depending only on  $\theta^*$  and on  $\Omega$  such that*

$$\|\phi\|_{T,P} \leq C(\theta^*) \|\nabla_h^D \phi\|_D, \quad \|\psi\|_{T,P} \leq C(\theta^*) \|\nabla_h^D \psi\|_D. \quad (4.19)$$

## 5 Application to the Stokes equations

In this section, we are interested in the discretization of Eqs. (1.9), (1.10) and (1.2) supplemented with one of the following non-standard sets of conditions, which generalize conditions (1.3) to (1.6) to non-simply connected domains:

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} = \sigma \text{ over } \Gamma, & \omega|_{\Gamma_0} = \omega_d; \quad \omega|_{\Gamma_q} = \omega_d + c_q, \quad \forall q \in [1, Q], \\ \int_{\Gamma_q} \mathbf{u} \cdot \boldsymbol{\tau} = k_q, \quad \forall q \in [1, Q], & \int_{\Omega} p(\mathbf{x}) d\mathbf{x} = 0, \end{cases} \quad (5.1)$$

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} = \sigma \text{ over } \Gamma, & p|_{\Gamma_0} = p_d; \quad p|_{\Gamma_q} = p_d + c_q, \quad \forall q \in [1, Q], \\ \int_{\Gamma_q} \mathbf{u} \cdot \boldsymbol{\tau} = k_q, \quad \forall q \in [1, Q], & \int_{\Omega} \omega(\mathbf{x}) d\mathbf{x} = m_\omega, \end{cases} \quad (5.2)$$

$$\begin{cases} \mathbf{u} \cdot \boldsymbol{\tau} = \sigma \text{ over } \Gamma, & \omega|_{\Gamma_0} = \omega_d; \quad \omega|_{\Gamma_q} = \omega_d + c_q, \quad \forall q \in [1, Q], \\ \int_{\Gamma_q} \mathbf{u} \cdot \mathbf{n} = k_q, \quad \forall q \in [1, Q], & \int_{\Omega} p(\mathbf{x}) d\mathbf{x} = 0, \end{cases} \quad (5.3)$$

$$\begin{cases} \mathbf{u} \cdot \boldsymbol{\tau} = \sigma \text{ over } \Gamma, & p|_{\Gamma_0} = p_d; \quad p|_{\Gamma_q} = p_d + c_q, \quad \forall q \in [1, Q], \\ \int_{\Gamma_q} \mathbf{u} \cdot \mathbf{n} = k_q, \quad \forall q \in [1, Q], & \int_{\Omega} \omega(\mathbf{x}) d\mathbf{x} = m_\omega, \end{cases} \quad (5.4)$$

where  $\sigma$ ,  $p_d$  and  $\omega_d$  are given functions,  $m_\omega$  is a given real number,  $(k_q)_{q \in [1, Q]}$  is a set of given real numbers and the constants  $(c_q) \in \mathbb{R}^Q$  have to be determined.

Before going into details of the discretization of these equations, we discuss the compatibility conditions of the right-hand sides of these sets of equations, as was announced in the introduction of this article.

### 5.1 Compatibility conditions

Consider Eqs. (1.2), (1.9) and (1.10).

First, when associated with one of the sets of conditions (5.1) or (5.2), the data has to verify

$$\int_{\Omega} g(\mathbf{x}) d\mathbf{x} = \int_{\Gamma} \sigma(\xi) d\xi. \quad (5.5)$$

Then, when associated with the set (5.4), integration of (1.10) over  $\Omega$  and application of the Green formula and of the boundary conditions yield

$$\int_{\Gamma} \sigma(\xi) d\xi = m_{\omega}. \quad (5.6)$$

At last, in the case of Eqs. (1.2), (1.9) and (1.10) associated with the set (5.3), the situation is more involved. Indeed, we infer from (1.10) that  $\int_{\Gamma} \sigma(\xi) d\xi = \int_{\Omega} \omega(\mathbf{x}) d\mathbf{x}$ . However, the quantity  $\int_{\Omega} \omega(\mathbf{x}) d\mathbf{x}$  is not a data given by (5.3), but is a result of the computation of  $\omega$  through the Hodge decomposition of  $\mathbf{f} + \nabla g$  expressed by Eq. (1.9) associated with the conditions over  $p$  and  $\omega$  expressed in the set (5.3). This may be interpreted as an implicit compatibility condition between the boundary conditions  $\omega_d$  and  $\sigma$  in (5.3). Further details on how to handle this will be given in subsection 5.2.

## 5.2 Discretization of the Stokes equations in vorticity-velocity-pressure formulation.

In this subsection, we are interested in the approximation of the continuous problem given by (1.2)-(1.9)-(1.10) associated with one of the sets of conditions (5.1) to (5.4). We choose to approach the solution of this problem by a vector  $(\mathbf{u}_j)$ , with  $j \in [1, J]$ , which discretizes the velocity field by values defined over the diamond cells of the mesh, and by scalars  $(\omega_i^T, \omega_k^P)$  and  $(p_i^T, p_k^P)$ , with  $i \in [1, I + J^{\Gamma}]$ ,  $k \in [1, K]$ , which discretize the vorticity and the pressure fields by values defined over the primal and dual cells of the mesh. The problem will be solved in two steps. In the first, we use the Hodge decomposition of  $\mathbf{f} + \nabla g$  (see prop. 4.4) to solve for  $p$  and  $\omega$ . In the second, we solve a div-curl problem for  $\mathbf{u}$ .

**Step 1:** The discrete Hodge decomposition of the data  $\mathbf{f} + \nabla g$  reads:  
find  $p = (p_i^T, p_k^P)_{i \in [1, I + J^{\Gamma}], k \in [1, K]}$ ,  $\omega = (\omega_i^T, \omega_k^P)_{i \in [1, I + J^{\Gamma}], k \in [1, K]}$  and  $(c_q^T, c_q^P)_{q \in [1, Q]}$  such that

$$(\nabla_h^D p)_j + (\nabla_h^D \times \omega)_j = \mathbf{f}_j^D + (\nabla g)_j^D, \quad \forall j \in [1, J], \quad (5.7)$$

together with one of the following sets of conditions

$$\left\{ \begin{array}{l} \omega_i^T = \omega_d(G_i) \quad \forall i \in \Gamma_0, \quad \omega_i^T = \omega_d(G_i) + c_q^T \quad \forall i \in \Gamma_q, \quad \forall q, \\ \omega_k^P = \omega_d(S_k) \quad \forall k \in \Gamma_0, \quad \omega_k^P = \omega_d(S_k) + c_q^P \quad \forall k \in \Gamma_q, \quad \forall q, \\ \sum_{i \in [1, I]} |T_i| p_i^T = \sum_{k \in [1, K]} |P_k| p_k^P = 0, \end{array} \right. \quad (5.8)$$

in the case of a given (up to constants to be determined on each internal boundary) vorticity field  $\omega_d$  on the boundary (see the corresponding equations in the sets (5.1) and (5.3)), or

$$\left\{ \begin{array}{l} p_i^T = p_d(G_i) \quad \forall i \in \Gamma_0, \quad p_i^T = p_d(G_i) + c_q^T \quad \forall i \in \Gamma_q, \quad \forall q, \\ p_k^P = p_d(S_k) \quad \forall k \in \Gamma_0, \quad p_k^P = p_d(S_k) + c_q^P \quad \forall k \in \Gamma_q, \quad \forall q, \\ \sum_{i \in [1, I]} |T_i| \omega_i^T = \sum_{k \in [1, K]} |P_k| \omega_k^P = m_{\omega}, \end{array} \right. \quad (5.9)$$

in the case of a given (up to constants to be determined on each internal boundary) pressure field  $p_d$  on the boundary (see the corresponding equations in the sets (5.2) and (5.4)).

In (5.7), the following definitions have been used

$$\begin{aligned}\mathbf{f}_j^D &= \frac{1}{|D_j|} \int_{D_j} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} \quad \forall j \in [1, J], \\ (\nabla g)_j^D &= \frac{1}{|D_j|} \int_{D_j} \nabla g(\mathbf{x}) \, d\mathbf{x} \quad \forall j \in [1, J].\end{aligned}\tag{5.10}$$

The two problems involving Eqs. (5.7) and (5.8) on the one hand, and Eqs. (5.7) and (5.9) on the other hand are solved in a very similar way, thus we only detail the solution of (5.7)-(5.8).

**Proposition 5.1.** *Problem (5.7)-(5.8) may be split into two independent subproblems: setting  $\mathbf{s} := (\mathbf{s}_j^D)_{j \in [1, J]}$  with  $\mathbf{s}_j^D := \mathbf{f}_j^D + (\nabla g)_j^D$ , find  $(\omega_i^T, \omega_k^P)_{i \in [1, I+J^\Gamma], k \in [1, K]}$  and  $(c_q^T, c_q^P)_{q \in [1, Q]}$  such that*

$$\left\{ \begin{array}{l} -(\nabla_h^T \cdot \nabla_h^D \omega)_i = (\nabla_h^T \times \mathbf{s})_i, \quad \forall i \in [1, I], \\ -(\nabla_h^P \cdot \nabla_h^D \omega)_k = (\nabla_h^P \times \mathbf{s})_k, \quad \forall k \in [1, K - J^\Gamma], \\ (\nabla_h^D \omega \cdot \mathbf{n}, 1)_{\Gamma_q, h} = -(\mathbf{s} \cdot \boldsymbol{\tau}, 1)_{\Gamma_q, h}, \quad \forall q \in [1, Q], \\ -\sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \cdot \nabla_h^D \omega)_k = \sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \times \mathbf{s})_k, \quad \forall q \in [1, Q], \\ \omega_i^T = \omega_d(G_i), \quad \forall i \in \Gamma_0, \quad \omega_k^P = \omega_d(S_k), \quad \forall k \in \Gamma_0, \\ \omega_i^T = \omega_d(G_i) + c_q^T, \quad \forall i \in \Gamma_q, \quad \forall q \in [1, Q], \quad \omega_k^P = \omega_d(S_k) + c_q^P, \quad \forall k \in \Gamma_q, \quad \forall q \in [1, Q], \end{array} \right.\tag{5.11}$$

and, once  $\omega$  has been computed, find  $(p_i^T, p_k^P)_{i \in [1, I+J^\Gamma], k \in [1, K]}$  such that

$$\left\{ \begin{array}{l} (\nabla_h^T \cdot \nabla_h^D p)_i = (\nabla_h^T \cdot \mathbf{s})_i, \quad \forall i \in [1, I], \\ (\nabla_h^P \cdot \nabla_h^D p)_k = (\nabla_h^P \cdot (\mathbf{s} - \nabla_h^D \times \omega))_k, \quad \forall k \in [1, K], \\ (\nabla_h^D p)_j \cdot \mathbf{n}_j = (\mathbf{s}_j - (\nabla_h^D \times \omega)_j) \cdot \mathbf{n}_j, \quad \forall j \in [J - J^\Gamma + 1, J], \\ \sum_{i \in [1, I]} |T_i| p_i^T = \sum_{k \in [1, K]} |P_k| p_k^P = 0. \end{array} \right.\tag{5.12}$$

*Proof.* Applying the discrete vector curl operator to (5.7) on any primal cell and on any inner dual cell yields the first two lines of (5.11), thanks to (4.9) and (4.10).

Next, for a given  $q \in [1, Q]$ , we consider the element  $\psi \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$  which has the following values:  $\psi_i^T = 1, \forall i \in \Gamma_q$  and  $\psi_i^T = 0$  everywhere else, and  $\psi_k^P = 0$  everywhere. Then, we compute the scalar product (4.1) of Eq. (5.7) with  $\nabla_h^D \times \psi$ :

$$(\nabla_h^D \times \omega, \nabla_h^D \times \psi)_D + (\nabla_h^D p, \nabla_h^D \times \psi)_D = (\mathbf{s}, \nabla_h^D \times \psi)_D.\tag{5.13}$$

Using the orthogonality of  $\nabla_h^D p$  and  $\nabla_h^D \times \psi$  (see last line of Prop. 4.4), using the discrete Green formula (4.7) and Eq. (4.10), we infer

$$\begin{aligned} -(\nabla_h^{T,P} \cdot \nabla_h^D \omega, \psi)_{T,P} - (\nabla_h^D \times \omega \cdot \boldsymbol{\tau}, \tilde{\psi})_{\Gamma, h} = \\ (\nabla_h^{T,P} \times \mathbf{s}, \psi)_{T,P} - (\mathbf{s} \cdot \boldsymbol{\tau}, \tilde{\psi})_{\Gamma, h}. \end{aligned}\tag{5.14}$$

Further, using the fact that  $\psi$  vanishes everywhere but on the boundary points  $G_i$ , the first term in the left-hand side and the first term in the right-hand side of (5.14) vanish. Finally, since  $\nabla_h^D \times \omega \cdot \boldsymbol{\tau} = -\nabla_h^D \omega \cdot \mathbf{n}$ , and using the definition (4.3) and the fact that the values of  $\psi$



on the boundaries imply that  $\tilde{\psi}_j = 1/2$  for all  $j \in \Gamma_q$  and  $\tilde{\psi}_j = 0$  for all  $j \in \Gamma'_q$  for any  $q' \neq q$ , Equ. (5.14) implies

$$-(\nabla_h^D \omega \cdot \mathbf{n}, \frac{1}{2})_{\Gamma_q, h} = (\mathbf{s} \cdot \boldsymbol{\tau}, \frac{1}{2})_{\Gamma_q, h}$$

and thus the third line of (5.11).

Last, we consider the element  $\psi \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$  which has the following values:  $\psi_i^T = 1$ ,  $\forall i \in \Gamma_q$  and  $\psi_k^P = 1$ ,  $\forall k \in \Gamma_q$  and vanishing everywhere else. Then, the dot product of Eq. (5.7) with  $\nabla_h^D \times \psi$  yields (5.13) and (5.14) again. Now, we use the fact that the first term in the left-hand side of (5.14) equals  $-\frac{1}{2} \sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \cdot \nabla_h^D \omega)_k$  while the first term in the right-hand side of (5.14) equals  $\frac{1}{2} \sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \times \mathbf{s})_k$ . Using the previously proved third line of (5.11), we obtain the fourth line of (5.11).

Once  $\omega$  has been computed, the derivation of (5.12) from (5.7)-(5.8) is obvious. We stress that in the second equation of (5.12), there holds  $(\nabla_h^P \cdot \nabla_h^D \times \omega)_k = 0$  for all inner dual cells ( $k \in [1, K - J^\Gamma]$ ) but that this property might not necessarily be true for boundary dual cells ( $k \in [K - J^\Gamma + 1, K]$ ), see Prop. 4.2.  $\square$

Step 1 ends with the fact that (5.11) and (5.12) are well-posed:

**Lemma 5.1.** *It was shown in [21, Proposition 5.2] that systems of the type (5.11) and (5.12) both have a unique solution.*

Now, we shall describe Step 2 of the calculations, which consists in solving for the velocity.

**Step 2:** Once  $(\omega_i^T, \omega_k^P)_{i \in [1, I], k \in [1, K]}$  has been computed through Step 1, we solve a div-curl problem for  $\mathbf{u}$ : given  $(k_q)_{q \in [1, Q]}$ , find  $(\mathbf{u}_j)_{j \in [1, J]}$  such that

$$\left\{ \begin{array}{l} (\nabla_h^{T,P} \cdot \mathbf{u})_{i,k} = g_{i,k}^{T,P}, \quad \forall i \in [1, I], \forall k \in [1, K], \\ (\nabla_h^{T,P} \times \mathbf{u})_{i,k} = \omega_{i,k}^{T,P}, \quad \forall i \in [1, I], \forall k \in [1, K - J^\Gamma], \\ \mathbf{u}_j \cdot \mathbf{n}_j = \sigma_j, \quad \forall j \in [J - J^\Gamma + 1, J], \\ (\mathbf{u} \cdot \boldsymbol{\tau}, 1)_{\Gamma_q, h} = k_q, \quad \forall q \in [1, Q], \\ \sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \times \mathbf{u})_k = \sum_{k \in \Gamma_q} |P_k| \omega_k^P, \quad \forall q \in [1, Q], \end{array} \right. \quad (5.15)$$

in the case of a given normal velocity field on the boundary (Eqs. (5.1) or (5.2)) or

$$\left\{ \begin{array}{l} (\nabla_h^{T,P} \cdot \mathbf{u})_{i,k} = g_{i,k}^{T,P}, \quad \forall i \in [1, I], \forall k \in [1, K - J^\Gamma], \\ (\nabla_h^{T,P} \times \mathbf{u})_{i,k} = \omega_{i,k}^{T,P}, \quad \forall i \in [1, I], \forall k \in [1, K], \\ \mathbf{u}_j \cdot \boldsymbol{\tau}_j = \sigma_j, \quad \forall j \in [J - J^\Gamma + 1, J], \\ (\mathbf{u} \cdot \mathbf{n}, 1)_{\Gamma_q, h} = k_q, \quad \forall q \in [1, Q], \\ \sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \cdot \mathbf{u})_k = \sum_{k \in \Gamma_q} |P_k| g_k^P, \quad \forall q \in [1, Q], \end{array} \right. \quad (5.16)$$

in the case of a given tangential velocity field on the boundary (Eqs. (5.3) or (5.4)).

In (5.15) and (5.16), we have set

$$g_i^T = \frac{1}{|T_i|} \int_{T_i} g(\mathbf{x}) \, d\mathbf{x} \quad \forall i \in [1, I], \quad (5.17)$$

$$g_k^P = \frac{1}{|P_k|} \int_{P_k} g(\mathbf{x}) \, d\mathbf{x} \quad \forall k \in [1, K], \quad (5.18)$$

$$\sigma_j = \frac{1}{|A_j|} \int_{A_j} \sigma(\xi) \, d\xi \quad \forall j \in [J - J^\Gamma + 1, J]. \quad (5.19)$$

Moreover, the right-hand sides in (5.15) and (5.16) have to verify compatibility conditions. Indeed, it is readily seen from the definition of the discrete divergence and curl operators (3.3) and (3.4) that the following equalities hold

$$\sum_{i \in [1, I]} |T_i| (\nabla_h^T \cdot \mathbf{u})_i = \sum_{k \in [1, K]} |P_k| (\nabla_h^P \cdot \mathbf{u})_k = \sum_{j \in [J - J^\Gamma + 1, J]} |A_j| \mathbf{u}_j \cdot \mathbf{n}_j \quad (5.20)$$

and

$$\sum_{i \in [1, I]} |T_i| (\nabla_h^T \times \mathbf{u})_i = \sum_{k \in [1, K]} |P_k| (\nabla_h^P \times \mathbf{u})_k = \sum_{j \in [J - J^\Gamma + 1, J]} |A_j| \mathbf{u}_j \cdot \boldsymbol{\tau}_j. \quad (5.21)$$

Then, because of (5.20), the right-hand sides in (5.15) must satisfy

$$\sum_{i \in [1, I]} |T_i| g_i^T = \sum_{k \in [1, K]} |P_k| g_k^P = \sum_{j \in [J - J^\Gamma + 1, J]} |A_j| \sigma_j. \quad (5.22)$$

This relation is true thanks to the definitions (5.17), (5.18) and (5.19) since

$$\sum_{i \in [1, I]} |T_i| g_i^T = \sum_{k \in [1, K]} |P_k| g_k^P = \int_{\Omega} g(\mathbf{x}) \, d\mathbf{x}$$

and

$$\sum_{j \in [J - J^\Gamma + 1, J]} |A_j| \sigma_j = \int_{\Gamma} \sigma(\xi) \, d\xi.$$

That the right-hand sides of the previous two equalities are identical follows from (5.5).

Further, because of (5.21), the right-hand sides in (5.16) must satisfy

$$\sum_{i \in [1, I]} |T_i| \omega_i^T = \sum_{k \in [1, K]} |P_k| \omega_k^P = \sum_{j \in [J - J^\Gamma + 1, J]} |A_j| \sigma_j. \quad (5.23)$$

In the case of the set of conditions given by (5.9), the first two terms in (5.23) are equal to  $m_\omega$  and the last term in (5.23) is equal to  $\int_{\Gamma} \sigma(\xi) \, d\xi$ . These two quantities are identical thanks to (5.6).

On the other hand, in the case of the set of conditions given by (5.8), the values of the first two terms in (5.23) are never imposed, but, rather, are results of the computations involved in the first step of our procedure (see Eq. (5.11)), so that the compatibility condition (5.23) may

not be verified in general. A possible way to overcome this problem is to change  $\omega_i^T$  into  $\omega_i^T + c^T$  and  $\omega_k^P$  into  $\omega_k^P + c^P$  for all  $i \in [1, I + J^\Gamma]$  and all  $k \in [1, K]$  in (5.16), with two constants  $c^T$  and  $c^P$  computed so that (5.23) holds. Note that doing so does not change the value of  $\nabla_h^D \omega$ , so that the modified  $\omega$  still verifies system (5.11), but with modified boundary conditions. Another possible way to deal with this issue is to change uniformly the values of  $\sigma_j$  to  $\sigma_j + c^D$ , with a value of  $c^D$  chosen such that (5.23) holds. In any case, this may be interpreted as an implicit compatibility condition between the boundary conditions  $\omega_d$  and  $\sigma$  in (5.3).

Now using the discrete Hodge decomposition of  $(\mathbf{u}_j)_{j \in [1, J]}$ , each of the problems (5.15) and (5.16) may be split into two independent subproblems involving the potentials. We only detail the resulting systems for problem (5.15) using the Hodge decomposition with boundary conditions (4.12) to (4.14). A similar result holds for problem (5.16) using the Hodge decomposition with boundary conditions (4.16) to (4.18).

**Proposition 5.2.** *Problem (5.15) may be split into two independent problems: find  $(\phi_i^T, \phi_k^P)_{i \in [1, I + J^\Gamma], k \in [1, K]}$  such that*

$$\begin{cases} (\nabla_h^T \cdot \nabla_h^D \phi)_i = g_i^T, & \forall i \in [1, I], \\ (\nabla_h^P \cdot \nabla_h^D \phi)_k = g_k^P, & \forall k \in [1, K], \\ (\nabla_h^D \phi)_j \cdot \mathbf{n}_j = \sigma_j, & \forall j \in [J - J^\Gamma + 1, J], \\ \sum_{i \in [1, I]} |T_i| \phi_i^T = \sum_{k \in [1, K]} |P_k| \phi_k^P = 0, \end{cases} \quad (5.24)$$

and find  $(\psi_i^T, \psi_k^P)_{i \in [1, I + J^\Gamma], k \in [1, K]}$  and  $(c_q^T, c_q^P)_{q \in [1, Q]}$  such that

$$\begin{cases} -(\nabla_h^T \cdot \nabla_h^D \psi)_i = \omega_i^T, & \forall i \in [1, I], \\ -(\nabla_h^P \cdot \nabla_h^D \psi)_k = \omega_k^P, & \forall k \in [1, K - J^\Gamma], \\ -(\nabla_h^D \psi \cdot \mathbf{n}, 1)_{\Gamma_q, h} = k_q, & \forall q \in [1, Q], \\ -\sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \cdot \nabla_h^D \psi)_k = \sum_{k \in \Gamma_q} |P_k| \omega_k^P, & \forall q \in [1, Q], \\ \psi_i^T = \psi_k^P = 0, & \forall i \in \Gamma_0, \forall k \in \Gamma_0, \\ \forall q \in [1, Q], \psi_i^T = c_q^T, \quad \psi_k^P = c_q^P, & \forall i \in \Gamma_q, \forall k \in \Gamma_q. \end{cases} \quad (5.25)$$

*Proof.* The proof is given in [21, Proposition 5.1]. □

Step 2 ends with the fact that (5.24) and (5.25) are well-posed:

**Lemma 5.2.** *It has been shown in [21, Proposition 5.2] that systems of the type (5.24) and (5.25) both have a unique solution.*

Once these two subproblems have been solved, the vector  $\mathbf{u}$  is then reconstructed by  $\mathbf{u}_j = (\nabla_h^D \phi)_j + (\nabla_h^D \times \psi)_j$ .

## 6 Error estimates

Obtaining error estimates usually relies on regularity assumptions on the solution of the problem. In order to apply results given in [21, 22], we shall assume more regularity on the vorticity and

pressure fields  $(\hat{\omega}, \hat{p})$  given by Proposition 6.1 below, and on the velocity potentials given by Proposition 6.2 below.

**Proposition 6.1.** *Let  $(\mathbf{f}, g, \sigma, \omega_d)$  belong to  $L^2(\Omega)^2 \times H^1(\Omega) \times H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ , and let  $(k_q)_{q \in [1, Q]}$  be a set of given real numbers. Then, system (1.9)-(1.10)-(1.2)-(5.1) associated with (5.5) may be split into two subproblems, where  $(\hat{p}, \hat{\omega}) \in H^1(\Omega)^2$  and a set of real numbers  $(C_q)_{q \in [1, Q]}$  are the exact solution of the Hodge decomposition of  $\mathbf{f} + \nabla g$ :*

$$\begin{cases} \nabla \times \hat{\omega} + \nabla \hat{p} = \mathbf{f} + \nabla g & \text{in } \Omega, \\ \hat{\omega}|_{\Gamma_0} = \omega_d; \quad \hat{\omega}|_{\Gamma_q} = \omega_d + C_q & \forall q \in [1, Q], \\ \int_{\Omega} \hat{p}(x) dx = 0, \end{cases} \quad (6.1)$$

and, once  $\hat{\omega}$  has been determined,  $\hat{\mathbf{u}} \in H\text{div}(\Omega) \cap H\text{curl}(\Omega)$  is the solution of the div-curl problem:

$$\begin{cases} \nabla \cdot \hat{\mathbf{u}} = g & \text{in } \Omega, \\ \nabla \times \hat{\mathbf{u}} = \hat{\omega} & \text{in } \Omega, \\ \hat{\mathbf{u}} \cdot \mathbf{n} = \sigma & \text{on } \Gamma, \\ \int_{\Gamma_q} \hat{\mathbf{u}} \cdot \boldsymbol{\tau} = k_q, & \forall q \in [1, Q]. \end{cases} \quad (6.2)$$

**Hypothesis 6.1.** *We assume that the vorticity  $\hat{\omega}$  and the pressure  $\hat{p}$  given by Proposition 6.1 belong to  $H^2(\Omega)$ .*

The velocity field  $\hat{\mathbf{u}}$ , solution of (6.2) may be found by the following Hodge decomposition.

**Proposition 6.2.** *Let  $(g, \hat{\omega}, \sigma)$  belong to  $H^1(\Omega) \times H^2(\Omega) \times H^{1/2}(\Gamma)$ , and let  $(k_q)_{q \in [1, Q]}$  be a set of given real numbers; let  $\hat{\mathbf{u}}$  be the exact solution of problem (6.2). Then, there exist  $\hat{\phi}$  and  $\hat{\psi}$  both in  $H^1(\Omega)$  and a set of real numbers  $(C_q)_{q \in [1, Q]}$  such that*

$$\hat{\mathbf{u}} = \nabla \hat{\phi} + \nabla \times \hat{\psi},$$

where  $\hat{\phi}$  is the solution of

$$\begin{cases} \Delta \hat{\phi} = \nabla \cdot \hat{\mathbf{u}} = g & \text{in } \Omega, \\ \nabla \hat{\phi} \cdot \mathbf{n} = \hat{\mathbf{u}} \cdot \mathbf{n} = \sigma & \text{on } \Gamma, \\ \int_{\Omega} \hat{\phi} = 0, \end{cases} \quad (6.3)$$

and  $\hat{\psi}$  is the solution of

$$\begin{cases} -\Delta \hat{\psi} = \nabla \times \hat{\mathbf{u}} = \hat{\omega} & \text{in } \Omega, \\ \hat{\psi}|_{\Gamma_0} = 0; \quad \hat{\psi}|_{\Gamma_q} = C_q & \forall q \in [1, Q], \\ \int_{\Gamma_q} \nabla \hat{\psi} \cdot \mathbf{n} = -k_q. \end{cases} \quad (6.4)$$

*Proof.* The Hodge decomposition of  $\hat{\mathbf{u}}$  and the determination of  $\hat{\phi}$  and  $\hat{\psi}$  through (6.3) and (6.4) are direct consequences of [30, Theorem 3.2 and Corollary 3.1].  $\square$

**Hypothesis 6.2.** *We suppose that the potentials  $\hat{\phi}$  and  $\hat{\psi}$  given by Proposition 6.2 belong to  $H^2(\Omega)$ .*

We remark that due to re-entrant corners related to the internal polygonal boundaries  $\Gamma_q$ , the  $H^2$  regularity of the potentials is not a consequence of the regularity of the data  $(g, \hat{\omega}, \sigma)$ .

## 6.1 Error estimates in the $H^1$ semi-norm for the pressure and the vorticity

**Definition 6.1.** We define, for any continuous function  $v$ , the following element  $\Pi v$ , by

$$\begin{aligned} \forall i \in [1, I + J^\Gamma], \quad (\Pi v)_i^T &= v(G_i), \\ \forall k \in [1, K], \quad (\Pi v)_k^P &= v(S_k). \end{aligned}$$

We note that, under Hypothesis 6.1,  $\hat{\omega}$  and  $\hat{p}$  belong to  $H^2(\Omega)$ , which implies they are continuous, so that considering  $\Pi\hat{\omega}$  and  $\Pi\hat{p}$  makes sense.

**Theorem 6.1.** Let  $p$  and  $\omega$  be the solutions of the numerical scheme (5.7)–(5.8), and let  $(\hat{p}, \hat{\omega})$  be the exact solutions of (6.1). Then, if all diamond-cells are convex and under assumptions 2.1 and 6.1, there exists a constant  $C(\theta^*)$ , independent of  $h$ , such that

$$\|\nabla_h^D(p - \Pi\hat{p})\|_D + \|\nabla_h^D(\omega - \Pi\hat{\omega})\|_D \leq C(\theta^*) h (\|\hat{p}\|_{2,\Omega} + \|\hat{\omega}\|_{2,\Omega}). \quad (6.5)$$

*Proof.* Firstly, taking the mean-value of each term of the first line of Equ. (6.1) on a diamond cell  $D_j$  and using (5.10), we get:

$$\frac{1}{|D_j|} \int_{D_j} (\nabla\hat{p} + \nabla \times \hat{\omega})(\mathbf{x}) d\mathbf{x} = \mathbf{f}_j^D + (\nabla g)_j^D, \quad \forall j \in [1, J]. \quad (6.6)$$

Then, since (6.6) and (5.7) have the same right-hand side, we infer that:

$$(\nabla_h^D p)_j + (\nabla_h^D \times \omega)_j = \frac{1}{|D_j|} \int_{D_j} (\nabla\hat{p} + \nabla \times \hat{\omega})(\mathbf{x}) d\mathbf{x}, \quad \forall j \in [1, J]. \quad (6.7)$$

Setting

$$\varepsilon_p := p - \Pi\hat{p} \quad \text{and} \quad \varepsilon_\omega := \omega - \Pi\hat{\omega}, \quad (6.8)$$

Equ. (6.7) implies the following equality:

$$\begin{aligned} (\nabla_h^D \varepsilon_p)_j + (\nabla_h^D \times \varepsilon_\omega)_j &= \frac{1}{|D_j|} \int_{D_j} (\nabla\hat{p}(\mathbf{x}) - (\nabla_h^D \Pi\hat{p})_j) d\mathbf{x} \\ &+ \frac{1}{|D_j|} \int_{D_j} (\nabla \times \hat{\omega}(\mathbf{x}) - (\nabla_h^D \times \Pi\hat{\omega})_j) d\mathbf{x}, \quad \forall j \in [1, J]. \end{aligned} \quad (6.9)$$

According to (5.8) and the second line of (6.1), the error  $\varepsilon_\omega$  satisfies on the boundary

$$\begin{cases} (\varepsilon_\omega)_i^T = 0, \quad \forall i \in \Gamma_0 & , \quad (\varepsilon_\omega)_i^T = c_q^T - C_q, \quad \forall i \in \Gamma_q, \quad \forall q, \\ (\varepsilon_\omega)_k^P = 0, \quad \forall k \in \Gamma_0 & , \quad (\varepsilon_\omega)_k^P = c_q^P - C_q, \quad \forall k \in \Gamma_q, \quad \forall q. \end{cases} \quad (6.10)$$

Therefore,  $(\nabla_h^D \varepsilon_p)$  and  $(\nabla_h^D \times \varepsilon_\omega)$  are orthogonal for the scalar product  $(\cdot, \cdot)_D$ , as recalled at the end of Proposition 4.4. Consequently, multiplying (6.9) by  $|D_j|(\nabla_h^D \varepsilon_p)_j$  and summing for all  $j \in [1, J]$ , we obtain:

$$\begin{aligned} \|\nabla_h^D \varepsilon_p\|_D^2 &= \sum_{j \in [1, J]} \int_{D_j} (\nabla\hat{p}(\mathbf{x}) - (\nabla_h^D \Pi\hat{p})_j) \cdot (\nabla_h^D \varepsilon_p)_j d\mathbf{x} \\ &+ \sum_{j \in [1, J]} \int_{D_j} (\nabla \times \hat{\omega}(\mathbf{x}) - (\nabla_h^D \times \Pi\hat{\omega})_j) \cdot (\nabla_h^D \varepsilon_p)_j d\mathbf{x}. \end{aligned} \quad (6.11)$$

In the very same way, multiplying (6.9) by  $|D_j|(\nabla_h^D \times \varepsilon_\omega)_j$  and summing for all  $j \in [1, J]$ , we obtain:

$$\begin{aligned} \|\nabla_h^D \varepsilon_\omega\|_D^2 &= \sum_{j \in [1, J]} \int_{D_j} (\nabla \hat{p}(\mathbf{x}) - (\nabla_h^D \Pi \hat{p})_j) \cdot (\nabla_h^D \times \varepsilon_\omega)_j d\mathbf{x} \\ &+ \sum_{j \in [1, J]} \int_{D_j} (\nabla \times \hat{\omega}(\mathbf{x}) - (\nabla_h^D \times \Pi \hat{\omega})_j) \cdot (\nabla_h^D \times \varepsilon_\omega)_j d\mathbf{x}. \end{aligned} \quad (6.12)$$

The right-hand sides of (6.11) and (6.12) may be bounded using the traditional  $P^1$  Lagrange interpolations of  $\hat{p}$  and  $\hat{\omega}$  on a submesh of the diamond mesh, obtained by splitting each diamond cell along one of its diagonals into two triangles, as shown on Fig. 5.

The details of the calculations may be found in [22], starting with inequality (62) of that reference, and then using Lemma 5.11, where the norm in the right-hand side of inequality (66) has to be replaced by the  $H^2(\Omega)$  norm of  $\hat{p}$  and  $\hat{\omega}$ .  $\square$

## 6.2 Error estimates in the $L^2$ -norm for the pressure and the vorticity

We shall use the discrete Poincaré inequalities recalled in Proposition 4.5 to infer error estimates for the discrete  $L^2$  norm of the errors in the pressure and the vorticity.

On the one hand, it can be applied directly to the error  $\varepsilon_\omega$  because it verifies (6.10), which are exactly conditions (4.13) and (4.14) that Proposition 4.5 requires. Thus, we infer from (6.5) the following theorem

**Theorem 6.2.** *There exists a constant  $C(\theta^*)$  that does not depend on  $h$  such that*

$$\|\omega - \Pi \hat{\omega}\|_{T, P} \leq C(\theta^*) h (\|\hat{p}\|_{2, \Omega} + \|\hat{\omega}\|_{2, \Omega}). \quad (6.13)$$

On the other hand, since  $(p - \Pi \hat{p})$  does not in general verify the vanishing mean-value condition (4.12) (because  $\Pi \hat{p}$  doesn't), we may not apply the discrete Poincaré inequality in a straightforward way. However, defining

$$c^T := \frac{\sum_{i \in [1, I]} |T_i| (\Pi \hat{p})_i^T}{|\Omega|}, \quad (\tilde{\Pi} \hat{p})_i^T := (\Pi \hat{p})_i^T - c^T, \quad \forall i \in [1, I + J^\Gamma], \quad (6.14)$$

$$c^P := \frac{\sum_{k \in [1, K]} |P_k| (\Pi \hat{p})_k^P}{|\Omega|}, \quad (\tilde{\Pi} \hat{p})_k^P := (\Pi \hat{p})_k^P - c^P, \quad \forall k \in [1, K], \quad (6.15)$$

we have that  $\tilde{\Pi} \hat{p}$  verifies the vanishing mean-value condition (4.12) and

$$\nabla_h^D (\tilde{\Pi} \hat{p}) := \nabla_h^D (\Pi \hat{p}) \quad (6.16)$$

because  $(\tilde{\Pi} \hat{p})$  and  $(\Pi \hat{p})$  only differ by a constant on the primal and dual meshes, and it is seen from the definition (3.1) of the discrete gradient that this does not affect its values. Next, we have

$$\|p - \Pi \hat{p}\|_{T, P} \leq \|p - \tilde{\Pi} \hat{p}\|_{T, P} + \|\Pi \hat{p} - \tilde{\Pi} \hat{p}\|_{T, P}. \quad (6.17)$$

Estimating the first term in the right-hand-side of (6.17) may be performed through the discrete Poincaré inequality applied to  $(p - \tilde{\Pi} \hat{p})$ .

**Lemma 6.1.** *There exists a constant  $C(\theta^*)$  that does not depend on  $h$  such that*

$$\|p - \tilde{\Pi}\hat{p}\|_{T,P} \leq C(\theta^*) h (\|\hat{p}\|_{2,\Omega} + \|\hat{\omega}\|_{2,\Omega}). \quad (6.18)$$

*Proof.* From Prop. 4.5, it holds that

$$\|p - \tilde{\Pi}\hat{p}\|_{T,P} \leq C(\theta^*) \|\nabla_h^D(p - \tilde{\Pi}\hat{p})\|_D = C(\theta^*) \|\nabla_h^D(p - \Pi\hat{p})\|_D \quad (6.19)$$

because of (6.16). Applying (6.5), we get (6.18).  $\square$

To estimate the second term in the right-hand-side of (6.17), we first recall that these two elements only differ by a constant on the primal and dual meshes (see (6.14) and (6.15)), so that

$$\|\Pi\hat{p} - \tilde{\Pi}\hat{p}\|_{T,P}^2 = \frac{|\Omega|}{2} [(c^T)^2 + (c^P)^2]. \quad (6.20)$$

**Lemma 6.2.** *Let  $c^T$  be defined by (6.14); then*

$$|c^T| \leq |\Omega|^{-1/2} \left( \sum_i \int_{T_i} [(\Pi^T \hat{p})_i - \hat{p}]^2(\mathbf{x}) d\mathbf{x} \right)^{1/2}. \quad (6.21)$$

*Proof.* From (6.14), we have that

$$|\Omega|c^T = \sum_i |T_i| (\Pi^T \hat{p})_i = \sum_i \int_{T_i} [(\Pi^T \hat{p})_i - \hat{p}](\mathbf{x}) d\mathbf{x}$$

because  $\sum_i \int_{T_i} \hat{p}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \hat{p}(\mathbf{x}) d\mathbf{x} = 0$ . So, by a continuous and then a discrete Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\Omega| |c^T| &\leq \sum_i |T_i|^{1/2} \left( \int_{T_i} [(\Pi^T \hat{p})_i - \hat{p}]^2(\mathbf{x}) d\mathbf{x} \right)^{1/2} \\ &\leq |\Omega|^{1/2} \left( \sum_i \int_{T_i} [(\Pi^T \hat{p})_i - \hat{p}]^2(\mathbf{x}) d\mathbf{x} \right)^{1/2}, \end{aligned}$$

which implies (6.21).  $\square$

So, what remains to evaluate is the  $L^2$  norm of  $(\Pi^T \hat{p})_i - \hat{p}$  on  $T_i$ .

**Lemma 6.3.** *There exists a constant  $C(\theta^*)$  that does not depend on  $h$  such that*

$$\|\hat{p} - (\Pi^T \hat{p})_i\|_{L^2(T_i)} \leq C(\theta^*) h \|\hat{p}\|_{2,T_i}. \quad (6.22)$$

*Proof.* First, we split the integral on the subtriangles  $t_{i,j,k}$  with vertices  $G_i M_j S_k$  (see Fig. 6), where we recall that  $M_j$  is the midpoint of the edge  $A_j \subset \partial T_i$  and  $S_k$  is one of the vertices of  $A_j$ :

$$\int_{T_i} [(\Pi^T \hat{p})_i - \hat{p}]^2(\mathbf{x}) d\mathbf{x} = \sum_{t_{i,j,k} \subset T_i} \int_{t_{i,j,k}} [(\Pi^T \hat{p})_i - \hat{p}]^2(\mathbf{x}) d\mathbf{x}. \quad (6.23)$$

We define  $\pi_{i,j,k}\hat{p}$  as the standard  $P^1$  Lagrange interpolation of  $\hat{p}$  on the triangle  $t_{i,j,k}$  and we have

$$\int_{t_{i,j,k}} [(\Pi^T \hat{p})_i - \hat{p}]^2(\mathbf{x}) d\mathbf{x} \leq 2 \int_{t_{i,j,k}} [(\Pi^T \hat{p})_i - \pi_{i,j,k}\hat{p}]^2(\mathbf{x}) d\mathbf{x} + 2 \int_{t_{i,j,k}} [\pi_{i,j,k}\hat{p} - \hat{p}]^2(\mathbf{x}) d\mathbf{x}. \quad (6.24)$$

It is a standard result (see Theorem 3.1.5 of [14]) that there exists a constant  $C$ , not depending on  $t_{i,j,k}$  such that

$$\|\pi_{i,j,k}\hat{p} - \hat{p}\|_{L^2(t_{i,j,k})} \leq C \text{diam}(t_{i,j,k})^2 \|\hat{p}\|_{H^2(t_{i,j,k})}, \quad (6.25)$$

which evaluates the second term in the right-hand side of (6.24). As far as the first term in the right-hand side of (6.24) is concerned, since  $(\Pi^T \hat{p})_i - \pi_{i,j,k}\hat{p}(G_i) = 0$ , and since  $\pi_{i,j,k}\hat{p}$  is a  $P^1$  function, we have that, for all  $\mathbf{x} \in t_{i,j,k}$

$$(\Pi^T \hat{p})_i - \pi_{i,j,k}\hat{p}(\mathbf{x}) = \nabla \pi_{i,j,k}\hat{p} \cdot (G_i - \mathbf{x}),$$

which ensures that

$$|(\Pi^T \hat{p})_i - \pi_{i,j,k}\hat{p}(\mathbf{x})| \leq \text{diam}(t_{i,j,k}) \|\nabla \pi_{i,j,k}\hat{p}\|$$

for all  $\mathbf{x} \in t_{i,j,k}$  and thus

$$\|\pi_{i,j,k}\hat{p} - (\Pi^T \hat{p})_i\|_{L^2(t_{i,j,k})} \leq \text{diam}(t_{i,j,k}) \|\nabla \pi_{i,j,k}\hat{p}\|_{L^2(t_{i,j,k})}. \quad (6.26)$$

Using a triangular inequality

$$\|\nabla \pi_{i,j,k}\hat{p}\|_{L^2(t_{i,j,k})} \leq \|\nabla(\pi_{i,j,k}\hat{p} - \hat{p})\|_{L^2(t_{i,j,k})} + \|\nabla \hat{p}\|_{L^2(t_{i,j,k})}$$

and using again Theorem 3.1.5 of [14], we have that there exists a constant  $C$  depending only the regularity parameter of the subtriangulation (and thus on  $\theta^*$ ) such that

$$\|\nabla \pi_{i,j,k}\hat{p}\|_{L^2(t_{i,j,k})} \leq (1 + C \text{diam}(t_{i,j,k})) \|\hat{p}\|_{H^2(t_{i,j,k})}. \quad (6.27)$$

Gathering (6.23), (6.24), (6.25), (6.26) and (6.27) implies (6.22).  $\square$

We are now able to estimate the second term in the right-hand side of (6.17)

**Proposition 6.3.** *There exists a constant  $C(\theta^*)$  that does not depend on  $h$  such that*

$$\|\Pi \hat{p} - \tilde{\Pi} \hat{p}\|_{T,P} \leq C(\theta^*) h \|\hat{p}\|_{2,\Omega}. \quad (6.28)$$

*Proof.* Bounds (6.21) and (6.22) imply that

$$|c^T| \leq C(\theta^*) h \|\hat{p}\|_{2,\Omega}. \quad (6.29)$$

Using (6.20), (6.29) and a similar bound that can be obtained in the same way for  $c^P$ , we obtain (6.28).  $\square$

Finally, using (6.17), (6.18) and (6.28), we obtain the following theorem:

**Theorem 6.3.** *There exists a constant  $C(\theta^*)$  that does not depend on  $h$  such that*

$$\|p - \Pi \hat{p}\|_{T,P} \leq C(\theta^*) h (\|\hat{p}\|_{2,\Omega} + \|\hat{\omega}\|_{2,\Omega}). \quad (6.30)$$



### 6.3 Error estimate in the $L^2$ -norm for the velocity

Once the numerical approximation  $\omega = (\omega_i^T, \omega_k^P)_{i \in [1, I+J^\Gamma], k \in [1, K]}$  of the vorticity  $\hat{\omega}$  is known, we have to solve (5.15) in order to find the discrete velocity  $\mathbf{u} = (\mathbf{u}_j)_{j \in [1, J]}$ , with which we define a piecewise constant function  $\mathbf{u}_h$  by

$$\mathbf{u}_h(\mathbf{x}) = \mathbf{u}_j, \quad \forall \mathbf{x} \in D_j, \quad \forall j \in [1, J].$$

In order to estimate the error between  $\mathbf{u}_h$  and the exact velocity  $\hat{\mathbf{u}}$ , we first introduce an intermediary discrete velocity  $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}_j)_{j \in [1, J]}$ , solution of the discrete div-curl system

$$\left\{ \begin{array}{l} (\nabla_h^{T,P} \cdot \tilde{\mathbf{u}})_{i,k} = g_{i,k}^{T,P}, \quad \forall i \in [1, I], \forall k \in [1, K], \\ (\nabla_h^{T,P} \times \tilde{\mathbf{u}})_{i,k} = \langle \hat{\omega} \rangle_{i,k}^{T,P}, \quad \forall i \in [1, I], \forall k \in [1, K - J^\Gamma], \\ \tilde{\mathbf{u}}_j \cdot \mathbf{n}_j = \sigma_j, \quad \forall j \in [J - J^\Gamma + 1, J], \\ (\tilde{\mathbf{u}} \cdot \boldsymbol{\tau}, 1)_{\Gamma_q, h} = k_q, \quad \forall q \in [1, Q], \\ \sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \times \tilde{\mathbf{u}})_k = \sum_{k \in \Gamma_q} |P_k| \langle \hat{\omega} \rangle_k^P, \quad \forall q \in [1, Q], \end{array} \right. \quad (6.31)$$

where we have set

$$\langle \hat{\omega} \rangle_i^T := \frac{1}{|T_i|} \int_{T_i} \hat{\omega}(\mathbf{x}) \, d\mathbf{x} \quad \forall i \in [1, I], \quad (6.32)$$

$$\langle \hat{\omega} \rangle_k^P := \frac{1}{|P_k|} \int_{P_k} \hat{\omega}(\mathbf{x}) \, d\mathbf{x} \quad \forall k \in [1, K]. \quad (6.33)$$

Note that  $\tilde{\mathbf{u}}$  is of course never actually computed (because the exact vorticity  $\hat{\omega}$  is not known), but only serves for theoretical reasons. The following triangle inequality holds

$$\|\mathbf{u}_h - \hat{\mathbf{u}}\|_{0, \Omega} \leq \|\mathbf{u} - \tilde{\mathbf{u}}\|_D + \|\tilde{\mathbf{u}}_h - \hat{\mathbf{u}}\|_{0, \Omega}. \quad (6.34)$$

The estimation of the second term in the right-hand side of (6.34) is provided by [21, Theorem 5.22]:

**Proposition 6.4.** *If all diamond-cells are convex and under Hypotheses 2.1 and 6.2, there exists a constant  $C(\theta^*)$  independent of  $h$  such that*

$$\|\tilde{\mathbf{u}}_h - \hat{\mathbf{u}}\|_{0, \Omega} \leq C(\theta^*)h \left( \|g\|_{0, \Omega} + \|\hat{\omega}\|_{0, \Omega} + \|\hat{\phi}\|_{2, \Omega} + \|\hat{\psi}\|_{2, \Omega} \right). \quad (6.35)$$

The next step is the evaluation of the difference  $\mathbf{d}_u := \mathbf{u} - \tilde{\mathbf{u}}$ , which, using (5.15) and (6.31), is the solution of

$$\left\{ \begin{array}{l} (\nabla_h^{T,P} \cdot \mathbf{d}_u)_{i,k} = 0, \quad \forall i \in [1, I], \forall k \in [1, K], \\ (\nabla_h^{T,P} \times \mathbf{d}_u)_{i,k} = (d_\omega)_{i,k}^{T,P}, \quad \forall i \in [1, I], \forall k \in [1, K - J^\Gamma], \\ (\mathbf{d}_u)_j \cdot \mathbf{n}_j = 0, \quad \forall j \in [J - J^\Gamma + 1, J], \\ (\mathbf{d}_u \cdot \boldsymbol{\tau}, 1)_{\Gamma_q, h} = 0, \quad \forall q \in [1, Q], \\ \sum_{k \in \Gamma_q} |P_k| (\nabla_h^P \times \mathbf{d}_u)_k = \sum_{k \in \Gamma_q} |P_k| (d_\omega)_k^P, \quad \forall q \in [1, Q], \end{array} \right. \quad (6.36)$$

with  $d_\omega := \omega - \langle \hat{\omega} \rangle$ .

**Proposition 6.5.** *There exists a constant  $C(\theta^*)$  independent of  $h$  such that*

$$\|\mathbf{u} - \tilde{\mathbf{u}}\|_D \leq C(\theta^*) \|\omega - \hat{\omega}\|_{T,P}. \quad (6.37)$$

*Proof.* Using the discrete Hodge decomposition (see Proposition 4.4) of  $\mathbf{d}_u$  into  $\nabla_h^D \phi_{\mathbf{d}} + \nabla_h^D \times \psi_{\mathbf{d}}$ , we easily get, using Proposition 5.2 that  $\phi_{\mathbf{d}}$  vanishes, and we may thus evaluate  $\|\mathbf{d}_u\|_D = \|\nabla_h^D \times \psi_{\mathbf{d}}\|_D$  as follows. Let us first use the discrete Green formula (4.7):

$$\|\nabla_h^D \times \psi_{\mathbf{d}}\|_D^2 = (\nabla_h^{T,P} \times \nabla_h^D \times \psi_{\mathbf{d}}, \psi_{\mathbf{d}})_{T,P} - (\nabla_h^D \times \psi_{\mathbf{d}} \cdot \boldsymbol{\tau}, \tilde{\psi}_{\mathbf{d}})_{\Gamma,h}. \quad (6.38)$$

In (6.38), let us first treat the second term in the right-hand side. By definition, we have

$$(\nabla_h^D \times \psi_{\mathbf{d}} \cdot \boldsymbol{\tau}, \tilde{\psi}_{\mathbf{d}})_{\Gamma,h} = \sum_{q \in [0, Q]} (\nabla_h^D \times \psi_{\mathbf{d}} \cdot \boldsymbol{\tau}, \tilde{\psi}_{\mathbf{d}})_{\Gamma_q, h}.$$

But  $\psi_{\mathbf{d}}$  vanishes on  $\Gamma_0$  and  $(\psi_{\mathbf{d}})_i^T$  and  $(\psi_{\mathbf{d}})_k^P$  are equal to constants  $c_q^T$  and  $c_q^P$  on  $\Gamma_q$ , for  $q \in [1, Q]$  (see (4.13) and (4.14)). Thus:

$$(\nabla_h^D \times \psi_{\mathbf{d}} \cdot \boldsymbol{\tau}, \tilde{\psi}_{\mathbf{d}})_{\Gamma,h} = \sum_{q \in [1, Q]} \left( \frac{c_q^T + c_q^P}{2} \right) (\nabla_h^D \times \psi_{\mathbf{d}} \cdot \boldsymbol{\tau}, 1)_{\Gamma_q, h} = 0 \quad (6.39)$$

thanks to the fourth line of (6.36). Let us now turn to the first term in the right-hand side of (6.38). By definition of the scalar product (4.2), and using the second line in (6.36), it holds that

$$\begin{aligned} (\nabla_h^{T,P} \times \nabla_h^D \times \psi_{\mathbf{d}}, \psi_{\mathbf{d}})_{T,P} &= \frac{1}{2} \sum_{i \in [1, I]} |T_i| (d_\omega)_i^T (\psi_{\mathbf{d}})_i^T \\ &+ \frac{1}{2} \sum_{k \notin \Gamma} |P_k| (d_\omega)_k^P (\psi_{\mathbf{d}})_k^P + \frac{1}{2} \sum_{k \in \Gamma_0} |P_k| (\nabla_h^{T,P} \times \mathbf{d}_u)_k^P (\psi_{\mathbf{d}})_k^P \\ &+ \sum_{q \in [1, Q]} \frac{1}{2} \sum_{k \in \Gamma_q} |P_k| (\nabla_h^{T,P} \times \mathbf{d}_u)_k^P (\psi_{\mathbf{d}})_k^P. \end{aligned} \quad (6.40)$$

Using once again boundary condition (4.13), we have

$$\sum_{k \in \Gamma_0} |P_k| (\nabla_h^{T,P} \times \mathbf{d}_u)_k^P (\psi_{\mathbf{d}})_k^P = 0 = \sum_{k \in \Gamma_0} |P_k| (d_\omega)_k^P (\psi_{\mathbf{d}})_k^P. \quad (6.41)$$

Moreover, using boundary condition (4.14) and the last line in (6.36), we obtain

$$\begin{aligned} \sum_{k \in \Gamma_q} |P_k| (\nabla_h^{T,P} \times \mathbf{d}_u)_k^P (\psi_{\mathbf{d}})_k^P &= c_q^P \sum_{k \in \Gamma_q} |P_k| (\nabla_h^{T,P} \times \mathbf{d}_u)_k^P \\ &= c_q^P \sum_{k \in \Gamma_q} |P_k| (d_\omega)_k^P \\ &= \sum_{k \in \Gamma_q} |P_k| (d_\omega)_k^P (\psi_{\mathbf{d}})_k^P. \end{aligned} \quad (6.42)$$

Thus, gathering (6.40), (6.41) and (6.42), we get that

$$(\nabla_h^{T,P} \times \nabla_h^D \times \psi_{\mathbf{d}}, \psi_{\mathbf{d}})_{T,P} = (d_\omega, \psi_{\mathbf{d}})_{T,P}. \quad (6.43)$$

Finally, (6.38), (6.39) and (6.43), may be used to obtain

$$\|\nabla_h^D \times \psi_{\mathbf{d}}\|_D^2 = (d_\omega, \psi_{\mathbf{d}})_{T,P}. \quad (6.44)$$

The Cauchy-Schwarz inequality:  $(d_\omega, \psi_{\mathbf{d}})_{T,P} \leq \|d_\omega\|_{T,P} \|\psi_{\mathbf{d}}\|_{T,P}$  and the discrete Poincaré inequality (4.19) allow us to conclude (recalling that  $\mathbf{u} - \tilde{\mathbf{u}} = \mathbf{d}_u = \nabla_h^D \times \psi_{\mathbf{d}}$  and  $d_\omega = \omega - \langle \hat{\omega} \rangle$ ).  $\square$

Next, it remains to estimate the right-hand side of (6.37). We first link  $\|\omega - \langle \hat{\omega} \rangle\|_{T,P}$  to a previous result by the triangle inequality

$$\|\omega - \langle \hat{\omega} \rangle\|_{T,P} \leq \|\omega - \Pi\hat{\omega}\|_{T,P} + \|\langle \hat{\omega} \rangle - \Pi\hat{\omega}\|_{T,P}. \quad (6.45)$$

The first term in the right-hand side of (6.45) is bounded by (6.13). Bounding the second term requires some more analysis.

**Proposition 6.6.** *Under hypotheses 2.1 and 6.1, there exists a constant  $C(\theta^*)$  independent of  $h$  such that*

$$\|\langle \hat{\omega} \rangle - \Pi\hat{\omega}\|_{T,P} \leq C(\theta^*) h \|\hat{\omega}\|_{2,\Omega}. \quad (6.46)$$

*Proof.* First, we have

$$\|\langle \hat{\omega} \rangle - \Pi\hat{\omega}\|_{T,P}^2 = \frac{1}{2} \left[ \sum_i |T_i| (\langle \hat{\omega} \rangle_i^T - (\Pi^T \hat{\omega})_i)^2 + \sum_k |P_k| (\langle \hat{\omega} \rangle_k^P - (\Pi^P \hat{\omega})_k)^2 \right]. \quad (6.47)$$

Let us consider the first sum in the right-hand side of (6.47); the second sum will be treated in the same way, with however an important modification that will be underlined within the developments that follow.

Since  $\langle \hat{\omega} \rangle_i^T - (\Pi^T \hat{\omega})_i$  is a constant on  $T_i$ , we have

$$\begin{aligned} |T_i| (\langle \hat{\omega} \rangle_i^T - (\Pi^T \hat{\omega})_i)^2 &= \int_{T_i} (\langle \hat{\omega} \rangle_i^T - (\Pi^T \hat{\omega})_i)^2 d\mathbf{x} \\ &\leq 2 \left( \|\hat{\omega} - \langle \hat{\omega} \rangle_i^T\|_{L^2(T_i)}^2 + \|\hat{\omega} - (\Pi^T \hat{\omega})_i\|_{L^2(T_i)}^2 \right). \end{aligned} \quad (6.48)$$

Using arguments similar to those that led to (6.22), we have that

$$\|\hat{\omega} - (\Pi^T \hat{\omega})_i\|_{L^2(T_i)} \leq Ch \|\hat{\omega}\|_{2,T_i}. \quad (6.49)$$

Further, since  $T_i$  is an open bounded set which is star-shaped with respect to  $G_i$ , there exists a constant  $C(T_i)$  only depending on the shape of  $T_i$ , but not on its diameter such that

$$\|\hat{\omega} - \langle \hat{\omega} \rangle_i^T\|_{L^2(T_i)} \leq C(T_i) \text{diam}(T_i) \|\nabla \hat{\omega}\|_{L^2(T_i)}. \quad (6.50)$$

Since  $T_i$  is convex, a universal constant  $C(T_i)$  is given by  $\frac{1}{\pi}$  (see [41]). As far as dual cells  $P_k$  are concerned, a similar inequality holds because  $P_k$  is star-shaped with respect to  $S_k$ :

$$\|\hat{\omega} - \langle \hat{\omega} \rangle_k^P\|_{L^2(P_k)} \leq C(P_k) \text{diam}(P_k) \|\nabla \hat{\omega}\|_{L^2(P_k)}. \quad (6.51)$$

However, since  $P_k$  is not necessarily convex, the estimation of  $C(P_k)$  is less obvious, but we may use explicitly computable formulas given, for example, by [11, 43]. These formulas show that  $C(P_k)$  only depend on the angles of the subtriangulation mentioned in Hyp. 2.1. Gathering (6.48), (6.49) and (6.50), and similar inequalities on the dual cells  $P_k$ , the upper bound (6.46) is obtained from (6.47).  $\square$

Now, we are able to estimate the first term in the right-hand side of (6.34). With (6.45), (6.13) and (6.46), we obtain from (6.37) the following proposition:

**Proposition 6.7.** *Under hypotheses 2.1 and 6.1, there exists a constant  $C(\theta^*)$  independent of  $h$  such that*

$$\|\mathbf{u} - \tilde{\mathbf{u}}\|_D \leq C(\theta^*)h (\|\hat{\omega}\|_{2,\Omega} + \|\hat{p}\|_{2,\Omega}). \quad (6.52)$$

Finally, plugging (6.52) and (6.35) into (6.34) leads to the following theorem

**Theorem 6.4.** *If all diamond cells are convex and under hypotheses 2.1, 6.1 and 6.2, there exists a constant  $C(\theta^*)$  independent of  $h$  such that*

$$\|\mathbf{u}_h - \hat{\mathbf{u}}\|_{0,\Omega} \leq C(\theta^*)h \left( \|\hat{\omega}\|_{2,\Omega} + \|\hat{p}\|_{2,\Omega} + \|g\|_{0,\Omega} + \|\hat{\phi}\|_{2,\Omega} + \|\hat{\psi}\|_{2,\Omega} \right). \quad (6.53)$$

## 6.4 Superconvergence on some mesh families

In this section, we consider families of meshes satisfying the following property:

**Hypothesis 6.3.** *There exists a finite number  $L$  of sub-domains  $(\Omega_\ell)_{\ell \in [1,L]}$  included in  $\Omega$  and independent of the mesh step  $h$ , such that the diamond-cells which are not parallelograms are included in strips  $\delta_\ell$ , having width  $Ch$  and located along the boundaries of  $\Omega_\ell$ , where  $C$  is a constant independent of  $h$ .*

We can note that the diamond-cells located on  $\Gamma$  are triangles. Consequently, it is impossible to find a mesh in which all diamond-cells are parallelograms. Examples of families of meshes verifying Hypothesis 6.3 are families of uniformly refined meshes of rectangles, or any (possibly non-conforming) union of such meshes; see for example on Fig. (9) the second family of meshes used in section 7. In that case, each mesh of rectangles is an  $\Omega_\ell$ . Another example of such families of meshes is what is called ‘‘homothetically refined triangular meshes’’, obtained from a coarse triangular mesh by iteratively refining each triangle into four homothetic sub-triangles by joining the midpoints of its edges. In that case, each triangle of the coarse mesh is an  $\Omega_\ell$ .

**Hypothesis 6.4.** *We assume that the vorticity  $\hat{\omega}$  and the pressure  $\hat{p}$  given by Proposition 6.1 belong to  $H^3(\Omega)$ .*

We have the following superconvergence results:

**Theorem 6.5.** *If all diamond-cells are convex and under hypotheses 2.1, 6.3 and 6.4, there exists a constant  $C(\theta^*)$ , depending on the shape of the  $\Omega_\ell$  but not on  $h$ , such that*

$$\|\nabla_h^D(p - \Pi\hat{p})\|_D + \|\nabla_h^D(\omega - \Pi\hat{\omega})\|_D \leq C(\theta^*) h^{3/2} (\|\hat{p}\|_{3,\Omega} + \|\hat{\omega}\|_{3,\Omega}). \quad (6.54)$$

*Proof.* We only sketch the proof for the pressure (we can obtain the same results for the vorticity). Starting from (6.11), setting

$$\begin{aligned} \langle \nabla \hat{p} \rangle_j &:= \frac{1}{|D_j|} \int_{D_j} \nabla \hat{p}(\mathbf{x}) d\mathbf{x} \quad , \quad \langle \nabla \times \hat{\omega} \rangle_j := \frac{1}{|D_j|} \int_{D_j} \nabla \times \hat{\omega}(\mathbf{x}) d\mathbf{x} \\ (d_p)_j &:= \langle \nabla \hat{p} \rangle_j - (\nabla_h^D \Pi \hat{p})_j \quad , \quad (d_v)_j := \langle \nabla \times \hat{\omega} \rangle_j - (\nabla_h^D \times \Pi \hat{\omega})_j, \end{aligned} \quad (6.55)$$

recalling definition (6.8), and using the discrete Cauchy-Schwarz inequality leads to

$$\|\nabla_h^D \varepsilon_p\|_D^2 \leq 2 \sum_{j \in [1, J]} |D_j| [(d_p)_j^2 + (d_v)_j^2]. \quad (6.56)$$

In (6.56), we shall only estimate the sum of the  $(d_p)_j^2$ , the other term being treated in the same way. In order to obtain the refined estimates, we shall distinguish between the cells  $D_j$  which are not parallelograms (we shall write  $j \in \mathcal{J}$ ) and those which are parallelograms (we shall write  $j \notin \mathcal{J}$ ). Indeed, when  $D_j$  is a parallelogram, then the quantity  $(d_p)_j$  vanishes when  $\hat{p}$  is a  $P^2$  polynomial function because both quantities in the right-hand side of (6.55) are equal to the gradient of the  $P^2$  function, evaluated at the intersection of the diagonals of  $D_j$ . Then, standard numerical analysis results (the Bramble-Hilbert lemma and a scaling argument in parallelograms) show that there exists a constant  $C$ , depending only on the shape of  $D_j$ , and thus only on its angles, and thus only on  $\theta^*$ , such that for any  $\hat{p}$  in  $H^3(D_j)$ , there holds

$$(d_p)_j^2 \leq Ch^2 \|\hat{p}\|_{H^3(D_j)}^2,$$

which implies that

$$\sum_{j \notin \mathcal{J}} |D_j| (d_p)_j^2 \leq Ch^4 \|\hat{p}\|_{H^3(\Omega)}^2. \quad (6.57)$$

On the other hand, when the cell  $D_j$  is not a parallelogram, then

$$(d_p)_j^2 \leq C \|\hat{p}\|_{H^2(D_j)}^2,$$

which implies that

$$\sum_{j \in \mathcal{J}} |D_j| (d_p)_j^2 \leq Ch^2 \sum_{\ell \in [1, L]} \|\hat{p}\|_{H^2(\delta_\ell)}^2. \quad (6.58)$$

Since Ilin's inequality (see, for example, [12, Formula (2.2)]) implies, under Hypotheses 6.3 and 6.4 that  $\|\hat{p}\|_{H^2(\delta_\ell)} \leq (Ch)^{1/2} \|\hat{p}\|_{H^3(\Omega_\ell)}$ , we finally obtain

$$\sum_{j \in \mathcal{J}} |D_j| (d_p)_j^2 \leq Ch^3 \|\hat{p}\|_{H^3(\Omega)}^2. \quad (6.59)$$

Gathering (6.56), (6.57) and (6.59) ends the proof.  $\square$

## 7 Numerical results

In this section, we test the finite volume method applied to the vorticity-velocity-pressure formulation against known analytical solutions and plot convergence curves (in log-log scale) for several quantities. The domain of computation is  $\Omega = ] - 1/2; 1/2[^2$  and the data are chosen so that the exact solution

$$\hat{\mathbf{u}} = \begin{pmatrix} \exp(x) \cos(\pi y) \\ x \sin(\pi y) + \cos(\pi x) \end{pmatrix} \quad \text{and} \quad \hat{p} = xy \exp(x) \cos(\pi y) \quad (7.1)$$

illustrate section 5.2 based on the Hodge decomposition of  $\mathbf{u}$  with boundary conditions given by  $\hat{\mathbf{u}} \cdot \mathbf{n} = \sigma$ ,  $\int_{\Omega} \hat{p} = 0$  and  $\hat{\omega} = \omega_d$ .

Four families of increasingly fine meshes are used. The first family is a family of standard triangular meshes, see Fig. 8(a). The second family has very localized non-conformities, see Fig. 9(a), and is obtained in the following way: the first mesh is obtained by dividing the domain into  $8 \times 8$  identical squares, and the 4 squares at the center of the mesh are further refined into  $4 \times 4$  sub-squares. Then the subsequent meshes are obtained by dividing each cell of the previous mesh into  $2 \times 2$  square cells. The third family has non-conformities spread over the entire domain, see Fig. 10(a), since every other cell is refined into  $4 \times 4$  sub-cells. Of course, the third family of meshes is not of practical use but illustrates well the ability of the scheme to deal with heavily non-conforming meshes. The fourth family is a family of triangular meshes for the non-simply connected domain  $\Omega = ] - 1/2; 1/2[^2 \setminus ] - 1/6; 1/6[^2$ , see Fig. 11(a).

We have proved (see Sections 6.1 and 6.2) that  $p$ ,  $\omega$ , as well as their gradients converge to the exact solution of the Stokes problem. We are thus interested here in the numerical convergence of  $p$ ,  $\nabla_h^D p$ ,  $\omega$  and  $\nabla_h^D \omega$ , which we measure by the following errors

$$(e0p)^2(h) := \frac{\frac{1}{2} (\sum_i |T_i| (p_i^T - (\Pi\hat{p})_i^T)^2 + \sum_k |P_k| (p_k^P - (\Pi\hat{p})_k^P)^2)}{\frac{1}{2} (\sum_i |T_i| ((\Pi\hat{p})_i^T)^2 + \sum_k |P_k| ((\Pi\hat{p})_k^P)^2)},$$

where  $\forall i \in [1, I]$ ,  $(\Pi\hat{p})_i^T = \hat{p}(G_i)$  and  $\forall k \in [1, K]$ ,  $(\Pi\hat{p})_k^P = \hat{p}(S_k)$ , and

$$(e1p)^2(h) := \frac{\sum_j |D_j| |(\nabla_h^D p)_j - (\Pi\nabla\hat{p})_j|^2}{\sum_j |D_j| |(\Pi\nabla\hat{p})_j|^2},$$

where  $\forall j \in [1, J]$ ,  $(\Pi\nabla\hat{p})_j = (\nabla\hat{p})(B_j)$ , where  $B_j$  is the center of gravity of the diamond cell  $D_j$ . The same definitions hold for  $\omega$  by replacing  $p$  by  $\omega$  in the previous formulae. From the numerical results given in [22], we may expect second-order accuracy for  $p$  and  $\omega$  (although we were able to prove only first-order accuracy in section 6), and first-order accuracy for  $\nabla_h^D p$  and  $\nabla_h^D \omega$  on general meshes. However, on meshes with diamond-cells which are almost all parallelograms, which is the case for the second family of meshes, we expect a convergence order of 1.5 for  $\nabla_h^D p$  and  $\nabla_h^D \omega$ .

We also proved the convergence of the velocity field  $\mathbf{u}$  to the solution of the Stokes problem. The discrete relative  $L^2$  error on the diamond cells for the velocity is measured by the following quantity:

$$e^2(h) := \frac{\sum_j |D_j| |\mathbf{u}_j - (\Pi\hat{\mathbf{u}})_j|^2}{\sum_j |D_j| |(\Pi\hat{\mathbf{u}})_j|^2},$$

where  $(\Pi\hat{\mathbf{u}})_j$  is the value of the exact solution  $\hat{\mathbf{u}}$  at the midpoint of the edge  $A_j$  (denoted by  $M_j$ ):

$$\forall j \in [1, J], (\Pi\hat{\mathbf{u}})_j = \hat{\mathbf{u}}(M_j).$$

We expect first-order convergence of the velocity field on general meshes like those of the first and third families. On the second family of meshes, since almost all diamond cells are parallelograms, we may expect from the numerical results of [21] an order of convergence of at least 1.5. In Finite Element methods, one is also usually concerned with the convergence of  $\nabla\mathbf{u}$ , since  $\mathbf{u}$  belongs to  $H^1(\Omega)$  and since the term  $\int_{\Omega} \nabla\mathbf{u} : \nabla\mathbf{v}d\mathbf{x}$  appears in the bilinear form associated with the variational formulation of the Stokes problem. In our formulation, we used formula (1.7), so that the natural norm induced by the variational formulation is  $\left(\|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)}^2 + \|\nabla \times \mathbf{u}\|_{L^2(\Omega)}^2\right)^{1/2}$ . Since  $\nabla_h^{T,P} \cdot \mathbf{u}$  is always exactly imposed through the first equation of (5.15) or (5.16), we measure the errors on the derivatives of  $\mathbf{u}$  through the error on  $\omega = \nabla_h^{T,P} \times \mathbf{u}$ .

## 7.1 Triangular meshes

We first consider standard triangular meshes, as shown in Fig. 8(a). On this type of meshes,  $\mathbf{u}$ ,  $\nabla_h^D p$  and  $\nabla_h^D \omega$  are all first-order accurate, while  $p$  and  $\omega$  are second-order accurate, as respectively displayed in Fig. 8(b), (d), (f) and (c) and (e). These are the expected orders of convergence, as explained above.

## 7.2 Locally refined meshes

On the second family of meshes (see Fig. 9)(a), we observe a super-convergence of order 1.5 of  $\nabla_h^D p$  and  $\nabla_h^D \omega$ , as expected. Moreover, as far as  $\mathbf{u}$  is concerned, we observe in practice an order of convergence which is better than expected since it is slightly lower than 2. On the third family of meshes, see Fig. 10(a), we recover the same orders of convergence as those obtained on triangular meshes, as expected.

## 7.3 Non simply-connected meshes

Here, the domain of computation is  $\Omega = ]-1/2; 1/2[^2 \setminus ]-1/6; 1/6[^2$ . We compute the numerical solution on a family of five increasingly fine triangular meshes, the coarser of which is displayed in Fig. 11(a). On this type of meshes,  $\mathbf{u}$ ,  $\nabla_h^D p$  and  $\nabla_h^D \omega$  are all first-order accurate, while  $p$  and  $\omega$  are second-order accurate, as respectively displayed in Fig. 11(b), (d), (f) and (c) and (e). These are the expected orders of convergence.

# 8 Conclusion

We have proposed a finite volume method for the two-dimensional Stokes equations with the non-standard boundary conditions (5.1) to (5.4). These non-standard boundary conditions are treated through the vorticity-velocity-pressure formulation of the Stokes equations, for which the finite volume method was successfully applied over unstructured and non-conforming meshes. Numerical results show a first-order convergence for the velocity, the pressure gradient and the

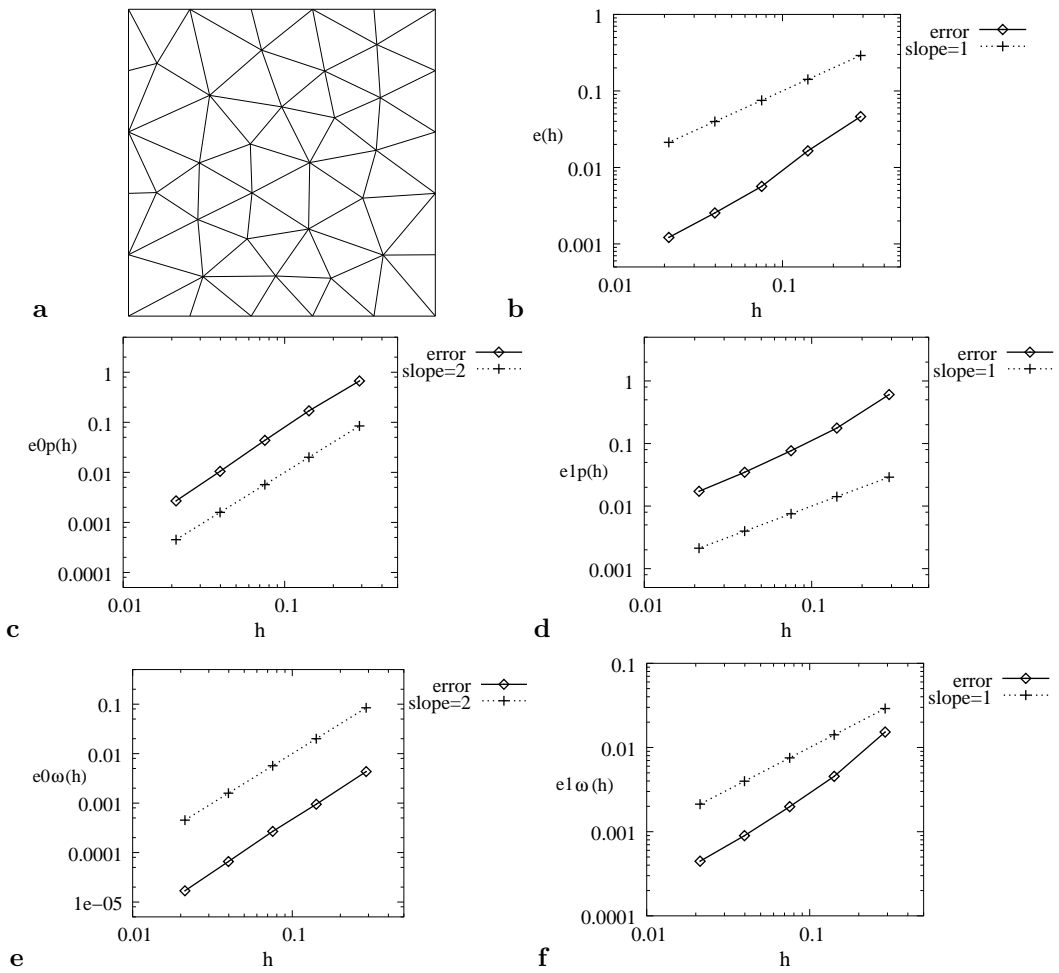


Figure 8: (a) Triangular mesh. (b) Error on the velocity. (c) Error on the pressure. (d) Error on the pressure gradient. (e) Error on the vorticity. (f) Error on the vorticity gradient.



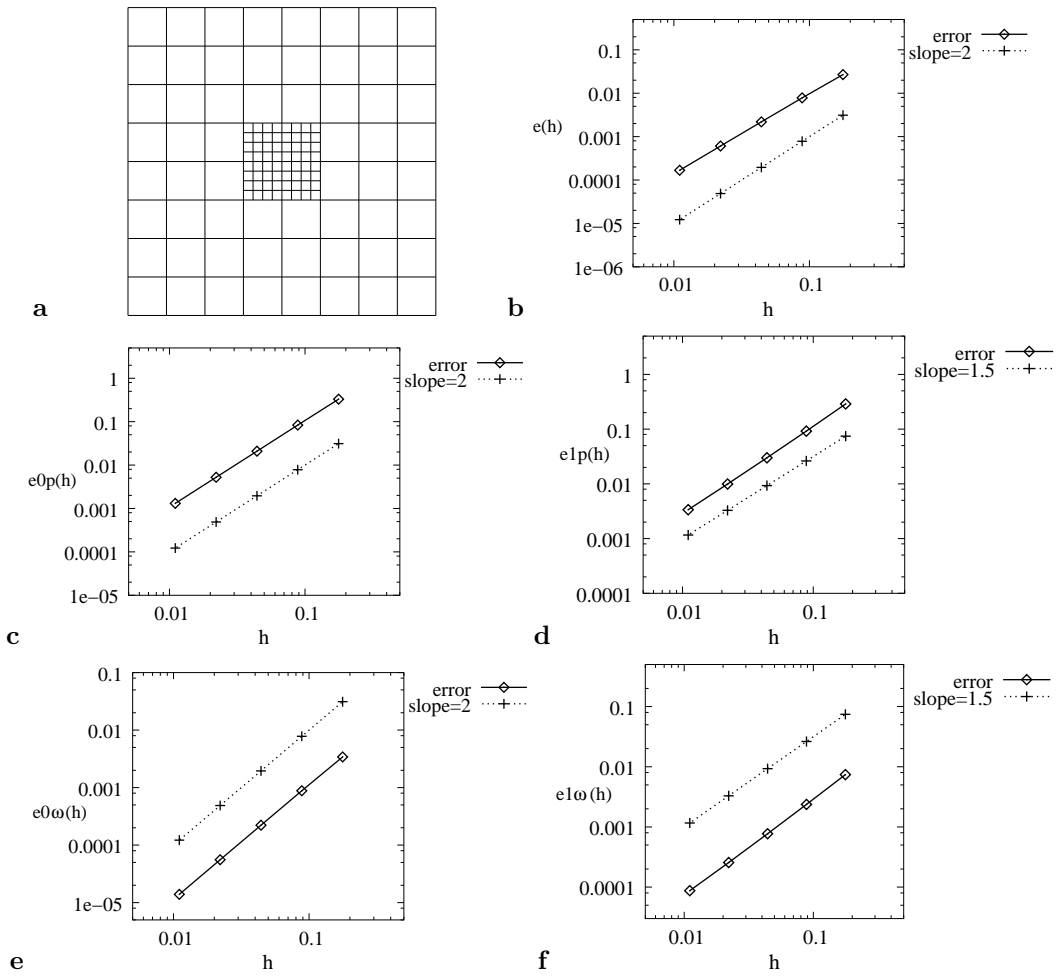


Figure 9: (a) Non-conforming mesh. (b) Error on the velocity. (c) Error on the pressure. (d) Error on the pressure gradient. (e) Error on the vorticity. (f) Error on the vorticity gradient.

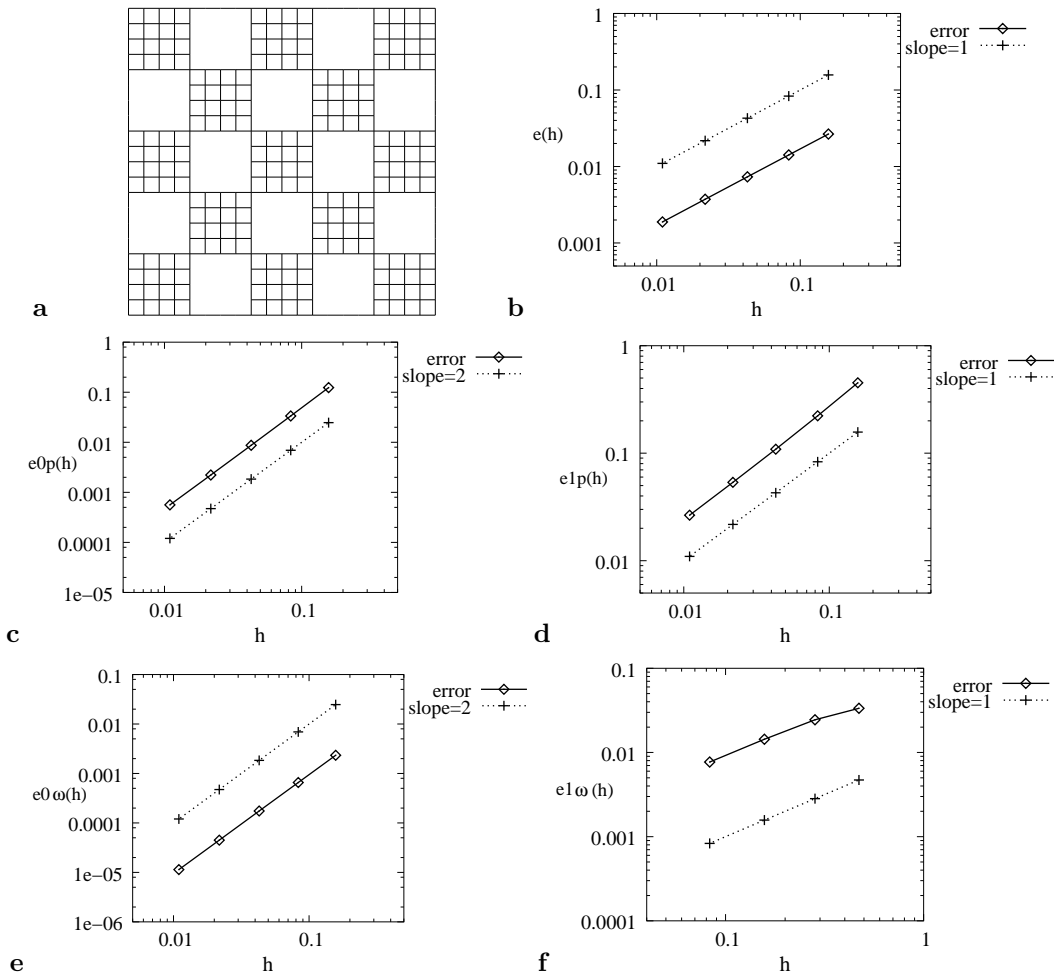


Figure 10: (a) Non-conforming mesh. (b) Error on the velocity. (c) Error on the pressure. (d) Error on the pressure gradient. (e) Error on the vorticity. (f) Error on the vorticity gradient.

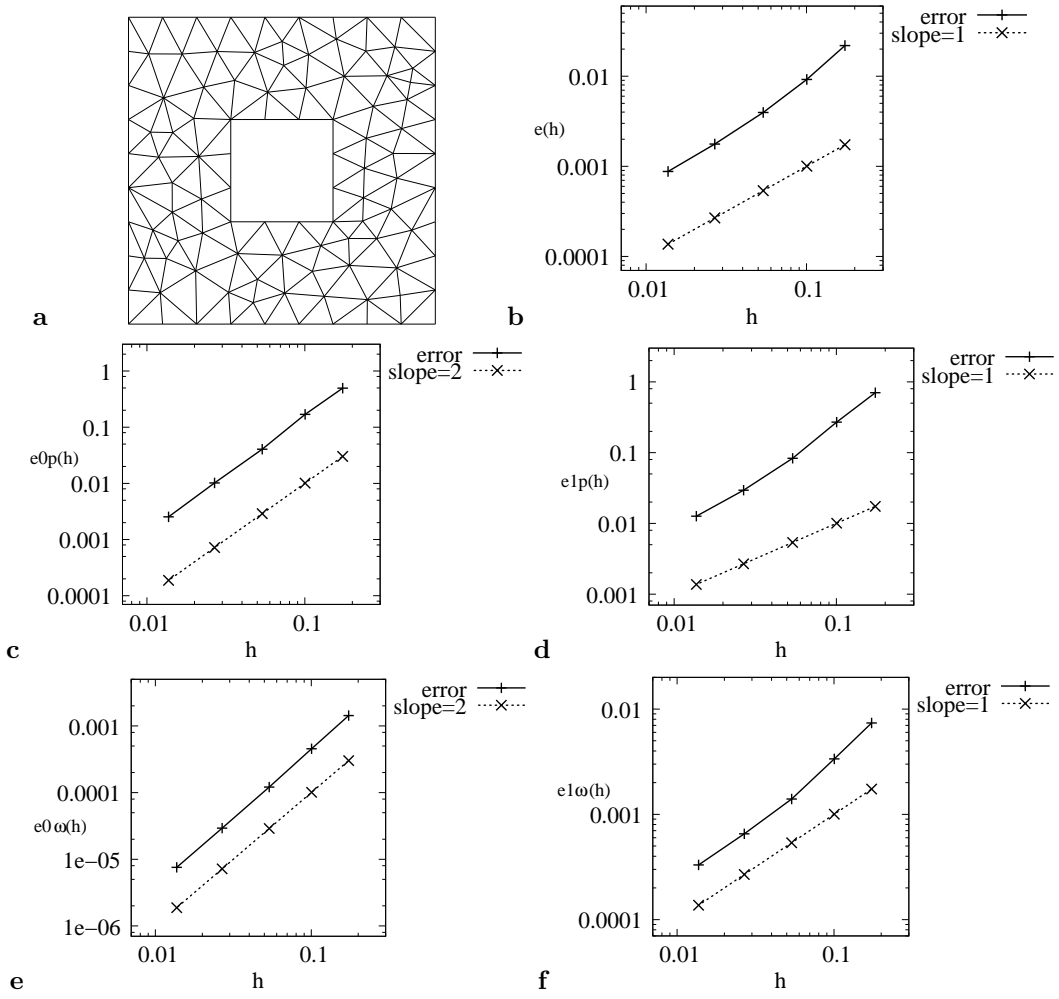


Figure 11: (a) Triangular mesh. (b) Error on the velocity. (c) Error on the pressure. (d) Error on the pressure gradient. (e) Error on the vorticity. (f) Error on the vorticity gradient.

vorticity gradient and a second order convergence for the pressure and the vorticity, while a superconvergence order of 1.5 for the pressure gradient and the vorticity gradient and an order two for the velocity are obtained on regular (but possibly locally non-conforming) meshes. Some of these convergence orders were proved through the theoretical analysis we led in this paper, while only suboptimal orders were obtained for pressure and vorticity on general meshes and for velocity on regular meshes.

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