An a posteriori error estimation for the discrete duality finite volume discretization of the Stokes equations

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AN A POSTERIORI ERROR ESTIMATION FOR THE DISCRETE DUALITY FINITE VOLUME DISCRETIZATION OF THE STOKES EQUATIONS

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Abstract. We derive an a posteriori error estimation for the discrete duality finite volume (DDFV) discretization of the stationary Stokes equations on very general twodimensional meshes, when a penalty term is added in the incompressibility equation to stabilize the variational formulation. Two different estimators are provided: one for the error on the velocity and one for the error on the pressure. They both include a contribution related to the error due to the stabilization of the scheme, and a contribution due to the discretization itself. The estimators are globally upper as well as locally lower bounds for the errors of the DDFV discretization. They are fully computable as soon as a lower bound for the inf-sup constant is available. Numerical experiments illustrate the theoretical results and we especially consider the influence of the penalty parameter on the error for a fixed mesh and also of the mesh size for a fixed value of the penalty parameter. A global error reducing strategy that mixes the decrease of the penalty parameter and adaptive mesh refinement is described.

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Introduction

Let Ω be a two-dimensional simply connected polygonal domain with boundary Γ. We consider the Stokes equations

$$-\Delta\widehat{\mathbf{u}} + \nabla\widehat{p} = \mathbf{f} \text{ in } \Omega, \tag{1}$$

$$\nabla \cdot \widehat{\mathbf{u}} = 0 \text{ in } \Omega, \tag{2}$$

$$\widehat{\mathbf{u}} = \mathbf{g} \text{ on } \Gamma, \tag{3}$$

$$\int_\Omega \widehat{p}(x)dx = 0, \tag{4}$$

where $\widehat{\mathbf{u}}$ is the fluid velocity, $\widehat{p}$ the pressure, $\mathbf{f}$ the body forces per unit mass, and the function $\mathbf{g}$ satisfies $\int_\Gamma \mathbf{g}(\sigma)\cdot\mathbf{n}d\sigma = 0$. With $\mathbf{f} \in H^{-1}(\Omega)$ and $\mathbf{g} \in H^{1/2}(\Gamma)$, this problem is well-posed (see [18]) due to the so-called

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inf-sup condition: there exists $\beta > 0$ such that:

$$\beta = \inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{v} \in (H^1_0(\Omega))^2} \frac{\int_\Omega q \nabla \cdot \mathbf{v}(\mathrm{x}) d\mathrm{x}}{\|\mathbf{v}\|_{(H^1_0(\Omega))^2} \|q\|_{L^2(\Omega)}}, \tag{5}$$

in which $L^2_0(\Omega)$ is the set of L^2 functions over Ω verifying (4).

Our purpose in this work is to compute an a posteriori error estimation between the exact solution $\widehat{\mathbf{u}}$, $\widehat{p}$ of (1)–(4) and its numerical approximation by the penalized discrete duality finite volume scheme (DDFV) as presented in [23]. Originally developed for linear diffusion equations [17, 20, 21], the DDFV method has been extended to nonlinear diffusion [2, 4, 10], convection-diffusion [11], electro-cardiology [1, 12], drift-diffusion and energy-transport models [6], electro- and magnetostatics [15], electromagnetism [22], and Stokes flows [14,16,23]. The originality of the DDFV method is that it is able to treat very general meshes, including very distorted, degenerating, or highly non-conforming meshes (see the numerical tests in [17]). The name of the method comes from the definition of discrete gradient and divergence operators which verify a discrete Green formula.

Like for other equations, the development of a posteriori error estimations for the Stokes problem has followed the a priori investigation of numerical methods. As far as finite elements methods are concerned, R. Verfürth [29] made one the very first contributions by getting two a posteriori error estimations for the mini-element discretization: one is based on a suitable evaluation of the residual, the other is based on the solution of local Stokes problems. Later on, R. Verfürth [30] generalized the first estimator developed in [29] to the non-conforming Crouzeix–Raviart finite element method, neglecting however the consistency error in the estimator. It was shown however in E. Dari et al. [13] that this consistency error may not always be neglected, and, in order to properly take it into account, the authors of [13] use a Helmholtz-Hodge like decomposition (adapted to the Stokes problem) of the velocity error. In the resulting error estimator, this gives rise to terms related to the jumps of the tangential velocity components from one cell to another, in addition to the usual jumps of the normal components of the stress tensor. The case of the non-conforming Fortin–Soulie quadratic elements is also treated in [13].

All the above-cited finite element methods satisfy a uniform discrete inf-sup condition. However, it is often found useful in practice to consider discretizations (especially low-order ones) that do not verify such a uniform discrete inf-sup condition. In this context, C. Bernardi et al. [3] consider the finite element approximation of the Stokes equations when a penalty term is added to stabilize the variational formulation. The a posteriori error estimation they obtain includes two contributions: one related to the discretization on a given mesh, the other related to the penalty term. Based on these two contributions, the mesh refinement and the decrease of the penalty term are linked within an adaptive process.

A very recent contribution by A. Hannukainen et al. [19] sets a general framework for obtaining a posteriori error estimations for the discretization of the Stokes equations. The method is based on the reconstruction of postprocessed H^1_0 conforming velocity and $H - div$ conforming stress tensor fields deduced from the numerical approximation, and it may be applied to various conforming and conforming stabilized finite element methods, the discontinuous Galerkin method, the Crouzeix–Raviart non-conforming finite element method, the mixed finite element method, and a general class of finite volume methods.

However, as far as finite volume methods are concerned, the use of arbitrary meshes in [19] requires first to solve local Stokes problems on a conforming subtriangulation of each control volume, and then to apply the above-cited reconstruction on this subtriangulation. Instead, we would like to obtain error estimates for the solution of the DDFV scheme presented in [23] without having to solve any local problem or to compute any reconstruction. To do this, we shall adapt to the Stokes problem the a posteriori error estimation investigated in [25] for the DDFV discretization of the Laplace equation, using the discrete variational formulation verified by this scheme. The non-conformity of the method is dealt with using the Helmholtz-Hodge like decomposition introduced in [13]. Our estimator also includes a contribution related to the stabilization term in the incompressibility equation, which allows to monitor the amplitude of the penalization coefficient with respect to the mesh refinement process.

Although a DDFV scheme for the Stokes equations was constructed and analyzed in the three dimensional case [24], we limit the discussion in the present article to two space dimensions, because, as emphasized in [24], the 3-D scheme is not a simple extension to three spatial dimensions of the 2-D scheme originally developed in [23], because it is based on a construction for the dual meshes and the diamond mesh that is very specific to 3-D.

This article is organized as follows. Section 1 sets some notations and definitions related to the meshes, to discrete differential operators, and to discrete functions. In section 2, we present the DDFV scheme, and its equivalent variational formula is recalled. In section 3, representations of the errors are elaborated. This is used in section 4 to find a computable upper bound of these errors, provided a lower bound for the inf-sup constant in (5) is known. Such estimations are available in [7–9]. In section 5, the local efficiency of the error estimators is verified. Numerical experiments are presented in section 6.

1. Notations and Definitions

Let Ω be covered by a primal mesh with polygonal cells denoted by T_i, $i \in [1, I]$. We associate with each T_i a point G_i located in the interior of T_i. With any vertex S_k of the primal mesh, with $k \in [1, K]$, we associate a dual cell P_k by joining points G_i associated with the primal cells surrounding S_k to the midpoints of the edges of which S_k is a node. The notations are summarized in Fig. 1 and 2.

With any primal edge A_j with $j \in [1, J]$, we associate a so-called diamond-cell D_j obtained by joining the vertices $S_{k_1(j)}$ and $S_{k_2(j)}$ of A_j to the points $G_{i_1(j)}$ and $G_{i_2(j)}$ associated with the primal cells that share A_j as a part of their boundaries. When A_j is a boundary edge (there are J^Γ such edges), the associated diamond-cell is a flat quadrilateral (i.e. a triangle) and we denote by $G_{i_2(j)}$ the midpoint of A_j (thus, there are J^Γ such additional points G_i). The unit normal vector to A_j is $\mathbf{n}_j$ and points from $G_{i_1(j)}$ to $G_{i_2(j)}$. We denote by A'_{j1} (resp. A'_{j2}) the segment joining $G_{i_1(j)}$ (resp. $G_{i_2(j)}$) and the midpoint of A_j. Its associated unit normal vector, pointing from $S_{k_1(j)}$ to $S_{k_2(j)}$, is denoted by $\mathbf{n}'_{j1}$ (resp. $\mathbf{n}'_{j2}$). In the case of a boundary diamond-cell, A'_{j2} reduces to $\{G_{i_2(j)}\}$ and does not play any role. Finally, for any diamond-cell D_j, we shall denote by $M_{i_\alpha k_\beta}$ the midpoint of $[G_{i_\alpha(j)} S_{k_\beta(j)}]$, with $(\alpha, \beta) \in \{1; 2\}^2$. With $\mathbf{n}_j$, $\mathbf{n}'_{j1}$ and $\mathbf{n}'_{j2}$, we associate orthogonal unit vectors $\boldsymbol{\tau}_j$, $\boldsymbol{\tau}'_{j1}$ and $\boldsymbol{\tau}'_{j2}$, such that the corresponding orthonormal bases are positively oriented. For any primal cell T_i such that $A_j \subset \partial T_i$, we shall define $\mathbf{n}_{ji} := \mathbf{n}_j$ if $i = i_1(j)$ and $\mathbf{n}_{ji} := -\mathbf{n}_j$ if $i = i_2(j)$, so that $\mathbf{n}_{ji}$ is always exterior to T_i. With $\mathbf{n}_{ji}$, we associate $\boldsymbol{\tau}_{ji}$ such that $(\mathbf{n}_{ji}, \boldsymbol{\tau}_{ji})$ is positively oriented. Similarly, when A'_{j1} and A'_{j2} belong to ∂P_k, we define $(\mathbf{n}'_{jk1}, \boldsymbol{\tau}'_{jk1})$ and $(\mathbf{n}'_{jk2}, \boldsymbol{\tau}'_{jk2})$ so that $\mathbf{n}'_{jk1}$ and $\mathbf{n}'_{jk2}$ are orthogonal to A'_{j1} and A'_{j2} and exterior to P_k.

NB: by a slight abuse of notations, we shall write $i \in \Gamma$ (respectively $j \in \Gamma$ and $k \in \Gamma$) to mean $G_i \in \Gamma$, (resp. $A_j \subset \Gamma$ and $S_k \in \Gamma$). The same convention will be used for any set of points other than Γ; e.g. for ∂T_i. We shall write $j \in \partial P_k$ to mean that A'_{j1} and and A'_{j2} are subsets of ∂P_k.

For $\mathbf{v} \in (H^2(\Omega))^2$ with $\mathbf{v} = (v_1, v_2)^t$, we define

$$\nabla \mathbf{v} = \begin{pmatrix} \partial v_1/\partial x & \partial v_1/\partial y \\ \partial v_2/\partial x & \partial v_2/\partial y \end{pmatrix}, \quad \nabla \times \mathbf{v} = \begin{pmatrix} \partial v_1/\partial y & -\partial v_1/\partial x \\ \partial v_2/\partial y & -\partial v_2/\partial x \end{pmatrix},$$

$$\Delta \mathbf{v} = \begin{pmatrix} \Delta v_1 \\ \Delta v_2 \end{pmatrix}.$$

If A and B are two matrices with dimension M, we define the inner product

$$A : B = \sum_{i,j=1}^{M} A_{ij} B_{ij}.$$

For future use, we recall Green's formulae

$$\int_\Omega \Delta \mathbf{v} \cdot \mathbf{w} dx = -\int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w} + \int_{\partial\Omega} (\nabla \mathbf{v}\ \mathbf{n}) \cdot \mathbf{w} ds, \tag{6}$$

$$\int_\Omega \nabla \mathbf{v} : \nabla \times \mathbf{w} dx = -\int_{\partial\Omega} (\nabla \mathbf{v} \boldsymbol{\tau}) \cdot \mathbf{w} ds, \tag{7}$$

for any $\mathbf{v} \in (H^2(\Omega))^2$ and $\mathbf{w} \in (H^1(\Omega))^2$. Here, $\mathbf{n}$ is the outward normal to $\partial\Omega$ and $\boldsymbol{\tau}$ is the tangent vector to $\partial\Omega$ such that $(\mathbf{n}, \boldsymbol{\tau})$ is positively oriented.

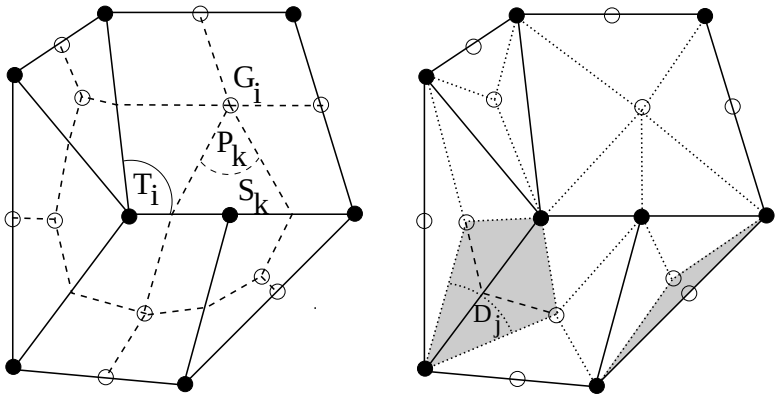

FIGURE 1. A non-conforming primal mesh (solid lines) and its associated dual mesh (left, dashed lines) and diamond mesh (right, dotted lines).

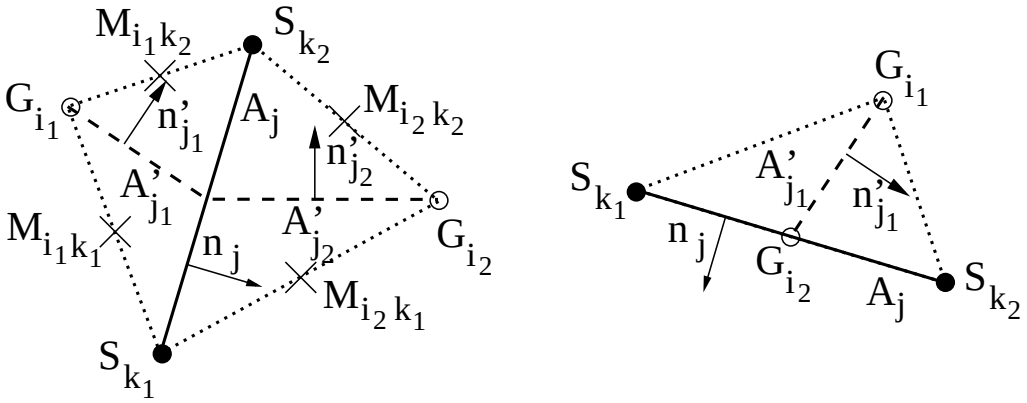

FIGURE 2. Notations for an inner diamond-cell (left) and a boundary diamond cell (right).

In the definition of the DDFV scheme, we shall associate the velocity unknowns to the points G_i and S_k and the pressure unknowns to the diamond-cells. Moreover the gradient and divergence of the velocity will be defined on the diamond-cells. This leads us to the following definitions.

Definition 1.1. Let $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P)$, and $\mathbf{v} = (\mathbf{v}_i^T, \mathbf{v}_k^P)$ be in $(\mathbb{R}^2)^I \times (\mathbb{R}^2)^K$. Let $\Phi = (\Phi_j)$ and $\Psi = (\Psi_j)$ be in $(\mathbb{R}^{2\times 2})^J$. Let $p = (p_j)$ and $q = (q_j)$ be in $\mathbb{R}^J$. We define the following scalar products

$$(\mathbf{u}, \mathbf{v})_{T,P} := \frac{1}{2}\left(\sum_{i\in[1,I]} |T_i| \mathbf{u}_i^T \cdot \mathbf{v}_i^T + \sum_{k\in[1,K]} |P_k| \mathbf{u}_k^P \cdot \mathbf{v}_k^P \right), \tag{8}$$

$$(\Phi, \Psi)_D := \sum_{j\in[1,J]} |D_j| \Phi_j : \Psi_j \quad \text{and} \quad (p,q)_D = \sum_{j\in[1,J]} |D_j| p_j q_j. \tag{9}$$

Definition 1.2. Let $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P)$ be in $(\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$. For any boundary edge A_j, with the notations of Fig. 2 (right), we define $\tilde{\mathbf{u}}_j$ as the trace of $\mathbf{u}$ over A_j by

$$\tilde{\mathbf{u}}_j = \frac{1}{4}\left(\mathbf{u}_{k_1(j)}^P + 2\mathbf{u}_{i_2(j)}^T + \mathbf{u}_{k_2(j)}^P \right). \tag{10}$$

Let $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P)$ be in $(\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$ and let $\mathbf{w} = (\mathbf{w}_j)$ be defined on the boundary Γ. We define the following boundary scalar product

$$(\mathbf{w}, \tilde{\mathbf{u}})_{\Gamma_h} := \sum_{j\in\Gamma} |A_j| \mathbf{w}_j \cdot \tilde{\mathbf{u}}_j. \tag{11}$$

Definition 1.3. Let $\Phi = (\Phi_j)$ be in $(\mathbb{R}^{2\times 2})^J$. We define divergences and curls of the tensor field Φ on the primal and dual cells by

$$(\nabla_h^T \cdot \Phi)_i := \frac{1}{|T_i|} \sum_{j\in\partial T_i} |A_j| \Phi_j \mathbf{n}_{ji},$$

$$(\nabla_h^P \cdot \Phi)_k := \frac{1}{|P_k|}\left(\sum_{j\in\partial P_k} (|A'_{j_1}| \Phi_j \mathbf{n'}_{j_1} + |A'_{j_2}| \Phi_j \mathbf{n'}_{j_2}) + \sum_{j\in\partial P_k\cap\Gamma} \frac{|A_j|}{2} \Phi_j \mathbf{n}_j \right),$$

$$(\nabla_h^T \times \Phi)_i := \frac{1}{|T_i|} \sum_{j\in\partial T_i} |A_j| \Phi_j \boldsymbol{\tau}_{ji},$$

$$(\nabla_h^P \times \Phi)_k := \frac{1}{|P_k|}\left(\sum_{j\in\partial P_k} (|A'_{j_1}| \Phi_j \boldsymbol{\tau}'_{j_1} + |A'_{j_2}| \Phi_j \boldsymbol{\tau}'_{j_2}) + \sum_{j\in\partial P_k\cap\Gamma} \frac{|A_j|}{2} \Phi_j \boldsymbol{\tau}_j \right).$$

Definition 1.4. Let $u_1 = ((u_1)_i^T, (u_1)_k^P)$ be in $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ and $u_2 = ((u_2)_i^T, (u_2)_k^P)$ be in $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, and $\mathbf{u} = (u_1, u_2)$; the discrete gradient $\nabla_h^D \mathbf{u}$ and the discrete curl $\nabla_h^D \times \mathbf{u}$ are defined by their values in the diamond-cells D_j by

$$(\nabla_h^D \mathbf{u})_j = \begin{pmatrix} (\nabla_h^D u_1)^t{}_j \\ (\nabla_h^D u_2)^t{}_j \end{pmatrix}, \ (\nabla_h^D \times \mathbf{u})_j = \begin{pmatrix} (\nabla_h^D \times u_1)^t{}_j \\ (\nabla_h^D \times u_2)^t{}_j \end{pmatrix},$$

where, for $\phi \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, we define

$$(\nabla_h^D \phi)_j := \frac{1}{2|D_j|}\left\{ [\phi_{k_2(j)}^P - \phi_{k_1(j)}^P](|A'_{j_1}| \mathbf{n'}_{j_1} + |A'_{j_2}| \mathbf{n'}_{j_2}) + [\phi_{i_2(j)}^T - \phi_{i_1(j)}^T] |A_j| \mathbf{n}_j \right\},$$

$$(\nabla_h^D \times \phi)_j := -\frac{1}{2|D_j|}\left\{ [\phi_{k_2(j)}^P - \phi_{k_1(j)}^P](|A'_{j_1}| \boldsymbol{\tau}'_{j_1} + |A'_{j_2}| \boldsymbol{\tau}'_{j_2}) + [\phi_{i_2(j)}^T - \phi_{i_1(j)}^T] |A_j| \boldsymbol{\tau}_j \right\}.$$

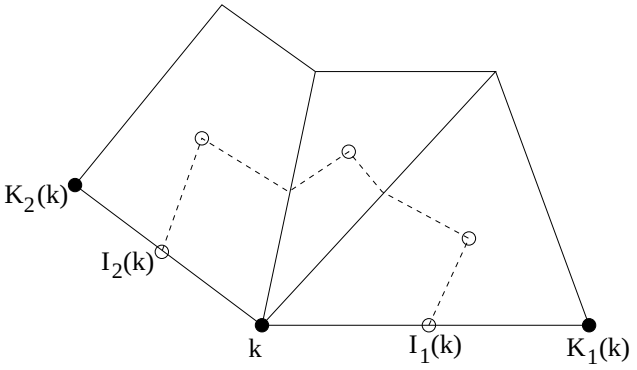

FIGURE 3. Notation for a boundary dual cell in formula (19).

We also need a discrete divergence of a vector, which is defined using the discrete gradient

$$(\nabla_h^D \cdot \mathbf{u})_j = \mathrm{Trace}((\nabla_{\mathrm{h}}^{\mathrm{D}}\mathbf{u})_{\mathrm{j}}).$$

From basic geometrical arguments, we obtain some properties of the discrete gradient:

$$\phi^P_{k_2(j)} - \phi^P_{k_1(j)} = (\nabla_h^D\phi)_j \cdot \overrightarrow{S_{k_1(j)}S_{k_2(j)}} \quad , \quad \phi^T_{i_2(j)} - \phi^T_{i_1(j)} = (\nabla_h^D\phi)_j \cdot \overrightarrow{G_{i_1(j)}G_{i_2(j)}}. \tag{12}$$

For the penalization of the scheme, we need to define the following (non-consistent) Laplacian-type operator.

Definition 1.5. Let $p = (p_j) \in \mathbb{R}^J$, we define:

$$(\Delta_h^D p)_j = \frac{1}{|D_j|} \sum_{j' \in \partial D_j} \frac{d_j^2 + d_{j'}^2}{d_j^2}(p_{j'} - p_j), \tag{13}$$

where ∂D_j is the set of indexes of diamond cells which have a common boundary segment with D_j and $d_j = diam(D_j)$ and $d_{j'} = diam(D_{j'})$.

Proposition 1.6. *For $\Phi \in (\mathbb{R}^{2\times 2})^J$ and $= (\mathbf{u}^T, \mathbf{u}^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$ and $p \in \mathbb{R}^J$, the following discrete Green formula hold:*

$$(\nabla_h^{T,P} \cdot \Phi, \mathbf{u})_{T,P} = -(\nabla_h^D \mathbf{u}, \Phi)_D + (\Phi\mathbf{n}, \tilde{\mathbf{u}})_{\Gamma,h}, \tag{14}$$

$$(\nabla_h^{T,P} \times \Phi, \mathbf{u})_{T,P} = (\nabla_h^D \times \mathbf{u}, \Phi)_D + (\Phi\boldsymbol{\tau}, \tilde{\mathbf{u}})_{\Gamma,h}, \tag{15}$$

$$(\nabla_h^{T,P} \cdot pI_2, \mathbf{u})_{T,P} = -(\nabla_h^D \cdot \mathbf{u}, p)_D + (p\mathbf{n}, \tilde{\mathbf{u}})_{\Gamma,h}, \tag{16}$$

where I_2 is the 2×2 identity matrix and $\nabla_h^{T,P}\cdot$ and $\nabla_h^{T,P}\times$ stand for $\nabla_h^T\cdot$ and $\nabla_h^T\times$ on the primal cells and for $\nabla_h^P\cdot$ and $\nabla_h^P\times$ on the dual cells.

Formula (14) is called the discrete Stokes formula and is proved in [23]; its componentwise counterpart can be found in [17]. Formulae (15) and (16) can be demonstrated in the same way. Proposition 1.7 below may be found componentwise in [15].

Proposition 1.7. *For all $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$, it holds that*

$$(\nabla_h^T \times (\nabla_h^D \mathbf{u}))_i = 0 \ , \ \forall i \in [1, I], \tag{17}$$

$$(\nabla_h^P \times (\nabla_h^D \mathbf{u}))_k = 0 \ , \ \forall k \notin \Gamma. \tag{18}$$

In addition, for $k \in \Gamma$, the following equality holds (see Fig. 3 for the notations)

$$(\nabla_h^P \times (\nabla_h^D \mathbf{u}))_k = \frac{1}{|P_k|}\left[(\mathbf{u}^T_{I_2(k)} - \mathbf{u}^T_{I_1(k)}) + \frac{1}{2}(\mathbf{u}^P_{K_1(k)} - \mathbf{u}^P_{K_2(k)})\right]. \tag{19}$$

The derivation of the a posteriori error estimates is based on the reformulation of the scheme under a variational form which uses functions associated to the discrete unknowns.

Definition 1.8. With any $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$, we associate the function $\mathbf{u}_h$ defined by

$$(\mathbf{u}_h)|_{D_j} \in (P^1(D_j))^2 \ , \ \forall j \in [1, J],$$
$$\mathbf{u}_h(M_{i_\alpha(j)k_\beta(j)}) = \frac{1}{2}(\mathbf{u}^T_{i_\alpha(j)} + \mathbf{u}^P_{k_\beta(j)}) \ , \ \forall j \in [1, J], \ (\alpha, \beta) \in \{1, 2\}^2.$$

In addition, for all $p = (p_j) \in \mathbb{R}^J$, we construct piecewise constant functions corresponding to the approximate pressure, penalty term and to the diamond-cell diameter d_j:

$$p_h(\mathrm{x}) = p_j, \ \forall \ \mathrm{x} \in D_j, \ j \in [1, J], \tag{20}$$
$$(\Delta_h^D p)_h(\mathrm{x}) = (\Delta_h^D p)_j, \ \forall \ \mathrm{x} \in D_j, j \in [1, J], \tag{21}$$
$$d_h(\mathrm{x}) = d_j, \ \forall \ \mathrm{x} \in D_j, j \in [1, J]. \tag{22}$$

The validity of the definition of $\mathbf{u}_h$ by its values in four different points and the proof of the fundamental properties below, which allow to reformulate the scheme under a variational form, may be found in [17].

Proposition 1.9. *Let $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$ and let $\mathbf{u}_h$ be defined by Definition 1.8. It holds that*

$$(\nabla_h^D \mathbf{u})_j = \nabla \mathbf{u}_h|_{D_j}, \ \forall \ j \in [1, J], \tag{23}$$
$$(\nabla_h^D \cdot \mathbf{u})_j = \nabla \cdot \mathbf{u}_h|_{D_j}, \ \forall \ j \in [1, J]. \tag{24}$$

Definition 1.10. Let $\mathbf{u}_h$ be a broken affine function on the diamond mesh: $(\mathbf{u}_h)|_{D_j} \in (P^1(D_j))^2, \ \forall j \in [1, J]$, and $\mathbf{u}_h$ is not necessarily continuous over the interfaces of neighboring diamond-cells. We define its broken gradient and divergence over Ω by:

$$\nabla_h \mathbf{u}_h(\mathrm{x}) = \nabla \mathbf{u}_h|_{D_j}(\mathrm{x}) \text{ and } \nabla_h \cdot \mathbf{u}_h(\mathrm{x}) = \nabla \cdot \mathbf{u}_h|_{D_j}(\mathrm{x}), \ \forall \mathrm{x} \in D_j, \ j \in [1, J]. \tag{25}$$

2. The Finite Volume Scheme on General Meshes

The finite volume scheme used for the numerical approximation of equations (1)–(4) is constructed on the basis of the discrete differential operators defined in section 1. It is very similar to that studied in [23], where proofs of existence, uniqueness, stability with respect to the data and a priori error estimates can be found as soon as $\varepsilon > 0$.

$$(\nabla_h^T \cdot (-\nabla_h^D \mathbf{u} + pI_2))_i = \mathbf{f}_i^T \ , \ \forall i \in [1, I], \tag{26}$$
$$(\nabla_h^P \cdot (-\nabla_h^D \mathbf{u} + pI_2))_k = \mathbf{f}_k^P \ , \ \forall k \in [1, K], \tag{27}$$
$$(\nabla_h^D \cdot \mathbf{u})_j - \varepsilon d_j^2 (\Delta_h^D p)_j = 0 \ , \ \forall j \in [1, J], \tag{28}$$
$$\frac{\mathbf{u}^P_{k_1(j)} + 2\mathbf{u}^T_{i_2(j)} + \mathbf{u}^P_{k_2(j)}}{4} = \mathbf{g}_j \ , \ \forall j \in \Gamma, \tag{29}$$
$$\sum_{j=1}^{J} |D_j| p_j = 0. \tag{30}$$

We suppose that $\mathbf{g}$ is regular enough, so that we can set $\mathbf{g}_j = \mathbf{g}(G_{i_2(j)})$ in (29), while in (26) and (27), $\mathbf{f}_i^T$ and $\mathbf{f}_k^P$ are the mean values of $\mathbf{f}$ over T_i and P_k, respectively:

$$\mathbf{f}_i^T = \frac{1}{|T_i|}\int_{T_i} \mathbf{f}(\mathrm{x})d\mathrm{x} \quad \text{and} \quad \mathbf{f}_k^P = \frac{1}{|P_k|}\int_{P_k} \mathbf{f}(\mathrm{x})d\mathrm{x}. \tag{31}$$

We can prove that the solution of the scheme verifies a discrete variational formulation:

Proposition 2.1. *Let $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P)$ and $p = (p_j)_{j\in[1,J]}$ be the solution of the scheme (26)-(30). Let $\mathbf{v} = (\mathbf{v}_i^T, \mathbf{v}_k^P)$ such that $\tilde{\mathbf{v}}_j = 0$ for all $j \in \Gamma$. Let $\mathbf{u}_h$ and $\mathbf{v}_h$ be the solution associated to $\mathbf{u}$ and $\mathbf{v}$ by Definition 1.8. Let us set in addition*

$$\mathbf{v}_h^*(\mathrm{x}) := \frac{1}{2}\left(\sum_{i\in[1,I]} \mathbf{v}_i^T\theta_i^T(\mathrm{x}) + \sum_{k\in[1,K]} \mathbf{v}_k^P\theta_k^P(\mathrm{x})\right), \tag{32}$$

where θ_i^T and θ_k^P are respectively the characteristic function of the cells T_i and P_k. Then, it holds that

$$\sum_j \int_{D_j} \nabla_h\mathbf{u}_h : \nabla_h\mathbf{v}_h(\mathrm{x})d\mathrm{x} - \sum_j \int_{D_j} \nabla_h\cdot\mathbf{v}_h\ p_h(\mathrm{x})d\mathrm{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h^*(\mathrm{x})d\mathrm{x}. \tag{33}$$

Proof. Starting from Eq. (26) and (27), we have

$$-(\nabla_h^T\cdot(\nabla_h^D\mathbf{u})_i\cdot\mathbf{v}_i^T + (\nabla_h^T\cdot(pI_2)_i)\cdot\mathbf{v}_i^T = \mathbf{f}_i^T\cdot\mathbf{v}_i^T \ \ , \ \ \forall i\in[1,I], \tag{34}$$

$$-(\nabla_h^P\cdot(\nabla_h^D\mathbf{u}))_i\cdot\mathbf{v}_k^P + (\nabla_h^P\cdot(pI_2))_k\cdot\mathbf{v}_k^P = \mathbf{f}_k^P\cdot\mathbf{v}_k^P \ \ , \ \ \forall k\in[1,K]. \tag{35}$$

Multiplying (34) by $|T_i|$ and (35) by $|P_k|$ and summing over all i and all k, we obtain

$$-(\nabla_h^{T,P}\cdot(\nabla_h^D\mathbf{u}),\mathbf{v})_{T,P} + (\nabla_h^{T,P}\cdot(pI_2),\mathbf{v})_{T,P} = (\mathbf{f},\mathbf{v})_{T,P}.$$

We can apply (14), (16) and $\tilde{\mathbf{v}}_j = 0$ for all $j\in\Gamma$ and we obtain

$$(\nabla_h^D\mathbf{u},\nabla_h^D\mathbf{v})_D - (\nabla_h^D\cdot\mathbf{v},p)_D = (\mathbf{f},\mathbf{v})_{T,P}.$$

Using (23)–(24) and definitions (31)–(32), we obtain (33). □

3. A representation of the error

3.1. A representation of the velocity error

The variational formulation of (1) reads:

$$\int_\Omega \nabla\widehat{\mathbf{u}} : \nabla\mathbf{v}d\mathrm{x} - \int_\Omega \widehat{p}\nabla\cdot\mathbf{v}d\mathrm{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}(\mathrm{x})d\mathrm{x}, \tag{36}$$

for all $\mathbf{v}\in(H_0^1(\Omega))^2$. We shall estimate the H^1 semi norm of the error between the exact solution $\widehat{\mathbf{u}}$ and the function $\mathbf{u}_h$ associated to the solution of the DDFV scheme. For this, we shall denote by $\mathbf{e} := \widehat{\mathbf{u}} - \mathbf{u}_h$ and $e_p := \widehat{p} - p_h$ the error in the velocity and pressure, respectively. We have

$$\|\nabla_h\mathbf{e}\|_{L^2(\Omega)} = \left(\sum_j \int_{D_j} |\nabla\widehat{\mathbf{u}} - \nabla_h\mathbf{u}_h|^2(\mathrm{x})d\mathrm{x}\right)^{1/2}. \tag{37}$$

Since Ω is a simply connected domain and since $\nabla_h \mathbf{e} = \nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h$ belongs to $(L^2(\Omega))^{2\times 2}$, we may decompose it in the following way (see Lemma 3.2 in [13]):

$$\nabla_h \mathbf{e} = \nabla \widehat{\Phi} - qI_2 + \nabla \times \widehat{\Psi}, \tag{38}$$

where $q \in L^2_0(\Omega)$, $\widehat{\Phi} \in (H^1_0(\Omega))^2$ with $\nabla \cdot \widehat{\Phi} = 0$ and $\widehat{\Psi} \in (H^1(\Omega))^2$ with $\int_\Omega \hat{\Psi}(\mathrm{x})d\mathrm{x} = 0$, with the following estimations

$$\begin{aligned} \|\nabla \widehat{\Phi}\|_{L^2(\Omega)} &\leq \|\nabla_h e\|_{L^2(\Omega)}, \\ \|q\|_{L^2(\Omega)} &\leq \frac{2}{\beta} \|\nabla_h e\|_{L^2(\Omega)}, \\ \|\nabla \times \widehat{\Psi}\|_{L^2(\Omega)} &\leq (1 + \frac{2\sqrt{2}}{\beta}) \|\nabla_h e\|_{L^2(\Omega)}, \end{aligned} \tag{39}$$

where β is defined by (5).

Now, we estimate the velocity error using decomposition (38). First observe that

$$\int_\Omega \nabla_h \mathbf{e} : I_2 q(\mathrm{x})d\mathrm{x} = \int_\Omega \nabla_h \cdot \mathbf{e}\, q(\mathrm{x})d\mathrm{x} = \int_\Omega (\nabla \cdot \widehat{\mathbf{u}} - \nabla_h \cdot \mathbf{u}_h) q(\mathrm{x})d\mathrm{x}.$$

From (2) and (28), we have

$$\int_\Omega \nabla_h \mathbf{e} : I_2 q(\mathrm{x})d\mathrm{x} = \varepsilon \int_\Omega d_h^2(\mathrm{x})(\Delta_h^D p)_h q(\mathrm{x})d\mathrm{x}, \tag{40}$$

where we recall that the function d_h is defined through (22). Multiplying the term $\nabla_h \mathbf{e}(\mathrm{x})$ with (38) side by side and integrating over Ω, it holds that

$$\begin{aligned} \|\nabla_h \mathbf{e}\|^2_{L^2(\Omega)} &= \int_\Omega \nabla_h \mathbf{e} : (\nabla \widehat{\Phi} + \nabla \times \hat{\Psi} - qI_2)d\mathrm{x} \\ &= i_1 + i_2 - \varepsilon \int_\Omega d_h^2(\mathrm{x})(\Delta_h^D p)_h q(\mathrm{x})d\mathrm{x}, \end{aligned} \tag{41}$$

where

$$i_1 = \sum_j \int_{D_j} (\nabla \widehat{\mathbf{u}} - \nabla \mathbf{u}_h) : \nabla \hat{\Phi}(\mathrm{x})d\mathrm{x}$$

and

$$i_2 = \sum_j \int_{D_j} (\nabla \widehat{\mathbf{u}} - \nabla \mathbf{u}_h) : \nabla \times \hat{\Psi}(\mathrm{x})d\mathrm{x}.$$

In order to find a suitable representation of i_1 and i_2, we shall need the following definitions

Definition 3.1. The boundary ∂D_j of any diamond-cell D_j is composed of the four segments $[G_{i_\alpha(j)} S_{k_\beta(j)}]$ with $(\beta, \alpha) \in \{1, 2\}$ (see Fig. 2). Let us define by S the set of these edges when j runs over the whole set of diamond-cells and $\overset{\circ}{S}$ those edges that do not lie in the boundary Γ. Each $s \in \overset{\circ}{S}$ is thus a segment that we shall denote by $[G_{i(s)} S_{k(s)}]$. We shall also write $s \in \overset{\circ}{T_i}$ (resp. $s \in \overset{\circ}{P_k}$) if $s \subset T_i$ (resp. $s \subset P_k$) and $s \not\subset \Gamma$. Finally, we shall denote by $\mathbf{n}_s$ one of the two unit normal vectors to s, arbitrarily chosen among the two possible choices

but then fixed in what follows, and $[(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s$ is the jump of the normal component of $\nabla_h \mathbf{u}_h - p_h I_2$ through segment s. Moreover, $\boldsymbol{\tau}_s$ will be such that $(\mathbf{n}_s, \boldsymbol{\tau}_s)$ is a positively oriented orthonormal basis of $\mathbb{R}^2$ and $[(\nabla_h \mathbf{u}_h)\boldsymbol{\tau}_s]_s$ is the jump of the tangential component of $\nabla_h \mathbf{u}_h$ through segment s.

Proposition 3.2. *Let $\widehat{\Phi}$ be defined in equation (38). Let $\Phi = (\Phi_i^T, \Phi_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$ be such that*

$$\tilde{\Phi}_j = 0 \text{ for all } j \in \Gamma. \tag{42}$$

The values (Φ_i^T, Φ_k^P) are up to now unrelated to $\widehat{\Phi}$, they will be chosen in section 4.2. Then, it holds that

$$\begin{aligned} i_1 = &\frac{1}{2}\sum_{i\in[1,I]} \int_{T_i} \mathbf{f}\cdot(\widehat{\Phi} - \Phi_i^T)(\mathbf{x})d\mathbf{x} + \frac{1}{2}\sum_{k\in[1,K]} \int_{P_k} \mathbf{f}\cdot(\widehat{\Phi} - \Phi_k^P)(\mathbf{x})d\mathbf{x} \\ &- \frac{1}{2}\sum_{i\in[1,I]} \sum_{s\subset \overset{\circ}{T}_i} \int_s [(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s \cdot (\widehat{\Phi} - \Phi_i^T)(\sigma)d\sigma \\ &- \frac{1}{2}\sum_{k\in[1,K]} \sum_{s\subset \overset{\circ}{P}_k} \int_s [(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s \cdot (\widehat{\Phi} - \Phi_k^P)(\sigma)d\sigma. \end{aligned} \tag{43}$$

Proof. First, since $\widehat{\Phi} \in (H_0^1(\Omega))^2$ and $\nabla \cdot \widehat{\Phi} = 0$, using (36) yields:

$$\begin{aligned} i_1 &= \sum_j \int_{D_j} \nabla\widehat{\mathbf{u}} : \nabla\hat{\Phi}(\mathrm{x})d\mathrm{x} - \sum_j \int_{D_j} \nabla_h \mathbf{u}_h : \nabla\hat{\Phi}(\mathrm{x})d\mathrm{x} \\ &= \int_\Omega \nabla\widehat{\mathbf{u}} : \nabla\hat{\Phi}(\mathrm{x})d\mathrm{x} - \sum_j \int_{D_j} \nabla_h \mathbf{u}_h : \nabla\hat{\Phi}(\mathrm{x})d\mathrm{x} \\ &= \int_\Omega \mathbf{f}\cdot\widehat{\Phi}(\mathrm{x})d\mathrm{x} - \sum_j \int_{D_j} \nabla_h \mathbf{u}_h : \nabla\widehat{\Phi}(\mathrm{x})d\mathrm{x}. \end{aligned}$$

For any $\Phi = (\Phi_i^T, \Phi_k^P)$ satisfying (42), formula (33) leads to

$$\begin{aligned} i_1 = &\int_\Omega \mathbf{f}\cdot(\hat{\Phi} - \Phi_h^*)(\mathrm{x})d\mathrm{x} - \sum_j \int_{D_j} p_h \nabla_h \cdot \Phi_h(\mathrm{x})d\mathrm{x} \\ &- \sum_j \int_{D_j} \nabla_h \mathbf{u}_h : (\nabla\widehat{\Phi} - \nabla_h\Phi_h)(\mathrm{x})d\mathrm{x}. \end{aligned}$$

We know that $p_h \nabla_h \cdot \Phi_h = p_h I_2 : \nabla_h \Phi_h$ and $p_h I_2 : \nabla\widehat{\Phi} = 0$ (since $\nabla \cdot \widehat{\Phi} = 0$), then

$$i_1 = \int_\Omega \mathbf{f}\cdot(\hat{\Phi} - \Phi_h^*)(\mathrm{x})d\mathrm{x} - \sum_j H_1(j), \tag{44}$$

where

$$H_1(j) = \int_{D_j} (\nabla_h \mathbf{u}_h - p_h I_2) : (\nabla\widehat{\Phi} - \nabla_h\Phi_h)(\mathrm{x})d\mathrm{x}. \tag{45}$$

Let us consider a diamond-cell D_j. Since $\nabla_h \mathbf{u}_h - p_h I_2$ is constant over D_j, we may write, using Green's formula over D_j,

$$H_1(j) = \int_{\partial D_j} (\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_{\partial D_j} \cdot (\widehat{\Phi} - \Phi_h)(\sigma) d\sigma,$$

where $\mathbf{n}_{\partial D_j}$ is the unit normal vector exterior to D_j on its boundary. Moreover, let s be any of the four boundary edges of D_j, the function Φ_h belongs to P^1 over s and the quantity $\nabla_h \mathbf{u}_h - p_h I_2$ is a constant; the integral of $(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_{\partial D_j} \cdot \Phi_h$ along this edge may thus exactly be computed by the midpoint rule; using the definition of Φ_h, this function equals $\frac{1}{2}(\Phi^T_{i(s)} + \Phi^T_{k(s)})$ at the midpoint of s. Thus, it holds that:

$$H_1(j) = \sum_{s \subset \partial D_j} \int_s (\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_{s,j} \cdot \left[\widehat{\Phi} - \frac{1}{2}\left(\Phi^T_{i(s)} + \Phi^P_{k(s)}\right)\right](\sigma) d\sigma, \tag{46}$$

where $\mathbf{n}_{s,j}$ is the unit normal vector exterior to D_j on s.

In (44), in the sum of the $H_1(j)$ over $j \in [1, J]$, there are two types of edges s: those in $\overset{\circ}{S}$ and those included in Γ. First, each $s \in \overset{\circ}{S}$ is the common edge of two diamond-cells; then, in the sum, there are two corresponding integrals over s, in which we can factorize by $\left[\widehat{\Phi} - \frac{1}{2}\left(\Phi^T_{i(s)} + \Phi^P_{k(s)}\right)\right](\sigma)$. Indeed, the jump of this function through s vanishes because $\widehat{\Phi} \in (H^1_0(\Omega))^2$. This implies:

$$\begin{aligned}\sum_j \sum_{\substack{s \subset \partial D_j \\ s \not\subset \Gamma}} \int_s (\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_{s,j} \cdot \left[\widehat{\Phi} - \frac{1}{2}\left(\Phi^T_{i(s)} + \Phi^P_{k(s)}\right)\right](\sigma) d\sigma \quad &= \\ \sum_{s \in \overset{\circ}{S}} \int_s [(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s \cdot \left[\widehat{\Phi} - \frac{1}{2}(\Phi^T_{i(s)} + \Phi^P_{k(s)})\right](\sigma) d\sigma.&\end{aligned} \tag{47}$$

Secondly, each diamond-cell D_j whose boundary intersects Γ has two edges of equal length $s = [G_{i_2(j)} S_{k_\beta(j)}]$ with $\beta \in \{1, 2\}$ which are included in Γ, and their union is exactly A_j. Since $(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_j$ is a constant on A_j, and since $\sum_{\beta \in \{1,2\}} \int_{[G_{i_2(j)} S_{k_\beta(j)}]} (\widehat{\Phi} - \frac{1}{2}(\Phi^T_{i_2(j)} + \Phi^P_{k_\beta(j)}))(\sigma) d\sigma = \int_{A_j} (\widehat{\Phi} - \tilde{\Phi}_h)(\sigma) d\sigma$, we have

$$\begin{aligned}&\sum_{s \in \partial D_j \cap \Gamma} \int_s (\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_{s,j} \cdot \left[\widehat{\Phi} - \frac{1}{2}\left(\Phi^T_{i(s)} + \Phi^P_{k(s)}\right)\right](\sigma) d\sigma \\ &= \sum_{\beta \in \{1,2\}} \int_{[G_{i_2(j)} S_{k_\beta(j)}]} (\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_j \cdot \left[\widehat{\Phi} - \frac{1}{2}\left(\Phi^T_{i_2(j)} + \Phi^P_{k_\beta(j)}\right)\right](\sigma) d\sigma \\ &= \int_{A_j} (\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_j \cdot (\widehat{\Phi} - \tilde{\Phi}_h)(\sigma) d\sigma = 0,\end{aligned} \tag{48}$$

thanks to (42) and to the fact that $\widehat{\Phi} \in (H^1_0(\Omega))^2$. With (47) and (48), we can write

$$\sum_{j \in [1,J]} H_1(j) = \sum_{s \in \overset{\circ}{S}} \int_s [(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s \left[\widehat{\Phi} - \frac{1}{2}(\Phi^T_{i(s)} + \Phi^P_{k(s)})\right](\sigma) d\sigma. \tag{49}$$

Then, we may write $\widehat{\Phi} - \frac{1}{2}\left(\Phi^T_{i(s)} + \Phi^P_{k(s)}\right) = \frac{1}{2}\left[\left(\widehat{\Phi} - \Phi^T_{i(s)}\right) + \left(\widehat{\Phi} - \Phi^P_{k(s)}\right)\right]$. Summing in the right-hand side of (49) the various contributions of Φ^T_i for a fixed i and the various contributions of Φ^P_k for a fixed k, we obtain

the following formula

$$\begin{aligned}\sum_j H_1(j) = \frac{1}{2}\sum_{i\in[1,I]}\sum_{s\subset\overset{\circ}{T}_i}\int_s [(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s \cdot (\widehat{\Phi} - \Phi_i^T)(\sigma)d\sigma \\ + \frac{1}{2}\sum_{k\in[1,K]}\sum_{s\subset\overset{\circ}{P}_k}\int_s [(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s \cdot (\widehat{\Phi} - \Phi_k^P)(\sigma)d\sigma.\end{aligned} \tag{50}$$

Finally, according to (44) and definition (32) of Φ_h^*, we obtain (43). □

Now, we turn to a representation formula for i_2 in (41).

Proposition 3.3. *Let $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P)$ be the velocity component of the solution of the scheme (26)–(30) and $\mathbf{u}_h$ the function associated to $\mathbf{u}$ by Def. 1.8. Let $\widehat{\Psi}$ be defined in (38). Let $\Psi = (\Psi_i^T, \Psi_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$ and Ψ_h be its associated function. The values (Ψ_i^T, Ψ_k^P) are up to now unrelated to $\widehat{\Psi}$, they will be chosen in section 4.2. Then, the following representation holds*

$$\begin{aligned} i_2 = \frac{1}{2}\sum_{i\in[1,I]}\sum_{s\in\overset{\circ}{T}_i}\int_s [\nabla_h \mathbf{u}_h \boldsymbol{\tau}_s]_s \cdot (\widehat{\Psi} - \Psi_i^T)(\sigma)d\sigma \\ + \frac{1}{2}\sum_{k\in[1,K]}\sum_{s\in\overset{\circ}{P}_k}\int_s [\nabla_h \mathbf{u}_h \boldsymbol{\tau}_s]_s \cdot (\widehat{\Psi} - \Psi_k^P)(\sigma)d\sigma \\ - \sum_{k\in\Gamma}\int_{\partial P_k\cap\Gamma} (\nabla \mathbf{g}(\sigma) - \nabla_h \mathbf{u}_h(\sigma))\boldsymbol{\tau}_k \cdot (\widehat{\Psi}(\sigma) - \Psi_k^P)d\sigma.\end{aligned} \tag{51}$$

Proof. From (41), it holds that

$$\begin{aligned} i_2 &= \sum_j \int_{D_j} (\nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h) : \nabla \times \widehat{\Psi}(\mathrm{x})d\mathrm{x} \\ &= \int_\Omega \nabla \widehat{\mathbf{u}} : \nabla \times \widehat{\Psi}(\mathrm{x})d\mathrm{x} - \sum_j \int_{D_j} \nabla_h \mathbf{u}_h : \nabla_h \times \Psi_h(\mathrm{x})d\mathrm{x} - \sum_j H_2(j),\end{aligned} \tag{52}$$

where

$$H_2(j) = \int_{D_j} \nabla_h \mathbf{u}_h : (\nabla \times \widehat{\Psi} - \nabla_h \times \Psi_h)(\mathrm{x})d\mathrm{x}. \tag{53}$$

By application of the continuous Green formula, it holds that

$$\int_\Omega \nabla \widehat{\mathbf{u}} : \nabla \times \widehat{\Psi}(\mathrm{x})d\mathrm{x} = -\int_{\partial\Omega} \nabla \widehat{\mathbf{u}}\ \boldsymbol{\tau} \cdot \widehat{\Psi}(\sigma)d\sigma = -\int_\Gamma \nabla \mathbf{g}\ \boldsymbol{\tau} \cdot \widehat{\Psi}(\sigma)d\sigma. \tag{54}$$

We can evaluate the sum of $H_2(j)$ over j just like we evaluated the sum of $H_1(j)$ in Proposition 3.2. There are only two differences. The first is that the gradients of $\widehat{\Phi}$ and Φ_h are replaced by the curls of $\widehat{\Psi}$ and Ψ_h, which implies that normal vectors $\mathbf{n}_s$ are replaced by tangent vectors $-\boldsymbol{\tau}_s$. The second difference is that the boundary integrals do not vanish any more, but can be evaluated like in the discussion that leads to (48). Then, noting that

$$\sum_{j\in J^\Gamma}\int_{A_j} \nabla_h \mathbf{u}_h \boldsymbol{\tau}_j \cdot \widehat{\Psi}(\sigma)d\sigma = \sum_{k\in\Gamma}\int_{\partial P_k\cap\Gamma} \nabla_h \mathbf{u}_h \boldsymbol{\tau}_k \cdot \widehat{\Psi}(\sigma)d\sigma,$$

where $\boldsymbol{\tau}_k$ is the tangent vector to $\partial P_k \cap \Gamma$ which is positively oriented with respect to the unit normal vector exterior to $\partial P_k \cap \Gamma$, we obtain the following formula

$$\begin{aligned}\sum_j H_2(j) = &-\frac{1}{2}\sum_{i\in[1,I]}\sum_{s\in\overset{\circ}{T}_i}\int_s [\nabla_h \mathbf{u}_h \boldsymbol{\tau}_s]_s \cdot (\widehat{\Psi} - \Psi_i^T)(\sigma)d\sigma \\ &-\frac{1}{2}\sum_{k\in[1,K]}\sum_{s\in\overset{\circ}{P}_k}\int_s [\nabla_h \mathbf{u}_h \boldsymbol{\tau}_s]_s \cdot (\widehat{\Psi} - \Psi_k^P)(\sigma)d\sigma \\ &-\sum_{k\in\Gamma}\int_{P_k\cap\Gamma} \nabla_h \mathbf{u}_h \boldsymbol{\tau}_k \cdot \widehat{\Psi}(\sigma)d\sigma + (\nabla_h^D \mathbf{u}\boldsymbol{\tau}, \tilde{\Psi})_{\Gamma,h}.\end{aligned} \tag{55}$$

Moreover, it holds that (see the proof of Lemma 3.5 below):

$$\sum_j \int_{D_j} \nabla_h \mathbf{u}_h : \nabla_h \times \Psi_h(\mathrm{x})d\mathrm{x} = -\sum_{k\in\Gamma}\int_{\partial P_k\cap\Gamma} (\nabla \mathbf{g}(\sigma) - \nabla_h \mathbf{u}_h(\sigma))\boldsymbol{\tau}_k \cdot \Psi_k^P d\sigma - (\nabla_h^D \mathbf{u}\,\boldsymbol{\tau}, \tilde{\Psi})_{\Gamma,h}. \tag{56}$$

Combining (52), (54), (55), and (56), we obtain (51). □

In order to prove (56), we need some technical lemmas related to the $L^2(\Omega)$ scalar product of discrete gradients and curls.

Lemma 3.4. *Let $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P)$ be the velocity component of the solution of the scheme (26)–(30) and $\Psi = (\Psi_i^T, \Psi_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$. It holds that*

$$(\nabla_h^{T,P} \times (\nabla_h^D \mathbf{u}), \Psi)_{T,P} = -\sum_{k\in\Gamma}\int_{\partial P_k\cap\Gamma} (\nabla \mathbf{g}(\sigma) - \nabla_h \mathbf{u}_h(\sigma))\boldsymbol{\tau}_k \cdot \Psi_k^P d\sigma. \tag{57}$$

Proof. According to Eq. (17) and (18), it holds that

$$(\nabla_h^T \times (\nabla_h^D \mathbf{u}))_i = 0 \ , \ \forall i \in [1, I] \ \text{ and } \ (\nabla_h^P \times (\nabla_h^D \mathbf{u}))_k = 0 \ , \ \forall k \notin \Gamma. \tag{58}$$

On the other hand, since the solution of the discrete problem verifies (29), there holds, for $k \in \Gamma$, with the notations of Fig. 3:

$$\mathbf{u}_{I_1(k)}^T = 2\mathbf{g}(G_{I_1(k)}) - \frac{1}{2}(\mathbf{u}_k^P + \mathbf{u}_{K_1(k)}^P) \text{ and } \mathbf{u}_{I_2(k)}^T = 2\mathbf{g}(G_{I_2(k)}) - \frac{1}{2}(\mathbf{u}_k^P + \mathbf{u}_{K_2(k)}^P). \tag{59}$$

Following (19) and (59), we obtain that

$$(\nabla_h^P \times (\nabla_h^D \mathbf{u}))_k = \frac{1}{|P_k|}\left[2(\mathbf{g}(G_{I_2(k)}) - \mathbf{g}(G_{I_1(k)})) + (\mathbf{u}_{K_1(k)}^P - \mathbf{u}_{K_2(k)}^P)\right], \ \forall\ k \in \Gamma. \tag{60}$$

From (58), and using the definition of the scalar product in (8), we obtain

$$(\nabla_h^{T,P} \times (\nabla_h^D \mathbf{u}), \Psi)_{T,P} = \frac{1}{2}\sum_{k\in\Gamma} |P_k|(\nabla_h^P \times (\nabla_h^D \mathbf{u}))_k \cdot \Psi_k^P.$$

Using (60) leads to

$$(\nabla_h^{T,P} \times (\nabla_h^D \mathbf{u}), \Psi)_{T,P} = \sum_{k\in\Gamma}(\mathbf{g}(G_{I_2(k)}) - \mathbf{g}(G_{I_1(k)})) \cdot \Psi_k^P + \frac{1}{2}\sum_{k\in\Gamma}(\mathbf{u}_{K_1(k)}^P - \mathbf{u}_{K_2(k)}^P) \cdot \Psi_k^P. \tag{61}$$

In addition, we have

$$\mathbf{g}(G_{I_2(k)}) - \mathbf{g}(G_{I_1(k)}) = (\mathbf{g}(G_{I_2(k)}) - \mathbf{g}(S_k)) + (\mathbf{g}(S_k) - \mathbf{g}(G_{I_1(k)}),$$

so that

$$\mathbf{g}(G_{I_2(k)}) - \mathbf{g}(G_{I_1(k)}) = -\int_{\partial P_k \cap \Gamma} \nabla \mathbf{g}(\sigma) \boldsymbol{\tau}_k d\sigma, \tag{62}$$

In the same way, we have

$$\mathbf{u}^P_{K_1(k)} - \mathbf{u}^P_{K_2(k)} = (\mathbf{u}^P_{K_1(k)} - \mathbf{u}^P_k) + (\mathbf{u}^P_k - \mathbf{u}^P_{K_2(k)}).$$

Applying property (12) of the discrete gradient, it holds that

$$\mathbf{u}^P_{K_1(k)} - \mathbf{u}^P_k = |S_k S_{K_1(k)}| \nabla_h \mathbf{u}_h(\sigma) \boldsymbol{\tau}_k = 2|S_k G_{I_1(k)}| \nabla_h \mathbf{u}_h(\sigma) \boldsymbol{\tau}_k, \ \forall \sigma \in [S_k S_{K_1(k)})] \tag{63}$$

and

$$\mathbf{u}^P_k - \mathbf{u}^P_{K_2(k)} = |S_{K_2(k)} S_k| \nabla_h \mathbf{u}_h(\sigma) \boldsymbol{\tau}_k = 2|S_k G_{I_2(k)}| \nabla_h \mathbf{u}_h(\sigma) \boldsymbol{\tau}_k, \ \forall \sigma \in [S_{K_2(k)} S_k)]. \tag{64}$$

As a consequence, it holds that

$$\mathbf{u}^P_{K_1(k)} - \mathbf{u}^P_{K_2(k)} = 2 \int_{\partial P_k \cap \Gamma} \nabla_h \mathbf{u}_h(\sigma) \boldsymbol{\tau}_k d\sigma. \tag{65}$$

Combining (61) with (62) and (65), we obtain (57). □

Lemma 3.5. *Let* $\mathbf{u} = (\mathbf{u}_i^T, \mathbf{u}_k^P)$ *be the velocity component of the solution of the scheme (26)–(30) and* $\Psi = (\Psi_i^T, \Psi_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$. *Let* $\mathbf{u}_h$ *and* Ψ_h *be their associated functions through Def. 1.8. Then formula (56) is true.*

Proof. Applying the discrete Green formula (15), it holds that

$$\begin{aligned} \sum_j \int_{D_j} \nabla_h \mathbf{u}_h : \nabla_h \times \Psi_h(\mathrm{x}) d\mathrm{x} &= (\nabla_h^D \mathbf{u}, \nabla_h^D \times \Psi)_D \\ &= (\nabla_h^{T,P} \times (\nabla_h^D \mathbf{u}), \Psi)_{T,P} - (\nabla_h^D \mathbf{u}\, \boldsymbol{\tau}, \tilde{\Psi})_{\Gamma,h}. \end{aligned}$$

Using the result of Lemma 3.4, we obtain (56). □

3.2. **A representation of the pressure error**

We shall estimate the L^2 norm of the error between the exact solution $\hat{p}$ and the function associated to the solution of the DDFV scheme by (20). We recall that $e_p = \hat{p} - p_h$ and $\mathbf{e} = \hat{\mathbf{u}} - \mathbf{u}_h$.

Proposition 3.6. *Let $\widehat{\mathbf{v}} \in (H_0^1(\Omega))^2$ and $\mathbf{v} = (\mathbf{v}_i^T, \mathbf{v}_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$ be such that $\tilde{\mathbf{v}}_j = 0$ for all $j \in \Gamma$. The values $(\mathbf{v}_i^T, \mathbf{v}_k^P)$ are up to now unrelated to $\widehat{\mathbf{v}}$, they will be chosen in section 4.3. We have that*

$$\begin{aligned}\int_\Omega e_p \nabla\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x} = \int_\Omega \nabla_h\mathbf{e} : \nabla\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x} - \frac{1}{2}\sum_{i\in[1,I]}\int_{T_i}\mathbf{f}\cdot(\widehat{\mathbf{v}}-\mathbf{v}_i^T)(\mathbf{x})d\mathbf{x} - \frac{1}{2}\sum_{k\in[1,K]}\int_{P_k}\mathbf{f}\cdot(\widehat{\mathbf{v}}-\mathbf{v}_k^P)(\mathbf{x})d\mathbf{x}\\ + \frac{1}{2}\sum_{i\in[1,I]}\sum_{s\subset\mathring{T}_i}\int_s[(\nabla_h\mathbf{u}_h - p_hI_2)\mathbf{n}_s]_s\cdot(\widehat{\mathbf{v}}-\mathbf{v}_i^T)(\sigma)d\sigma\\ + \frac{1}{2}\sum_{k\in[1,K]}\sum_{s\subset\mathring{P}_k}\int_s[(\nabla_h\mathbf{u}_h - p_hI_2)\mathbf{n}_s]_s\cdot(\widehat{\mathbf{v}}-\mathbf{v}_k^P)(\sigma)d\sigma.\end{aligned} \tag{66}$$

Proof. We can use formula (36) to obtain

$$\begin{aligned}\int_\Omega e_p\nabla\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x} &= \int_\Omega \hat{p}\nabla\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x} - \int_\Omega p_h\nabla\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x}\\ &= \int_\Omega\nabla_h\mathbf{e}:\nabla\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x} - \int_\Omega\mathbf{f}\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x} + \int_\Omega\nabla_h\mathbf{u}_h:\nabla\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x} - \int_\Omega p_h\nabla\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x}.\end{aligned} \tag{67}$$

Using Eq. (33), we have

$$\int_\Omega e_p\nabla\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x}(\mathrm{x})d\mathrm{x} = \int_\Omega\nabla_h\mathbf{e}:\nabla\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x} - \int_\Omega\mathbf{f}\cdot(\widehat{\mathbf{v}}-\mathbf{v}_h^*)(\mathrm{x})d\mathrm{x} + \int_\Omega(\nabla_h\mathbf{u}_h - p_hI_2):(\nabla\widehat{\mathbf{v}}-\nabla_h\mathbf{v}_h)(\mathrm{x})d\mathrm{x}. \tag{68}$$

Just like in Proposition 3.2 (see (45) and (50)), we have

$$\begin{aligned}\int_\Omega(\nabla_h\mathbf{u}_h - p_hI_2):(\nabla\widehat{\mathbf{v}}-\nabla_h\mathbf{v}_h)(\mathrm{x})d\mathrm{x} = \frac{1}{2}\sum_{i\in[1,I]}\sum_{s\subset\mathring{T}_i}\int_s[(\nabla_h\mathbf{u}_h - p_hI_2)\mathbf{n}_s]_s\cdot(\widehat{\mathbf{v}}-\mathbf{v}_i^T)(\sigma)d\sigma\\ + \frac{1}{2}\sum_{k\in[1,K]}\sum_{s\subset\mathring{P}_k}\int_s[(\nabla_h\mathbf{u}_h - p_hI_2)\mathbf{n}_s]_s\cdot(\widehat{\mathbf{v}}-\mathbf{v}_k^P)(\sigma)d\sigma.\end{aligned} \tag{69}$$

From (68) and (69), we obtain (66). □

4. A Computable Error Bound

4.1. Preliminaries

In this subsection, we will present some Poincaré-type inequalities which are useful to obtain a computable error bound.

Lemma 4.1. *Let ω be an open bounded set which is star-shaped with respect to one of its points. Let $\mathbf{u} \in (H^1(\omega))^2$ and let $\overline{\mathbf{u}}_\omega$ be the mean-value of $\mathbf{u}$ over ω. Then,*

$$\exists C(\omega),\ s.t\ .\ \|\mathbf{u}-\overline{\mathbf{u}}_\omega\|_{L^2(\omega)} \le C(\omega)diam(\omega)\|\nabla\mathbf{u}\|_{L^2(\omega)}. \tag{70}$$

Remark 4.2. In what follows, we shall use Lemma 4.1 on primal and dual cells. Since primal cells are (usually) convex, it is useful to note that when ω is convex, a universal constant $C(\omega)$ is given by $\frac{1}{\pi}$ (see [26]). On the other hand, a dual cell P_k may be non-convex, but the way it is constructed implies that it is star-shaped with respect to the vertex S_k it is associated to, and that it is a union of triangles having S_k as a vertex. Bounds for $C(\omega)$ in

such a case were investigated in [5, 28, 32] and we may use explicit expressions given in these references, which show that the constant $C(P_k)$ only depends on the shape of P_k, not on its diameter, meaning that if P_k is the image of some $P_{k'}$ through the composition of a translation, a rotation and a homothety, then $C(P_k) = C(P_{k'})$. The influence of the shape of P_k over $C(P_k)$ and how this affects the efficiency of the estimators is discussed in section 5.

Finally, we will also need a trace inequality (see [25]).

Lemma 4.3. *Let T be a triangle and let E be one of its edges; let ρ be the distance from E to the vertex of T opposite to E, and let σ be the longest among the two other sides of T. Let $\varepsilon > 0$ be an arbitrary real valued number; then for all $\mathbf{u} \in (H^1(T))^2$, it holds that*

$$\|\mathbf{u}\|^2_{L^2(E)} \leq \frac{1}{\rho}\left((2+\varepsilon^{-2})\|\mathbf{u}\|^2_{L^2(T)} + \varepsilon^2\sigma^2\|\nabla\mathbf{u}\|^2_{L^2(T)}\right). \tag{71}$$

4.2. **A computable bound for the velocity error**

The main result of this Section is the following:

Theorem 4.4. *Let $\|\nabla_h \mathbf{e}\|_{L^2(\Omega)}$ be defined by (37). Let us define the data oscillations by*

$$osc(\mathbf{f}, T, \Omega) = \left(\sum_{i\in[1,I]} (C(T_i)h_i^T)^2\|\mathbf{f} - \mathbf{f}_i^T\|^2_{L^2(T_i)}\right)^{1/2}, \tag{72}$$

$$osc(\mathbf{f}, P, \Omega) = \left(\sum_{k\in[1,K]} (C(P_k)h_k^P)^2\|\mathbf{f} - \mathbf{f}_k^P\|^2_{L^2(P_k)}\right)^{1/2} \tag{73}$$

where $\mathbf{f}_i^T$ and $\mathbf{f}_k^P$ are defined by (31), $h_i^T := diam(T_i)$, $h_k^P := diam(P_k)$ and the constants $C(T_i)$ and $C(P_k)$ are those involved in Lemma 4.1 and Remark 4.2.

We define the local and global error estimators related to the primal mesh:

$$(\eta_i^T)^2 = \inf_{\mu>0}\left[\chi_i(\mu)\sum_{s\in\overset{\circ}{T_i}} C_s(\mu)\|[(\nabla_h\mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s\|^2_{L^2(s)}\right] \text{ and } (\eta^T)^2 = \sum_i(\eta_i^T)^2, \tag{74}$$

$$(\eta'^T_i)^2 = \inf_{\mu>0}\left[\chi_i(\mu)\sum_{s\in\overset{\circ}{T_i}} C_s(\mu)\|[\nabla_h\mathbf{u}_h\boldsymbol{\tau}_s]_s\|^2_{L^2(s)}\right] \text{ and } (\eta'^T)^2 = \sum_i(\eta'^T_i)^2, \tag{75}$$

where the functions χ_i and C_s are defined by (88) and (87) below.

We define the local and global error estimators related to the dual mesh:

$$(\eta_k^P)^2 = \inf_{\mu>0}\left[\chi_k(\mu)\sum_{s\in\overset{\circ}{P_k}} C_s(\mu)\|[(\nabla_h\mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s\|^2_{L^2(s)}\right] \text{ and } (\eta^P)^2 = \sum_k(\eta_k^P)^2, \tag{76}$$

$$(\eta'^P_k)^2 = \inf_{\mu>0}\left[\chi_k(\mu)\sum_{s\in\overset{\circ}{P_k}} C_s(\mu)\|[\nabla_h\mathbf{u}_h\boldsymbol{\tau}_s]_s\|^2_{L^2(s)}\right] \text{ and } (\eta'^P)^2 = \sum_k(\eta'^P_k)^2, \tag{77}$$

where the function χ_k is defined by (92) below.

We define the local and global error estimator related to the boundary:

$$(\zeta_k^P)^2 = \inf_{\mu}\left[\lambda_k(\mu)\sum_{\alpha=1}^{2} C_\alpha(\mu)\|(\nabla \mathbf{g} - \nabla \mathbf{u}_h)\boldsymbol{\tau}_{j_\alpha(k)}\|^2_{L^2(b_{j_\alpha(k)})}\right] \text{ and } (\zeta^P)^2 = \sum_{k\in\Gamma}(\zeta_k^P)^2, \tag{78}$$

where the functions C_α and λ_k and the boundary segment $b_{j_\alpha(k)}$ are defined in Proposition 4.10 below.

Let us define the indicator related to the penalization:

$$\zeta_\varepsilon = \varepsilon\|d_h^2(\Delta_h^D p)_h\|_{L^2(\Omega)}. \tag{79}$$

We have

$$\|\nabla_h \mathbf{e}\|_{L^2(\Omega)} \le \eta = \eta_h + \eta_\epsilon. \tag{80}$$

where

$$\eta_h = \frac{1}{2}\Big(osc(f,T,\Omega) + osc(f,P,\Omega) + \eta^T + \eta^P\Big) + \left(\frac{1}{2} + \frac{\sqrt{2}}{\beta}\right)\Big(\eta'^T + \eta'^P + 2\zeta^P\Big), \tag{81}$$

$$\eta_\epsilon = \frac{2}{\beta}\zeta_\varepsilon. \tag{82}$$

The quantities η, η_h, and η_ϵ are respectively called the total estimator, the discretization estimator, and the penalization estimator for the velocity.

Proof. The result is obtained using the sum (41), expressions (43) and (51), the bounds (85)–(86), (89)–(90), (93)–(94), (97) and (101) below and the relations (39). □

In order to find bounds for the expressions (43) of i_1 and (51) of i_2, the values of (Φ_i^T, Φ_k^P) and (Ψ_i^T, Ψ_k^P) in these expressions have to be specified, since they were up to now arbitrary, except for the boundary midpoint values of (Φ_i^T, Φ_k^P) that were chosen so that (42) holds. This is performed in the definition that follows.

Definition 4.5. Since $\widehat{\Phi}$, $\widehat{\Psi}$ are not necessarily more regular than $(H^1(\Omega))^2$, we choose as an interpolation their L^2 projection on the primal and dual cells:

$$\Phi_i^T = \frac{1}{|T_i|}\int_{T_i}\widehat{\Phi}(\mathrm{x})d\mathrm{x}\ ,\ \forall i\in[1,I]\ ;\ \Phi_k^P = \frac{1}{|P_k|}\int_{P_k}\widehat{\Phi}(\mathrm{x})d\mathrm{x}\ ,\ \forall k\in[1,K], \tag{83}$$

$$\Psi_i^T = \frac{1}{|T_i|}\int_{T_i}\widehat{\Psi}(\mathrm{x})d\mathrm{x}\ ,\ \forall i\in[1,I]\ ;\ \Psi_k^P = \frac{1}{|P_k|}\int_{P_k}\widehat{\Psi}(\mathrm{x})d\mathrm{x}\ ,\ \forall k\in[1,K]. \tag{84}$$

In order to complete the definition of (Φ_i^T, Φ_k^P), for any $i\in\Gamma$, the boundary value of Φ_i^T is given so that (42) holds. Note that it is not necessary to define the value of Ψ_i^T for all $i\in\Gamma$.

Proposition 4.6. *Let $h_i^T := diam(T_i)$, $h_k^P := diam(P_k)$ and let definitions (72) and (73) hold. Then it holds that*

$$\left|\sum_{i\in[1,I]}\int_{T_i}\mathbf{f}\cdot(\widehat{\Phi} - \Phi_i^T)(\mathrm{x})d\mathrm{x}\right| \le osc(\mathbf{f},T,\Omega)\|\nabla\widehat{\Phi}\|_{L^2(\Omega)}, \tag{85}$$

$$\left|\sum_{k\in[1,K]}\int_{P_k}\mathbf{f}\cdot(\widehat{\Phi} - \Phi_k^P)(\mathrm{x})d\mathrm{x}\right| \le osc(\mathbf{f},P,\Omega)\|\nabla\widehat{\Phi}\|_{L^2(\Omega)}. \tag{86}$$

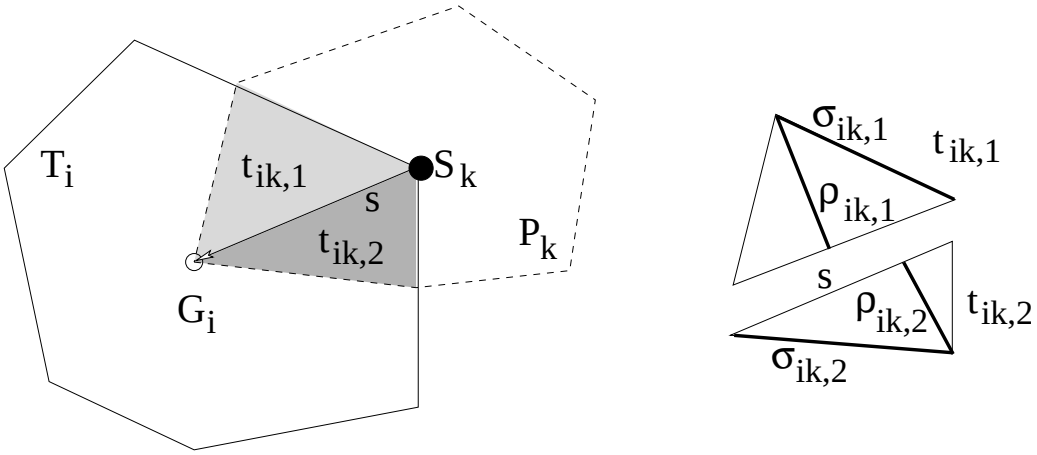

FIGURE 4. For each cell T_i and each vertex S_k of T_i, $T_i \cap P_k$ is split in two triangles $t_{ik,1}$ and $t_{ik,2}$.

Proof. Since Φ_i^T was chosen as the mean-value of $\widehat{\Phi}$ over T_i (see (83)), we have

$$\int_{T_i} \mathbf{f} \cdot (\widehat{\Phi} - \Phi_i^T)(\mathrm{x}) d\mathrm{x} = \int_{T_i} (\mathbf{f} - \mathbf{f}_i^T) \cdot (\widehat{\Phi} - \Phi_i^T)(\mathrm{x}) d\mathrm{x}.$$

Applying the Cauchy-Schwarz inequality, Lemma 4.1 to $\widehat{\Phi}$ over T_i and the discrete Cauchy-Schwarz inequality, we obtain (85). Similarly, we also obtain (86). □

Propositions 4.7 and 4.9 below are proved just like Propositions 5.9 and 5.10 in [25].

Proposition 4.7. *For any primal cell T_i and any dual P_k such that $T_i \cap P_k \neq \emptyset$, Let $s = [G_i S_k]$ and $t_{ik,1}$ and $t_{ik,2}$ be the triangles defined in Fig. 4 such that $t_{ik,1} \cup t_{ik,2} = T_i \cap P_k$. Let $\rho_{ik,\alpha}$ be the distance from s to the vertex of $t_{ik,\alpha}$ opposite to s and $\sigma_{ik,\alpha}$ be the length of the longest among the two other edges of $t_{ik,\alpha}$. $C(T_i)$ is the constant that appears in (70). For any strictly positive μ, let us define*

$$C_s(\mu) = \frac{\left(1 + \sqrt{1 + \frac{\sigma_{ik,1}^2}{\mu}}\right)\left(1 + \sqrt{1 + \frac{\sigma_{ik,2}^2}{\mu}}\right)}{\left(1 + \sqrt{1 + \frac{\sigma_{ik,1}^2}{\mu}}\right)\rho_{ik,2} + \left(1 + \sqrt{1 + \frac{\sigma_{ik,2}^2}{\mu}}\right)\rho_{ik,1}}, \tag{87}$$

$$\chi_i(\mu) = (C(T_i) h_i^T)^2 + \mu. \tag{88}$$

Let η^T and ${\eta'}^T$ be defined by (74)–(75). With these definitions, it holds that:

$$\left| \sum_{i \in [1,I]} \sum_{s \in \overset{\circ}{T_i}} \int_s [(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s \cdot (\hat{\Phi} - \Phi_i^T)(\sigma) d\sigma \right| \leq \eta^T \|\nabla \widehat{\Phi}\|_{L^2(\Omega)}, \tag{89}$$

$$\left| \sum_{i \in [1,I]} \sum_{s \in \overset{\circ}{T_i}} \int_s [\nabla_h \mathbf{u}_h \boldsymbol{\tau}_s]_s \cdot (\hat{\Psi} - \Psi_i^T)(\sigma) d\sigma \right| \leq {\eta'}^T \|\nabla \widehat{\Psi}\|_{L^2(\Omega)}. \tag{90}$$

Remark 4.8. The minimization in (74) is performed numerically when we effectively compute the estimators. Although we have no formal proof that the function to be minimized is unimodal, we used in the numerical tests of section 6 the so-called "golden section search" to perform this minimization (with no claim that this is the most efficient method), starting with the interval $[10^{-2}h_{T_i}^2, 10^2 h_{T_i}^2]$. Moreover, we may already get an idea of the behaviour of η_i^T by choosing $\mu = h_{T_i}^2$ to evaluate it. By definition of $\sigma_{ik,\alpha}$, this length is lower than the diameter of T_i, which implies

$$C_s\left(\left(h_i^T\right)^2\right) \leq \frac{(1+\sqrt{2})^2}{2(\rho_{ik,1}+\rho_{ik,2})}. \tag{91}$$

Under the hypothesis that the ratios $\frac{\rho_{ik,\alpha}}{h_i^T}$ are all bounded by below by the same constant which does not depend on the mesh, we obtain the following bound

$$\eta_i^T \leq K h_i^T \sum_{s \in \overset{\circ}{T}_i} \|[(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s\|_{L^2(s)}^2,$$

where the constant K does not depend on the mesh. The same remark holds for ${\eta'}_i^T$.

Proposition 4.9. *Let us set the same notations as in Prop. 4.7. Let C_s be defined by (87). Let $C(P_k)$ be the constant involved in (70). Let us define for any strictly positive μ,*

$$\chi_k(\mu) = (C(P_k)h_k^P)^2 + \mu. \tag{92}$$

Let η^P and ${\eta'}^P$ be defined by (76)–(77). With these definitions, it holds that:

$$\left|\sum_{k\in[1,K]} \sum_{s\in\overset{\circ}{P}_k} \int_s [(\nabla_h \mathbf{u}_h - p_h I_2)\mathbf{n}_s]_s \cdot (\widehat{\Phi} - \Phi_k^P)(\sigma) d\sigma\right| \leq \eta^P \|\nabla\widehat{\Phi}\|_{L^2(\Omega)}, \tag{93}$$

$$\left|\sum_{k\in[1,K]} \sum_{s\in\overset{\circ}{P}_k} \int_s [\nabla_h \mathbf{u}_h \boldsymbol{\tau}_s]_s \cdot (\widehat{\Psi} - \Psi_k^P)(\sigma) d\sigma\right| \leq {\eta'}^P \|\nabla\widehat{\Psi}\|_{L^2(\Omega)}. \tag{94}$$

The next proposition bounds the boundary terms in the expression (51) of i_2.

Proposition 4.10. *For any $k \in \Gamma$, let us denote by $D_{j_1(k)}$ and $D_{j_2(k)}$ the two diamond cells whose boundary intersect Γ and which have S_k as a vertex. Let $q_{j_1(k)} = P_k \cap D_{j_1(k)}$, $q_{j_2(k)} = P_k \cap D_{j_2(k)}$ and the segment $b_{j_\alpha(k)}$ be the intersection between $\partial q_{j_\alpha(k)}$ and Γ; see Fig. 5. Let $\rho_{j_\alpha(k)}$ be the distance from $b_{j_\alpha(k)}$ to the vertex of $q_{j_\alpha(k)}$ opposite to $b_{j_\alpha(k)}$ and $\sigma_{j_\alpha(k)}$ be the length of the longest among the two other edges of $q_{j_\alpha(k)}$. Let $C(P_k)$ be the constant that appears in (70). For any strictly positive μ, let us define*

$$C_\alpha(\mu) = \frac{2 + \frac{\sigma_{j_\alpha(k)}^2}{\mu + \sqrt{\mu^2 + \mu\sigma_{j_\alpha(k)}^2}}}{\rho_{j_\alpha(k)}}, \tag{95}$$

$$\lambda_k(\mu) = (C(P_k)h_k^P)^2 + \mu. \tag{96}$$

Let ζ^P be defined by (78). With these definitions, it holds that:

$$\sum_{k\in\Gamma} \int_{\partial P_k \cap \Gamma} \left|(\nabla\mathbf{g} - \nabla_h \mathbf{u}_h)\boldsymbol{\tau}_k \cdot (\widehat{\Psi} - \Psi_k^P)(\sigma)\right| d\sigma \leq \zeta^P \|\nabla\widehat{\Psi}\|_{L^2(\Omega)}. \tag{97}$$

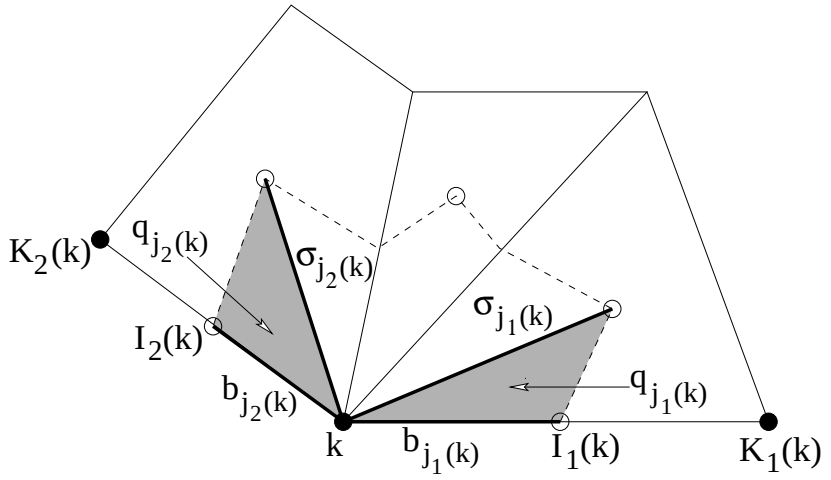

FIGURE 5. For any $k \in \Gamma$, S_k is the common vertex of $q_{j_1(k)}$ and $q_{j_2(k)}$.

Proof. By application of the Cauchy-Schwarz inequality on each edge $b_{j_\alpha(k)}$ and the weighted discrete Cauchy-Schwarz inequality, we obtain for any set of strictly positive real-valued numbers C_α^P

$$\int_{\partial P_k \cap \Gamma} \left|(\nabla \mathbf{g} - \nabla_h \mathbf{u}_h)\boldsymbol{\tau}_k \cdot (\widehat{\Psi} - \Psi_k^P)(\sigma)\right| d\sigma = \sum_{\alpha=1}^{2} \int_{b_{j_\alpha(k)}} \left|(\nabla \mathbf{g} - \nabla_h \mathbf{u}_h)\boldsymbol{\tau}_{j_\alpha(k)} \cdot (\widehat{\Psi} - \Psi_k^P)(\sigma)\right| d\sigma$$
$$\leq \left[\sum_{\alpha=1}^{2} C_\alpha^P \|(\nabla \mathbf{g} - \nabla_h \mathbf{u}_h)\boldsymbol{\tau}_{j_\alpha(k)}\|^2_{L^2(b_{j_\alpha(k)})}\right]^{1/2} \left[\sum_{\alpha=1}^{2} \frac{1}{C_\alpha^P} \|\widehat{\Psi} - \Psi_k^P\|^2_{L^2(b_{j_\alpha(k)})}\right]^{1/2}. \tag{98}$$

Now, for each segment $b_{j_\alpha(k)}$, we can apply the trace inequality (71) on each triangle $q_{j_\alpha(k)}$, for all $\alpha \in \{1, 2\}$ and for all strictly positive $\varepsilon_{j_\alpha(k)}$. With $C_{1,j_\alpha(k)} = \frac{2+\varepsilon^{-2}_{j_\alpha(k)}}{\rho_{j_\alpha(k)}}$ and $C_{2,j_\alpha(k)} = \frac{\varepsilon^2_{j_\alpha(k)}\sigma^2_{j_\alpha(k)}}{\rho_{j_\alpha(k)}}$, we obtain

$$\|\widehat{\Psi} - \Psi_k^P\|^2_{L^2(b_{j_\alpha(k)})} \leq C_{1,j_\alpha(k)} \|\widehat{\Psi} - \Psi_k^P\|^2_{L^2(q_{j_\alpha(k)})} + C_{2,j_\alpha(k)} \|\nabla \widehat{\Psi}\|^2_{L^2(q_{j_\alpha(k)})}.$$

Let $\mu > 0$ be arbitrary. For $b_{j_\alpha(k)}$ for $\alpha \in \{1, 2\}$, let us choose $\varepsilon_{j_\alpha(k)}$ so that

$$\varepsilon^2_{j_\alpha(k)} = \frac{\mu + \sqrt{\mu^2 + \mu\sigma^2_{j_\alpha(k)}}}{\sigma^2_{j_\alpha(k)}} \iff C_{2,j_\alpha(k)} = \mu C_{1,j_\alpha(k)} \tag{99}$$

and $C_\alpha^P = C_{1,j_\alpha(k)}$. It holds that:

$$\begin{aligned}
\sum_{\alpha=1}^{2} \frac{1}{C_\alpha^P} \|\widehat{\Psi} - \Psi_k^P\|^2_{L^2(b_{j_\alpha(k)})} &\leq \sum_{\alpha=1}^{2} \left(\|\widehat{\Psi} - \Psi_k^P\|^2_{L^2(q_{j_\alpha(k)})} + \mu \|\nabla \widehat{\Psi}\|^2_{L^2(q_{j_\alpha(k)})} \right) \\
&\leq \|\widehat{\Psi} - \Psi_k^P\|^2_{L^2(P_k)} + \mu \|\nabla \widehat{\Psi}\|^2_{L^2(P_k)} \\
&\leq \left[(C(P_k) h_k^P)^2 + \mu \right] \|\nabla \widehat{\Psi}\|^2_{L^2(P_k)}.
\end{aligned} \tag{100}$$

Plugging (100) into (98) and applying the discrete Cauchy-Schwarz inequality leads to (97). □

The final term to be evaluated in the right-hand side of (41) is related to the penalization. The proposition that follows is a simple consequence of the Cauchy-Schwarz inequality.

Proposition 4.11. *Let definition (79) of ζ_ε hold. Then, we have*

$$\left|\varepsilon \int_\Omega d_h^2(\Delta_h^D p)_h q(\mathrm{x})d\mathrm{x}\right| \leq \zeta_\varepsilon \|q\|_{L^2(\Omega)}. \tag{101}$$

This ends the various proofs of bounds that were necessary to prove Theorem 4.4.

4.3. **A computable bound for the pressure error**

Proposition 4.12. *The following estimate holds:*

$$\|e_p\|_{L^2(\Omega)} \leq \frac{1}{2\beta}\left(2\|\nabla_h \mathbf{e}\|_{L^2(\Omega)} + osc(\mathbf{f},T,\Omega) + osc(\mathbf{f},P,\Omega) + \eta^T + \eta^P\right). \tag{102}$$

Proof. Since $e_p \in L^2_0(\Omega)$, there exists $\widehat{\mathbf{v}} \in (H^1_0(\Omega))^2$, such that

$$\|e_p\|_{L^2(\Omega)} \leq \frac{1}{\beta}\frac{\int_\Omega e_p \nabla\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x}}{\|\nabla\widehat{\mathbf{v}}\|_{L^2(\Omega)}}. \tag{103}$$

With this given $\widehat{\mathbf{v}}$, we associate $\mathbf{v} = (\mathbf{v}_i^T, \mathbf{v}_k^P) \in (\mathbb{R}^2)^{I+J^\Gamma} \times (\mathbb{R}^2)^K$ such that

$$\mathbf{v}_i^T = \frac{1}{|T_i|}\int_{T_i} \widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x}\ \forall i \in [1,I],\ \mathbf{v}_k^P = \frac{1}{|P_k|}\int_{P_k}\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x}\ \forall k \in [1,K] \tag{104}$$

and the boundary values of $\mathbf{v}_i$, $i \in \Gamma$ are chosen so that $\tilde{\mathbf{v}}_j = 0$ for all $j \in \Gamma$. We may then apply (66) and follow calculations like those involved in propositions 4.6, 4.7 and 4.9. We finally obtain

$$\left|\int_\Omega e_p \nabla\cdot\widehat{\mathbf{v}}(\mathrm{x})d\mathrm{x}\right| \leq \frac{1}{2}\left(2\|\nabla_h\mathbf{e}\|_{L^2(\Omega)} + osc(\mathbf{f},T,\Omega) + osc(\mathbf{f},P,\Omega) + \eta^T + \eta^P\right)\|\nabla\widehat{\mathbf{v}}\|_{L^2(\Omega)}. \tag{105}$$

Taking (105) into (103) proves (102). □

5. Efficiency of the estimators

We state the main result of this Section:

Theorem 5.1. *For any primal cell T_i, let $h_i^T := diam(T_i)$ and $\mathbf{f}_i^T$ be the mean-value of $\mathbf{f}$ over T_i. Let η_i^T (resp. $\eta_i'^T$) be defined in (74) (resp. in (75)). For any dual cell P_k, let $h_k^P := diam(P_k)$ and $\mathbf{f}_k^P$ be the mean-value of $\mathbf{f}$ over P_k. Let η_k^P (resp. $\eta_k'^P$) be defined in (76) (resp. in (77)). For any boundary dual cell P_k, let ζ_k^P be defined in (78). Let Hypothesis 5.5 below hold. Then, there exists a constant C independent of the mesh such that*

$$(\eta_i^T)^2 \leq C\left(\|\nabla_h\mathbf{u}_h - \nabla\widehat{\mathbf{u}}\|^2_{L^2(T_i)} + \|p_h - \widehat{p}\|^2_{L^2(T_i)}\right) + C(h_i^T)^2\|\mathbf{f} - \mathbf{f}_i^T\|^2_{L^2(T_i)}, \tag{106}$$

$$(\eta_i'^T)^2 \leq C\|\nabla_h\mathbf{u}_h - \nabla\widehat{\mathbf{u}}\|^2_{L^2(T_i)}, \tag{107}$$

$$(\eta_k^P)^2 \leq C\left(\|\nabla_h\mathbf{u}_h - \nabla\widehat{\mathbf{u}}\|^2_{L^2(P_k)} + \|p_h - \widehat{p}\|^2_{L^2(P_k)}\right) + C(h_k^P)^2\|\mathbf{f} - \mathbf{f}_k^P\|^2_{L^2(P_k)}, \tag{108}$$

$$(\eta_k'^P)^2 \leq C\|\nabla_h\mathbf{u}_h - \nabla\widehat{\mathbf{u}}\|^2_{L^2(P_k)}. \tag{109}$$

Moreover, in the case $\mathbf{g} = \mathbf{0}$, there exists a constant C independent of the mesh such that

$$(\zeta_k^P)^2 \leq C\|\nabla_h\mathbf{u}_h - \nabla\widehat{\mathbf{u}}\|^2_{L^2(P_k)}. \tag{110}$$

The proof of this theorem is postponed after some technical properties.

In order to prove local efficiency of the estimators, we shall follow the bubble function technique as presented in [31]. Since the estimator η_i^T involves jumps of $\nabla_h \mathbf{u}_h - p_h I_2$ through the common edge $s = [G_i S_k]$ of two neighboring diamond-cells, we shall use functions with a support included in the set $T_i \cap P_k = \cup_{\alpha=1,2} t_{ik,\alpha}$, where triangles $t_{ik,\alpha}$ with $\alpha = 1$ or 2 are depicted in Fig. 4. Since we consider a fixed s in what follows, we simplify the notations to t_1 and t_2. For any triangle t in $\{t_1, t_2\}$, we denote by $\lambda_{t,\beta}$ the barycentric coordinates associated with the three vertices of t, with $\beta \in \{1, 2, 3\}$. We suppose that the vertices of t_1 and t_2 are locally numbered so that the two nodes of the edge s are the vertices 1 and 2 of each of the triangles t_1 and t_2.

Definition 5.2. We define the following bubble functions

$$b_t = \begin{cases} 27\lambda_{t,1}\lambda_{t,2}\lambda_{t,3} \text{ on } t, \\ 0 \text{ elsewhere.} \end{cases} \tag{111}$$

$$b_s = \begin{cases} 4\lambda_{t_\alpha,1}\lambda_{t_\alpha,2} \text{ on } t_\alpha, \ \alpha = \{1, 2\} \\ 0 \text{ elsewhere.} \end{cases} \tag{112}$$

It holds that $\omega_t = supp(b_t) \subset t$ and $\omega_s := supp(b_s) = T_i \cap P_k = t_1 \cup t_2$. The following propositions are given for example, in [31].

Proposition 5.3. *It holds that*

$$0 \leq b_t \leq 1, \ 0 \leq b_s \leq 1, \tag{113}$$

$$\int_s b_s(\sigma) d\sigma = \frac{2}{3}|s|. \tag{114}$$

Proposition 5.4. *There exists a constant $C > 0$ only depending on the minimal angle in the couple (t_1, t_2) such that, for $t = t_1$ or $t = t_2$ and $h_t = diam(t)$*

$$\frac{1}{C}h_t^2 \leq \int_t b_t(\mathrm{x}) d\mathrm{x} = \frac{9}{20}|t| \leq C h_t^2, \tag{115}$$

$$\frac{1}{C}|s|^2 \leq \int_t b_s(\mathrm{x}) d\mathrm{x} = \frac{1}{3}|t| \leq C|s|^2, \tag{116}$$

$$\|\nabla b_t\|_{L^2(t)} \leq C h_t^{-1} \|b_t\|_{L^2(t)}, \tag{117}$$

$$\|\nabla b_s\|_{L^2(t)} \leq C|s|^{-1} \|b_s\|_{L^2(t)}. \tag{118}$$

In order to prove the local efficiency of the error estimator we shall make the following hypothesis:

Hypothesis 5.5. *We suppose that the triangulation of Ω composed of all the triangles $t_{ik,\alpha}$ is regular in the sense that the minimum angles in those triangles are bounded by below independently of the mesh.*

From this hypothesis, we derive the following propositions.

Proposition 5.6. *For any primal cell T_i and any dual cell P_k such that $T_i \cap P_k \neq \emptyset$, let $s = [G_i S_k]$ and $t_{ik,1}$ and $t_{ik,2}$ be the triangles in Fig. 4 such that $t_{ik,1} \cup t_{ik,2} = T_i \cap P_k$. Let $h_i^T = diam(T_i)$, $h_k^P = diam(P_k)$ and $S_{ik} = |T_i \cap P_k|$. Let Hypothesis 5.5 hold. Then, there exists a constant C independent of the mesh such that*

$$(h_i^T)^2 S_{ik}^{-1} \leq C \text{ and } (h_k^P)^2 S_{ik}^{-1} \leq C.$$

Proof. We will only prove the first inequality, since the second one can be treated in the same way.

Let $\alpha_0 > 0$ be the lower bound of all the angles of all the triangles $t_{ik,\alpha}$.

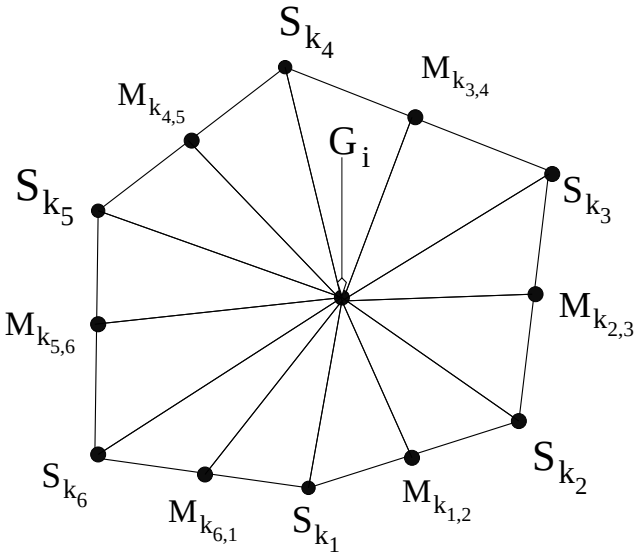

FIGURE 6. Notations of Prop. 5.6.

For any $i \in [1, I]$, let V_i be the number of vertices S_{k_ℓ}, with $\ell \in [1, V_i]$ of the primal cell T_i; see Fig. 6 for the notations. First, we note that

$$V_i \leq V := \frac{2\pi}{2\alpha_0}, \text{ for all } i \in [1, I]. \tag{119}$$

Let $M_{k_{\ell,\ell+1}}$ be the midpoint of segment $[S_{k_\ell} S_{k_{\ell+1}}]$, with the convention that $k_{V_i+1} = k_1$. Then

$$S_{ik_\ell} = |T_i \cap P_{k_\ell}| = |G_i S_{k_\ell} M_{k_{\ell,\ell+1}}| + |G_i S_{k_\ell} M_{k_{\ell-1,\ell}}|. \tag{120}$$

Now, let us estimate the area of triangle $G_i S_{k_\ell} M_{k_{\ell,\ell+1}}$. Following Hypothesis 5.5, all the angles of triangle $G_i S_{k_\ell} M_{k_{\ell,\ell+1}}$ are greater than α_0. Let h_{G_i} be the the maximum distance from point G_i to the boundary of T_i, i.e.,

$$h_{G_i} := \max\{|G_i S_{k_\ell}|, \ \ell \in [1, V_i]\}. \tag{121}$$

We have

$$|G_i S_{k_\ell} M_{k_{\ell,\ell+1}}| = \frac{1}{2} \sin(S_{k_\ell} \widehat{G_i M_{k_{\ell,\ell+1}}}) |G_i S_{k_\ell}| |G_i M_{k_{\ell,\ell+1}}|. \tag{122}$$

By a calculation on triangles $G_i S_{k_\ell} M_{k_{\ell,\ell+1}}$ and $G_i S_{k_{\ell+1}} M_{k_{\ell,\ell+1}}$, it holds that

$$|G_i M_{k_{\ell,\ell+1}}| \geq |G_i S_{k_\ell}| \sin\alpha_0 \quad \text{and} \quad |G_i S_{k_{\ell+1}}| \geq |G_i M_{k_{\ell,\ell+1}}| \sin\alpha_0. \tag{123}$$

From (123), we have the recurrence formula

$$|G_i S_{k_\ell}| \geq (\sin\alpha_0)^2 |G_i S_{k_{\ell+1}}|. \tag{124}$$

Starting from the vertex S_k which reaches the max in definition (121), the shortest way to go to a given vertex S_{k_ℓ} contains at most $V_i/2$ neighboring vertices for which we may apply (124), and we obtain:

$$|G_i S_{k_\ell}| \geq (\sin\alpha_0)^{V_i} h_{G_i}. \tag{125}$$

Then, from (123), we get that

$$|G_i S_{k_\ell}||G_i M_{k_{\ell,\ell+1}}| \geq (\sin\alpha_0)^{2V_i+1} h_{G_i}^2. \tag{126}$$

Combining (119), (122) and (126), and noting that from the definition of h_{G_i}, then $h_{G_i} \geq \frac{h_i^T}{2}$, we obtain

$$|G_i S_{k_\ell} M_{k_{\ell,\ell+1}}| \geq (\sin\alpha_0)^{2V+2} \frac{(h_i^T)^2}{8}. \tag{127}$$

In the same way, we have

$$|G_i S_{k_\ell} M_{k_{\ell-1,\ell}}| \geq (\sin\alpha_0)^{2V+2} \frac{(h_i^T)^2}{8}. \tag{128}$$

Using (120), (127)–(128), we obtain:

$$S_{ik_\ell} \geq (\sin\alpha_0)^{2V+2} \frac{(h_i^T)^2}{4}. \tag{129}$$

Thus, the inequality is proved with $C = 4\,(\sin\alpha_0)^{-\frac{2\pi}{\alpha_0}-2}$. □

Proposition 5.7. *Under Hypothesis 5.5, the positive constants $C(T_i)$ and $C(P_k)$ are bounded independently of the mesh, and the constant C in Prop. 5.4 is bounded by above and by below independently of the mesh.*

Proof. The constants $C(T_i)$, $C(P_k)$ coming from (70) were bounded explicitly in [32]. From these expressions, it is easily seen that they are bounded if Hyp. 5.5 holds. Moreover, it is proved in [31] that C in Prop. 5.4 depends only on the regularity of the triangles $t_{ik,\alpha}$. □

Now, we shall prove the efficiency of the estimators (Proof of Theorem 5.1).

Proof. Let us consider an element T_i of the primal mesh and a diamond edge s in $\overset{\circ}{T}_i$. Let us recall that by definition, such an edge s does not belong to Γ. Let us consider the function $\mathbf{w}_s = [(\nabla_h \mathbf{u}_h - I_2 p_h)\mathbf{n}_s]_s b_s$, where b_s is defined by (112). This function belongs to $(H_0^1(\Omega))^2$ and we may thus apply (36), which, taking into account the support of $\mathbf{w}_s$, reduces to

$$\int_{\omega_s} (\nabla\widehat{\mathbf{u}} - I_2\widehat{p}) : \nabla\mathbf{w}_s(\mathrm{x})d\mathrm{x} = \int_{\omega_s} \mathbf{f}\cdot\mathbf{w}_s(\mathrm{x})d\mathrm{x}. \tag{130}$$

Moreover, $\mathbf{u}_h|_{D_j}$ belongs to $(P^1(D_j))^2$, p_h is a constant in each D_j and $\mathbf{w}_s$ vanishes on Γ. Thus, it holds that:

$$\begin{aligned}
\int_\Omega (\nabla_h\mathbf{u}_h - I_2 p_h) : \nabla\mathbf{w}_s(\mathrm{x})d\mathrm{x} &= \sum_j \int_{D_j} (\nabla_h\mathbf{u}_h - I_2 p_h) : \nabla\mathbf{w}_s(\mathrm{x})d\mathrm{x} \\
&= \sum_j \int_{\partial D_j} \big((\nabla_h\mathbf{u}_h - I_2 p_h)\mathbf{n}_{\partial D_j}\big)\cdot\mathbf{w}_s(\sigma)d\sigma \\
&= \sum_i \sum_{s'\subset\overset{\circ}{T}_i} \int_{s'} [(\nabla_h\mathbf{u}_h - I_2 p_h)\mathbf{n}_{s'}]_{s'}\cdot\mathbf{w}_s(\sigma)d\sigma.
\end{aligned}$$

But since $\mathbf{w}_s$ vanishes on all the other edges $s' \neq s$, taking into account the definition of $\mathbf{w}_s$ and the property of b_s in (114), it holds that

$$\begin{aligned}\int_\Omega (\nabla_h \mathbf{u}_h - I_2 p_h) : \nabla \mathbf{w}_s(\mathrm{x}) d\mathrm{x} &= \int_s [(\nabla_h \mathbf{u}_h - I_2 p_h)\mathbf{n}_s]_s \cdot \mathbf{w}_s(\sigma) d\sigma \\ &= |[(\nabla_h \mathbf{u}_h - I_2 p_h)\mathbf{n}_s]_s|^2 \int_s b_s(\sigma)\sigma \\ &= \frac{2}{3}|s|\ |[(\nabla_h \mathbf{u}_h - I_2 p_h)\mathbf{n}_s]_s|^2 \\ &= \frac{2}{3}\|[(\nabla_h \mathbf{u}_h - I_2 p_h)\mathbf{n}_s]_s\|^2_{L^2(s)}.\end{aligned} \tag{131}$$

Defining $M := \|[(\nabla_h \mathbf{u}_h - I_2 p_h)\mathbf{n}_s]_s\|_{L^2(s)}$, and taking into account (130), we have

$$\begin{aligned}M^2 &= \frac{3}{2}\int_\Omega (\nabla_h \mathbf{u}_h - I_2 p_h) : \nabla \mathbf{w}_s(\mathrm{x}) d\mathrm{x} = \frac{3}{2}\int_{\omega_s} (\nabla_h \mathbf{u}_h - I_2 p_h) : \nabla \mathbf{w}_s(\mathrm{x}) d\mathrm{x} \\ &= \frac{3}{2}\left[\int_{\omega_s} (\nabla_h \mathbf{u}_h - \nabla \widehat{\mathbf{u}}) : \nabla \mathbf{w}_s(\mathrm{x}) d\mathrm{x} - \int_{\omega_s} (p_h - \widehat{p})\nabla \cdot \mathbf{w}_s(\mathrm{x}) d\mathrm{x} + \int_{\omega_s} \mathbf{f} \cdot \mathbf{w}_s(\mathrm{x}) d\mathrm{x}\right].\end{aligned} \tag{132}$$

Using the Cauchy-Schwarz inequality leads to

$$M^2 \leq \frac{3}{2}\left[\left(\|\nabla_h \mathbf{u}_h - \nabla \widehat{\mathbf{u}}\|_{L^2(\omega_s)} + \sqrt{2}\|p_h - \widehat{p}\|_{L^2(\omega_s)}\right)\|\nabla \mathbf{w}_s\|_{L^2(\omega_s)}\right] + \frac{3}{2}\|\mathbf{f}\|_{L^2(\omega_s)}\|\mathbf{w}_s\|_{L^2(\omega_s)}. \tag{133}$$

Let us now bound $\|\nabla \mathbf{w}_s\|_{L^2(\omega_s)}$ and $\|\mathbf{w}_s\|_{L^2(\omega_s)}$. Thanks to (118), it holds that

$$\|\nabla \mathbf{w}_s\|_{L^2(\omega_s)} = |[(\nabla \mathbf{u}_s - I_2 p_h)\mathbf{n}_s]_s| \|\nabla b_s\|_{L^2(\omega_s)} \leq |[(\nabla \mathbf{u}_s - I_2 p_h)\mathbf{n}_s]_s|\ C|s|^{-1}\|b_s\|_{L^2(\omega_s)}, \tag{134}$$

$$\|\mathbf{w}_s\|_{L^2(\omega_s)} = |[(\nabla \mathbf{u}_s - I_2 p_h)\mathbf{n}_s]_s| \|b_s\|_{L^2(\omega_s)}. \tag{135}$$

So there remains to find a bound for $\|b_s\|_{L^2(\omega_s)}$. In order to do this, we first infer from (113) that $b_s^2 \leq b_s$. This implies, using (116)

$$\|b_s\|_{L^2(\omega_s)} = \left[\|b_s\|^2_{L^2(t_1)} + \|b_s\|^2_{L^2(t_2)}\right]^{1/2} \leq \left[\int_{t_1 \cup t_2} b_s(\mathrm{x}) d\mathrm{x}\right]^{1/2} \leq C|s|. \tag{136}$$

Taking into account that $|[(\nabla_h \mathbf{u}_h - I_2 p_h)\mathbf{n}_s]_s| = |s|^{-1/2} M$ and considering (133) to (136), we obtain

$$M \leq C\left[|s|^{-1/2}(\|\nabla_h \mathbf{u}_h - \nabla \widehat{\mathbf{u}}\|_{L^2(\omega_s)} + \sqrt{2}\|p_h - \widehat{p}\|_{L^2(\omega_s)}) + |s|^{1/2}\|\mathbf{f}\|_{L^2(\omega_s)}\right]. \tag{137}$$

One usually expresses $\|\mathbf{f}\|_{L^2(\omega_s)}$ as a function of $\|\nabla_h \mathbf{u}_h - \nabla \widehat{\mathbf{u}}\|_{L^2(\omega_s)} + \|p_h - \widehat{p}\|_{L^2(\omega_s)}$ and of higher order terms. Let $t = t_1$ or t_2, and let us denote by $\mathbf{f}_t$ the mean value of $\mathbf{f}$ over t. Then,

$$\|\mathbf{f}\|_{L^2(t)} \leq \|\mathbf{f} - \mathbf{f}_t\|_{L^2(t)} + \|\mathbf{f}_t\|_{L^2(t)}. \tag{138}$$

Then, consider $\mathbf{w}_t = \mathbf{f}_t b_t$, where b_t is defined by (111). The function $\mathbf{w}_t$ belongs to $(H_0^1)^2$. Thus, taking into account the support of b_t, Eq. (36) reduces to

$$\int_t (\nabla \widehat{\mathbf{u}} - I_2 \widehat{p}) : \nabla \mathbf{w}_t(\mathrm{x}) d\mathrm{x} = \int_t \mathbf{f} \cdot \mathbf{w}_t(\mathrm{x}) d\mathrm{x}. \tag{139}$$

Moreover, since $\nabla_h \mathbf{u}_h - I_2 p_h$ is a constant over each t, and since $\mathbf{w}_t$ vanishes on the boundary of t, it holds that

$$\int_t (\nabla_h \mathbf{u}_h - I_2 p_h) : \nabla \mathbf{w}_t \,(\mathrm{x}) d\mathrm{x} = 0. \tag{140}$$

Since $\mathbf{f}_t$ is a constant over t, there holds, thanks to (115), (139) and (140),

$$\begin{aligned}
\|\mathbf{f}_t\|^2_{L^2(t)} &= |t| \,(\mathbf{f}_t)^2 = C \,(\mathbf{f}_t)^2 \int_t b_t(\mathrm{x}) d\mathrm{x} = C \int_t \mathbf{f}_t \cdot \mathbf{w}_t(\mathrm{x}) d\mathrm{x} \\
&= C \left[\int_t \mathbf{f} \cdot \mathbf{w}_t(\mathrm{x}) d\mathrm{x} + \int_t (\mathbf{f}_t - \mathbf{f}) \cdot \mathbf{w}_t(\mathrm{x}) d\mathrm{x} \right] \\
&= C \left[\int_t (\nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h) : \nabla \mathbf{w}_t \,(\mathrm{x}) d\mathrm{x} - \int_t (\widehat{p} - p_h) \nabla \cdot \mathbf{w}_t(\mathrm{x}) d\mathrm{x} + \int_t (\mathbf{f}_t - \mathbf{f}) \cdot \mathbf{w}_t(\mathrm{x}) d\mathrm{x} \right] \\
&\leq C \left(\|\nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h\|_{L^2(t)} + \sqrt{2} \|\widehat{p} - p_h\|_{L^2(t)} \right) \|\nabla \mathbf{w}_t\|_{L^2(t)} + C \,\|\mathbf{f}_t - \mathbf{f}\|_{L^2(t)} \,\|\mathbf{w}_t\|_{L^2(t)} \,,
\end{aligned} \tag{141}$$

with $C = 20/9$ in the above expressions. Let us now bound $\|\mathbf{w}_t\|_{L^2(t)}$ and $\|\nabla \mathbf{w}_t\|_{L^2(t)}$. With (117), it holds that

$$\|\nabla \mathbf{w}_t\|_{L^2(t)} = |\mathbf{f}_t| \,\|\nabla b_t\|_{L^2(t)} \quad \leq \quad |\mathbf{f}_t| \, C h_t^{-1} \,\|b_t\|_{L^2(t)} \,, \tag{142}$$

$$\|\mathbf{w}_t\|_{L^2(t)} \quad = \quad |\mathbf{f}_t| \,\|b_t\|_{L^2(t)} \,. \tag{143}$$

The remaining term that has to be bounded is $\|b_t\|_{L^2(t)}$. For this, we first infer from (113) that $b_t^2(\mathrm{x}) \leq b_t(\mathrm{x})$ and then

$$|\mathbf{f}_t| \,\|b_t\|_{L^2(t)} \leq |\mathbf{f}_t| \left(\int_t b_t(\mathrm{x}) d\mathrm{x} \right)^{1/2} \leq C \,|\mathbf{f}_t| \,|t|^{1/2} = C \,\|\mathbf{f}_t\|_{L^2(t)} \,, \tag{144}$$

in which $C = \sqrt{9/20}$. Combining (141)–(142)–(143)–(144), we finally get

$$\|\mathbf{f}_t\|_{L^2(t)} \leq C \left(\|\mathbf{f}_t - \mathbf{f}\|_{L^2(t)} + h_t^{-1} \,\|\nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h\|_{L^2(t)} + h_t^{-1} \|\widehat{p} - p_h\|_{L^2(t)} \right).$$

Since s is an edge of t, it holds that $|s| \leq h_t$; applying (138), we obtain

$$\|\mathbf{f}\|_{L^2(t)} \leq C \left(\|\mathbf{f}_t - \mathbf{f}\|_{L^2(t)} + |s|^{-1} \,\|\nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h\|_{L^2(t)} + |s|^{-1} \|\widehat{p} - p_h\|_{L^2(t)} \right).$$

Thus, taking into account that $\omega_s = t_1 \cup t_2$, it holds that

$$\begin{aligned}
\|\mathbf{f}\|_{L^2(\omega_s)} &\leq \|\mathbf{f}\|_{L^2(t_1)} + \|\mathbf{f}\|_{L^2(t_2)} \\
&\leq C \left(\|\mathbf{f}_{t_1} - \mathbf{f}\|_{L^2(t_1)} + \|\mathbf{f}_{t_2} - \mathbf{f}\|_{L^2(t_2)} \right) + C|s|^{-1} \Big[\|\nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h\|_{L^2(t_1)} \\
&\quad + \left(\|\widehat{p} - p_h\|_{L^2(t_1)} + \|\nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h\|_{L^2(t_2)} + \|\widehat{p} - p_h\|_{L^2(t_2)} \right) \Big] \\
&\leq C|s|^{-1} \left(\|\nabla \widehat{\mathbf{u}} - \nabla_h \mathbf{u}_h\|_{L^2(\omega_s)} + \|\widehat{p} - p_h\|_{L^2(\omega_s)} \right) + C \,\|\mathbf{f}_{\omega_s} - \mathbf{f}\|_{L^2(\omega_s)} \,.
\end{aligned} \tag{145}$$

In this sequence of inequalities, we have used the fact that $\mathbf{f}_t$ minimizes $\|\mathbf{f} - \mathbf{c}\|_{L^2(t)}$ when $\mathbf{c}$ runs over $\mathbb{R}^2$; in particular, $\|\mathbf{f}_t - \mathbf{f}\|_{L^2(t)} \leq \|\mathbf{f}_{\omega_s} - \mathbf{f}\|_{L^2(t)}$, where $\mathbf{f}_{\omega_s}$ is the mean value of $\mathbf{f}$ over ω_s. Combining (137) and (145), we obtain

$$M \leq C|s|^{1/2} \,\|\mathbf{f} - \mathbf{f}_{\omega_s}\|_{L^2(\omega_s)} + C|s|^{-1/2} \left(\|\nabla_h \mathbf{u}_h - \nabla \widehat{\mathbf{u}}\|_{L^2(\omega_s)} + \|\widehat{p} - p_h\|_{L^2(\omega_s)} \right). \tag{146}$$

By definition, the local estimator $(\eta_i^T)^2$ is lower than the value taken by the function in (74) in $\mu = (h_i^T)^2$. In (74), we may bound $C(T_i)$ by $1/\pi$ since the primal cells have been supposed to be convex, and with (91) and (146), we obtain

$$\left(\eta_i^T\right)^2 \leq C\left(h_i^T\right)^2 \sum_{s\in\overset{\circ}{T}_i} \frac{|s|^{-1}}{\rho_{ik,1}+\rho_{ik,2}} \left(\|\nabla_h \mathbf{u}_h - \nabla\widehat{\mathbf{u}}\|^2_{L^2(\omega_s)} + \|p_h - \widehat{p}\|^2_{L^2(\omega_s)}\right)$$
$$+ C\left(h_i^T\right)^2 \sum_{s\in\overset{\circ}{T}_i} \frac{|s|}{\rho_{ik,1}+\rho_{ik,2}} \|\mathbf{f} - \mathbf{f}_{\omega_s}\|^2_{L^2(\omega_s)}\,. \tag{147}$$

Using Prop. 5.6, and since by definition $S_{ik} = \frac{1}{2}|s|\,(\rho_{ik,1}+\rho_{ik,2})$, we have that $\left(h_i^T\right)^2 |s|^{-1}(\rho_{ik,1}+\rho_{ik,2})^{-1}$ is bounded by a constant that does not depend on the mesh under Hyp. 5.5. Moreover, under Hyp. 5.5, the ratio $|s|\rho^{-1}_{ik,\alpha}$ is also bounded by a constant that does not depend on the mesh. So (147) leads to (106).

As far as (107) is concerned, let us consider the function $\mathbf{v}_s = [\nabla_h \mathbf{u}_h \boldsymbol{\tau}_s]_s b_s$. There obviously holds

$$\int_\Omega \nabla\widehat{\mathbf{u}} : \nabla\times\mathbf{v}_s(\mathrm{x})\,d\mathrm{x} = \int_{\omega_s} \nabla\widehat{\mathbf{u}} : \nabla\times\mathbf{v}_s(\mathrm{x})\,d\mathrm{x} = 0\,. \tag{148}$$

Eq. (148) and the calculations that previously led to (131) may be used to yield

$$\begin{aligned}\|[\nabla_h \mathbf{u}_h \boldsymbol{\tau}_s]_s\|^2_{L^2(s)} &= \frac{3}{2}\int_{\omega_s} \nabla_h\mathbf{u}_h : \nabla\times\mathbf{v}_s(\mathrm{x})d\mathrm{x}\\ &= \frac{3}{2}\int_{\omega_s} (\nabla_h\mathbf{u}_h - \nabla\widehat{\mathbf{u}}) : \nabla\times\mathbf{v}_s(\mathrm{x})d\mathrm{x}\\ &\leq \frac{3}{2}\|\nabla_h\mathbf{u}_h - \nabla\widehat{\mathbf{u}}\|_{L^2(\omega_s)}\|\nabla\mathbf{v}_s\|_{L^2(\omega_s)}\,.\end{aligned} \tag{149}$$

Just like (133) led to (137) and then to (106), inequality (149) leads to (107). The dual inequalities (108), (109) and (110) may be obtained in the same way. The proof of (110) is very similar to that of (107) and (109), but some definitions have to be changed because the segment $b_{j_\alpha(k)}$ in the definition (78) is a boundary segment, and is thus the edge of only one triangle t; the function b_s is thus defined only in that triangle t. □

6. Numerical results

First, we study the influence of the parameter ε for a fixed mesh and then the influence of the mesh size for a fixed value of the penalty parameter. Secondly, we give an overall process to recursively adapt the value of the penalty parameter and adaptively refine the mesh.

6.1. **Influence of the penalty parameter**

In this subsection, we will work on the domain $\Omega = [0;1]^2$. A triangular mesh with rather uniform triangles is used. The exact solution $(\widehat{\mathbf{u}}, \widehat{p})$ is regular with $\widehat{\mathbf{u}} = (\partial_y\varphi, -\partial_x\varphi)$ given by

$$\varphi(x,y) = 100x^2y^2(1-x)^2(1-y)^2 \text{ and } \widehat{p}(x,y) = 10(x^2+y^2-\frac{2}{3}). \tag{150}$$

Fig. 7 presents the plots of the errors and the estimators when the penalty parameter ε goes from 10^{-5} to 10^2. They include the actual errors in the $H^1(\Omega)$ and $L^2(\Omega)$ norms for the velocity, i.e. the error in the velocity gradient $\|\nabla\widehat{\mathbf{u}} - \nabla_h\mathbf{u}_h\|_{L^2(\Omega)}$ and in the velocity $\|\widehat{\mathbf{u}} - \mathbf{u}_h\|_{L^2(\Omega)}$, the total estimator, the discretization estimator and the penalization estimator which are given by Theorem 4.4 when we estimate the velocity error. The left

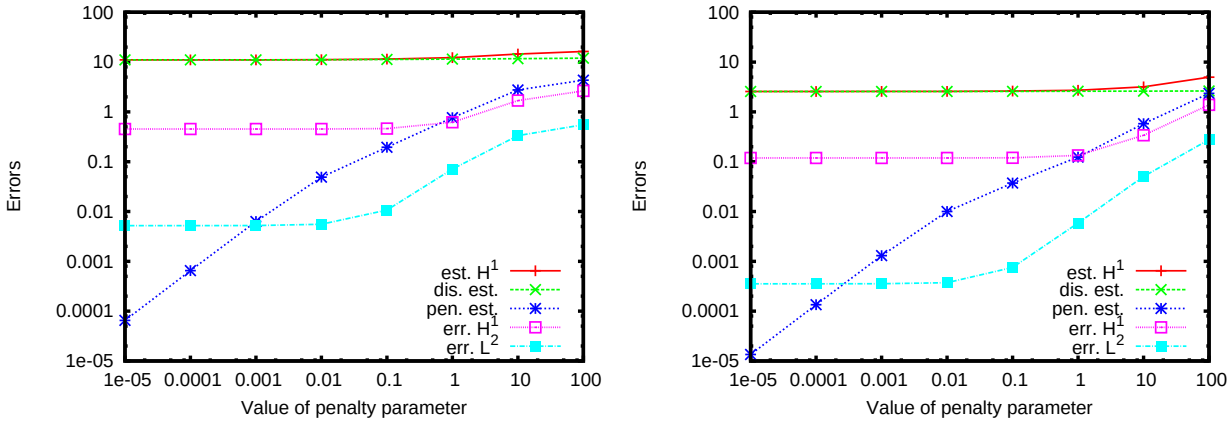

FIGURE 7. Actual errors in $H^1(\Omega)$ and $L^2(\Omega)$ norms, total estimator, discretization estimator and penalization estimator for the velocity. Left: $h = 10^{-1}$, right: $h = 3.125 \times 10^{-2}$.

(resp. right) figure corresponds to the mesh size $h = 10^{-1}$ (resp. $h = 3.125 \times 10^{-2}$). We see that for a given mesh, the ratio between the penalization estimator and the penalty parameter ε asymptotically tends to a constant when ε tends to 0, while the discretization estimator is nearly independent of ε when $\varepsilon \leq 1$. Moreover, the actual errors decrease with ε until a certain level. Then, the discretization error is the dominant error and decreasing ε further does not have any influence on the overall error. It is noticeable that the influence of ε is more important on the L^2 norm than on the H^1 semi-norm.

6.2. **Influence of the mesh size**

On the same square domain Ω and with the same exact solution as previously, we work with the following values: $\varepsilon = 10^2$, $\varepsilon = 1$ and $\varepsilon = 10^{-2}$. Since the solution is regular, only uniformly refined triangular meshes will be considered. Figure 8 presents the same curves as Fig. 7, but now as a function of h, varying from 0.25 to 1.6×10^{-2}. For the test $\varepsilon = 10^2$, when h is large, εh^2 is so large that the solution of the penalized model is very

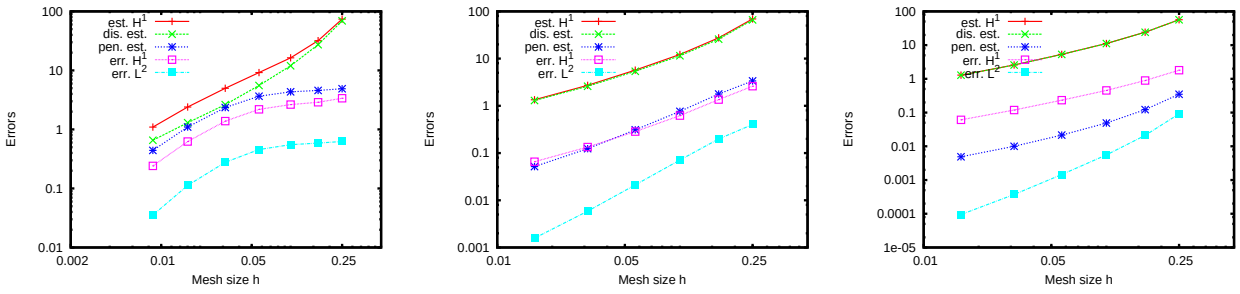

FIGURE 8. Actual errors in $H^1(\Omega)$ and $L^2(\Omega)$ norms, total estimator, discretization estimator and penalization estimator for the velocity. Left: $\varepsilon = 10^2$, center: $\varepsilon = 1$, right: $\varepsilon = 10^{-2}$.

different from the solution of the original, non-penalized model. It is then not surprising that the exact error is strongly correlated to the penalization estimator rather than to the total estimator. Moreover, we observe what is called a "numerical locking": for large h, the error almost does not decrease when h is decreased; it only starts decreasing when h is small enough. However, the values $\varepsilon = 10^{-2}$ or even $\varepsilon = 1$ are small enough so that no such locking occurs. The penalization estimator is small with respect to the total estimator and we

observe that the actual H^1 semi-norm of the error, the total estimator and the discretization estimator decrease roughly like h when h decreases. As above, we notice that the L^2 norm of the error is more influenced than the H^1 semi-norm when decreasing ε from 1 to 10^{-2}.

6.3. **Adaptive penalty parameter and mesh refinement**

We propose the following computational process. We start with a given coarse mesh and an initial value of ε, and we fix some ratio $0 < \gamma \leq 1$.

Then, we compute the numerical solution, and we get η_h and η_ϵ. Then,

- If $\eta_\epsilon \geq \gamma\eta_h$, we adapt a new ε by multiplying the previous value of ε with the ratio $\frac{\gamma\eta_h}{2\eta_\epsilon}$ and keep the same mesh for a new computation. This has the effect of maintaining the error due to the penalization below a certain ratio of the error due to the discretization.
- Otherwise, we adaptively refine the mesh based on the discretization estimator η_h. For this, on the given mesh, we compute local discretization estimators $\eta_{i,h}$ on each primal cell T_i, obtained by adding η_i, η_i' (respectively defined by (74) and (75)) to the contribution of T_i in the oscillation term (72), and by redistributing the sums that appear in the dual estimators (76), (77) and (73) and in the boundary estimators (78) to the cells T_i that have an intersection with P_k. More specifically, in (73), the integral over P_k is split into the sum (over all T_i that intersect P_k) of the integrals over $T_i \cap P_k$, which are then redistributed to the corresponding T_i. Moreover, each local dual estimator (76) and (77) is a sum over diamond edges $s = [G_iS_k]$ of quantities which are thus redistributed to the corresponding T_i. Finally, each boundary estimator (78) is a sum of two quantities that can be related each to a given primal cell (see Fig. 5) to which the corresponding contribution in (78) is redistributed. Once all $\eta_{i,h}$ have been computed, we require the refinement of a given primal cell T_i by a factor 4 in terms of area if $\eta_{i,h} \geq (\max_i \eta_{i,h})/2$. Specifically, we use the Triangle mesh generator [27] and its mesh refinement feature which remeshes the domain in a way that user-specified local area constraints are fulfilled. Note that simply dropping the dual and boundary estimators instead of redistributing them onto the primal cells as explained above had almost no influence on the general shape of the convergence curves in Figures 9 and 10 below.

The test we present to illustrate this strategy is also on the domain $\Omega = [0;1]^2$, The exact solution $(\widehat{\mathbf{u}}, \widehat{p})$ is regular with $\widehat{\mathbf{u}} = (\varphi_y, -\varphi_x)$, and φ is given by

$$\varphi(x,y) = x^2(1-x)^2y^2(1-y)^2 \text{ and } \widehat{p}(x,y) = 5(x^2+y^2-\frac{2}{3}).$$

For accuracy reasons, the ratio γ may be chosen so that the penalization error is much lower than the discretization error like in the left part of Fig. 9 obtained with $\gamma = 1/500$. We chose this value of γ because Fig. 7 showed that the actual H^1 error stops decreasing when the penalization estimator is around two orders of magnitude lower than the discretization estimator, while the L^2 error stops decreasing when this ratio is around three orders of magnitude. The first four computations on the coarsest mesh are used to tune the value of ε: starting with $\varepsilon = 10^2$, its value is decreased by the adaptive process described above to 2.5×10^{-3}. We observe (see the overlapping squares on the left part of the H^1 semi-norm error curve) that the first two decreasing steps of ε have the effect of decreasing the actual error. The last decreasing step of ε from 2.9×10^{-2} to 2.5×10^{-3} does not affect the total error and we are thus sure that the total error does not suffer from any error introduced by a too high value of ε. Then, the value of ε does not change any more when we refine the mesh. Along this process, the total estimator is kept almost equal to the discretization estimator.

Moreover, we made a test with $\gamma = 1$ and we present the result on the right part of Fig. 9. This value of γ is obviously too high because we observe the numerical locking discussed above.

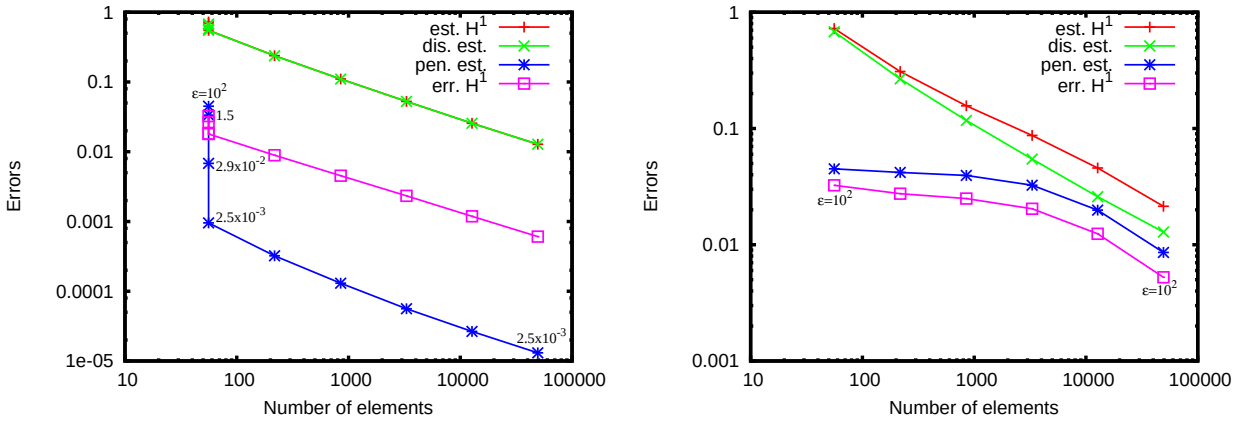

FIGURE 9. Actual errors in the $H^1(\Omega)$ semi-norm, total estimator, discretization estimator and penalization estimator for the velocity. Left: $\gamma = 1/500$, right: $\gamma = 1$.

The next test is inspired from [19]. Our test is in the domain $\Omega = [0,1[^2$ and the exact couple solution $(\widehat{\mathbf{u}}, \widehat{p})$ is singular with $\widehat{\mathbf{u}} = (\varphi_y, -\varphi_x)$, and φ is

$$\varphi(x,y) = x^{\frac{7}{4}}(1-x)^2y^2(1-y)^2 \text{ and } \widehat{p}(x,y) = \frac{x+y-1}{10}.$$

We observe that the velocity $\widehat{\mathbf{u}}$ is in $[H^{\frac{5}{4}}(\Omega)]^2$ and there is a boundary singularity on the edge $x = 0$.

We compare the exact $H^1(\Omega)$ semi-norm of the velocity error with the total velocity error estimator for uniform and adaptive refinements. For uniform refinements, only ε is adapted, while for adaptive refinements, both ε and the meshes are adapted by the penalty - mesh refinement process described above. In both cases, we choose $\gamma = 1/500$. In Fig. 10, we start with $\varepsilon = 0.1$. The first computation of the adaptive process

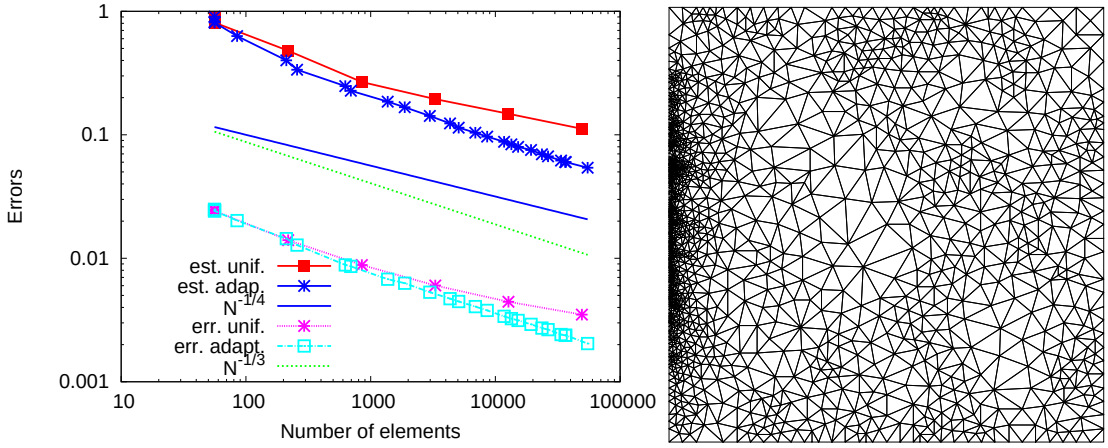

FIGURE 10. Estimated and exact errors in uniform/adaptive refinement (left) and an adapted mesh (right).

sets ε to 1.8×10^{-3}, and then ε remains unchanged in the rest of the adaptive process for both adaptive and

uniform refinements. The convergence curve corresponding to the uniform mesh refinement is observed to be asymptotically parallel to the $N^{-1/4}$ curve, while the convergence curve corresponding to the adaptive mesh refinement is parallel to the $N^{-1/3}$ curve. Moreover, the effectivity of the error estimators for both types of refinements is around 25.

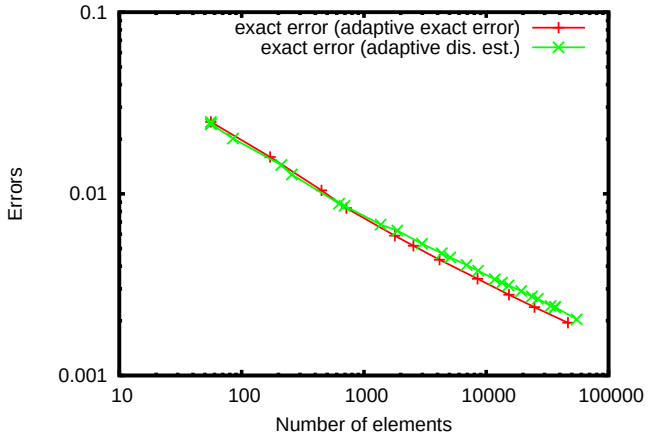

FIGURE 11. Exact errors in the adaptive process using the discretization estimator and exact error.

We could have expected the plot of the exact error corresponding to an adaptive mesh refinement to be parallel to the optimal $N^{-1/2}$ straight line, but this is not the case, just like in [19]. We would like to determine whether this problem is caused by our discretization estimator or not. For this, in Fig. 11, we compare the exact errors corresponding to two different adaptive refinement process: one is driven by our discretization estimator, while the other is driven by the exact error (which we may compute exactly in this test case). Clearly, the exact error is hardly affected by the applied adaptive process.

REFERENCES

[1] B. Andreianov, M. Bendahmane, K. H. Karlsen and C. Pierre, Convergence of discrete duality finite volume schemes for the cardiac bidomain model. *Networks and Heterogeneous Media* **6** (2011) 195–240.
[2] B. Andreianov, F. Boyer and F. Hubert, Discrete duality finite volume schemes for Leray-Lions-type elliptic problems on general 2D meshes. *Numer. Methods Partial Differ. Equations* **23** (2007) 145–195.
[3] C. Bernardi, V. Girault and F. Hecht, A Posteriori Analysis of a Penalty Method and Application to the Stokes Problem. *Mathematical Models and Methods in Applied Sciences* **13** (2003) 1599–1628.
[4] F. Boyer and F. Hubert, Finite volume method for 2D linear and nonlinear elliptic problems with discontinuities. *SIAM J. Numer. Anal.* **46** (2008) 3032–3070.
[5] C. Carstensen and S. Funken, Constants in Clément-interpolation error and residual based *a posteriori* estimates in finite element methods. *East-West J. Numer. Math.* **8** (2000) 153–175.
[6] C. Chainais-Hillairet, Discrete duality finite volume schemes for two-dimensional drift-diffusion and energy-transport models. *Internat. J. Numer. Methods Fluids* **59** (2009) 239–257.
[7] E.V. Chizhonkov and M.A. Olshanskii, On the domain geometry dependence of the LBB condition. *M2AN Math. Model. Numer. Anal.* **34** (2000) 935–951.
[8] M. Dobrowolski, On the LBB condition in the numerical analysis of the Stokes equations. *Appl. Numer. Math.* **54** (2005) 314–323.
[9] S. Zsuppn, On the domain dependence of the infsup and related constants via conformal mapping. *J. Math. Anal. Appl.* 382 (2011) 856–863.
[10] Y. Coudière and F. Hubert, A 3D discrete duality finite volume method for nonlinear elliptic equations. *SIAM J. Sci. Comput.* **33** (2011) 1739–1764.
[11] Y. Coudière and G. Manzini, The discrete duality finite volume method for convection-diffusion problems. *SIAM J. Numer. Anal.* **47** (2010) 4163–4192.

[12] Y. Coudière, C. Pierre, O. Rousseau and R. Turpault, A 2D/3D Discrete Duality Finite Volume Scheme. Application to ECG simulation. *International Journal On Finite Volumes* **6** (1) (2009), electronic only.
[13] E. Dari, R. Durán, and C. Padra, Error Estimators for Nonconforming Finite Element Approximations of the Stokes Problem. *Math. Comp.* **64** (1995) 1017–1033.
[14] S. Delcourte, Développement de méthodes de volumes finis pour la mécanique des fluides. Ph.D. Thesis (in French), University of Toulouse III, France, 2007. available at http://tel.archives-ouvertes.fr/tel-00200833/fr/
[15] S. Delcourte, K. Domelevo and P. Omnes, A Discrete Duality Finite Volume Approach to Hodge Decomposition and Div–Curl Problems on Almost Arbitrary Two–Dimensional Meshes. *Siam J. Numer. Anal. Vol.* **45** (2007) 1142–1174.
[16] S. Delcourte and P. Omnes, A discrete duality finite volume discretization of the vorticity-velocity-pressure stokes problem on almost arbitrary two-dimensional grids. *Numer. Methods Partial Differential Eq.*. doi: 10.1002/num.21890
[17] K. Domelevo and P. Omnes, A finite volume method for the Laplace equation on almost arbitrary two-dimensional grids. *M2AN Math. Model. Numer. Anal.* **39** (2005) 1203–1249.
[18] Y. Girault, and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations*. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo (1986).
[19] A. Hannukainen, R. Stenberg, and M. Vohralík, A Unified Framework for a Posteriori Error Estimation for the Stokes Problem. *Numer. Math.* **122** (2012) 725–769.
[20] F. Hermeline, A finite volume method for the approximation of diffusion operators on distorted meshes. *J. Comput. Phys.* **160** (2000) 481–499.
[21] F. Hermeline, Approximation of diffusion operators with discontinuous tensor coefficients on distorted meshes. *Comput. Methods Appl. Mech. Eng.* **192** (2003) 1939–1959.
[22] F. Hermeline, S. Layouni and P. Omnes, A finite volume method for the approximation of Maxwell's equations in two space dimensions on arbitrary meshes. *J. Comput. Phys.* **227** (2008) 9365–9388.
[23] S. Krell, Stabilized DDFV Schemes for Stokes Problem with Variable Viscosity on General 2D Meshes. *Numer. Methods Partial Differential Eq.* **27** (2011) 1666–1706.
[24] S. Krell and G. Manzini, The discrete duality finite volume method for Stokes equations on three-dimensional polyhedral meshes. *SIAM J. Numer. Anal.* **50** (2012) 808–837.
[25] P. Omnes, Y. Penel, and Y. Rosenbaum, A Posteriori Error Estimation for the Discrete Duality Finite Volume Discretization of the Laplace Equation. *SIAM J. Numer. Anal.* **47** (2009) 2782–2807.
[26] L.E. Payne, and H.F. Weinberger, An optimal Poincaré inequality for convex domain. *Arch. Rational Mech. Anal.* **5** (1960) 286–292.
[27] J.R. Shewchuk, Triangle: Engineering a 2D Quality Mesh Generator and Delaunay Triangulator, in *Applied Computational Geometry: Towards Geometric Engineering*, Ming C. Lin and Dinesh Manocha, editors, Lecture Notes in Computer Science, Vol. 1148, Springer-Verlag, Berlin (1996) 203–222. `http://www.cs.cmu.edu/~quake/triangle.html`
[28] A. Veeser and R. Verfürth, Poincaré constants for finite element stars. *IMA J. Numer. Anal.* **32** (2012) 30–47.
[29] R. Verfürth, A Posteriori Error Estimation for the Stokes Equations. *Numer. Math.* **55** (1989) 309–325.
[30] R. Verfürth, A Posteriori Error Estimation for the Stokes Equations II non-conforming discretizations. *Numer. Math.* **60** (1991) 235–249.
[31] R. Verfürth, *A review of a posteriori error estimation and adaptive mesh-refinement techniques.* Teubner–Wiley, Stuttgart (1996).
[32] R. Verfürth, Error estimates for some quasi-interpolation operators. *ESAIM:M2AN* **33** (1999) 695–713.