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# ON THE SECOND-ORDER CONVERGENCE OF FINITE VOLUME METHODS FOR THE LAPLACE EQUATION ON DELAUNAY-VORONOI MESHES

PASCAL OMNES<sup>1,2</sup>

**Abstract.** Cell-centered and vertex-centered finite volume schemes for the Laplace equation with homogeneous Dirichlet boundary conditions are considered on triangular meshes and their Voronoi duals. On a two-dimensional convex polygonal domain, it is shown that a suitable combination of the solutions of these two schemes converges with second-order accuracy towards the exact solution in the  $L^2$  norm, under the sufficient condition that the right-hand side of the Laplace equation belongs to  $H^1(\Omega)$ .

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## INTRODUCTION

Finite volume schemes are popular methods to obtain approximations of the solutions of various types of partial differential equations. In the present work, we consider approximations by such schemes of the Laplace equation  $-\Delta u = f$  in a two-dimensional convex polygonal bounded domain  $\Omega$ , associated with homogeneous Dirichlet boundary conditions. Starting from a given mesh covering  $\Omega$ , we may distinguish three families of finite volume schemes. First, the principle of the so-called “cell-centered” schemes is to associate discrete unknowns with the cells of the mesh and to integrate the Laplace equation on each cell. Among various approaches, (which have been developed mainly for anisotropic diffusion, but which may of course be applied to the Laplace equation), we may cite [1,2,7,10,17,26,31–33]. The principle of the second family, the so-called “vertex-centered” schemes, is to associate discrete unknowns with the vertices of the primal mesh, and then integrate the Laplace equation on the cells of a dual mesh, centered on the vertices [5,6,11,12,25,37]. More recently, a third family of schemes has emerged, which combines the previous two approaches, since these schemes associate unknowns with both the cells and the vertices of the mesh, and integrate the Laplace equation on both the cells of the primal and dual meshes [4,14,16,19,20,27,28,35]. The originality of these schemes is that they work well on all kind of meshes, including very distorted, degenerating, or highly nonconforming meshes (see the numerical tests in [20]). Since these schemes are based on the definition of discrete gradient and divergence operators which verify a discrete Green formula, they are called “discrete duality finite volume” (DDFV) schemes.

In this work, we shall be interested in the convergence analysis in the  $L^2$  norm of a combination of the first two families of schemes when the primal mesh is triangular, and the dual mesh its associated Voronoi diagram, thanks to tools introduced for the third family in [20].

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*Keywords and phrases:* finite volume method, Laplace equation, Delaunay meshes, Voronoi meshes, convergence, error estimates

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Let us first review how the above-mentioned three families of schemes are constructed. First, both sides of the Laplace equation are integrated on the cells of the primal and/or dual mesh; for each cell, the resulting left-hand side is then transformed into the integral of  $-\nabla u \cdot \mathbf{n}$  on the boundary of the cell, thanks to the Green formula. The evaluation of this boundary integral in terms of the discrete unknowns of the scheme is a key issue in the construction and analysis of any of these schemes.

As far as cell-centered schemes are concerned, a simple answer to this question is obtained in the case of so-called “admissible meshes” [23, Definition 9.1]. Roughly speaking, a mesh is said to be admissible if one may associate a point  $\mathbf{x}_K$  to each cell  $K$  of the mesh and a point  $\mathbf{x}_\sigma$  to each boundary edge  $\sigma$  of the mesh such that

- The point  $\mathbf{x}_K$  lies inside  $K$ .
- For all pairs of neighboring cells  $K$  and  $L$  the segment  $[\mathbf{x}_K \mathbf{x}_L]$  is orthogonal to the common edge  $\partial K \cap \partial L$  of  $K$  and  $L$ .
- For any boundary cell  $K$  which has an edge  $\sigma$  on the boundary, the segment  $[\mathbf{x}_K \mathbf{x}_\sigma]$  is orthogonal to  $\sigma$ .

In that case, on the interface  $\partial K \cap \partial L$ , the value of  $\nabla u \cdot \mathbf{n}$  is approached in terms of the unknowns  $u_K$  and  $u_L$  associated to the cells  $K$  and  $L$ , respectively, by the finite difference  $\frac{u_L - u_K}{\|\mathbf{x}_K \mathbf{x}_L\|}$ , if  $\mathbf{n}$  is oriented from  $K$  to  $L$ . On the boundary edge  $\sigma$ , with  $\sigma \subset \partial K \cap \partial \Omega$ , the approximation of  $\nabla u \cdot \mathbf{n}$  is given by  $\frac{-u_K}{\|\mathbf{x}_K \mathbf{x}_\sigma\|}$  in the case of homogeneous Dirichlet boundary conditions. The case of non admissible meshes, and/or anisotropic diffusion has recently drawn much attention as reported in the above-cited papers, but is out of the scope of this work. In the case of admissible meshes, the resulting scheme has as many unknowns as equations (one per cell) and is shown to possess a unique solution which converges to the solution with first-order accuracy in a discrete energy norm (if the solution itself belongs to  $H^2(\Omega)$  and with some additional constraints on the mesh, see [23, Definition 9.4]), and, as a consequence of the discrete Poincaré inequality (see [23, Lemma 9.1]), in the discrete  $L^2$  norm as well, [23, Theorem 9.4]. Related results in the energy norm have been shown in [34, 38]. On the other hand, deriving necessary and/or sufficient conditions to obtain second-order accuracy in the discrete  $L^2$  norm is still an open issue on general admissible meshes. This issue has been positively answered in a very special case, namely for rectangular Cartesian grids when the point  $\mathbf{x}_K$  is the center of the rectangle  $K$  and under various regularity assumptions over the solution of the Laplace equation. We refer for example to [24, 30, 39]. On Delaunay triangular meshes, when the points  $\mathbf{x}_K$  are the circumcenters of the triangles  $K$ , the answer is believed to be true by some authors (see, e.g. [8]), based on numerical evidence. However, it has been shown in [36], by means of one-dimensional counter-examples, that second-order convergence in the discrete  $L^2$  norm may be lost if the right-hand side of the Laplace equation does not belong to  $H^1(\Omega)$ , or if the points  $\mathbf{x}_k$  associated to the one-dimensional segments  $K$  are not properly chosen.

As far as vertex-centered schemes are concerned, a simple way to evaluate  $\nabla u \cdot \mathbf{n}$  on the boundaries of the dual cells has been proposed in what is known as the Finite Volume Element (FVE) scheme, also named “box method” and may be explained in the following way in the case of a triangular primal mesh: each segment of the boundary of any dual cell is included in a triangle, in which  $u$  is approached locally by a standard Lagrange  $P^1$  finite element function  $u_h$  constructed with the help of the three unknowns located at the vertices of the triangle. This way to proceed leads to a linear system with as many unknowns as equations (one per vertex) and is shown to possess a unique solution  $u_h$  which converges to the solution with first-order accuracy in the standard energy norm, as reported in [6, 11, 12]. Second order convergence of  $u_h$  in the  $L^2$  norm has been shown in the special case in which the dual cell is the barycentric dual cell constructed by connecting the barycenter of each triangle to the midpoints of its edges. Sufficient hypotheses for this result to hold are that the solution of the Laplace equation is in  $H^2(\Omega)$ , and the right-hand  $f$  is in  $H^1(\Omega)$ . The proofs of second-order accuracy in the  $L^2$  norm explicitly use properties of the barycentric dual cell, and, thus, may not be extended to other constructions of the dual cells (see in particular [15, Assumption (1.4)], [22, page 1873], [21, Page 297]). Especially, second-order accuracy when the dual mesh is the Voronoi diagram associated to the vertices of the triangular primal mesh is an open issue.

Finally, DDFV schemes use a four-point gradient formula defined in [17] on the so-called “diamond cells”, whose diagonals are the primal and associated dual edges. Such schemes for the Laplace equation have been shown to converge in [20] on very general meshes, with first-order accuracy in the broken energy norm, as well

as in the discrete  $L^2(\Omega)$  norm, provided the solution of the Laplace equation belongs to  $H^2(\Omega)$ . Additional convergence results for anisotropic and/or non linear diffusion and/or discontinuous coefficients may be found in [4, 9], see also [35]. For such schemes, an almost second-order accuracy result in the  $L^2$  norm was shown (see [20, Theorem 7.2]) for homothetically refined triangular grids (see the definition in section 7 of [20]) in which the points  $\mathbf{x}_K$  associated to the primal cells  $K$  are their isobarycenters, under the supplementary assumption that the right-hand side of the Laplace equation belongs to  $H^1(\Omega)$ . Since the main argument in the proof is that for homothetically refined triangular grids, almost all diamond-cells are parallelograms, the proof of [20, Theorem 7.2] may be adapted to the case of homothetically refined triangular grids in which the point  $x_K$  associated to a triangle  $K$  is the circumcenter of  $K$ . However, second-order accuracy in the  $L^2$  norm on more general meshes is an open issue, in particular on families of non homothetically refined triangular meshes.

The aim of this article is to show that, when  $\Omega$  is a two-dimensional convex polygonal bounded domain covered by a family of finer and finer triangular primal meshes (with some restrictions on the angles of the triangles), a suitably defined combination of the solutions of the associated cell-centered and vertex-centered schemes (on the Voronoi duals) converges to the solution of the Laplace equation with second-order accuracy, under the sufficient condition that the right-hand side  $f$  belongs to  $H^1(\Omega)$ . Therefore, though this result does not give a complete answer to the above-mentioned open issues, it constitutes an important improvement over what was previously known. The tool we shall use to prove this, is the combination of the above-mentioned two schemes into a single DDFV scheme, which, in turn, is shown to be equivalent to a finite element-like scheme (only the right-hand side of the resulting linear system slightly differs). Then, the traditional Aubin-Nitsche lemma allows us to prove second-order convergence. The additional difficulty with respect to genuine finite element schemes is the lack of Galerkin orthogonality associated to finite volume schemes. The three main points in the proof are, first, that  $f$  is locally well approached by its mean value since it is regular, second, that the diamond-cells are symmetric with respect to the dual edges, and, third, that since the point  $\mathbf{x}_K$  associated to a primal cell  $K$  is its circumcenter, it is equidistant from the vertices of  $K$ .

This article is constructed as follows. In section 1, we construct the primal and dual meshes and introduce some notations. In section 2, we present the finite volume schemes on the primal and dual meshes, while in section 3, we combine them into a single finite element-like scheme. This allows us to perform the error analysis in section 4. Conclusions are drawn in section 5.

## 1. THE PRIMAL AND DUAL MESHES AND ASSOCIATED NOTATIONS

We shall consider in what follows a two-dimensional bounded convex polygonal domain  $\Omega$  covered by a family of triangulations  $\mathcal{T}$  characterized by  $h := \sup_{K \in \mathcal{T}} \text{diam}(K)$ .

Due to some technicalities in the proofs of our results, we shall state the following hypothesis

**Hypothesis 1.1.** *We suppose that there exists an angle  $\theta^* > 0$ , not depending on  $h$ , such that any angle of any triangle  $K$  in the triangulation is lower than (or equal to)  $\pi/2 - \theta^*$ .*

Note that this hypothesis immediately implies that, choosing the circumcenter of  $K$  for the point  $\mathbf{x}_K$  associated to the triangle  $K$  and the midpoint of  $\sigma$  for the point  $\mathbf{x}_\sigma$  associated to the boundary edge  $\sigma$ , properly defines an admissible mesh as defined in the introduction.

We shall use the following notations, summarized on Fig. 1, 2 and 3: For any  $K \in \mathcal{T}$ , we shall denote by  $m(K)$  the area of  $K$ ; we shall call  $\mathcal{E}_K$  the set of three edges of  $K$ . Then,  $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$  is the set of all edges in the mesh. Further,  $\mathcal{E}_{ext}$  is the set of boundary edges, while  $\mathcal{E}_{int}$  is the set of interior edges. For any edge  $\sigma \in \mathcal{E}$ , we shall call  $\mathbf{x}_\sigma$  the midpoint of  $\sigma$ . If  $\sigma \in \mathcal{E}_{int}$ , we shall name  $K$  and  $L$  the two triangles such that  $\sigma = \partial K \cap \partial L$ . We shall often write  $\sigma = K|L$ . The unit normal vector to  $\sigma = K|L$  pointing outward  $K$  will be denoted by  $\mathbf{n}_{KL}$  or, equivalently, by  $\mathbf{n}_{K\sigma}$ . Note that  $\mathbf{n}_{KL} = -\mathbf{n}_{LK}$ . If  $\sigma \in \mathcal{E}_{ext}$ , the unit normal vector pointing outward  $K$  will be denoted by  $\mathbf{n}_{K\sigma}$ . The one-dimensional measure of  $\sigma$  will be denoted by  $m(\sigma)$ . We shall call  $\mathbf{x}_K$  the intersection of the orthogonal bisectors of the three edges of  $K$  and  $R_K$  the distance from  $\mathbf{x}_K$  to the vertices of  $K$ , which is the radius of the circle in which  $K$  is inscribed. For any  $\sigma \in \mathcal{E}_K$ , we shall denote by  $\mathcal{V}_{K,\sigma}$  the convex hull of  $\sigma \cup \{\mathbf{x}_K\}$ . The distance between  $\mathbf{x}_K$  and  $\mathbf{x}_\sigma$  will be denoted by  $d_{K\sigma}$ . If  $\sigma = K|L$ , we shall

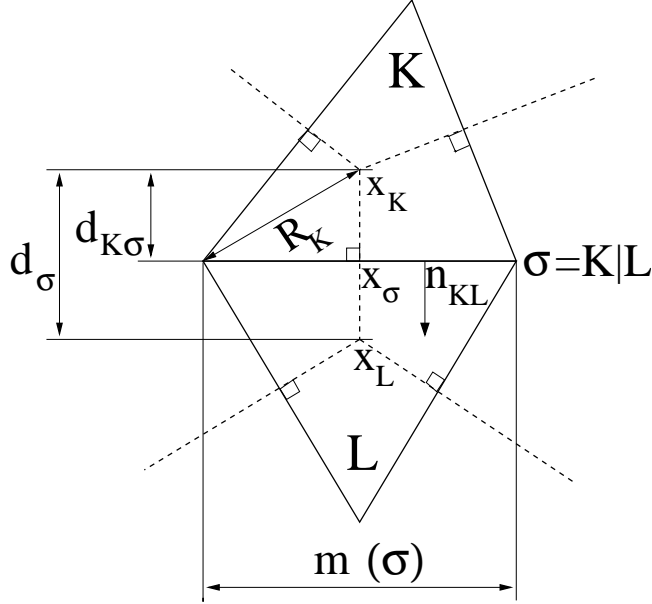


FIGURE 1. Notations for two neighboring primal cells

denote by  $d_\sigma = d_{K\sigma} + d_{L\sigma}$  the norm of  $(\mathbf{x}_K - \mathbf{x}_L)$ . If  $\sigma \in \mathcal{E}_{ext}$ ,  $d_\sigma = d_{K\sigma}$  is the norm of  $(\mathbf{x}_K - \mathbf{x}_\sigma)$ . We also consider the Voronoi (dual) mesh  $\mathcal{T}^*$  associated with the primal triangulation  $\mathcal{T}$  of  $\Omega$ . This mesh is obtained by joining the points  $\mathbf{x}_K$  to the points  $\mathbf{x}_L$  as soon as  $K$  and  $L$  are neighboring triangles and the points  $\mathbf{x}_K$  to the points  $\mathbf{x}_\sigma$  as soon as  $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$ . The elements of  $\mathcal{T}^*$  are denoted by  $K^*$ ,  $L^*$ , etc .... Any  $K^* \in \mathcal{T}^*$  is associated to a vertex  $\mathbf{x}_{K^*}$  of the primal mesh. We define  $\mathcal{T}_{int}^*$  as the set of dual elements which are such that  $\mathbf{x}_{K^*} \notin \Gamma$  and we set  $\mathcal{T}_{ext}^* = \mathcal{T}^* \setminus \mathcal{T}_{int}^*$ . For any  $K^* \in \mathcal{T}^*$ , we shall denote by  $m(K^*)$  the area of  $K^*$ ; we shall call  $\mathcal{E}_{K^*}$  the set of edges of  $K^*$ . Then,  $\mathcal{E}^* = \cup_{K^* \in \mathcal{T}^*} \mathcal{E}_{K^*}$  is the set of all edges of the dual mesh. Further,  $\mathcal{E}_{ext}^*$  is the set of boundary edges of the dual mesh, while  $\mathcal{E}_{int}^*$  is the set of interior dual edges. There is an obvious one-to-one correspondence between one given primal edge  $\sigma \in \mathcal{E}$  and an interior dual edge  $\sigma^* \in \mathcal{E}_{int}^*$ . Indeed, for any  $\sigma = K|L \in \mathcal{E}_{int}$ , we may associate  $\sigma^* = [\mathbf{x}_K \mathbf{x}_L]$ , while if  $\sigma \subset \partial K \cap \Gamma \in \mathcal{E}_{ext}$ , we may associate  $\sigma^* = [\mathbf{x}_K \mathbf{x}_\sigma]$ . For any  $\sigma \in \mathcal{E}$  we shall denote by  $\mathcal{V}_{\sigma, \sigma^*}$  the convex hull of  $\sigma \cup \sigma^*$ . We note that if  $\sigma = K|L \in \mathcal{E}_{int}$  then  $\mathcal{V}_{\sigma, \sigma^*} = \mathcal{V}_{K, \sigma} \cup \mathcal{V}_{L, \sigma}$ . And if  $\sigma \subset \partial K \cap \Gamma \in \mathcal{E}_{ext}$ , then  $\mathcal{V}_{\sigma, \sigma^*} = \mathcal{V}_{K, \sigma}$ . For any  $\sigma^* \in \mathcal{E}_{int}^*$ , with  $\sigma^* = \partial K^* \cap \partial L^*$ , we shall write  $\sigma^* = K^*|L^*$ . The unit normal vector directed from  $K^*$  to  $L^*$  will be denoted by  $\mathbf{n}_{K^*L^*}$ , and we shall denote by  $\mathcal{V}_{K^*, \sigma^*}$  the convex hull of  $\sigma^* \cup \{\mathbf{x}_{K^*}\}$ . Then there holds  $\mathcal{V}_{\sigma, \sigma^*} = \mathcal{V}_{K^*, \sigma^*} \cup \mathcal{V}_{L^*, \sigma^*}$ . We shall denote by  $m(\mathcal{V}_{\sigma, \sigma^*})$  the area of  $\mathcal{V}_{\sigma, \sigma^*}$  and remark that

$$m(\mathcal{V}_{\sigma, \sigma^*}) = \frac{1}{2} m(\sigma) d_\sigma. \quad (1)$$

We first prove a lemma that will be useful in the error estimations

**Lemma 1.2.** *Let Hyp. 1.1 hold. Then:*

- Any angle in any triangle  $K$  is greater than (or equal to)  $2\theta^*$ .
- There exists a constant  $C(\theta^*)$ , not depending on  $h$ , such that for any triangle  $K$ , there holds

$$\frac{\text{diam}(K)^2}{m(K)} \leq C(\theta^*).$$

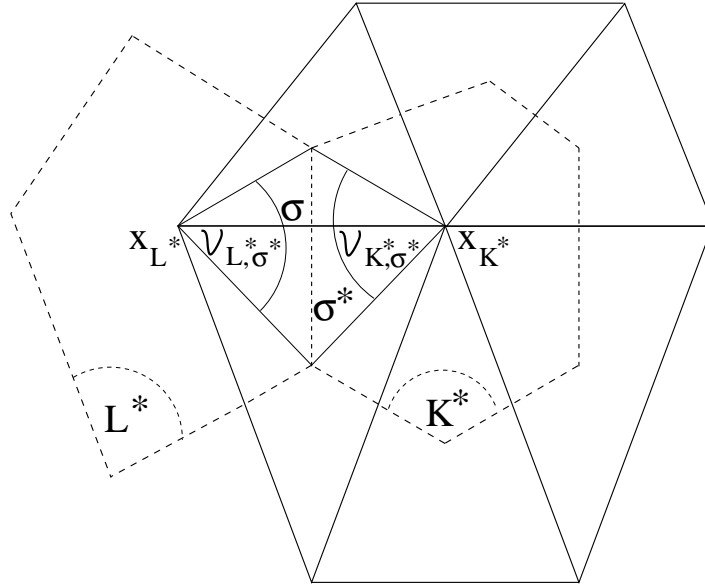


FIGURE 2. Notations for neighboring dual cells

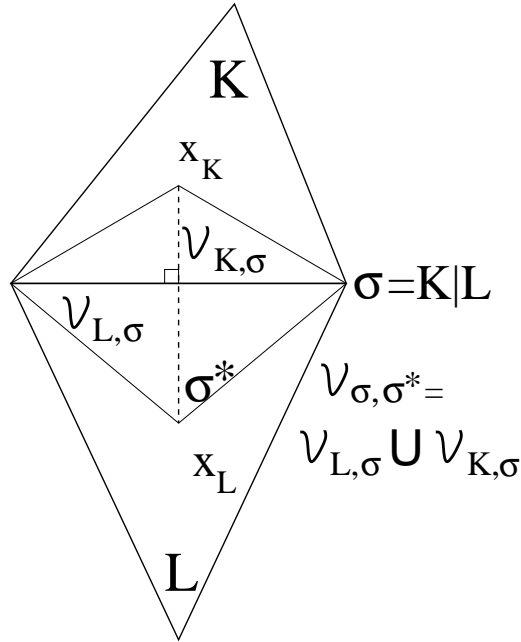


FIGURE 3. Notations for a diamond-cell

• Let  $\sigma = K|L$  or  $\sigma \subset \partial K \cap \Gamma$  be any primal edge and let  $\sigma^* = K^*|L^*$  be its associated interior dual edge. Then the smallest angle in the triangles  $\mathbf{x}_\sigma \mathbf{x}_{K^*} \mathbf{x}_K$  and  $\mathbf{x}_\sigma \mathbf{x}_{L^*} \mathbf{x}_L$  and in the triangles  $\mathbf{x}_\sigma \mathbf{x}_K \mathbf{x}_L$  and  $\mathbf{x}_\sigma \mathbf{x}_{L^*} \mathbf{x}_L$  when they exist (i.e. if  $\sigma$  is an interior primal edge), is bounded by below by a strictly positive angle which depends only on  $\theta^*$ , and thus independently of  $h$ .

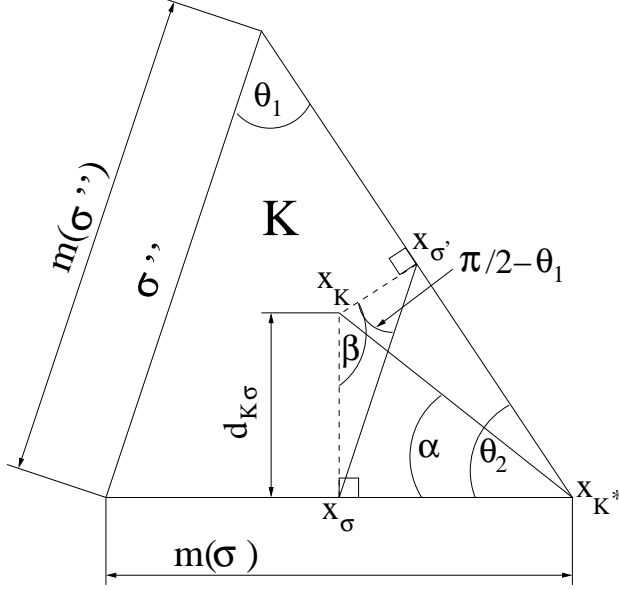


FIGURE 4. Notations for lemma 1.2.

• There exists a constant  $C(\theta^*)$ , not depending on  $h$ , such that for any triangle  $K$  and any of its edges  $\sigma$  there holds

$$\frac{m(\sigma)}{d_{K\sigma}} \leq C(\theta^*).$$

*Proof.* The first two points of the lemma are obvious. For the remaining two points, we refer to Fig. 4 for the notations, and we prove the third point for the triangle  $\mathbf{x}_{\sigma}\mathbf{x}_{K^*}\mathbf{x}_K$  only, since the proof is the same for the other three cases.

Let us start by considering the triangle  $\mathbf{x}_K\mathbf{x}_{\sigma}\mathbf{x}_{\sigma'}$ . Since  $\mathbf{x}_{\sigma}\mathbf{x}_{\sigma'}$  is parallel to  $\sigma''$ , there holds

$$\frac{d_{K\sigma}}{\sin(\pi/2 - \theta_1)} = \frac{\|\mathbf{x}_{\sigma}\mathbf{x}_{\sigma'}\|}{\sin \beta} = \frac{m(\sigma'')}{2 \sin \beta}$$

so that

$$d_{K\sigma} \geq \frac{m(\sigma'') \sin(\pi/2 - \theta_1)}{2 \sin \beta} \geq \frac{m(\sigma'') \sin(\theta^*)}{2}. \quad (2)$$

Now, consider the triangle  $\mathbf{x}_{\sigma}\mathbf{x}_{K^*}\mathbf{x}_K$ . There holds

$$\tan(\alpha) = \frac{2d_{K\sigma}}{m(\sigma)} \geq \frac{m(\sigma'')}{m(\sigma)} \sin(\theta^*) = \frac{\sin(\theta_2)}{\sin(\theta_1)} \sin(\theta^*) \geq \sin(2\theta^*) \sin(\theta^*), \quad (3)$$

thanks to (2) and since  $\sin(\theta_2) \geq \sin(2\theta^*)$ , as noticed in the first point of the lemma. Moreover, the other acute angle in the triangle  $\mathbf{x}_{\sigma}\mathbf{x}_{K^*}\mathbf{x}_K$  is equal to  $\pi/2 - \alpha \geq \theta^*$  since  $\alpha \leq \theta_2 \leq \pi/2 - \theta^*$ . This proves the third point in the lemma, while the fourth point easily follows from the estimation (3) on  $\tan(\alpha)$ .  $\square$

## 2. THE FINITE VOLUME SCHEMES

We shall consider two finite volume schemes which approach the solution  $\hat{u} \in H_0^1(\Omega)$  of the Laplace equation  $-\Delta u = f$  associated to homogeneous Dirichlet boundary conditions on  $\Gamma$ . The first scheme has unknowns  $u_K$

associated to the elements  $K$  of the primal mesh  $\mathcal{T}$ . The second scheme has unknowns  $u_{K^*}$  associated to the elements  $K^*$  of the dual mesh  $\mathcal{T}^*$ . Both of them are constructed by integrating the Laplace equation over the elements of their respective (primal or dual) mesh and by using a Green formula, which leads to the evaluation of the normal gradient of  $u$  at the interfaces between neighboring elements, or on the boundary  $\Gamma$ . Thanks to the orthogonality between the edges of the primal and dual meshes, these normal derivatives may be approached by simple expressions. More precisely, the first scheme reads

$$-\frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = f_K, \forall K \in \mathcal{T}, \quad (4)$$

with the fluxes

$$F_{K,\sigma} = m(\sigma) \frac{(u_L - u_K)}{d_\sigma}, \text{ if } \sigma \in \mathcal{E}_{int}, \sigma = K|L, \quad (5)$$

$$F_{K,\sigma} = m(\sigma) \frac{(u_\sigma - u_K)}{d_\sigma}, \text{ if } \sigma \in \mathcal{E}_{ext}, \sigma \subset \partial K \cap \Gamma. \quad (6)$$

In (4), the right-hand side is the mean-value of  $f$  over  $K$ :

$$f_K = \frac{1}{m(K)} \int_K f(\mathbf{x}) d\mathbf{x}. \quad (7)$$

In (6), we set  $u_\sigma = 0$  according to the homogeneous Dirichlet boundary condition.

The second scheme reads

$$-\frac{1}{m(K^*)} \sum_{\sigma^* \in \mathcal{E}_{K^*}} F_{K^*,\sigma^*} = f_{K^*}, \forall K^* \in \mathcal{T}_{int}^*, \quad (8)$$

with the fluxes

$$F_{K^*,\sigma^*} = d_\sigma \frac{(u_{L^*} - u_{K^*})}{m(\sigma)}, \text{ if } \sigma^* \in \mathcal{E}_{int}^*, \sigma^* = K^*|L^* \quad (9)$$

and boundary conditions

$$u_{K^*} = 0, \forall K^* \in \mathcal{T}_{ext}^*. \quad (10)$$

Note that in Eq. (9), the quantities  $d_\sigma$  and  $m(\sigma)$  refer to the primal edge  $\sigma$  associated to the internal dual edge  $\sigma^*$  as explained previously. Moreover, boundary fluxes over  $\partial K^* \cap \partial\Omega$  are not needed in the definition of the second scheme, since Eq. (8) is only written for interior dual cells.

In (8), the right-hand side is the mean-value of  $f$  over  $K^*$ :

$$f_{K^*} = \frac{1}{m(K^*)} \int_{K^*} f(\mathbf{x}) d\mathbf{x}. \quad (11)$$

Note that although the values of  $f_{K^*}$  when  $K^* \in \mathcal{T}_{ext}^*$  are not needed by the scheme (8)–(10), we may still define the mean-values of  $f$  over these boundary dual cells by Eq. (11).

### 3. AN EQUIVALENT FINITE ELEMENT-LIKE SCHEME

First, we shall prove that a certain combination of the schemes considered above verify a discrete variational formulation. For this, we shall first define the set in which the unknowns of the above two schemes are to be searched

**Definition 3.1.** We set

$$V_0(\mathcal{T}) := \left\{ v = ((v_K)_{K \in \mathcal{T}}, (v_\sigma)_{\sigma \in \mathcal{E}_{ext}}, (v_{K^*})_{K^* \in \mathcal{T}^*}), \text{ s.t. } \right. \\ \left. v_\sigma = 0, \forall \sigma \in \mathcal{E}_{ext}, v_{K^*} = 0, \forall K^* \in \mathcal{T}_{ext}^* \right\} \quad (12)$$



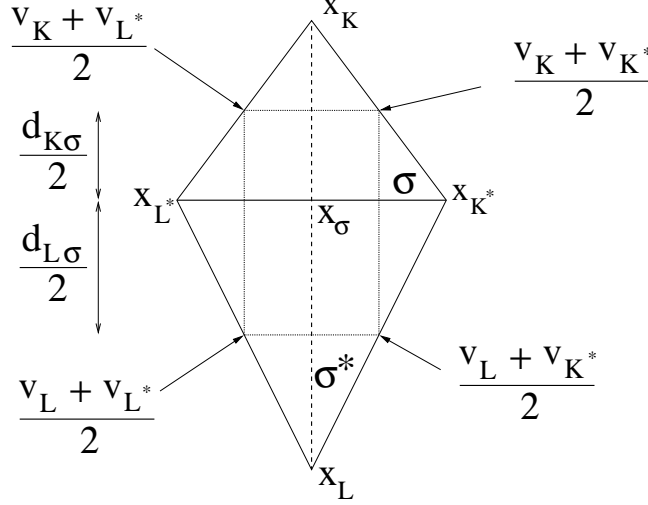


FIGURE 5. Values of the  $P^1$  function  $v_h$  on the diamond-cell  $\mathcal{V}_{\sigma, \sigma^*}$ .

With these discrete values, we shall associate two functions. We start with

**Definition 3.2.** Let  $\theta_K$  be the characteristic function of  $K$ . Identically, let  $\theta_{K^*}$  be the characteristic function of  $K^*$ . Let  $v = ((v_K)_{K \in \mathcal{T}}, (v_{\sigma})_{\sigma \in \mathcal{E}_{ext}}, (v_{K^*})_{K^* \in \mathcal{T}^*})$  be in  $V_0(\mathcal{T})$  defined above. We define the function  $v_h^{\Delta, *}$  as follows

$$v_h^{\Delta, *} := \frac{1}{2} \left( \sum_{K \in \mathcal{T}} v_K \theta_K + \sum_{K^* \in \mathcal{T}^*} v_{K^*} \theta_{K^*} \right). \quad (13)$$

In the notation  $v_h^{\Delta, *}$ , the superscript  $\Delta$  stands for the primal (triangular) mesh and the superscript  $*$  stands for the dual mesh.

The second function is defined by its restrictions on the diamond-cells of the mesh

**Definition 3.3.** Let  $v = ((v_K)_{K \in \mathcal{T}}, (v_{\sigma})_{\sigma \in \mathcal{E}_{ext}}, (v_{K^*})_{K^* \in \mathcal{T}^*})$  be in  $V_0(\mathcal{T})$  defined above. With these values, we define a function  $v_h$  constructed in the following way. Let us first consider an inner primal edge  $\sigma \in \mathcal{E}_{int}$  with  $\sigma = K|L$  and  $\sigma^* = K^*|L^*$  its associated inner dual edge. The restriction of  $v_h$  on  $\mathcal{V}_{\sigma, \sigma^*}$  is defined as the only  $P^1$  function over  $\mathcal{V}_{\sigma, \sigma^*}$  which is such that (see Fig. 5)

$$\begin{aligned} v_h \left( \frac{\mathbf{x}_K + \mathbf{x}_{K^*}}{2} \right) &= \frac{v_K + v_{K^*}}{2}, & v_h \left( \frac{\mathbf{x}_K + \mathbf{x}_{L^*}}{2} \right) &= \frac{v_K + v_{L^*}}{2}, \\ v_h \left( \frac{\mathbf{x}_L + \mathbf{x}_{L^*}}{2} \right) &= \frac{v_L + v_{L^*}}{2}, & v_h \left( \frac{\mathbf{x}_L + \mathbf{x}_{K^*}}{2} \right) &= \frac{v_L + v_{K^*}}{2}. \end{aligned} \quad (14)$$

A similar formula defines  $v_h$  on  $\mathcal{V}_{\sigma, \sigma^*}$  if  $\sigma \in \mathcal{E}_{ext}$  with  $\sigma \subset \partial K \cap \Gamma$  associated to the inner dual edge  $\sigma^* = K^*|L^*$ :

$$\begin{aligned} v_h \left( \frac{\mathbf{x}_K + \mathbf{x}_{K^*}}{2} \right) &= \frac{v_K + v_{K^*}}{2}, & v_h \left( \frac{\mathbf{x}_K + \mathbf{x}_{L^*}}{2} \right) &= \frac{v_K + v_{L^*}}{2}, \\ v_h \left( \frac{\mathbf{x}_{\sigma} + \mathbf{x}_{L^*}}{2} \right) &= \frac{v_{\sigma} + v_{L^*}}{2}, & v_h \left( \frac{\mathbf{x}_{\sigma} + \mathbf{x}_{K^*}}{2} \right) &= \frac{v_{\sigma} + v_{K^*}}{2}. \end{aligned} \quad (15)$$

Of course, the definition of a  $P^1$  function by four of its values is in general impossible. However, it may be checked that, in the present case, such a function exists and is unique thanks to the fact that the quadrangle

$\frac{\mathbf{x}_K + \mathbf{x}_{K^*}}{2}$ ,  $\frac{\mathbf{x}_K + \mathbf{x}_{L^*}}{2}$ ,  $\frac{\mathbf{x}_L + \mathbf{x}_{L^*}}{2}$ ,  $\frac{\mathbf{x}_L + \mathbf{x}_{K^*}}{2}$  is a rectangle (actually, a parallelogram would be enough) and since

$$v_h \left( \frac{\mathbf{x}_K + \mathbf{x}_{K^*}}{2} \right) + v_h \left( \frac{\mathbf{x}_L + \mathbf{x}_{L^*}}{2} \right) = v_h \left( \frac{\mathbf{x}_K + \mathbf{x}_{L^*}}{2} \right) + v_h \left( \frac{\mathbf{x}_L + \mathbf{x}_{K^*}}{2} \right).$$

Note that the function  $v_h$  is non-conforming since it is only continuous at the midpoints of the boundaries of the cells  $\mathcal{V}_{\sigma, \sigma^*}$ . Note that thanks to the second line in (15), and since  $v_\sigma = 0$  for all  $\sigma \in \mathcal{E}_{ext}$  and  $v_{K^*} = 0$  for all  $K^* \in \mathcal{T}_{ext}^*$ , the function  $v_h$  vanishes on  $\Gamma$ .

**Definition 3.4.** We call  $\mathcal{L}$  the linear operator which associates to the element  $v \in V_0(\mathcal{T})$  the function  $v_h$  defined above. Next, we define

$$V_{h0} := \mathcal{L}(V_0(\mathcal{T}))$$

the set of all possible functions  $v_h$  defined by (14) and (15).

A direct calculation leads to the following proposition

**Proposition 3.5.** *Let  $v_h$  be in  $V_{h0}$ . Then its broken (diamond-cell per diamond-cell) gradient  $\nabla_h v_h$  has the following expression*

$$(\nabla_h v_h)_{\mathcal{V}_{\sigma, \sigma^*}} = \begin{cases} \frac{(v_L - v_K)}{d_\sigma} \mathbf{n}_{KL} + \frac{(v_{L^*} - v_{K^*})}{m(\sigma)} \mathbf{n}_{K^*L^*} & \text{if } \sigma \in \mathcal{E}_{int} \\ \frac{(v_\sigma - v_K)}{d_\sigma} \mathbf{n}_{K\sigma} + \frac{(v_{L^*} - v_{K^*})}{m(\sigma)} \mathbf{n}_{K^*L^*} & \text{if } \sigma \in \mathcal{E}_{ext} \end{cases}, \forall \mathcal{V}_{\sigma, \sigma^*}. \quad (16)$$

We may now state the main result of this section

**Proposition 3.6.** *The finite volume formulations (4)–(11) may be combined into a single finite element-like formulation which reads: Find  $u_h$  in  $V_{h0}$  such that for all  $v_h$  in  $V_{h0}$ ,*

$$a(u_h, v_h) = \ell(v_h^{\Delta, *}), \quad (17)$$

where

$$a(u_h, v_h) = \sum_{\mathcal{V}_{\sigma, \sigma^*}} \int_{\mathcal{V}_{\sigma, \sigma^*}} \nabla_h u_h \cdot \nabla_h v_h \, d\mathbf{x}$$

and

$$\ell(v_h^{\Delta, *}) = \int_{\Omega} f v_h^{\Delta, *}(\mathbf{x}) \, d\mathbf{x}. \quad (18)$$

Moreover, there exists a constant  $C$  not depending on the mesh such that, if  $\hat{u}$  is in  $H^2(\Omega)$ , there holds

$$|\hat{u} - u_h|_{1,h} := a(\hat{u} - u_h, \hat{u} - u_h)^{1/2} \leq Ch \|\hat{u}\|_{H^2(\Omega)}. \quad (19)$$

Note that the definitions of the bilinear form  $a$  and of the linear form  $\ell$  may be extended to functions belonging to  $H^1(\Omega)$ .

*Proof.* The equivalence between the finite element like formulation and the finite volume schemes is a particular case of the proof given in [20, Prop. 4.4]. As an illustration, we shall only show how to derive (17) from (4)–(11).

Consider any vector  $v = ((v_K)_{K \in \mathcal{T}}, (v_\sigma)_{\sigma \in \mathcal{E}_{ext}}, (v_{K^*})_{K^* \in \mathcal{T}^*})$  in  $V_0(\mathcal{T})$ . Thanks to (4) and (7), there holds

$$-v_K \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = m(K) v_K f_K = \int_{\Omega} f(\mathbf{x}) v_K \theta_K(\mathbf{x}) \, d\mathbf{x}, \forall K \in \mathcal{T}.$$

Then, thanks to (8) and (11), there holds for all  $K^* \in \mathcal{T}_{int}^*$

$$-v_{K^*} \sum_{\sigma^* \in \mathcal{E}_{K^*}} F_{K^*, \sigma^*} = m(K^*) v_{K^*} f_{K^*} = \int_{\Omega} f(\mathbf{x}) v_{K^*} \theta_{K^*}(\mathbf{x}) d\mathbf{x}. \quad (20)$$

But since  $v_{K^*}$  vanishes for all  $K^* \in \mathcal{T}_{ext}^*$ , we may also write, for  $K^* \in \mathcal{T}_{ext}^*$

$$-v_{K^*} \sum_{\sigma^* \in \mathcal{E}_{K^*} \cap \mathcal{E}_{int}^*} F_{K^*, \sigma^*} = \int_{\Omega} f(\mathbf{x}) v_{K^*} \theta_{K^*}(\mathbf{x}) d\mathbf{x}.$$

Thus, for any vector  $v$  in  $V_0(\mathcal{T})$ , there holds

$$-\frac{1}{2} \left( \sum_{K \in \mathcal{T}} v_K \sum_{\sigma \in \mathcal{E}_K} F_{K, \sigma} + \sum_{K^* \in \mathcal{T}^*} v_{K^*} \sum_{\sigma^* \in \mathcal{E}_{K^*} \cap \mathcal{E}_{int}^*} F_{K^*, \sigma^*} \right) = \int_{\Omega} f v_h^{\Delta, *}(\mathbf{x}) d\mathbf{x}. \quad (21)$$

Now the sums in the left-hand side of Eq. (21) can be reorganized in the following way. Let us first consider a given  $\sigma \in \mathcal{E}_{int}$  and its associated  $\sigma^* \in \mathcal{E}_{int}^*$ . They both appear twice in the sums in the left-hand side of Eq. (21). Since  $F_{K, \sigma} = -F_{L, \sigma}$  and  $F_{K^*, \sigma^*} = -F_{L^*, \sigma^*}$  if  $\sigma = K|L$  and  $\sigma^* = K^*|L^*$ , we may write, thanks to the expressions of  $F_{K, \sigma}$  and  $F_{K^*, \sigma^*}$ , respectively given by (5) and (9)

$$\begin{aligned} & -\frac{1}{2} (v_K F_{K, \sigma} + v_L F_{L, \sigma} + v_{K^*} F_{K^*, \sigma^*} + v_{L^*} F_{L^*, \sigma^*}) = \\ & \frac{1}{2} m(\sigma) \frac{(u_L - u_K)}{d_\sigma} (v_L - v_K) + \frac{1}{2} d_\sigma \frac{(u_{L^*} - u_{K^*})}{m(\sigma)} (v_{L^*} - v_{K^*}) = \\ & m(\mathcal{V}_{\sigma, \sigma^*}) \left[ \frac{(u_L - u_K)}{d_\sigma} \frac{(v_L - v_K)}{d_\sigma} + \frac{(u_{L^*} - u_{K^*})}{m(\sigma)} \frac{(v_{L^*} - v_{K^*})}{m(\sigma)} \right], \end{aligned}$$

thanks to (1). Let us now consider a given  $\sigma \in \mathcal{E}_{ext}$  and its associated  $\sigma^* \in \mathcal{E}_{int}^*$ . In Eq. (21),  $\sigma$  appears only once since it is a boundary edge. On the other hand,  $\sigma^*$  still appears twice and we may write

$$\begin{aligned} & -\frac{1}{2} (v_K F_{K, \sigma} + v_{K^*} F_{K^*, \sigma^*} + v_{L^*} F_{L^*, \sigma^*}) = \\ & -\frac{1}{2} m(\sigma) \frac{(u_\sigma - u_K)}{d_\sigma} v_K + \frac{1}{2} d_\sigma \frac{(u_{L^*} - u_{K^*})}{m(\sigma)} (v_{L^*} - v_{K^*}) = \\ & m(\mathcal{V}_{\sigma, \sigma^*}) \left[ \frac{(u_\sigma - u_K)}{d_\sigma} \frac{(v_\sigma - v_K)}{d_\sigma} + \frac{(u_{L^*} - u_{K^*})}{m(\sigma)} \frac{(v_{L^*} - v_{K^*})}{m(\sigma)} \right], \end{aligned}$$

since  $v_\sigma$  vanishes. Finally, since  $\mathbf{n}_{KL} \cdot \mathbf{n}_{K^*L^*} = 0$ , Eq. (21) may be rewritten as

$$\sum_{\mathcal{V}_{\sigma, \sigma^*}} m(\mathcal{V}_{\sigma, \sigma^*}) (\nabla_h u_h)_{\mathcal{V}_{\sigma, \sigma^*}} \cdot (\nabla_h v)_{\mathcal{V}_{\sigma, \sigma^*}} = \sum_{\mathcal{V}_{\sigma, \sigma^*}} \int_{\mathcal{V}_{\sigma, \sigma^*}} \nabla_h u_h \cdot \nabla_h v_h = \int_{\Omega} f v_h^{\Delta, *}(\mathbf{x}) d\mathbf{x},$$

which is the desired result.

Moreover, the error estimation (19) is inferred from [20, Th. 5.20], in which the angle  $\tau^*$  is always equal to  $\pi/2$  in the special case of Delaunay-Voronoi meshes. Related convergence results in a discrete norm for each component (primal or dual) of the gradient may be found in [23, 34, 38].  $\square$

#### 4. ERROR ESTIMATION IN THE $L^2$ NORM

Using the equivalent (non-conforming) finite element formulation, we shall derive an estimation in the  $L^2$  norm using the traditional Aubin-Nitsche lemma. An additional difficulty will arise due to the fact that the right-hand side in (17) is given by (18) instead of the more traditional term  $\int_{\Omega} f v_h(\mathbf{x}) d\mathbf{x}$  which would arise in a genuine finite element method. In order to evaluate errors coming from the difference between these two terms, we shall state a regularity hypothesis on  $f$ , which is the same as that involved in the studies concerning vertex-centered finite volume element schemes, see, e.g., [22] and one-dimensional cell-centered finite volume schemes, see [36].

**Hypothesis 4.1.** *We suppose that the function  $f$  belongs to  $H^1(\Omega)$ .*

**Remark 4.2.** The first consequence of this hypothesis (actually,  $f$  in  $L^2(\Omega)$  would be enough for this) is that, since  $\Omega$  has been supposed to be a convex polygonal domain, the exact solution  $\hat{u}$  of the Laplace equation belongs to  $H^2(\Omega)$  and  $\|\hat{u}\|_{H^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$ . As a corollary, the error estimation (19) provides

$$|\hat{u} - u_h|_{1,h} \leq Ch \|f\|_{L^2(\Omega)}, \quad (22)$$

with a constant  $C$  that does not depend on the mesh.

##### 4.1. A representation formula for the error in the $L^2(\Omega)$ norm

We start by writing

$$\|\hat{u} - u_h\|_{L^2(\Omega)} = \sup_{g \in L^2(\Omega)} \frac{\int_{\Omega} (u_h - \hat{u})g(\mathbf{x})d\mathbf{x}}{\|g\|_{L^2(\Omega)}}. \quad (23)$$

Now, for a given  $g \in L^2(\Omega)$ , let us define  $\hat{\phi} \in H_0^1(\Omega)$  which is the unique solution of the following problem

$$\begin{cases} -\Delta \hat{\phi} = g & \text{in } \Omega \\ \hat{\phi} = 0 & \text{on } \Gamma. \end{cases} \quad (24)$$

Since  $g$  is in  $L^2(\Omega)$ , and since we have supposed that  $\Omega$  is a convex polygonal domain,  $\hat{\phi}$  belongs to  $H^2(\Omega)$  and there exists a constant  $C$  depending only on  $\Omega$  such that

$$\|\hat{\phi}\|_{H^2(\Omega)} \leq C \|g\|_{L^2(\Omega)}. \quad (25)$$

We may write the following representation formula

**Proposition 4.3.** *Let  $\phi = ((\phi_K)_{K \in \mathcal{T}}, (\phi_{\sigma})_{\sigma \in \mathcal{E}_{ext}}, (\phi_{K^*})_{K^* \in \mathcal{T}^*})$  be given in  $V_0(\mathcal{T})$  and let*

$$\phi_h = \mathcal{L}(\phi) \in V_{h0} \quad (26)$$

*be the function associated to  $\phi$  through Def. 3.4. There holds*

$$\begin{aligned} \int_{\Omega} (u_h - \hat{u})g(\mathbf{x})d\mathbf{x} &= a(u_h - \hat{u}, \hat{\phi} - \phi_h) - \int_{\Omega} f(\phi_h - \phi_h^{\Delta,*})(\mathbf{x})d\mathbf{x} \\ &- \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\partial\mathcal{V}_{\sigma,\sigma^*}} \nabla \hat{u} \cdot \mathbf{n} \phi_h d\sigma \\ &- \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\partial\mathcal{V}_{\sigma,\sigma^*}} (u_h - \hat{u}) \nabla \hat{\phi} \cdot \mathbf{n} d\sigma. \end{aligned} \quad (27)$$

*Proof.* Through Eq. (24), there holds

$$\begin{aligned} \int_{\Omega} (u_h - \hat{u})g(\mathbf{x})d\mathbf{x} &= - \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\mathcal{V}_{\sigma,\sigma^*}} (u_h - \hat{u}) \Delta \hat{\phi}(\mathbf{x}) d\mathbf{x} \\ &= a(u_h - \hat{u}, \hat{\phi}) - \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\partial\mathcal{V}_{\sigma,\sigma^*}} (u_h - \hat{u}) \nabla \hat{\phi} \cdot \mathbf{n} d\sigma, \end{aligned} \quad (28)$$

thanks to a Green formula on each  $\mathcal{V}_{\sigma,\sigma^*}$ , and where  $\mathbf{n}$  is the unit exterior normal vector on  $\partial\mathcal{V}_{\sigma,\sigma^*}$ . Now let us consider an arbitrary  $\phi$  be given in  $\in V_0(\mathcal{T})$  and let  $\phi_h = \mathcal{L}(\phi)$  be its associated function. There holds, by definition of the bilinear form  $a$

$$a(u_h - \hat{u}, \hat{\phi}) = a(u_h - \hat{u}, \hat{\phi} - \phi_h) + a(u_h, \phi_h) - \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\mathcal{V}_{\sigma,\sigma^*}} \nabla \hat{u} \cdot \nabla_h \phi_h(\mathbf{x}) d\mathbf{x}. \quad (29)$$

Thanks to (17), we have

$$a(u_h, \phi_h) = \int_{\Omega} f \phi_h^{\Delta,*}(\mathbf{x}) d\mathbf{x}. \quad (30)$$

On the other hand, since  $-\Delta \hat{u} = f$ , a Green formula on each  $\mathcal{V}_{\sigma,\sigma^*}$  provides

$$- \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\mathcal{V}_{\sigma,\sigma^*}} \nabla \hat{u} \cdot \nabla_h \phi_h(\mathbf{x}) d\mathbf{x} = - \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\mathcal{V}_{\sigma,\sigma^*}} f \phi_h(\mathbf{x}) d\mathbf{x} - \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\partial\mathcal{V}_{\sigma,\sigma^*}} \nabla \hat{u} \cdot \mathbf{n} \phi_h d\sigma. \quad (31)$$

Gathering (28)–(31), the result (27) is obtained.  $\square$

Up to now, the values of  $\phi$  are arbitrary, but since they will play a key role in the evaluation of the various terms in (27), we shall precise them now.

#### 4.2. Choosing $\phi$

Of course, we shall choose  $\phi$  so that the associated function  $\phi_h = \mathcal{L}(\phi)$  (see Def. 3.4) will be a good approximation of  $\hat{\phi}$ . We propose to choose the values  $((\phi_K)_{K \in \mathcal{T}}, (\phi_{\sigma})_{\sigma \in \mathcal{E}_{ext}}, (\phi_{K^*})_{K^* \in \mathcal{T}^*})$  as the solutions of the primal and dual finite volume schemes of section 2, associated to the Laplace equation (24) satisfied by  $\hat{\phi}$ . More precisely, we write

$$- \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = \int_K g(\mathbf{x}) d\mathbf{x}, \quad \forall K \in \mathcal{T}, \quad (32)$$

with the fluxes

$$F_{K,\sigma} = m(\sigma) \frac{(\phi_L - \phi_K)}{d_{\sigma}}, \quad \text{if } \sigma \in \mathcal{E}_{int}, \sigma = K|L, \quad (33)$$

$$F_{K,\sigma} = m(\sigma) \frac{(\phi_{\sigma} - \phi_K)}{d_{\sigma}}, \quad \text{if } \sigma \in \mathcal{E}_{ext}, \sigma \subset \partial K \cap \Gamma. \quad (34)$$

In (34), we set

$$\phi_{\sigma} = 0 \quad (35)$$

according to the homogeneous Dirichlet boundary condition.

The second scheme reads

$$- \sum_{\sigma^* \in \mathcal{E}_{K^*}} F_{K^*,\sigma^*} = \int_{K^*} g(\mathbf{x}) d\mathbf{x}, \quad \forall K^* \in \mathcal{T}_{int}^*, \quad (36)$$

with the fluxes

$$F_{K^*,\sigma^*} = d_{\sigma} \frac{(\phi_{L^*} - \phi_{K^*})}{m(\sigma)}, \quad \text{if } \sigma^* \in \mathcal{E}_{int}^*, \sigma^* = K^*|L^* \quad (37)$$

and boundary conditions

$$\phi_{K^*} = 0, \forall K^* \in \mathcal{T}_{ext}^*. \quad (38)$$

Note that Eqs (35) and (38) ensure that  $\phi$  is indeed in  $V_0(\mathcal{T})$  as required in Prop. 4.3.

In particular, two points will be important in what follows. First, since  $\hat{\phi}$  is in  $H^2(\Omega)$ , we may apply the error estimate given by (19), in which we replace  $\hat{u}$  and  $u_h$  by  $\hat{\phi}$  and  $\phi_h$ , so that there holds, taking into account (25)

$$\left| \hat{\phi} - \phi_h \right|_{1,h} \leq Ch \|g\|_{L^2(\Omega)}. \quad (39)$$

with a constant  $C$  not depending on the mesh. Moreover, it is clear from the definitions (16) and from (32) and (33), that  $\phi_h$  verifies

$$-\sum_{\sigma \in \mathcal{E}_K} m(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma} = \int_K g(\mathbf{x}) d\mathbf{x}, \forall K \in \mathcal{T}. \quad (40)$$

For this, we recall that we have set  $\mathbf{n}_{KL} = \mathbf{n}_{K\sigma}$  if  $\sigma = K|L$ .

#### 4.3. Estimations of the various terms in (27)

The technique used to evaluate the last two terms in Eq. (27) is classical and dates back to [18]. It is based on [18, lemma 3], in which we choose  $m = 0$ :

**Lemma 4.4.** *Let  $T$  be a triangle and let  $T'$  be any of its edges; there exists a constant  $C$  independent of  $T$  such that for all  $v$  in  $H^1(T)$ , and for all  $\varphi$  in  $H^1(T)$ , there holds*

$$\left| \int_{T'} \varphi (v - M_{T'} v) d\sigma \right| \leq C \sigma(T) \text{diam}(T) |\varphi|_{1,T} |v|_{1,T}, \quad (41)$$

where  $M_{T'} v := \frac{1}{m(T')} \int_{T'} v d\sigma$  is the mean value of  $v$  over  $T'$  and where  $\sigma(T) := \frac{\text{diam}(T)}{\rho(T)}$  is classically the ratio of the diameter of  $T$  to the diameter of the biggest circle inscribed in  $T$ .

Let us start by the last term in (27).

**Lemma 4.5.** *There exists a constant  $C$  depending only on  $\theta^*$  such that*

$$\left| \sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\partial \mathcal{V}_{\sigma,\sigma^*}} (u_h - \hat{u}) \nabla \hat{\phi} \cdot \mathbf{n} d\sigma \right| \leq Ch^2 \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \quad (42)$$

*Proof.* The function  $u_h$  is  $P^1$  and continuous at the midpoint of each edge of the diamond mesh and vanishes on the boundary  $\Gamma$ . Moreover,  $\hat{\phi}$  is in  $H^2(\Omega)$ . Thus, there holds

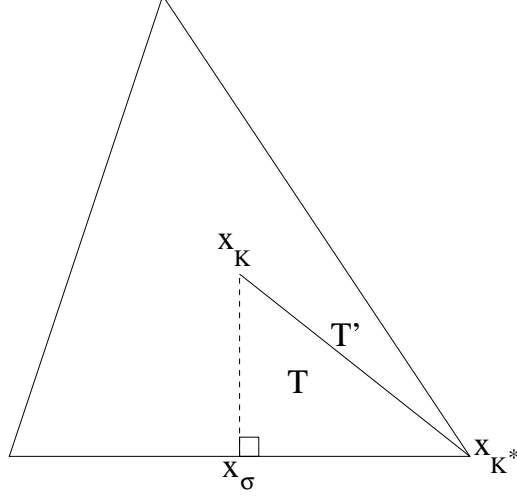
$$\sum_{\mathcal{V}_{\sigma,\sigma^*}} \sum_{T' \subset \partial \mathcal{V}_{\sigma,\sigma^*}} u_h M_{T'} \nabla \hat{\phi} \cdot \mathbf{n} = 0.$$

Moreover, since  $\hat{u}$  is in  $H^2(\Omega)$  and vanishes on  $\Gamma$

$$\sum_{\mathcal{V}_{\sigma,\sigma^*}} \sum_{T' \subset \partial \mathcal{V}_{\sigma,\sigma^*}} \hat{u} M_{T'} \nabla \hat{\phi} \cdot \mathbf{n} = 0.$$

Therefore,

$$\sum_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\partial \mathcal{V}_{\sigma,\sigma^*}} (u_h - \hat{u}) \nabla \hat{\phi} \cdot \mathbf{n} d\sigma = \sum_{\mathcal{V}_{\sigma,\sigma^*}} \sum_{T' \subset \partial \mathcal{V}_{\sigma,\sigma^*}} \int_{T'} (u_h - \hat{u}) (\nabla \hat{\phi} \cdot \mathbf{n} - M_{T'} \nabla \hat{\phi} \cdot \mathbf{n}) d\sigma.$$

FIGURE 6. Definition of the triangle  $T$  and its edge  $T'$ .

Next, for each  $\mathcal{V}_{\sigma, \sigma^*}$  and each edge  $T' \subset \partial\mathcal{V}_{\sigma, \sigma^*}$ , we shall apply lemma 4.4 on a triangle  $T$  defined to be the convex hull of  $T' \cup \{\mathbf{x}_\sigma\}$  (see Fig. 6) with  $v = \nabla \hat{\phi} \cdot \mathbf{n} \in H^1(T)$  (since  $\hat{\phi} \in H^2(\Omega)$ ) and  $\varphi = (u_h - \hat{u}) \in H^1(T)$  (since  $u_h$  is in  $P^1(T)$  and  $\hat{u} \in H^2(\Omega)$ ). Since  $\text{diam}(T) \leq h$  and since it is well-known that  $\sigma(T) \leq \frac{2}{\sin \theta(T)}$  where  $\theta(T)$  is the smallest angle in  $T$ , the third point of lemma 1.2 and (41) lead to the existence of a constant  $C$  depending only on  $\theta^*$  such that

$$\left| \int_{T'} (u_h - \hat{u}) (\nabla \hat{\phi} \cdot \mathbf{n} - M_{T'} \nabla \hat{\phi} \cdot \mathbf{n}) d\sigma \right| \leq Ch \|\nabla(\hat{u} - u_h)\|_{L^2(T)} \|\hat{\phi}\|_{H^2(T)}.$$

Since the set of such triangles  $T$  constitutes a partition of  $\Omega$ , a discrete Cauchy-Schwarz inequality, together with (25) and (22) leads to (42).  $\square$

Now, we turn to the third term in the right-hand side of (27).

**Lemma 4.6.** *There exists a constant  $C$  depending only on  $\theta^*$  such that*

$$\left| \sum_{\mathcal{V}_{\sigma, \sigma^*}} \int_{\partial\mathcal{V}_{\sigma, \sigma^*}} \nabla \hat{u} \cdot \mathbf{n} \phi_h d\sigma \right| \leq Ch^2 \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \quad (43)$$

*Proof.* The third term in the right-hand side of (27) may be transformed into

$$- \sum_{\mathcal{V}_{\sigma, \sigma^*}} \int_{\partial\mathcal{V}_{\sigma, \sigma^*}} \nabla \hat{u} \cdot \mathbf{n} (\phi_h - \hat{\phi}) d\sigma \quad (44)$$

since  $\hat{\phi}$  is continuous and vanishes along  $\Gamma$  and since there is no jump of  $\nabla \hat{u} \in H^1(\Omega)$  across  $\partial\mathcal{V}_{\sigma, \sigma^*}$ . Now, the technique we have used above to obtain (42) may be applied to evaluate (44) and we end up with

$$\left| \sum_{\mathcal{V}_{\sigma, \sigma^*}} \int_{\partial\mathcal{V}_{\sigma, \sigma^*}} (\phi_h - \hat{\phi}) \nabla \hat{u} \cdot \mathbf{n} d\sigma \right| \leq Ch \|\hat{\phi} - \phi_h\|_{1,h} \|f\|_{L^2(\Omega)}$$

with a constant depending only on  $\theta^*$ , and we conclude with (39).  $\square$

Next, bounding the first term in the right-hand side of (27) is performed by the Cauchy-Schwarz inequality and by (22) and (39). We obtain

**Lemma 4.7.** *There exists a constant  $C$  not depending on the mesh such that*

$$\left| a(u_h - \hat{u}, \hat{\phi} - \phi_h) \right| \leq Ch^2 \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \quad (45)$$

Now, the term which remains to be evaluated in (27) is that coming from the fact that (17)-(18) is not a genuine finite element formulation, like explained in the introduction of section 4.

We shall first define the following functions

**Definition 4.8.** Let  $((\phi_K)_{K \in \mathcal{T}}, (\phi_\sigma)_{\sigma \in \mathcal{E}_{ext}}, (\phi_{K^*})_{K^* \in \mathcal{T}^*})$  be given, we define  $\phi_h^\Delta$  and  $\phi_h^*$  by

$$\phi_h^\Delta|_K(\mathbf{x}) := \frac{\phi_K}{2}, \quad \forall \mathbf{x} \in K, \forall K \in \mathcal{T}, \quad (46)$$

$$\phi_h^*|_{K^*}(\mathbf{x}) := \frac{\phi_{K^*}}{2}, \quad \forall \mathbf{x} \in K^*, \forall K^* \in \mathcal{T}^*, \quad (47)$$

**Definition 4.9.** Let  $((\phi_K)_{K \in \mathcal{T}}, (\phi_\sigma)_{\sigma \in \mathcal{E}_{ext}}, (\phi_{K^*})_{K^* \in \mathcal{T}^*})$  be given, we define  $\phi_h^1$  and  $\phi_h^2$  by

$$\phi_h^1(\mathbf{x})|_{\mathcal{V}_{\sigma, \sigma^*}} := \begin{cases} \frac{d_{K\sigma}\phi_L + d_{L\sigma}\phi_K}{2d_\sigma} + (\mathbf{x} - \mathbf{x}_\sigma) \cdot \left( \frac{\phi_L - \phi_K}{d_\sigma} \right) \mathbf{n}_{KL}, \forall \mathbf{x} \in \mathcal{V}_{\sigma, \sigma^*} & \text{if } \sigma = K|L \in \mathcal{E}_{int} \\ \frac{\phi_\sigma}{2} + (\mathbf{x} - \mathbf{x}_\sigma) \cdot \left( \frac{\phi_\sigma - \phi_K}{d_\sigma} \right) \mathbf{n}_{K\sigma}, \forall \mathbf{x} \in \mathcal{V}_{\sigma, \sigma^*} & \text{if } \sigma \in \mathcal{E}_{ext} \end{cases}, \quad (48)$$

and

$$\phi_h^2(\mathbf{x})|_{\mathcal{V}_{\sigma, \sigma^*}} := \frac{\phi_{L^*} + \phi_{K^*}}{4} + (\mathbf{x} - \mathbf{x}_\sigma) \cdot \left( \frac{\phi_{L^*} - \phi_{K^*}}{m(\sigma)} \right) \mathbf{n}_{K^*L^*}, \quad \forall \mathbf{x} \in \mathcal{V}_{\sigma, \sigma^*}, \text{ with } \sigma^* = K^*|L^*, \quad (49)$$

With these definitions, there holds

**Lemma 4.10.**

$$\phi_h - \phi_h^{\Delta, *} = (\phi_h^1 - \phi_h^\Delta) + (\phi_h^2 - \phi_h^*). \quad (50)$$

*Proof.* From (13), (46) and (47), there holds

$$\phi_h^{\Delta, *}(\mathbf{x}) = \phi_h^\Delta(\mathbf{x}) + \phi_h^*(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega. \quad (51)$$

Moreover, the following equality may also be easily checked by simple interpolation (see Fig. 5)

$$\phi_h(\mathbf{x}_\sigma) = \begin{cases} \frac{d_{K\sigma}\phi_L + d_{L\sigma}\phi_K}{2d_\sigma} + \frac{\phi_{L^*} + \phi_{K^*}}{4} & \text{if } \sigma = K|L \in \mathcal{E}_{int}, \sigma^* = K^*|L^* \\ \frac{\phi_\sigma}{2} + \frac{\phi_{L^*} + \phi_{K^*}}{4} & \text{if } \sigma \in \mathcal{E}_{ext}, \sigma^* = K^*|L^* \end{cases}$$

so that, since  $\phi_h$  is a  $P^1$  function in  $\mathcal{V}_{\sigma, \sigma^*}$ ,

$$\phi_h(\mathbf{x})|_{\mathcal{V}_{\sigma, \sigma^*}} = \begin{cases} \frac{d_{K\sigma}\phi_L + d_{L\sigma}\phi_K}{2d_\sigma} + \frac{\phi_{L^*} + \phi_{K^*}}{4} + (\mathbf{x} - \mathbf{x}_\sigma) \cdot \nabla_h \phi_h, \forall \mathbf{x} \in \mathcal{V}_{\sigma, \sigma^*} & \text{if } \sigma = K|L \in \mathcal{E}_{int} \\ \frac{\phi_\sigma}{2} + \frac{\phi_{L^*} + \phi_{K^*}}{4} + (\mathbf{x} - \mathbf{x}_\sigma) \cdot \nabla_h \phi_h, \forall \mathbf{x} \in \mathcal{V}_{\sigma, \sigma^*} & \text{if } \sigma \in \mathcal{E}_{ext} \end{cases},$$

with, in both cases,  $\sigma^* = K^*|L^*$ . Recalling that  $\nabla_h \phi_h$  is given by (16), and with the definitions (48) and (49), there holds

$$\phi_h(\mathbf{x})|_{\mathcal{V}_{\sigma, \sigma^*}} = \phi_h^1(\mathbf{x})|_{\mathcal{V}_{\sigma, \sigma^*}} + \phi_h^2(\mathbf{x})|_{\mathcal{V}_{\sigma, \sigma^*}},$$

which, together with (51), leads to (50).  $\square$



Moreover, from (48), (49) and (16), recalling that we have set  $\mathbf{n}_{KL} = \mathbf{n}_{K\sigma}$  if  $\sigma = K|L$ , there holds

**Lemma 4.11.**

$$(\nabla_h \phi_h^1)|_{\mathcal{V}_{\sigma,\sigma^*}} \cdot \mathbf{n}_{K\sigma} = \begin{cases} \left( \frac{\phi_L - \phi_K}{d_\sigma} \right) = (\nabla_h \phi_h)|_{\mathcal{V}_{\sigma,\sigma^*}} \cdot \mathbf{n}_{K\sigma} & \text{if } \sigma = K|L \in \mathcal{E}_{int} \\ \left( \frac{\phi_\sigma - \phi_K}{d_\sigma} \right) = (\nabla_h \phi_h)|_{\mathcal{V}_{\sigma,\sigma^*}} \cdot \mathbf{n}_{K\sigma} & \text{if } \sigma \in \mathcal{E}_{ext} \end{cases} \quad (52)$$

and

$$(\nabla_h \phi_h^2)|_{\mathcal{V}_{\sigma,\sigma^*}} \cdot \mathbf{n}_{K^*L^*} = \left( \frac{\phi_{L^*} - \phi_{K^*}}{m(\sigma)} \right) = (\nabla_h \phi_h)|_{\mathcal{V}_{\sigma,\sigma^*}} \cdot \mathbf{n}_{K^*L^*}. \quad (53)$$

With these definitions, the remaining term which has to be evaluated in (27) reads

$$\begin{aligned} \int_{\Omega} f(\phi_h - \phi_h^{\Delta,*}) (\mathbf{x}) d\mathbf{x} &= \int_{\Omega} (f - f^\Delta) (\phi_h^1 - \phi_h^\Delta) (\mathbf{x}) d\mathbf{x} \\ &+ \int_{\Omega} f^\Delta (\phi_h^1 - \phi_h^\Delta) (\mathbf{x}) d\mathbf{x} \\ &+ \int_{\Omega} (f - f^{\sigma\sigma^*}) (\phi_h^2 - \phi_h^*) (\mathbf{x}) d\mathbf{x} \\ &+ \int_{\Omega} f^{\sigma\sigma^*} (\phi_h^2 - \phi_h^*) (\mathbf{x}) d\mathbf{x}, \end{aligned} \quad (54)$$

where the following  $L^2$  projections have been used:

$$f^\Delta(\mathbf{x}) = f_K = \frac{1}{m(K)} \int_K f(\mathbf{x}) d\mathbf{x}, \quad \forall \mathbf{x} \in K, \forall K \in \mathcal{T}, \quad (55)$$

$$f^{\sigma\sigma^*}(\mathbf{x}) = f_{\mathcal{V}_{\sigma,\sigma^*}} = \frac{1}{m(\mathcal{V}_{\sigma,\sigma^*})} \int_{\mathcal{V}_{\sigma,\sigma^*}} f(\mathbf{x}) d\mathbf{x}, \quad \forall \mathbf{x} \in \mathcal{V}_{\sigma,\sigma^*}, \forall \mathcal{V}_{\sigma,\sigma^*}.$$

We shall first evaluate the first and third terms in the right-hand side of Eq. (54).

**Lemma 4.12.** *There exists a constant  $C$ , not depending on the mesh, such that*

$$\left| \int_{\Omega} (f - f^\Delta) (\phi_h^1 - \phi_h^\Delta) (\mathbf{x}) d\mathbf{x} + \int_{\Omega} (f - f^{\sigma\sigma^*}) (\phi_h^2 - \phi_h^*) (\mathbf{x}) d\mathbf{x} \right| \leq Ch^2 \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)}. \quad (56)$$

*Proof.* From (48), if  $\sigma \in \mathcal{E}_{int}$ , with  $\sigma = K|L$ , there holds

$$\begin{aligned} \phi_h^1 \left( \frac{\mathbf{x}_K + \mathbf{x}_\sigma}{2} \right) &= \frac{d_{K\sigma} \phi_L + d_{L\sigma} \phi_K}{2d_\sigma} + \left( \frac{\mathbf{x}_K - \mathbf{x}_\sigma}{2} \right) \cdot \left( \frac{\phi_L - \phi_K}{d_\sigma} \right) \mathbf{n}_{KL} \\ &= \frac{d_{K\sigma} \phi_L + d_{L\sigma} \phi_K}{2d_\sigma} - \frac{d_{K\sigma} (\phi_L - \phi_K)}{2d_\sigma} \\ &= \frac{d_{K\sigma} + d_{L\sigma}}{2d_\sigma} \phi_K = \frac{\phi_K}{2} = \phi_h^\Delta|_K, \end{aligned} \quad (57)$$

and the same equality holds if  $\sigma \in \mathcal{E}_{ext}$ , with  $\sigma \subset \partial K \cap \partial\Omega$ . This shows that the function  $\phi_h^\Delta$  interpolates the function  $\phi_h^1$  at  $\frac{\mathbf{x}_K + \mathbf{x}_\sigma}{2} \in \mathcal{V}_{K,\sigma}$ . Thus, since  $\phi_h^1 - \phi_h^\Delta$  is a  $P^1$  function in  $\mathcal{V}_{K,\sigma}$ , there holds, with (52)

$$\left\| \phi_h^1 - \phi_h^\Delta \right\|_{L^2(\mathcal{V}_{K,\sigma})}^2 \leq \text{diam}^2(\mathcal{V}_{K,\sigma}) \|\nabla_h \phi_h^1\|_{L^2(\mathcal{V}_{K,\sigma})}^2 \leq \text{diam}^2(\mathcal{V}_{K,\sigma}) \|\nabla_h \phi_h\|_{L^2(\mathcal{V}_{K,\sigma})}^2.$$

Summing over all  $\mathcal{V}_{K,\sigma}$  for  $\sigma \in \mathcal{E}_K$  and  $K \in \mathcal{T}$ , and since  $\text{diam}(\mathcal{V}_{K,\sigma}) \leq h$ , we obtain

$$\left\| \phi_h^1 - \phi_h^\Delta \right\|_{L^2(\Omega)} \leq h |\phi_h|_{1,h}.$$

In the same way, it may be shown from (49) that  $\phi_h^*$  interpolates  $\phi_h^2$  at  $\frac{\mathbf{x}_{K^*} + \mathbf{x}_\sigma}{2} \in \mathcal{V}_{K^*,\sigma^*}$ , so that

$$\left\| \phi_h^2 - \phi_h^* \right\|_{L^2(\Omega)} \leq h |\phi_h|_{1,h}.$$

On the other hand, since, through Hyp. 4.1,  $f \in H^1(\Omega)$ , and since every  $K$  and every  $\mathcal{V}_{\sigma,\sigma^*}$  are convex, there exists a constant  $C$  that does not depend on  $f$ ,  $K$  or  $\mathcal{V}_{\sigma,\sigma^*}$  such that

$$\left\| f - f^\Delta \right\|_{L^2(K)} \leq C \text{diam}(K) \|\nabla f\|_{L^2(K)}$$

and

$$\left\| f - f^{\sigma\sigma^*} \right\|_{L^2(\mathcal{V}_{\sigma,\sigma^*})} \leq C \text{diam}(\mathcal{V}_{\sigma,\sigma^*}) \|\nabla f\|_{L^2(\mathcal{V}_{\sigma,\sigma^*})}.$$

This leads to

$$\left\| f - f^\Delta \right\|_{L^2(\Omega)} \leq Ch \|\nabla f\|_{L^2(\Omega)}$$

and

$$\left\| f - f^{\sigma\sigma^*} \right\|_{L^2(\Omega)} \leq Ch \|\nabla f\|_{L^2(\Omega)}.$$

We conclude that

$$\left| \int_{\Omega} (f - f^\Delta) (\phi_h^1 - \phi_h^\Delta) (\mathbf{x}) d\mathbf{x} + \int_{\Omega} (f - f^{\sigma\sigma^*}) (\phi_h^2 - \phi_h^*) (\mathbf{x}) d\mathbf{x} \right| \leq Ch^2 \|\nabla f\|_{L^2(\Omega)} |\phi_h|_{1,h}, \quad (58)$$

Moreover, the triangle inequality and (39) lead to

$$|\phi_h|_{1,h} \leq \left\| \hat{\phi} \right\|_{H^1(\Omega)} + \left| \hat{\phi} - \phi_h \right|_{1,h} \leq \left\| \hat{\phi} \right\|_{H^2(\Omega)} + Ch \|g\|_{L^2(\Omega)}, \quad (59)$$

which, injected in (58), and taking (25) into account, lead to (56).  $\square$

We now evaluate the last term in Eq. (54).

**Lemma 4.13.** *There holds*

$$\int_{\Omega} f^{\sigma\sigma^*} (\phi_h^2 - \phi_h^*) (\mathbf{x}) d\mathbf{x} = 0. \quad (60)$$

*Proof.* Since  $f^{\sigma\sigma^*}$  is a constant over each  $\mathcal{V}_{\sigma,\sigma^*}$ , there holds

$$\int_{\Omega} f^{\sigma\sigma^*} (\phi_h^2 - \phi_h^*) (\mathbf{x}) d\mathbf{x} = \sum_{\mathcal{V}_{\sigma,\sigma^*}} f_{\mathcal{V}_{\sigma,\sigma^*}} \int_{\mathcal{V}_{\sigma,\sigma^*}} (\phi_h^2 - \phi_h^*) (\mathbf{x}) d\mathbf{x}. \quad (61)$$

Since  $\mathbf{x}_\sigma$  is the midpoint of  $\sigma$ , and by symmetry of  $\mathcal{V}_{\sigma,\sigma^*}$  with respect to  $[\mathbf{x}_K \mathbf{x}_L]$ , there holds

$$\int_{\mathcal{V}_{\sigma,\sigma^*}} (\mathbf{x} - \mathbf{x}_\sigma) \cdot \mathbf{n}_{K^*L^*} d\mathbf{x} = 0.$$

Thus, from Eq. (49), we infer that

$$\int_{\mathcal{V}_{\sigma,\sigma^*}} \phi_h^2(\mathbf{x}) d\mathbf{x} = m(\mathcal{V}_{\sigma,\sigma^*}) \frac{\phi_{L^*} + \phi_{K^*}}{4}. \quad (62)$$

Moreover, from Eq. (47)

$$\int_{\mathcal{V}_{\sigma,\sigma^*}} \phi_h^*(\mathbf{x}) d\mathbf{x} = \int_{\mathcal{V}_{K^*,\sigma^*}} \frac{\phi_{K^*}}{2} d\mathbf{x} + \int_{\mathcal{V}_{L^*,\sigma^*}} \frac{\phi_{L^*}}{2} d\mathbf{x}.$$

By symmetry of  $\mathcal{V}_{\sigma,\sigma^*}$  with respect to  $[\mathbf{x}_K \mathbf{x}_L]$ , there holds  $m(\mathcal{V}_{K^*,\sigma^*}) = m(\mathcal{V}_{L^*,\sigma^*}) = \frac{1}{2}m(\mathcal{V}_{\sigma,\sigma^*})$ . Thus,

$$\int_{\mathcal{V}_{\sigma,\sigma^*}} \phi_h^*(\mathbf{x}) d\mathbf{x} = m(\mathcal{V}_{\sigma,\sigma^*}) \frac{\phi_{L^*} + \phi_{K^*}}{4}. \quad (63)$$

Thus, (60) follows from (61), (62) and (63).  $\square$

Finally, there remains to evaluate the second term in the right-hand side of (54). The following lemma will be helpful for this

**Lemma 4.14.** *Recall that  $R_K$  is the radius of the circle in which the triangle  $K$  is inscribed. There holds*

$$\begin{aligned} \int_{\Omega} f^{\Delta} \left( \phi_h^1 - \phi_h^{\Delta} \right) (\mathbf{x}) d\mathbf{x} &= \sum_{K \in \mathcal{T}} f_K \frac{R_K^2}{12} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma} \\ &\quad - \frac{1}{48} \sum_{K \in \mathcal{T}} f_K \sum_{\sigma \in \mathcal{E}_K} m^3(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma}. \end{aligned} \quad (64)$$

*Proof.* By definition of  $\phi_h^{\Delta}$ , see (46), and of  $f^{\Delta}$ , see (55), there holds

$$\int_{\Omega} f^{\Delta} \left( \phi_h^1 - \phi_h^{\Delta} \right) (\mathbf{x}) d\mathbf{x} = \sum_{K \in \mathcal{T}} f_K \sum_{\sigma \in \mathcal{E}_K} \int_{\mathcal{V}_{K,\sigma}} \left( \phi_h^1 - \frac{\phi_K}{2} \right) (\mathbf{x}) d\mathbf{x}. \quad (65)$$

Since  $\phi_h^1$  is a  $P^1$  function over the triangle  $\mathcal{V}_{K,\sigma}$ , the following quadrature formula is exact

$$\int_{\mathcal{V}_{K,\sigma}} \phi_h^1(\mathbf{x}) d\mathbf{x} = \frac{m(\mathcal{V}_{K,\sigma})}{3} [\phi_h^1(\mathbf{x}_K) + 2\phi_h^1(\mathbf{x}_{\sigma})]. \quad (66)$$

But we also have

$$\phi_h^1(\mathbf{x}_K) + \phi_h^1(\mathbf{x}_{\sigma}) = 2\phi_h^1 \left( \frac{\mathbf{x}_K + \mathbf{x}_{\sigma}}{2} \right) \quad (67)$$

and

$$\phi_h^1(\mathbf{x}_{\sigma}) = \phi_h^1 \left( \frac{\mathbf{x}_K + \mathbf{x}_{\sigma}}{2} \right) + \frac{\mathbf{x}_{\sigma} - \mathbf{x}_K}{2} \cdot \nabla_h \phi_h^1. \quad (68)$$

Summing (67) and (68), and using (57), the fact that  $\mathbf{x}_{\sigma} - \mathbf{x}_K = d_{K\sigma} \mathbf{n}_{K\sigma}$  and (52), Eq. (66) writes

$$\int_{\mathcal{V}_{K,\sigma}} \phi_h^1(\mathbf{x}) d\mathbf{x} = \frac{m(\mathcal{V}_{K,\sigma})}{3} \left[ 3 \frac{\phi_K}{2} + \frac{d_{K\sigma}}{2} \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma} \right].$$

Since  $m(\mathcal{V}_{K,\sigma}) = \frac{d_{K\sigma} m(\sigma)}{2}$ , we finally have

$$\int_{\mathcal{V}_{K,\sigma}} \left( \phi_h^1 - \frac{\phi_K}{2} \right) (\mathbf{x}) d\mathbf{x} = \frac{d_{K\sigma}^2 m(\sigma)}{12} \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma}.$$

The next step is the calculation of

$$\sum_{\sigma \in \mathcal{E}_K} \int_{\mathcal{V}_{K,\sigma}} \left( \phi_h^1 - \frac{\phi_K}{2} \right) (\mathbf{x}) d\mathbf{x} = \sum_{\sigma \in \mathcal{E}_K} \frac{d_{K\sigma}^2 m(\sigma)}{12} \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma}.$$

This is performed using the fact that  $d_{K\sigma}^2 = R_K^2 - \frac{m(\sigma)^2}{4}$  (see Fig. 1). Thus,

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}_K} \int_{\mathcal{V}_{K,\sigma}} \left( \phi_h^1 - \frac{\phi_K}{2} \right) (\mathbf{x}) d\mathbf{x} &= \frac{R_K^2}{12} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma} \\ &- \frac{1}{48} \sum_{\sigma \in \mathcal{E}_K} m^3(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma}. \end{aligned} \quad (69)$$

Inserting (69) into (65) yields (64).  $\square$

Now, we bound the first term in the right-hand side of (64).

**Lemma 4.15.** *There exists a constant  $C$  such that*

$$\left| \sum_{K \in \mathcal{T}} f_K \frac{R_K^2}{12} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma} \right| \leq Ch^2 \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \quad (70)$$

*Proof.* Recall that  $\phi$  has been chosen so that (40) holds. This implies

$$R_K^2 \sum_{\sigma \in \mathcal{E}_K} m(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma} = -R_K^2 \int_K g(\mathbf{x}) d\mathbf{x}$$

so that, using a continuous and then a discrete Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \sum_{K \in \mathcal{T}} f_K \frac{R_K^2}{12} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma} \right| &\leq \frac{1}{12} \sum_{K \in \mathcal{T}} f_K R_K^2 \sqrt{m(K)} \|g\|_{L^2(K)} \\ &\leq \frac{1}{12} h^2 \left( \sum_{K \in \mathcal{T}} m(K) |f_K|^2 \right)^{1/2} \|g\|_{L^2(\Omega)}, \end{aligned}$$

since  $R_K \leq h$  by definition. Eq. (70) is then obtained since  $\left( \sum_{K \in \mathcal{T}} m(K) |f_K|^2 \right)^{1/2} \leq \|f\|_{L^2(\Omega)}$ .  $\square$

**Lemma 4.16.** *Under Hyp. 1.1 and 4.1, there exists a constant  $C$ , depending only on  $\theta^*$ , such that*

$$\left| \frac{1}{48} \sum_{K \in \mathcal{T}} f_K \sum_{\sigma \in \mathcal{E}_K} m^3(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{K\sigma} \right| \leq Ch^2 \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)}. \quad (71)$$

*Proof.* Recall that for any  $\sigma = K|L$ , there holds  $\nabla_h \phi_h \cdot \mathbf{n}_{KL} = -\nabla_h \phi_h \cdot \mathbf{n}_{LK}$ . Thus,

$$\begin{aligned} \sum_{K \in \mathcal{T}} f_K \sum_{\sigma \in \mathcal{E}_K} m^3(\sigma) \nabla_h \phi_h \cdot \mathbf{n}_{KL} &= \sum_{\sigma \in \mathcal{E}_{int}, \sigma = K|L} m^3(\sigma) (f_K - f_L) \nabla_h \phi_h \cdot \mathbf{n}_{KL} \\ &+ \sum_{\sigma \in \mathcal{E}_{ext}} m^3(\sigma) f_K \nabla_h \phi_h \cdot \mathbf{n}_{KL}. \end{aligned} \quad (72)$$

Since  $m(\mathcal{V}_{\sigma, \sigma^*}) = \frac{m(\sigma) d_\sigma}{2}$ , there holds

$$m^3(\sigma) = \sqrt{2} \sqrt{m(\mathcal{V}_{\sigma, \sigma^*})} m^2(\sigma) \sqrt{\frac{m(\sigma)}{d_\sigma}}.$$

According to the last point in lemma 1.2,  $\frac{m(\sigma)}{d_\sigma} \leq C$ , with  $C$  depending only on  $\theta^*$ . Since  $m(\sigma) \leq h$ , there holds

$$m^3(\sigma) \leq Ch^2 \sqrt{m(\mathcal{V}_{\sigma, \sigma^*})}, \quad (73)$$

so that, using a discrete Cauchy-Schwarz inequality yields

$$\left| \sum_{\sigma \in \mathcal{E}_{int}, \sigma=K|L} m^3(\sigma) (f_K - f_L) \nabla_h \phi_h \cdot \mathbf{n}_{KL} \right| \leq Ch^2 \left[ \sum_{\sigma \in \mathcal{E}_{int}, \sigma=K|L} (f_K - f_L)^2 \right]^{1/2} |\phi_h|_{1,h}.$$

Next, let

$$f_\sigma := \frac{1}{m(\sigma)} \int_\sigma f(\xi) d\xi,$$

there holds

$$(f_K - f_L)^2 \leq 2(f_K - f_\sigma)^2 + 2(f_\sigma - f_L)^2.$$

By definition of  $f_K$ , there holds

$$f_K - f_\sigma = \frac{1}{m(K)} \int_K [f(\mathbf{x}) - f_\sigma] d\mathbf{x}$$

and

$$|f_K - f_\sigma| \leq \frac{1}{m^{1/2}(K)} \|f - f_\sigma\|_{L^2(K)}.$$

Since, by definition of  $f_\sigma$ , the function  $f - f_\sigma$  vanishes over  $\sigma$  which is an edge of the triangle  $K$ , there holds (see, e.g., [3, Lemma 2.2])

$$\|f - f_\sigma\|_{L^2(K)} \leq C \text{diam}(K) \|\nabla f\|_{L^2(K)}. \quad (74)$$

In (74), the constant  $C$  does not depend on  $K$ . Thus,

$$(f_K - f_\sigma)^2 \leq C \frac{\text{diam}^2(K)}{m(K)} \|\nabla f\|_{L^2(K)}^2 \leq C \|\nabla f\|_{L^2(K)}^2,$$

and the constant  $C$  depends only on  $\theta^*$  according to the second point in lemma 1.2. Since  $\sigma$  is also an edge of  $L$ , a similar conclusion holds for  $(f_\sigma - f_L)^2$ , so that

$$\sum_{\sigma \in \mathcal{E}_{int}, \sigma=K|L} (f_K - f_L)^2 \leq C \|\nabla f\|_{L^2(\Omega)}^2,$$

since each  $K$  is counted at most three times in the sum over the edges. This shows that

$$\left| \sum_{\sigma \in \mathcal{E}_{int}, \sigma=K|L} m^3(\sigma) (f_K - f_L) \nabla_h \phi_h \cdot \mathbf{n}_{KL} \right| \leq Ch^2 \|\nabla f\|_{L^2(\Omega)} |\phi_h|_{1,h}.$$

Using (59) and taking (25) into account, this yields

$$\left| \sum_{\sigma \in \mathcal{E}_{int}, \sigma=K|L} m^3(\sigma) (f_K - f_L) \nabla_h \phi_h \cdot \mathbf{n}_{KL} \right| \leq Ch^2 \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)}. \quad (75)$$

Now the last term in (72) may be estimated in the following way

$$\sum_{\sigma \in \mathcal{E}_{ext}} m^3(\sigma) f_K \nabla_h \phi_h \cdot \mathbf{n}_{KL} = \sum_{\sigma \in \mathcal{E}_{ext}} \frac{m^3(\sigma)}{m^{1/2}(K) m^{1/2}(\mathcal{V}_{\sigma, \sigma^*})} m^{1/2}(K) f_K m^{1/2}(\mathcal{V}_{\sigma, \sigma^*}) \nabla_h \phi_h \cdot \mathbf{n}_{KL}. \quad (76)$$

Since for boundary triangles  $\mathcal{V}_{\sigma,\sigma^*} = \mathcal{V}_{K,\sigma} \subset K$ , there holds

$$m^{1/2}(K)m^{1/2}(\mathcal{V}_{\sigma,\sigma^*}) \geq m(\mathcal{V}_{K,\sigma}) = \frac{m(\sigma)d_{K\sigma}}{2},$$

so that

$$\frac{m^3(\sigma)}{m^{1/2}(K)m^{1/2}(\mathcal{V}_{\sigma,\sigma^*})} \leq 2\frac{m^2(\sigma)}{d_{K\sigma}} \leq Ch,$$

with a constant depending only on  $\theta^*$ , thanks to the last point in lemma 1.2. Taking this into account in (76) and applying a discrete Cauchy-Schwarz inequality, there holds

$$\left| \sum_{\sigma \in \mathcal{E}_{ext}} m^3(\sigma) f_K \nabla_h \phi_h \cdot \mathbf{n}_{KL} \right| \leq Ch \|\nabla_h \phi_h\|_{L^2(B_h)} \|f\|_{L^2(B_h)}, \quad (77)$$

since  $f_K$  is the  $L^2$  orthogonal projection of  $f$  over  $K$ . We have denoted by  $B_h$  the strip around  $\Gamma$  which contains all  $K$  such that  $m(\partial K \cap \Gamma) \neq 0$ . Note that this strip has a width of at most  $h$ , so that, according to Ilin's inequality (see, e.g. [13, Formula (2.2)]), and since  $\hat{\phi} \in H^2(\Omega)$ , there holds

$$\|\nabla \hat{\phi}\|_{L^2(B_h)} \leq Ch^{1/2} \|\hat{\phi}\|_{H^2(\Omega)},$$

which implies

$$\begin{aligned} \|\nabla_h \phi_h\|_{L^2(B_h)} &\leq \|\phi_h - \hat{\phi}\|_{1,h} + \|\nabla \hat{\phi}\|_{L^2(B_h)} \\ &\leq Ch \|g\|_{L^2(\Omega)} + Ch^{1/2} \|\hat{\phi}\|_{H^2(\Omega)} \leq Ch^{1/2} \|g\|_{L^2(\Omega)}, \end{aligned} \quad (78)$$

according to (39). Moreover, since by Hyp. 4.1,  $f$  belongs to  $H^1(\Omega)$ , we may apply Ilin's inequality again to obtain

$$\|f\|_{L^2(B_h)} \leq Ch^{1/2} \|f\|_{H^1(\Omega)}. \quad (79)$$

Inserting (78) and (79) into (77), we conclude that there exists a constant  $C$  such that

$$\left| \sum_{\sigma \in \mathcal{E}_{ext}} m^3(\sigma) f_K \nabla_h \phi_h \cdot \mathbf{n}_{KL} \right| \leq Ch^2 \|g\|_{L^2(\Omega)} \|f\|_{H^1(\Omega)}. \quad (80)$$

Gathering (80) and (75) into (72) yields (71).  $\square$

Starting from (23), we may now gather all the intermediary results (27), (42), (43), (45), (54), (56), (60), (64), (70) and (71) to get the main result of this article

**Theorem 4.17.** *Let  $\Omega$  be a two-dimensional convex polygonal domain. Let  $\hat{u}$  be the exact solution of the equation  $-\Delta \hat{u} = f$  in  $\Omega$ , with homogeneous Dirichlet boundary conditions. Let  $u = ((u_K)_{K \in \mathcal{T}}, (u_\sigma)_{\sigma \in \mathcal{E}_{ext}}, (u_{K^*})_{K^* \in \mathcal{T}^*})$  be the solution of the finite volumes schemes (4)–(11), and let  $u_h$  be the function in  $V_{h,0}$  associated to  $u$  through definitions 3.3 and 3.4. Then, under Hyp. 1.1 and 4.1, there exists a constant  $C$  depending only on  $\theta^*$ , such that*

$$\|\hat{u} - u_h\|_{L^2(\Omega)} \leq Ch^2 \|f\|_{H^1(\Omega)}.$$

## 5. CONCLUSION

In a two-dimensional convex polygonal domain, we have proved convergence in the  $L^2$  norm with second-order accuracy of a well chosen function constructed with the help of the solutions of two finite volume schemes for the Laplace equation, one defined on a (primal) triangular mesh and the other defined on the associated Voronoi dual mesh, under the sufficient condition that the right-hand side of the Laplace equation is in  $H^1(\Omega)$ .

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